On the testing-function spaces for distributions associated with the Kontorovich-Lebedev transform

Semyon B. Yakubovich *

June 11, 2005

Abstract

We construct a testing-function space, which is equipped with the topology that is generated by $L_{\nu,p}$ - multinorm of the differential operator

$$A_x = x^2 - x\frac{d}{dx}\left[x\frac{d}{dx}\right],$$

and its k-th iterates A_x^k, where $k = 0, 1, \ldots,$ and $A_x^0\varphi = \varphi$. Comparing with other testing-function spaces we introduce in its dual the Kontorovich-Lebedev transformation for distributions with respect to a complex index. The existence, uniqueness, imbedding and inversion properties are investigated. As an application we find a solution of the Dirichlet problem for a wedge for the harmonic equation in terms of the Kontorovich- Lebedev integral.

Keywords: *Testing-function spaces, distributions, Kontorovich-Lebedev transform, modified Bessel functions, Dirichlet problem for a wedge*
AMS subject classification: 46F12, 44A15, 33C10, 35J25

1 Introduction

Let $\mathbb{R}_+ = (0, +\infty),\ 2 \le p < \infty,\ \nu > 0$ and we consider a class $\mathcal{A}_{\nu,p}$ of complex-valued, smooth functions $\varphi(x)$ on $\mathbb{R}_+$ for which the following quantity

$$\alpha_{k,\nu,p}(\varphi) = \alpha_{0,\nu,p}(A_x^k\varphi) = \left(\int_0^\infty \left|A_x^k\varphi\right|^p x^{\nu p-1}dx\right)^{1/p} \tag{1.1}$$

*Work supported by *Fundação para a Ciência e a Tecnologia* (FCT) through the *Centro de Matemática da Universidade do Porto* (CMUP). Available as a PDF file from http://www.fc.up.pt/cmup.

is finite for each $k \in \mathbb{N}_0$. Here A_x^k, where $k = 0, 1, \ldots,$ is k-th iterate of the differential operator

$$A_x = x^2 - x\frac{d}{dx}\left[x\frac{d}{dx}\right]. \tag{1.2}$$

As it is known operator (1.2) has an eigenfunction $K_s(x),\ s = \mu + i\tau$, which is the modified Bessel function or the Macdonald function [1] of a complex index s and satisfies the property

$$A_x K_s(x) = -s^2 K_s(x). \tag{1.3}$$

It has the following asymptotic behaviour (cf. [1], [8])

$$K_s(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}[1 + O(1/z)], \qquad z \to \infty, \tag{1.4}$$

and near the origin

$$K_s(z) = O\left(z^{-|\mu|}\right),\ z \to 0, \tag{1.5}$$

$$K_0(z) = O(\log z),\ z \to 0. \tag{1.6}$$

The modified Bessel function can be represented for instance, by the integrals [1], [8]

$$K_s(z) = \int_0^\infty e^{-z\cosh u} \cosh su du = \frac{1}{2}\int_{i\delta-\infty}^{i\delta+\infty} e^{-z\cosh u + su} du, \tag{1.7}$$

where $\text{Re}z > 0, \delta \in \left[0, \frac{\pi}{2}\right)$. Hence it is not difficult to show that $K_s(z)$ is an even entire function with respect to s and it is analytic in a right half-plane with respect to z. Moreover by using (1.7) and relation (2.3.16.1) in [6, Vol. I] we obtain the estimate

$$|K_s(z)| \le e^{-\delta|\tau|} \left(\frac{\text{Re}z + \text{Im}z \tan\delta}{\text{Re}z - \text{Im}z \tan\delta}\right)^{\mu/2} K_\mu\left(\sqrt{[\text{Re}z\cos\delta]^2 - [\text{Im}z\sin\delta]^2}\right), \tag{1.8}$$

in the sector $|\arg z| < \frac{\pi}{2} - \delta,\ \delta \in \left[0, \frac{\pi}{2}\right)$. In particular, putting $\delta = 0$ we get the elementary inequality $|K_s(z)| \le K_\mu(\text{Re}z),\ \text{Re}z > 0, s = \mu + i\tau$.

The classical Kontorovich-Lebedev transform [8] is defined usually for a pure imaginary index $i\tau$ by the integral

$$K_{i\tau}[f] = \int_0^\infty K_{i\tau}(x) f(x) dx. \tag{1.9}$$

If $f \in L_{\nu,p}(\mathbb{R}_+),\ \nu < 1$, i.e. (cf. (1.1)) the norm

$$||f||_{\nu,p} = \left(\int_0^\infty |f(x)|^p x^{\nu p - 1} dx\right)^{1/p} < \infty, \tag{1.10}$$

then it is shown in [8, Ch. 2] that (1.9) exists as a Lebesgue integral and $K_{i\tau}[f]$ is bounded from $L_{\nu,p}(\mathbb{R}_+)$ into $L_r(\mathbb{R}_+)$, where $p, r \in [1, \infty)$ has no dependence. Moreover (see [8], [9]) its inversion can be written in terms of the singular integral

$$f(x) = \frac{2}{\pi^2} \lim_{\varepsilon \to 0+} \frac{1}{x} \int_0^\infty \tau \sinh(\pi - \varepsilon)\tau K_{i\tau}(x) K_{i\tau}[f] d\tau, \tag{1.11}$$

where the limit in (1.11) is understood with respect to the norm (1.10) in $L_{\nu,p}(\mathbb{R}_+)$, $0 < \nu < 1$.

In [12] it was first generalized the transformation (1.9) for distributions with compact support and later on [2], [3], [5] it was studied in larger spaces of generalized functions. In [4], [7] it was given an another approach to introduce the Kontorovich-Lebedev transform for distributions.

Our goal is to prove that the class $\mathcal{A}_{\nu,p}$ is a testing-function space, which will generalize some known testing-function spaces (see [2], [3], [4]) related to the Kontorovich-Lebedev transform. Furthermore, we will show that this space can be used to study the Kontorovich-Lebedev transform of a complex index for distributions from the dual space $\mathcal{A}'_{\nu,p}$ into the space of analytic functions in a vertical strip. Our goal is also to study its existence, uniqueness and inversion properties on a manner to be found in [13]. These results are finally applied to give a solution of the Dirichlet problem for a wedge for the Laplace equation in cylindrical coordinates, which is associated with operator (1.2) and its iterations. We note that such a problem is considered, for instance in [11], where the formal solution is found.

2 Properties of the space $\mathcal{A}_{\nu,p}$ and its dual

We begin to show that the class $\mathcal{A}_{\nu,p}$ is a testing function space, which is associated with the multinorm (1.1). Indeed, it is easily seen that $\mathcal{A}_{\nu,p}$ is a linear space, each $\alpha_{k,\nu}$ is a seminorm, and $\alpha_{0,\nu,p}$ is clearly a norm on $\mathcal{A}_{\nu,p}$. We equip $\mathcal{A}_{\nu,p}$ as usual with the topology that is generated by $\{\alpha_{k,\nu,p}\}_{k=0}^{\infty}$, and this makes $\mathcal{A}_{\nu,p}$ a countably multinormed space. Since

$$\left(\int_0^\infty \left| A_x^k \varphi \right|^p x^{\nu p - 1} dx \right)^{1/p} = \left(\int_0^\infty \left(1 + x^2\right) \left| A_x^k \varphi \right|^p \frac{x^{\nu p - 1}}{1 + x^2} dx \right)^{1/p}$$

$$\leq \sup_{x>0} \left(\left| A_x^k \varphi \right| x^\nu + \left| A_x^k \varphi \right| x^{\nu+2} \right) \left(\int_0^\infty \frac{x^{-1/p}}{1 + x^2} dx \right)^{1/p} \leq C_p \left[\sup_{x>0} \left(x^\nu \left| A_x^k \varphi \right| \right) \right.$$

$$\left. + \sup_{x>0} \left(x^{\nu+2} \left| A_x^k \varphi \right| \right) \right]$$

where $C_p > 0$ is a constant, then it follows that the space $\mathcal{A}_{\nu,p}$ contains functions from spaces like in [2], [3], [12]. Under this formulation $\mathcal{A}_{\nu,p}$ turns to be a testing -function space

[13]; this will be proved below. Furthermore, $\mathcal{A}_{\nu,p}$ is a subspace of the space $L_{\nu,p}(\mathbb{R}_+)$ [8], [9] and convergence in $\mathcal{A}_{\nu,p}$ implies convergence in $L_{\nu,p}(\mathbb{R}_+)$.

Lemma 1. $\mathcal{A}_{\nu,p}$ *is complete and therefore a Frechet space.*

Proof. Let $\{\varphi_m\}_{m=1}^{\infty}$ be a Cauchy sequence in $\mathcal{A}_{\nu,p}$. Then, for each k and some $\nu > 0$ we have that $\{\varphi_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $L_{\nu,p}(\mathbb{R}_+)$. By the completeness of $L_{\nu,p}(\mathbb{R}_+)$ there exists a function $\chi'_k \in L_{\nu,p}(\mathbb{R}_+)$, which is the limit in $L_{\nu,p}(\mathbb{R}_+)$ of $\{A_x^k\varphi_m\}_{m=1}^{\infty}$. We will show that χ'_k is almost everywhere on $\mathbb{R}_+$ equal to $A_x^k\chi_0$, where $\chi_0 \in \mathcal{A}_{\nu,p}$ is independent of k.

Let $x_1 > 0$ be a fixed point and x a variable point in $\mathbb{R}_+$. From (1.2) we have

$$x\frac{d}{dx}\left[x\frac{d}{dx}A_x^k\varphi_m\right] = x^2A_x^k\varphi_m - A_x^{k+1}\varphi_m. \tag{2.1}$$

Hence dividing by x and integrating with respect to x over the interval $[x_1, x]$ we obtain

$$x\frac{d}{dx}A_x^k\varphi_m = \int_{x_1}^{x}\left(xA_x^k\varphi_m - x^{-1}A_x^{k+1}\varphi_m\right)dx + a_m, \tag{2.2}$$

where $a_m = x_1\left[\frac{d}{dx}A_x^k\varphi_m\right]_{x=x_1}$ is a constant.

Meanwhile by using the Hölder and Minkowski inequalities on the interval $[x_1, x]$ we may write

$$\left|\int_{x_1}^{x}\left(xA_x^k(\varphi_m-\varphi_n) - x^{-1}A_x^{k+1}(\varphi_m-\varphi_n)\right)dx\right|$$
$$\le \left(\int_{x_1}^{x} x^{\left(1-\nu+\frac{1}{p}\right)q}dx\right)^{1/q}\left(\int_0^{\infty}\left|A_x^k(\varphi_m-\varphi_n)\right|^p x^{\nu p-1}dx\right)^{1/p}$$
$$+\left(\int_{x_1}^{x} x^{-\nu q-1}dx\right)^{1/q}\left(\int_0^{\infty}\left|A_x^{k+1}(\varphi_m-\varphi_n)\right|^p x^{\nu p-1}dx\right)^{1/p}, \tag{2.3}$$

where $q = \frac{p}{p-1}$. Clearly the first integrals in the products on the right-hand side of (2.3) are bounded smooth functions on every open interval I whose closure is compact in $\mathbb{R}_+$. The second integrals converge to zero as m and n tend to infinity independently. This shows that the left-hand side of (2.3) converges to zero uniformly on every such an interval I. In the same manner returning to (2.2) it is not difficult to get the equality

$$A_x^k\varphi_m = \int_{x_1}^{x}\frac{dx}{x}\int_{x_1}^{x}\left(xA_x^k\varphi_m - x^{-1}A_x^{k+1}\varphi_m\right)dx + a_m\log\frac{x}{x_1} + b_m, \tag{2.4}$$

where $b_m = A_{x_1}^k\varphi_m$ is a constant. Since $A_x^k\varphi_m$ converges in $L_{\nu,p}(I)$ for every I and the iterated integral in (2.3) converges uniformly on I as $m \to \infty$ (cf. in (2.3)), we conclude that the sequence $\{\psi_m\}_{m=1}^{\infty}$, where $\psi_m(x) = a_m\log\frac{x}{x_1} + b_m$ converges in $L_{\nu,p}(I)$. Further, since the measure of the interval I is finite and $p \ge 2$, it follows immediately that $\psi_m(x)$

converges in $L_{\nu,2}(I)$. Now by using the orthogonality properties of the latter Hilbert space it is easily seen that coefficients a_m, b_m tend to the limits, say a and b, correspondingly. Consequently, $\psi_m(x)$ and therefore $A_x^k\varphi_m$ converge uniformly on every I.

We denote by $\chi_k(x)$ the uniform limit of the sequence $\{A_x^k\varphi_m\}_{m=1}^\infty$, which is evidently a continuous function on I. Moreover, this is true for any k. Therefore passing to the limit in (2.4) when $m \to \infty$ we find

$$\chi_k(x) = \int_{x_1}^{x} \frac{dx}{x} \int_{x_1}^{x} \left(x\chi_k(x) - x^{-1}\chi_{k+1}(x)\right) dx + a \log \frac{x}{x_1} + b. \tag{2.5}$$

Hence we obtain that $\chi_k(x)$ is a smooth function and making necessary differentiation in (2.5) we derive that $\chi_{k+1}(x) = A_x\chi_k$. Thus $\chi_k(x) = A_x^k\chi_0$.

Meantime, $\chi_k'(x) = \chi_k(x)$ almost everywhere on I, since $\chi_k(x)$ is the uniform limit on every I of the sequence $\{A_x^k\varphi_m\}_{m=1}^\infty$ and $\chi_k'(x)$ is the limit in $L_{\nu,p}(\mathbb{R}_+)$ of $\{A_x^k\varphi_m\}_{m=1}^\infty$. Thus both $A_x^k\chi_0$ and $\chi_k'(x)$ are in the same equivalence class in $L_{\nu,p}(\mathbb{R}_+)$. It follows from (1.1) that for every k and some ν $\alpha_{k,\nu,p}(\chi_0) = \alpha_{0,\nu,p}(A_x^k\chi_0) < \infty$ and

$$\alpha_{k,\nu,p}(\chi_0 - \varphi_m) = \alpha_{0,\nu,p}(A_x^k\chi_0 - A_x^k\varphi_m) \to 0,$$

as $m \to \infty$. Lemma 1 is proved.

Denoting by $\mathcal{D}(\mathbb{R}_+), \mathcal{E}(\mathbb{R}_+)$ customary spaces of testing functions encountered in distribution theory [13] it is easily seen that $\mathcal{D}(\mathbb{R}_+) \subset \mathcal{A}_{\nu,p} \subset \mathcal{E}(\mathbb{R}_+)$. Since $\mathcal{D}(\mathbb{R}_+)$ is dense in $\mathcal{E}(\mathbb{R}_+)$, we have that $\mathcal{A}_{\nu,p}$ is also dense in $\mathcal{E}(\mathbb{R}_+)$. The following lemma will be used in the sequel.

Lemma 2. *Let $\varphi \in \mathcal{D}(\mathbb{R}_+)$. Then φ can be represented by the Lebedev integral*

$$\varphi(x) = \lim_{\varepsilon \to 0+} \frac{i}{\pi^2} \int_{\mu - i\infty}^{\mu + i\infty} s \sin(\pi - \varepsilon)s \ K_s(x) \int_0^\infty K_s(y)\varphi(y)\frac{dy}{y} ds, \tag{2.6}$$

where the limit is understood as a convergence in $\mathcal{A}_{\nu,p}$ with $0 < \nu < 1$.

Proof. As we have seen above for each s the modified Bessel function $K_s(z)$ is analytic at least in the sector $|\arg z| < \frac{\pi}{2} - \delta$, $\delta \in \left[0, \frac{\pi}{2}\right)$, which contains $\mathbb{R}_+$. Moreover, employing the estimate (1.8) it is not difficult to establish under condition of the lemma the uniform convergence of the outward integral (2.6) on every compact interval $[x_0, X_0] \subset \mathbb{R}_+$. Thus denoting by

$$\varphi_\varepsilon(x) = \frac{i}{\pi^2} \int_{\mu - i\infty}^{\mu + i\infty} s \sin(\pi - \varepsilon)s \ K_s(x) \int_0^\infty K_s(y)\varphi(y)\frac{dy}{y} ds, \tag{2.7}$$

we may repeatedly differentiate under the integral sign to obtain

$$A_x^k\varphi_\varepsilon = \frac{i}{\pi^2} \int_{\mu - i\infty}^{\mu + i\infty} s \sin(\pi - \varepsilon)s \ A_x^k K_s(x) \int_0^\infty K_s(y)\varphi(y)\frac{dy}{y} ds. \tag{2.8}$$

Hence invoking (1.3), we integrate by parts in the inner integral with respect to y, where integrated terms are vanishing since $\varphi \in \mathcal{D}(\mathbb{R}_+)$. Thus we arrive at the equality

$$A_x^k \varphi_\varepsilon = \frac{i}{\pi^2} \int_{\mu-i\infty}^{\mu+i\infty} s \sin(\pi-\varepsilon)s \ K_s(x) \int_0^\infty K_s(y) A_y^k \varphi \frac{dy}{y} ds. \tag{2.8}$$

Further, we change the order of integration in (2.8) by the Fubini theorem and we find

$$A_x^k \varphi_\varepsilon = \int_0^\infty \mathcal{K}(x,y) A_y^k \varphi \frac{dy}{y}. \tag{2.9}$$

where we denote by

$$\mathcal{K}(x,y) = \frac{i}{\pi^2} \int_{\mu-i\infty}^{\mu+i\infty} s \sin(\pi-\varepsilon)s \ K_s(x) K_s(y) ds. \tag{2.10}$$

We calculate integral (2.10) appealing to Cauchy's theorem and shifting the contour of integration to the imaginary axis. This is indeed possible since the integrand is analytic with respect to s in the strip $|\mathrm{Res}| < |\mu|$ and the following integral

$$\int_{\mu \pm iB}^{\pm iB} s \sin(\pi-\varepsilon)s \ K_s(x) K_s(y) ds \to 0, |B| \to \infty,$$

for each $\varepsilon \in (0,\pi), x > 0, y \in \mathrm{supp}\ A_y^k \varphi$ via inequality (1.8). Then using relation (2.16.51.8) in [6, Vol. II] we calculate the kernel (2.10) and we write (2.9) in the form

$$A_x^k \varphi_\varepsilon = \frac{x \sin\varepsilon}{\pi} \int_0^\infty \frac{K_1((x^2+y^2-2xy\cos\varepsilon)^{1/2})}{(x^2+y^2-2xy\cos\varepsilon)^{1/2}} A_y^k \varphi dy. \tag{2.11}$$

To end the proof we appeal to the properties of the singular integral (2.11) (see in [8], [9], [10]), which give the convergence $A_x^k \varphi_\varepsilon$ to $A_x^k \varphi$ with respect to the norm in $L_{\nu,p}(\mathbb{R}_+),\ 0 < \nu < 1, p \geq 1$ when $\varepsilon \to 0+$. Thus we derive

$$\alpha_{k,\nu,p}(\varphi_\varepsilon - \varphi) = \alpha_{0,\nu,p}(A_x^k \varphi_\varepsilon - A_x^k \varphi) \to 0,\ \varepsilon \to 0+.$$

Lemma 2 is proved.

As usual we denote by $\mathcal{A}'_{\nu,p}$ the dual of $\mathcal{A}_{\nu,p}$. It's equipped with the weak topology and represents a Hausdorff locally convex space of distributions. From the imbedding above we obtain that $\mathcal{E}'(\mathbb{R}_+) \subset \mathcal{A}'_{\nu,p}$. Since $\mathcal{A}_{\nu,p} \subset L_{\nu,p}(\mathbb{R}_+)$ we imbed the dual space $L_{1-\nu,q}(\mathbb{R}_+),\ q = \frac{p}{p-1}$ into $\mathcal{A}'_{\nu,p}$ as a subspace of regular distributions. They act upon elements φ from $\mathcal{A}_{\nu,p}$ according to

$$\langle f, \varphi \rangle := \int_0^\infty f(x)\varphi(x) dx. \tag{2.12}$$

The continuity of the linear functional (2.12) on $\mathcal{A}_{\nu,p}$ follows from the fact that if $\{\varphi_m\}_{m=1}^{\infty}$ converges in $\mathcal{A}_{\nu,p}$ to zero, then by the Hölder inequality

$$|\langle f, \varphi\rangle| \leq \alpha_{0,1-\nu,q}(f)\alpha_{0,\nu,p}(\varphi_m) \to 0,\ m \to \infty.$$

We note that this imbedding of $L_{1-\nu,q}(\mathbb{R}_+)$ into $\mathcal{A}'_{\nu,p}$ is one-to-one. Indeed, if two members f and g of $L_{1-\nu,q}(\mathbb{R}_+)$ become imbedded at the same element of $\mathcal{A}'_{\nu,p}$, then $\langle f, \varphi\rangle = \langle g, \varphi\rangle$ for every $\varphi \in \mathcal{D}(\mathbb{R}_+)$. But this will imply that $f = g$ almost everywhere on $\mathbb{R}_+$ (cf. in [13]). Finally from the general theory of continuous linear functionals on countably multinormed spaces follows that each element $f \in \mathcal{A}'_{\nu,p}$ there exists a nonnegative integer r and a positive constant C such that

$$|\langle f, \varphi\rangle| \leq C \max_{0\leq k\leq r} \alpha_{k,\nu,p}(\varphi) \tag{2.13}$$

for every $\varphi \in \mathcal{A}_{\nu,p}$. Here r, C depends on f but not on φ.

3 The Kontorovich-Lebedev Transformation

We introduce the Kontorovich-Lebedev transformation on distributions $f \in \mathcal{A}'_{\nu,p}$ in a similar way as in [12]. Namely, it is defined by

$$\mathcal{KL}[f](s) := \langle f, K_s(\cdot)\rangle,\ s \in \mathbb{C}. \tag{3.1}$$

It is easily seen from (1.4), (1.5), (1.8), (1.10) that $K_s(x) \in L_{\nu,p}(\mathbb{R}_+)$ when $|\text{Re}s| < \nu$. Moreover it belongs to $\mathcal{A}_{\nu,p}$ under the same condition since via (1.3) we have $|A_x^k K_s(x)| = |s|^{2k}|K_s(x)|$. Hence for regular distributions $f \in L_{1-\nu,q}(\mathbb{R}_+)$ the Kontorovich -Lebedev transformation $K_s[f]$ can be written in the form (2.12), which coincides with (1.9) when $s = i\tau$ is a pure imaginary index. In this case we immediately obtain that $K_s[f]$ represents an analytic function in the open vertical strip $|\text{Re}s| < \nu$ (cf. in [8, Theorem 2.5]).

As in the classical case, the Kontorovich-Lebedev transformation (3.1) is an analytic function in the strip of definition. More precisely, we have

Theorem 1. *For each $f \in \mathcal{A}'_{\nu,p}$ $\mathcal{KL}[f](s)$ is analytic on the strip $\Omega_\nu := \{s = \text{Re}s + i\tau, |\text{Re}s| < \nu\}$ and its derivative*

$$F'(s) := \frac{d}{ds}\mathcal{KL}[f](s) = \left\langle f, \frac{\partial}{\partial s}K_s(\cdot)\right\rangle,\ s \in \Omega_\nu. \tag{3.2}$$

Furthermore, the following estimate is true

$$|\mathcal{KL}[f](s)| \leq C_{f,\delta,p,\nu} max\{1, |s|^{2r}\}e^{-(\pi/2-\delta)|\tau|},\ s \in \Omega_\nu, \tag{3.3}$$

where $\delta \in \left(0, \frac{\pi}{2}\right], r \in \mathbb{N}$ and $C_{f,\delta,p,\nu} > 0$ is a constant.

Proof. Let s be an arbitrary fixed point in Ω_ν. We choose $0<\nu_0<\nu$ such that $s, s+\Delta s\in\Omega_{\nu_0}$, where Δs is a complex increment such that $|\Delta s|<r_0$. We show that $\mathcal{KL}[f](s)$ admits a derivative in each inner strip Ω_{ν_0}. In view of our freedom to choose ν_0 arbitrarily close to ν we will establish the analyticity of $\mathcal{KL}[f](s)$ on Ω_{ν_0}.

Since the modified Bessel function $K_s(x)$ is an entire function of s then with $\Delta s\neq 0$ we invoke the definition (3.1) of $\mathcal{KL}[f](s)$ to write

$$\frac{\mathcal{KL}[f](s+\Delta s)-\mathcal{KL}[f](s)}{\Delta s}-\left\langle f,\frac{\partial}{\partial s}K_s(\cdot)\right\rangle=\left\langle f,\Psi_{\Delta s}(\cdot)\right\rangle, \tag{3.4}$$

where

$$\Psi_{\Delta s}(x)=\frac{1}{\Delta s}\left[K_{s+\Delta s}(x)-K_s(x)\right]-\frac{\partial}{\partial s}K_s(x).$$

We will show that $\Psi_{\Delta s}(x)\in\mathcal{A}_{\nu,p}$ so that (3.4) has a sense. Moreover, we will prove that as $|\Delta s|\to 0$ $\Psi_{\Delta s}(x)$ converges in $\mathcal{A}_{\nu,p}$ to zero. Because $f\in\mathcal{A}'_{\nu,p}$ this will imply the right-hand side of (3.4) tends to zero. Therefore in view of (3.4) we will get (3.2).

To do this we find a circle C with center at s and radius r_1 where $0<r_0<r_1<\min(\nu_0+\mathrm{Re}s,\nu_0-\mathrm{Re}s)$. Hence we may interchange differentiation on s with differentiation on x and invoke Cauchy's integral formulas. Taking into account (1.3) as a result we obtain

$$(-1)^kA_x^k\Psi_{\Delta s}=\frac{(-1)^k}{\Delta s}\left[A_x^kK_{s+\Delta s}(x)-A_x^kK_s(x)\right]-(-1)^k\frac{\partial}{\partial s}A_x^kK_s(x)$$

$$=\frac{1}{\Delta s}\left[(s+\Delta s)^{2k}K_{s+\Delta s}(x)-s^{2k}K_s(x)\right]-\frac{\partial}{\partial s}\left[s^{2k}K_s(x)\right]$$

$$=\frac{1}{2\pi i\Delta s}\int_C\left(\frac{1}{t-s-\Delta s}-\frac{1}{t-s}\right)t^{2k}K_t(x)dt-\frac{1}{2\pi i}\int_C\frac{t^{2k}K_t(x)}{(t-s)^2}dt$$

$$=\frac{\Delta s}{2\pi i}\int_C\frac{t^{2k}K_t(x)}{(t-s-\Delta s)(t-s)^2}dt$$

Hence via (1.1), (1.8) with the generalized Minkowski inequality and since $|t-s-\Delta s|>r_1-r_0>0, |t-s|=r_1,\ |t|<|s|+r_1$ we deduce

$$\alpha_{k,\nu,p}\left(\Psi_{\Delta s}\right)\le\frac{|\Delta s|}{2\pi}\frac{(|s|+r_1)^{2k}}{(r_1-r_0)r_1^2}\int_C\left(\int_0^\infty K^p_{\mathrm{Re}t}(x)x^{\nu p-1}dx\right)^{1/p}|dt|$$

$$\le|\Delta s|\frac{(|s|+r_1)^{2k}}{(r_1-r_0)r_1}\left(\int_0^\infty K^p_{\nu_0}(x)x^{\nu p-1}dx\right)^{1/p}=B_{s,\nu,p,k}|\Delta s|\to 0,\ |\Delta s|\to 0,$$

where $B_{s,\nu,p,k}>0$ is a constant. Thus $\Psi_{\Delta s}(x)$ converges in $\mathcal{A}_{\nu,p}$ to zero.

In order to prove (3.3) we recall inequalities (1.8), (2.13). Then via (3.2) we write

$$|\mathcal{KL}[f](s)| \le C\max_{0\le k\le r}\alpha_{k,\nu,p}(K_s(x)) \le Ce^{-(\pi/2-\delta)|\tau|}\left(\int_0^\infty K^p_{\mathrm{Re}s}(x\sin\delta)x^{\nu p-1}dx\right)^{1/p}$$

$$\times\max_{0\le k\le r}|s|^{2k} \le C_{f,\delta,p,\nu}\max\{1,|s|^{2r}\}e^{-(\pi/2-\delta)|\tau|},\ s\in\Omega_\nu.$$

Theorem 1 is proved.

We are ready to prove now an inversion theorem for the transformation (3.1). Indeed, we have

Theorem 2. *Let* $f\in\mathcal{A}'_{\nu,p}$ *with* $0<\nu<1$. *Then*

$$f(x)=\lim_{\varepsilon\to 0+}\frac{i}{x\pi^2}\int_{\mu-i\infty}^{\mu+i\infty} s\sin(\pi-\varepsilon)sK_s(x)\mathcal{KL}[f](s)ds,\ |\mu|<\nu, \tag{3.5}$$

where the convergence is understood in $\mathcal{D}'(\mathbb{R}_+)$.

Proof. We observe that formula (3.5) means the following equality

$$\langle f,\varphi\rangle=\lim_{\varepsilon\to 0+}\left\langle\frac{i}{\cdot\pi^2}\int_{\mu-i\infty}^{\mu+i\infty} s\sin(\pi-\varepsilon)sK_s(\cdot)\mathcal{KL}[f](s)ds,\varphi\right\rangle \tag{3.6}$$

for every $\varphi\in\mathcal{D}(\mathbb{R}_+)$ having a support, let say, in the closed interval $[a,b]\subset\mathbb{R}_+$. By using our discussions above it is easily seen that the integral with respect to s in (3.6) is absolutely convergent for each $\varepsilon>0$ and can be treated as a Riemann improper integral. Furthermore with inequality (1.8) we show that the expression under the limit sign is a regular distribution. Therefore it is equal to

$$\frac{i}{\pi^2}\int_a^b y^{-1}\varphi(y)\int_{\mu-i\infty}^{\mu+i\infty} s\sin(\pi-\varepsilon)sK_s(y)\mathcal{KL}[f](s)dsdy. \tag{3.7}$$

Appealing to the Fubini theorem we change the order of integration in (3.7) and we write it in the form

$$\frac{i}{\pi^2}\int_{\mu-i\infty}^{\mu+i\infty} s\sin(\pi-\varepsilon)s\mathcal{KL}[f](s)\int_a^b y^{-1}\varphi(y)K_s(y)dyds$$

$$=\frac{i}{\pi^2}\lim_{T\to\infty}\int_{\mu-iT}^{\mu+iT} s\sin(\pi-\varepsilon)s\mathcal{KL}[f](s)\int_a^b y^{-1}\varphi(y)K_s(y)dyds. \tag{3.8}$$

Invoking (3.1) and the Riemann sums technique (cf. in [7], [12], [13]) it is not difficult to prove that

$$\frac{i}{\pi^2}\int_{\mu-iT}^{\mu+iT} s\sin(\pi-\varepsilon)s\mathcal{KL}[f](s)\int_a^b y^{-1}\varphi(y)K_s(y)dyds=\langle f,\Theta_{T,\varepsilon}\rangle,$$

where

$$\Theta_{T,\varepsilon}(x)=\frac{i}{\pi^2}\int_{\mu-iT}^{\mu+iT} s\sin(\pi-\varepsilon)sK_s(x)\int_a^b y^{-1}\varphi(y)K_s(y)dyds$$

is an element of $\mathcal{A}_{\nu,p}$. Meanwhile, we will show that $\Theta_{T,\varepsilon}(x)\to\varphi_\varepsilon(x)$ in $\mathcal{A}_{\nu,p}$ as $T\to\infty$, where $\varphi_\varepsilon(x)$ is defined by (2.7). Indeed, choosing $0<\delta<\frac{\varepsilon}{2}$ we employ (1.3), (1.8) and the generalized Minkowski inequality. Hence we have

$$\alpha_{k,\nu,p}\left(\Theta_{T,\varepsilon}-\varphi_\varepsilon\right)=\frac{1}{\pi^2}\left(\int_0^\infty x^{\nu p-1}dx\left|\int_{|\mathrm{Im}s|\geq T} s\sin(\pi-\varepsilon)sA_x^kK_s(x)\right.\right.$$

$$\left.\left.\times\int_a^b y^{-1}\varphi(y)K_s(y)dyds\right|^p\right)^{1/p}\leq\frac{1}{\pi^2}\left(\int_0^\infty K_\mu(x\sin\delta)x^{\nu p-1}dx\right)^{1/p}$$

$$\times\int_a^b y^{-1}|\varphi(y)|K_\mu(y\sin\delta)dy\int_{|\mathrm{Im}s|\geq T}|s|^{2k+1}|\sin(\pi-\varepsilon)s|e^{(-\pi+2\delta)|\mathrm{Im}s|}|ds|$$

$$\leq C_{\delta,\nu,p}\int_{|\mathrm{Im}s|\geq T}|s|^{2k+1}e^{(2\delta-\varepsilon)|\mathrm{Im}s|}|ds|\to 0, T\to\infty,$$

where $C_{\delta,\nu,p}>0$ is a constant. Thus combining with (3.8) we arrived at the equality

$$\frac{i}{\pi^2}\int_{\mu-i\infty}^{\mu+i\infty} s\sin(\pi-\varepsilon)s\mathcal{KL}[f](s)\int_a^b y^{-1}\varphi(y)K_s(y)dyds=\langle f,\varphi_\varepsilon\rangle. \tag{3.9}$$

To end the proof of the theorem we pass to the limit through (3.9) when $\varepsilon\to 0+$. Hence by using Lemma 2 we get (3.6) and we establish the inversion formula (3.5). Theorem 2 is proved.

By using Theorem 2 we can readily prove the uniqueness property for the Kontorovich-Lebedev transformation (3.1).

Corollary 1. *If* $\mathcal{KL}[f](s)=F(s)$ *and* $\mathcal{KL}[g](s)=G(s)$, $s\in\Omega_\nu$, $0<\nu<1$ *and if* $F(s)=G(s), s\in\Omega_\nu$ *then* $f=g$ *in the sense of equality in* $\mathcal{D}'(\mathbb{R}_+)$.

Proof. Under conditions of the corollary f and g must assign the same value for each $\varphi\in\mathcal{D}(\mathbb{R}_+)$. Thus by invoking Theorem 2 and equating f and g in (3.5) we immediately obtain $\langle f,\varphi\rangle=\langle g,\varphi\rangle$, which proves Corollary 1.

4 Dirichlet's problem for a wedge

As an application let us consider Dirichlet's problem for a wedge (r,θ) with the origin at the apex and the sides of the wedge along the radial lines $\theta=0$ and $\theta=\alpha$ $(0<\alpha\leq\pi)$.

The problem for the interior of this wedge is to find a function $u_k(r,\theta)$ that satisfies the following harmonic equation

$$A_r^* u = \frac{\partial^2 u}{\partial \theta^2},\ 0 < r < \infty,\ 0 < \theta < \alpha, \tag{4.1}$$

where A_r^* is the adjoint operator to A_r

$$A_r^* = r^2 - 1 - 3r\frac{\partial}{\partial r} - r^2\frac{\partial^2}{\partial r^2}. \tag{4.2}$$

We assume that $u(r,\theta)$ is twice differentiable with respect to θ in a sense of a conventional derivative (cf. [13, Section 2.6]).

We impose the following boundary conditions:

1. As $\theta \to 0+$, $u(r,\theta) \to f(r)$ in $\mathcal{A}'_{\nu,p},\ 0<\nu<1,\ p\geq 2$ for any $\varphi \in \mathcal{A}_{\nu,p}\cap\mathcal{A}_{\nu+1,p}$.
2. As $\theta \to \alpha-$, $u(r,\theta)$ converges to zero in $\mathcal{A}'_{\nu,p}$ for any $\varphi \in \mathcal{A}_{\nu,p}\cap\mathcal{A}_{\nu+1,p}$.

This problem can be solved through an operational technique by the Kontorovich-Lebedev transformation (3.1). Indeed, applying (3.1) to both sides of the equation (4.1) and appealing to definitions of the adjoint operator and a conventional derivative we arrive at the equality

$$\langle u(\cdot,\theta), A.K_s(\cdot)\rangle - \frac{\partial^2}{\partial\theta^2}\langle u(\cdot,\theta), K_s(\cdot)\rangle = 0. \tag{4.3}$$

Hence via (1.3) we obtain

$$s^2\mathcal{KL}[u(\cdot,\theta)](s) + \frac{\partial^2}{\partial\theta^2}\mathcal{KL}[u(\cdot,\theta)](s) = 0. \tag{4.4}$$

Solving this differential equation find

$$\mathcal{KL}[u(\cdot,\theta)](s) = A(s)e^{is\theta} + B(s)e^{-is\theta}, \tag{4.5}$$

where the unknown functions $A(s), B(s)$ do not depend on θ. To determine A and B we first transform the boundary conditions. Indeed, since $K_s(r) \in \mathcal{A}_{\nu,p}\cap\mathcal{A}_{\nu+1,p}$ when $|\mathrm{Re} s| < \nu$, then invoking (4.5), (3.1) we have

$$\lim_{\theta\to 0+}\mathcal{KL}[u(\cdot,\theta)](s) = \mathcal{KL}[f](s) = A(s) + B(s), \tag{4.6}$$

$$\lim_{\theta\to \alpha-}\mathcal{KL}[u(\cdot,\theta)](s) = 0 = A(s)e^{is\alpha} + B(s)e^{-is\alpha}. \tag{4.7}$$

Combining with (4.5), (4.6), (4.7) and making elementary calculations we derive

$$\mathcal{KL}[u(\cdot,\theta)](s) = \frac{\sin\left(s(\alpha-\theta)\right)}{\sin\alpha s}\mathcal{KL}[f](s). \tag{4.8}$$

Consequently, invoking Theorem 2 we obtain as our possible solution

$$u(r,\theta)=\lim_{\varepsilon\to 0+}\frac{i}{r\pi^2}\int_{\mu-i\infty}^{\mu+i\infty} s\sin(\pi-\varepsilon)s\ K_s(r)\frac{\sin\left(s(\alpha-\theta)\right)}{\sin\alpha s}\mathcal{KL}[f](s)ds, \tag{4.9}$$

where $|\mu|<\nu<1$. It is easily seen that the integrand in (4.9) is analytic on the strip Ω_ν (cf. (3.2)). Moreover, appealing to (1.8), (3.3) we get the uniform estimate on $\varepsilon\in[0,\pi]$ and $-\nu<\mu<\nu$

$$\left|s\sin(\pi-\varepsilon)s\ K_s(r)\frac{\sin\left(s(\alpha-\theta)\right)}{\sin\alpha s}\mathcal{KL}[f](s)\right|\le \text{const.}|\tau|^{2a+1}\exp\left((2\delta-\theta)|\tau|\right)$$

$$\times K_\mu(r_0\sin\delta), \tag{4.10}$$

where $\tau=\text{Im}s,\ a>0,\ 0<2\delta<\theta\le\alpha,\ 0<r_0<r<\infty$. Therefore one can pass to the limit under the integral sign in (4.9) when $\varepsilon\to 0+$. Hence we write our solution in the form

$$u(r,\theta)=\frac{i}{r\pi^2}\int_{\mu-i\infty}^{\mu+i\infty} s\sin\pi s\ K_s(r)\frac{\sin\left(s(\alpha-\theta)\right)}{\sin\alpha s}\mathcal{KL}[f](s)ds. \tag{4.11}$$

Our goal now is to prove that (4.11) is indeed a solution, which satisfies the differential equation (4.1) and the corresponding boundary conditions. In order to verify that (4.11) is a solution of (4.1) we use (4.10) and the fact that the integrand in (4.11) is analytic on the strip Ω_ν. Consequently, the differentiations may be interchanged with the integration. Moreover, by straightforward calculations we see that the function

$$K_s(r)\frac{\sin\left(s(\alpha-\theta)\right)}{r\sin\alpha s}$$

satisfies (4.1). Thus $u(r,\theta)$ is a solution of (4.1).

We turn now to the boundary conditions. First we show that $ru(r,\theta)\in\mathcal{A}_{\nu,p}\cap L_{1-\nu,q}(\mathbb{R}_+)$ for any $0<\theta\le\alpha,\ q=p/(p-1),\ 0<\nu<1$. Indeed, from the uniform convergence of the integral (4.11) with respect to $r\in\mathbb{R}_+$ we see that $ru(r,\theta)$ is a smooth function. Moreover, invoking (1.1), (1.8), (1.10), (3.3) and the generalized Minkowski inequality we obtain the estimate

$$||\cdot u(\cdot,\theta)||_{\xi,\omega}\le\text{const.}\int_{-\infty}^{\infty}|\tau|^{2a+1}\exp\left((2\delta-\theta)|\tau|\right)d\tau$$

$$\times\left(\int_0^\infty K_\mu^\omega(r\sin\delta)r^{\xi\omega-1}dr\right)^{1/\omega}<\infty,\ a>0,\ \omega>1,|\mu|<\xi,\ \xi>0, \tag{4.12}$$

for any θ, such that $0<2\delta<\theta\le\alpha$, where $\delta>0$ can be chosen as a sufficiently small number. Thus, in particular $ru(r,\theta)\in\mathcal{A}_{\nu,p}\cap L_{1-\nu,q}(\mathbb{R}_+)$ with $0<\theta\le\alpha,\ q=$

$p/(p-1),\ 0 < \nu < 1$. Furthermore for any $\varphi \in \mathcal{A}_{\nu,p} \cap \mathcal{A}_{\nu+1,p}$ via (2.12) we write

$$\langle \cdot u(\cdot,\theta), \varphi \rangle = \langle u(\cdot,\theta), \cdot\varphi \rangle = \int_0^\infty r u(r,\theta)\varphi(r) dr = \frac{i}{\pi^2} \int_0^\infty \varphi(r) \int_{\mu-i\infty}^{\mu+i\infty} s \sin \pi s \ K_s(r)$$

$$\times \frac{\sin (s(\alpha-\theta))}{\sin \alpha s} \mathcal{KL}[f](s) ds \ dr = \frac{i}{\pi^2} \int_{\mu-i\infty}^{\mu+i\infty} s \sin \pi s \int_0^\infty \varphi(r) K_s(r) dr$$

$$\times \frac{\sin (s(\alpha-\theta))}{\sin \alpha s} \mathcal{KL}[f](s) ds, \tag{4.13}$$

where the change of the order of integration in (4.13) is due to Fubini's theorem, and this fact can be easily motivated by (4.10), (4.12) with the Hölder inequality. Precisely we appeal to the estimate

$$\int_{\mu-i\infty}^{\mu+i\infty} |s \sin \pi s| \int_0^\infty |\varphi(r) K_s(r)| \left| \frac{\sin (s(\alpha-\theta))}{\sin \alpha s} \mathcal{KL}[f](s) ds \right| dr$$

$$\leq \alpha_{0,\nu,p}(\varphi) \int_{-\infty}^{\infty} |\tau|^{2a+1} \exp\left((2\delta-\theta)|\tau|\right) d\tau \left(\int_0^\infty K_\mu^q (r \sin \delta) r^{(1-\nu)q-1} dr \right)^{1/q} < \infty,$$

where $q = \frac{p}{p-1},\ a > 0,\ 0 < \delta < \theta/2$ and $|\mu| < \min(1-\nu, \nu)$. Hence in a similar manner it is not difficult to see that the latter iterated integral in (4.13) converges uniformly with respect to θ on every interval $\beta \leq \theta \leq \alpha$, where $\beta > 2\delta > 0$. Therefore we may take the limit under the integral sign in (4.13) as $\theta \to \alpha-$ to get

$$\lim_{\theta \to \alpha-} \langle u(\cdot,\theta), \cdot\varphi \rangle = 0.$$

Thus the second boundary condition is verified.

In order to verify the first boundary condition it is sufficient to show that for any $\varphi \in \mathcal{A}_{\nu,p} \cap \mathcal{A}_{\nu+1,p}$

$$\lim_{\theta \to 0+} \langle u(\cdot,\theta), \cdot\varphi \rangle = \langle f, \cdot\varphi \rangle.$$

Since the integrand in (4.11) is analytic on Ω_ν, we can put $\mu = 0$ shifting the contour of integration to the imaginary axis by using Cauchy's theorem and its asymptotic behavior at infinity. Hence after elementary substitutions we find

$$u(r,\theta) = \frac{1}{r\pi^2} \int_{-\infty}^{\infty} \tau e^{\pi\tau} \ K_{i\tau}(r) \frac{\sinh (\tau(\alpha-\theta))}{\sinh \alpha\tau} \mathcal{KL}[f](i\tau) d\tau. \tag{4.14}$$

Invoking the following relation

$$\frac{\sinh (\tau(\alpha-\theta))}{\sinh \alpha\tau} = e^{-\tau\theta} - e^{-\tau\alpha} \frac{\sinh \theta\tau}{\sinh \alpha\tau}$$

we obtain

$$r\ u(r,\theta)=\frac{1}{\pi^2}\int_{-\infty}^{\infty}\tau e^{(\pi-\theta)\tau}\ K_{i\tau}(r)\mathcal{KL}[f](i\tau)d\tau-\frac{1}{\pi^2}\int_{-\infty}^{\infty}\tau e^{(\pi-\alpha)\tau}$$

$$\times\ K_{i\tau}(r)\frac{\sinh\theta\tau}{\sinh\alpha\tau}\mathcal{KL}[f](i\tau)d\tau.$$

Thus

$$\langle u(\cdot,\theta),\cdot\varphi\rangle=\int_0^{\infty}ru(r,\theta)\varphi(r)dr=\frac{1}{\pi^2}\int_{-\infty}^{\infty}\tau e^{(\pi-\theta)\tau}\ \mathcal{KL}[f](i\tau)\int_0^{\infty}\varphi(r)K_{i\tau}(r)drd\tau$$

$$-\frac{1}{\pi^2}\int_{-\infty}^{\infty}\tau e^{(\pi-\alpha)\tau}\frac{\sinh\theta\tau}{\sinh\alpha\tau}\mathcal{KL}[f](i\tau)\int_0^{\infty}\varphi(r)K_{i\tau}(r)drd\tau. \tag{4.15}$$

In the same manner as in the proof of Theorem 2 we substitute in (4.15) the value of $\mathcal{KL}[f](i\tau)$ by formula (3.1) and we use the Riemann sums and their limits to treat integrals with respect to τ. Finally we arrive at the following relations

$$\frac{1}{\pi^2}\int_{-\infty}^{\infty}\tau e^{(\pi-\theta)\tau}\ \mathcal{KL}[f](i\tau)\int_0^{\infty}\varphi(r)K_{i\tau}(r)drd\tau$$

$$=\left\langle f,\frac{1}{\pi^2}\int_{-\infty}^{\infty}\tau e^{(\pi-\theta)\tau}\ K_{i\tau}(\cdot)\int_0^{\infty}\varphi(r)K_{i\tau}(r)drd\tau\right\rangle, \tag{4.16}$$

$$\frac{1}{\pi^2}\int_{-\infty}^{\infty}\tau e^{(\pi-\alpha)\tau}\frac{\sinh\theta\tau}{\sinh\alpha\tau}\mathcal{KL}[f](i\tau)\int_0^{\infty}\varphi(r)K_{i\tau}(r)drd\tau$$

$$=\left\langle f,\frac{1}{\pi^2}\int_{-\infty}^{\infty}\tau e^{(\pi-\alpha)\tau}\frac{\sinh\theta\tau}{\sinh\alpha\tau}\ K_{i\tau}(\cdot)\int_0^{\infty}\varphi(r)K_{i\tau}(r)drd\tau\right\rangle. \tag{4.17}$$

Denoting by

$$\varphi_1(y,\theta)=\frac{1}{\pi^2}\int_{-\infty}^{\infty}\tau e^{(\pi-\theta)\tau}\ K_{i\tau}(y)\int_0^{\infty}\varphi(r)K_{i\tau}(r)drd\tau,$$

we invert the order of integration by Fubini's theorem and we calculate the integral with respect to τ invoking the value of the kernel (2.10). As a result we deduce

$$\varphi_1(y,\theta)=\frac{y\sin\theta}{\pi}\int_0^{\infty}\frac{K_1((r^2+y^2-2ry\cos\theta)^{1/2})}{(r^2+y^2-2xy\cos\theta)^{1/2}}r\varphi(r)dr.$$

It is not difficult to see that $\varphi_1(y,\theta)$ is an element of $\mathcal{A}_{\nu,p}$ for each $\theta\in(0,\alpha]$ and it converges to $y\varphi(y)$ in $\mathcal{A}_{\nu,p}$ when $\theta\to0+$ (cf. (2.11)). Therefore from (4.16) we have

$$\lim_{\theta\to0+}\langle f,\ \varphi_1(\cdot,\theta)\rangle=\langle f,\ \cdot\varphi_1\rangle$$

and to satisfy the first boundary condition we have to show that

$$\lim_{\theta\to 0+} \langle f,\ \varphi_2(\cdot,\theta)\rangle = 0, \tag{4.18}$$

where

$$\varphi_2(y,\theta) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} \tau e^{(\pi-\alpha)\tau}\frac{\sinh\theta\tau}{\sinh\alpha\tau}\ K_{i\tau}(y)\int_0^{\infty}\varphi(r)K_{i\tau}(r)drd\tau.$$

Indeed, invoking (4.10) we prove that $\varphi_2(y,\theta)$ is an element of $\mathcal{A}_{\nu,p}$ for each $\theta \in (0,\alpha]$. Moreover, since for each $y > 0$ and $\delta \in \left(0, \frac{\pi}{2}\right] \cap (0,\ \alpha)$ (see (1.8))

$$|\varphi_2(y,\theta)| \leq \text{const.}K_0(y\sin\delta)\int_{-\infty}^{\infty}|\tau|e^{(2\delta-\alpha)\tau}\frac{\sinh\theta\tau}{\sinh\alpha\tau}d\tau\ \int_0^{\infty}|\varphi(r)|K_0(r\sin\delta)dr < \infty,$$

we take into account that $K_0(y\sin\delta) \in \mathcal{A}_{\nu,p}\cap\mathcal{A}_{\nu+1,p}$ and we write for all $\varphi \in \mathcal{A}_{\nu,p}\cap\mathcal{A}_{\nu+1,p}$ the estimate

$$|\langle f,\ \varphi_2(\cdot,\theta)\rangle| \leq \text{const.}\,|\langle f,\ K_0(\cdot\sin\delta)\rangle|\int_{-\infty}^{\infty}|\tau|e^{(2\delta-\alpha)\tau}\frac{\sinh\theta\tau}{\sinh\alpha\tau}d\tau \to 0,\ \theta\to 0+$$

due to the dominated convergence theorem. Thus we we establish (4.18) and the first boundary condition is satisfied.

References

1. A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions*, Vol. II, McGraw-Hill, New York, London and Toronto (1953).

2. G. Buggle, *Die Kontorovich-Lebedev Transformation und die Mehler- Fock Transformation für Klassen verallgemeinerter Functionen*, Dissertation, TH Darmstadt, 1977 (in German).

3. H.J. Glaeske and A.A.Heβ, A convolution connected with the Kontorovich-Lebedev transform, *Math. Z.* **193** (1986), 67-78.

4. B. Lisena, On the generalized Kontorovich-Lebedev transform, *Rend. di Matematica, Ser. VII*, **9** (1989), 87-101.

5. R.S. Pathak, I.N.Pandey, The Kontorovich-Lebedev transformation of distributions, *Math. Z.*, **165** (1979), 29-51.

6. A.P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series, Vol. I: Elementary Functions, Vol. II: Special Functions*, Gordon and Breach, New York (1986).

7. S.B. Yakubovich and B. Fisher, On the theory of the Kontorovich-Lebedev transformation on distributions, *Proc. of the Amer. Math. Soc.*, **122** (1994), N 3, 773-777.

8. S.B. Yakubovich, *Index Transforms*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong (1996).

9. S.B. Yakubovich, On the Kontorovich-Lebedev transformation, *J. of Integral Equations and Appl.* **15** (2003), N 1, 95-112.

10. S.B. Yakubovich, The Kontorovich-Lebedev transformation on the Sobolev type spaces, *Preprint CMUP* (2005), 21 pp.

11. A.I. Zayed, *Function and Generalized Function Transformations*, CRC Press, Boca Raton, FL (1996).

12. A.H. Zemanian, The Kontorovich-Lebedev transformation on distributions of compact support and its inversion, *Math. Proc. Cambridge Philos. Soc.* **77** (1975), 139- 143.

13. A.H. Zemanian, *Generalized Integral Transformations*, Dover Publ., New York (1987).

S.B.Yakubovich
Department of Pure Mathematics,
Faculty of Sciences,
University of Porto,
Campo Alegre st., 687
4169-007 Porto
Portugal
E-Mail: syakubov@fc.up.pt