SIMPLE CONJUGACY INVARIANTS FOR BRAIDS

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Abstract. We define simple conjugacy invariants of braids, which we call turning numbers, and investigate their properties. Since our motivation comes from the investigation of periodic orbits of orientation preserving disk homeomorphisms, turning numbers work best for braids with the cyclic permutation, especially for positive permutation cyclic braids.

1. Introduction

Our main motivation is the study of braid types of periodic orbits of orientation preserving disk homeomorphisms.

Let f and g be orientation preserving disk homeomorphisms with periodic orbits P and Q respectively (for simplicity, we will consider only orbits contained in the interior of the disk). Two such pairs (f, P) and (g, Q) are *equivalent* if f is conjugate (in the dynamical systems meaning) to some $\widetilde{g}$ via a homeomorphism that maps P to Q and $\widetilde{g}$ is isotopic to g relative to Q (that is, via an isotopy that fixes the points of Q). Equivalence classes are called *braid types* ([5]) or *patterns* ([8], [9]).

Thus, basically we are looking at the *mapping classes* of the homeomorphism relative to the periodic orbit. There is a well known connection with braids, explaining the name "braid types." Take the suspension flow of the homeomorphism with a periodic orbit P and then the trajectory of any point $x \in P$ (cut at the level 0) can be identified with a braid. Since we pass from a 3-dimensional picture, where the points of the orbit are in the interior of the disk, to a basically 2-dimensional one, where they are ordered on an interval, the braid corresponding to an orbit is defined only up to an algebraic conjugacy (that is, the conjugacy in the braid group; we will refer to it just as conjugacy). Therefore, in order to study braid types with the tools of the braid theory, we need some effective invariants of braid conjugacies.

There is a *forcing relation* on braid types [5]. A braid type A *forces* braid type B if every orientation preserving disk homeomorphism exhibiting A has to exhibit B. This relation is a partial ordering [5].

According to the *Nielsen-Thurston classification* [11], if f is an orientation preserving disk homeomorphism and P is a periodic orbit, then there is a homeomorphism g, isotopic to f relative to P (so in particular, $g|_P = f|_P$), which is one of the three types:

(a) finite order (we will call it *twist*),
(b) pseudo-Anosov,

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(c) reducible.

When studying forcing, it is important to know what type our orbit is. If it is a twist, then it does not force anything except a fixed point (and itself). If it is reducible, the problem of forcing can be also reduced to the study of forcing by two simpler orbits. Therefore we need to relate invariants of conjugacy of braids to the above classification.

From this point of view one useful invariant is known: the *exponent sum* (or *wraith*) $\mathrm{es}(B)$ of B (we define it later). The new invariants that we will define here, turning numbers, are refinements of the exponent sum.

Because of our motivation, we will concentrate mainly on the braids with cyclic permutations. We will call them *cyclic braids*.

In the sequel, we will assume that the reader knows the basic notions of the braid theory. For an exposition on braids see, for example, [2], [3] or [6].

In Section 2 we define the turning numbers of a braid and we prove the main result of this paper, that turning numbers are invariants of conjugacy. In Section 3 we define extensions of braids by other braids and show how to compute their turning numbers. In Section 4 we relate turning numbers with braids that come from periodic orbits of interval maps. In Section 5 we investigate turning numbers for twist braids and connect our results with those of [10]. In Section 6 we investigate connections between the first and the second turning numbers. In Section 7 we give an example of two relatively simple braids which have the same turning numbers but are not conjugate.

2. Definition and basic properties

Let B be a braid with n strands (an element of the n-th braid group) and permutation τ (that is, the i-th strand joins i in the bottom with $\tau(i)$ at the top; we assume that the strands go up – this is because of the suspension model). For each crossing of two strands we define its *sign* in a standard way: it is ± 1 depending on whether the left strand goes over the right one or vice versa. To fix notation, assume that in the former case it is $+1$, and in the latter case -1. We also agree that a strand does not cross itself. Now for $k = 1, 2, \dots, n-1$ we define $T_k(i)$ as the sum of signs of the crossings between the i-th and $\tau^k(i)$-th strands. Finally, we define the *k-th turning number of B* as

$$\mathrm{TN}_k(B) = \frac{1}{2} \sum_{i=1}^{n} T_k(i).$$

Theorem 2.1. *For each k, the k-th turning number of a braid is well defined.*

Proof. In order to prove it, we have to show that the Artin relations preserve the numbers $T_k(i)$. However, those relations do not change the number or signs of crossings of any given pair of strands, so the collection of the numbers $T_k(i)$ remains the same (although the i's can be permuted). □

Lemma 2.2. *Turning numbers are integers.*

Proof. If $\tau^k(i) - i$ and $\tau^{k+1}(i) - \tau(i)$ have the same signs, then the i-th and $\tau^k(i)$-th strands cross at even number of points (this number may be 0), so $T_k(i)$ is even. Similarly, if those signs are opposite, then $T_k(i)$ is odd. Finally, $\tau^k(i) = i$ is equivalent to $\tau^{k+1}(i) = \tau(i)$ and then $T_k(i) = 0$; we count it as "the same signs." Set $a(i) = 0$

if $\tau^k(i) - i \geq 0$ and $a(i) = 1$ if $\tau^k(i) - i < 0$. Then $T_k(i) \equiv a(i) - a(\tau(i))$ modulo 2. Since τ is a cyclic permutation, we get modulo 2

$$\sum_{i=1}^{n} T_k(i) \equiv \sum_{i=1}^{n} a(i) - \sum_{i=1}^{n} a(\tau(i)).$$

However, the right-hand side above is 0, so $\sum_{i=1}^{n} T_k(i)$ is even. □

Lemma 2.3. *If a braid B is cyclic then the k-th and $(n-k)$-th turning numbers of B are equal. Moreover, for every i the $(in+k)$-th and k-th turning numbers of B are equal.*

Proof. A crossing of the i-th and $\tau^k(i)$-th strands is also a crossing of the $\tau^k(i)$-th and $\tau^{n-k}(\tau^k(i))$-th strands. This proves the first statement. The second one follows from the fact that $\tau^{in+k} = \tau^k$. □

The *exponent sum* (or *wraith*) $\mathrm{es}(B)$ of B is defined as the sum of signs of all crossings in B.

Theorem 2.4. *If a braid B with n strands is cyclic then we have*

$$\mathrm{es}(B) = \sum_{k=1}^{n-1} \mathrm{TN}_k(B).$$

Proof. If we take the sum of the numbers $T_k(i)$ over $k = 1, \dots, n-1$ and $i = 1, \dots, n$, we get $2\,\mathrm{es}(B)$, because in the sum every crossing is counted twice (as the crossing of of the i-th and $\tau^k(i)$-th strands and as the crossing of the $\tau^k(i)$-th and $\tau^{n-k}(\tau^k(i))$-th strands). However, this sum is also equal to $2\sum_{k=1}^{n-1} \mathrm{TN}_k(B)$. □

In the next proof we will be using multiplication in the braid and permutation groups [6]. To avoid misunderstandings, let us fix the notation now. We will use the opposite order than for composition of functions. That is, if σ and τ are permutations, then $(\sigma\tau)(i) = \tau(\sigma(i))$. Similarly, if A and B are braids, then AB will be the braid which we get by following first the strands of the braid A, and then the strands of the braid B.

Theorem 2.5. *For each k, the k-th turning number of a braid is an invariant of conjugacy.*

Proof. To prove that TN_k is an invariant of conjugacy, consider the braid $C = D^{-1}BD$, where a braid D has permutation σ. Then the permutation of C is $\sigma^{-1}\tau\sigma$. To distinguish the numbers $T_k(i)$ for B and C let us add corresponding superscripts. We have $(\sigma^{-1}\tau\sigma)^k = \sigma^{-1}\tau^k\sigma$, so $T_k^C(i)$ is equal to the sum of the signs of the crossings of the i-th and $\sigma^{-1}\tau^k\sigma(i)$-th strands of C. Those crossings are of three types: the crossings of the i-th and $\sigma^{-1}\tau^k\sigma(i)$-th strands of D^{-1}, the crossings of the $\sigma^{-1}(i)$-th and $\sigma^{-1}\tau^k(i)$-th strands of B, and the crossings of the $\sigma^{-1}\tau(i)$-th and $\sigma^{-1}\tau^{k+1}(i)$-th strands of D. Respectively, the sum over i that gives us $\mathrm{TN}_k(C)$ can be written as the sum of three sums. The second of those three sums is clearly $\mathrm{TN}_k(B)$. The sum of the first and third sums is 0, because the crossings of the i-th and $\sigma^{-1}\tau^k\sigma(i)$-th strands of D^{-1} are in one-to-one correspondence with the crossings of the $\sigma^{-1}(i)$-th and $\sigma^{-1}\tau^k(i)$-th strands of D, but have opposite signs. Thus, $\mathrm{TN}_k(C) = \mathrm{TN}_k(B)$. □

Theorem 2.6. *If B is a cyclic braid with n strands then for each k and m we have* $\mathrm{TN}_k(B^m) = m\,\mathrm{TN}_{km}(B)$.

Proof. We can think of B^m as m copies of B stacked upon one another, forming m levels. Each crossing occurs on some level. The permutation of B^m is τ^m. Thus, counting the crossings occurring at a given level and contributing to $\mathrm{TN}_k(B^m)$ is the same as counting the crossings for B, contributing to $\mathrm{TN}_{km}(B)$. There are m levels, so we get $\mathrm{TN}_k(B^m) = m\,\mathrm{TN}_{km}(B)$. $\square$

Note that when applying the above theorem, we can take km modulo n, since the k-th and ℓ-th turning numbers are the same if $k - \ell$ is divisible by n.

3. Extensions

As we mentioned in the introduction, one of the possibilities in the Nielsen-Thurston classification of periodic orbits is that the orbit is reducible. In such a case the structure of the orbit (call it P) is as follows. Up to isotopy rel. P, one can find a system of disks that are permuted by the map and P is contained in their union. In each disk the number of elements of P is the same, and it is larger than 1 but smaller than the period of P (in particular, this is impossible if the period of P is a prime number). Then we can consider two orbits: the *outer* one (call it R) is obtained by collapsing each disk to a point; the *inner* one (call it Q) is obtained by looking at one disk and taking the smallest iterate of the homeomorphism that maps this disk back to itself. In such a case one calls P an *extension of R by Q*, and similarly, the braid type of P is the extension of the braid type of R by the braid type of Q. For more information, see [8] or [9].

Translating this idea to the language of braids, for braids B and C, where B is cyclic, we can produce an *extension of B by C* as follows (see Figure 1). We replace

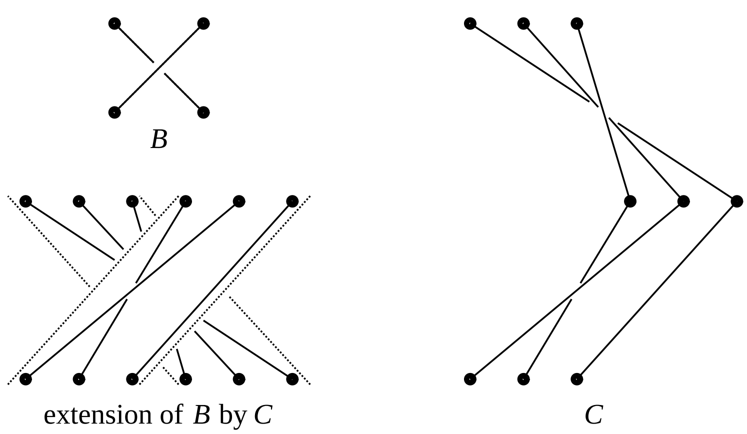

Figure 1. Construction of an extension.

each strand of B by a wide "tape" and insert into each tape some braid (skewed, because the top and the bottom of the tape are disjoint). Each of the braids in the tapes has the same number of strands, so we can multiply them in the order given by the permutation of B. In other words, we follow the tape from the bottom to the top. then we jump vertically down, etc., until we get back to the place from which we started. The braid that we get this way should be C.

Note that we have some freedom in the construction, but all braids obtained as an extension of B by C are conjugate. Therefore we will use the name "extension of B by C" for any braid conjugate to the ones obtained in the above construction.

Now we will show how to compute turning numbers of an extension of B by C from the turning numbers of B and C.

Theorem 3.1. *Let B be a cyclic braid with n strands, C a braid with r strands, and let D be an extension of B by C. If k is divisible by n then $\mathrm{TN}_k(D) = \mathrm{TN}_{k/n}(C)$. Otherwise, $\mathrm{TN}_k(D) = r\,\mathrm{TN}_k(B)$. Moreover, $\mathrm{es}(D) = r^2\,\mathrm{es}(B) + \mathrm{es}(C)$.*

Proof. We may assume that D is the braid produced in the construction described earlier in this section. Consider first the case when k is divisible by n. Then each crossing of strands number i and $\tau^k(i)$, where τ is the permutation of D, occurs inside of a tape. Therefore it can be identified with the crossing of some strands number j and $\eta^{k/n}(j)$ of C (where η is the permutation of C), with the same sign. This is a one-to-one correspondence, and therefore $\mathrm{TN}_k(D) = \mathrm{TN}_{k/n}(C)$.

Consider now the case when k is not divisible by n. Then each crossing of strands number i and $\tau^k(i)$ comes from a crossing of tapes, so it comes from the crossing of some strands number j and $\zeta^k(j)$ of B (where ζ is the permutation of B), with the same sign. There are r strands of D in a tape, so each such j works for r different numbers i. Therefore $\mathrm{TN}_k(D) = r\,\mathrm{TN}_k(B)$.

To compute $\mathrm{es}(D)$, observe that counting with signs, there are $r^2\,\mathrm{es}(B)$ crossings of strands of D that belong to different tapes, and $\mathrm{es}(C)$ crossings of strands that belong to the same tape. □

4. Connection with interval maps

If f is a continuous interval map with a periodic orbit P then a homeomorphism of a "thick interval," which is homeomorphic to a disk, can be associated to it. Let us call the corresponding periodic orbit of this homeomorphism P'. The interval defines the natural ordering on P, so when considering a braid associated to P', we have a natural choice of a braid. This braid is a *positive permutation braid* ([7], [10]), that is a braid with all crossings positive and each pair of strands crossing at most once. Its permutation (which is the same as the permutation of the points of the orbit P of the interval map) is cyclic. We will call those braids *positive permutation cyclic (ppc)* braids. Note that a positive permutation braid is uniquely determined by its permutation.

For a periodic orbit P of an interval map f, draw its picture by putting an arrow for every $p \in P$ above the interval if $f(p) < p$, and below if $f(p) > p$ (see Figure 2; Figure 3 explains the connection with the graph of an interval map). This defines a piecewise smooth closed curve. When we go around it, note what the vector normal to the curve does (to make its movement continuous, smoothen the curve in a natural way). Denote the winding number of this vector around the origin by m.

It is clear that m is equal to the number of times the arrow reverses its direction as we follow the curve, divided by 2. Observe that the strands of the braid obtained from (f, P) corresponding to p and $f(p)$ intersect if and only if the directions of the

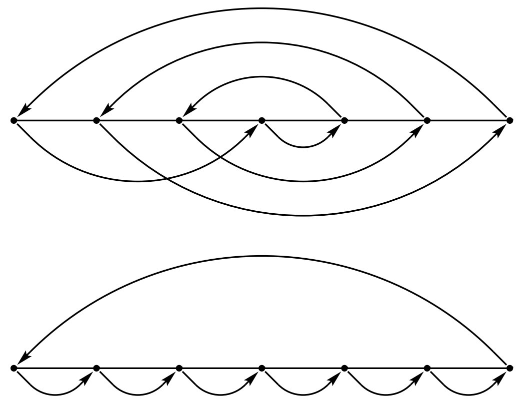

FIGURE 2. Two orbits of period 7 for interval maps.

arrows from p to $f(p)$ and from $f(p)$ to $f^2(p)$ are opposite. This shows that the first turning number of this braid is m.

This fact motivates the name "turning number," since the first turning number just counts how many times the curve defined above turns around.

Let us make another interesting observation. This number, one half of the number of times the arrows from p to $f(p)$ and from $f(p)$ to $f^2(p)$ have opposite directions as p runs over P, is exactly the *over-rotation number* of P [4].

For positive cyclic braids we can get more information about turning numbers, and even more is available for ppc braids.

Lemma 4.1. *Let B be a braid with n strands and cyclic permutation τ. If strands number j and i do not cross then $j - i$ and $\tau(j) - \tau(i)$ have the same sign. If additionally we assume that B is a ppc braid then we have equivalence: strands number j and i do not cross if and only if $j - i$ and $\tau(j) - \tau(i)$ have the same sign.*

Proof. The first part is obvious. The second one follows from the first one and the fact that two strands in a ppc braid cross at most once. □

Theorem 4.2. *Let B be a positive cyclic braid with n strands. Then $\mathrm{TN}_k(B)$ is positive for $k = 1, 2, \ldots, n-1$. If additionally we assume that B is a ppc braid then $\mathrm{TN}_k(B) \leq \lfloor n/2 \rfloor$.*

Proof. Let τ be the permutation of B. Since B is positive, $\mathrm{TN}_k(B) \geq 0$ for all k. By Lemma 4.1, if $T_k(i) = 0$ then the signs of $i - \tau^k(i)$ and $\tau(i) - \tau^{k+1}(i)$ are the same. Since τ is cyclic, we get the sign of $j - \tau^k(j)$ the same for all j. However, for $k = 1, 2, \ldots, n-1$, if j is the leftmost point, $\tau^k(j)$ is to its right and if j is the rightmost point, $\tau^k(j)$ is to its left, a contradiction. Hence, $\mathrm{TN}_k(B)$ cannot be 0.

If B is a ppc braid then $T_k(i) \leq 1$ for all i, so we get $\mathrm{TN}_k(B) \leq n/2$. However, by Lemma 2.2 $\mathrm{TN}_k(B)$ is an integer, so we get $\mathrm{TN}_k(B) \leq \lfloor n/2 \rfloor$. □

5. TWIST BRAIDS

Let F be the rotation of the unit disk by the angle $2\pi m/n$, where m and n are coprime. Then all periodic orbits of F in the interior of the disk (except its center) are

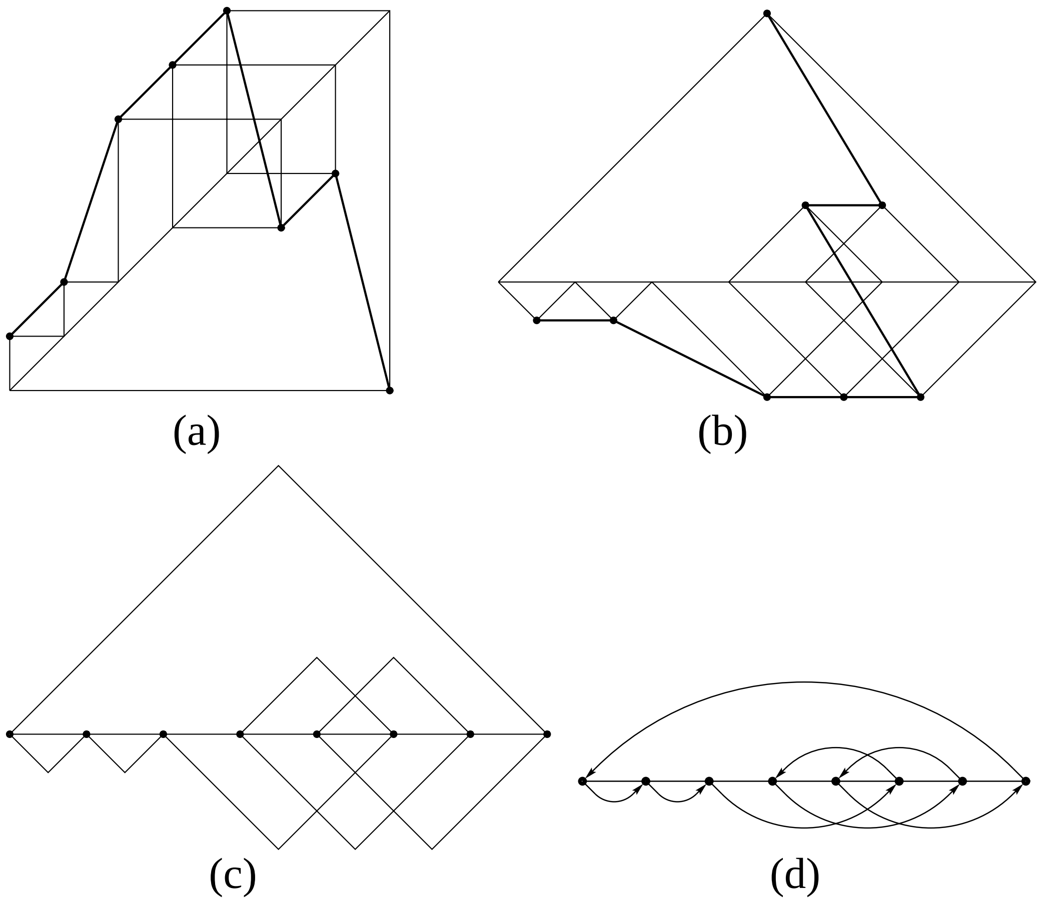

FIGURE 3. How to get a picture of an orbit from the graph of an interval map: (a) Draw the graph, mark the orbit and draw the usual cobweb for this orbit. (b) Apply the symmetry with respect to the diagonal and the clockwise rotation by 45 degrees. (c) Remove the graph and mark the points of the orbit on the (former) diagonal. (d) smoothen the curves joining marked points and make them arrows, upper ones pointing left and lower ones pointing right.

twist orbits of period n. We call them *m/n-twists*, and the corresponding braids *m/n-twist braids* (for a given m and n there are many such braids, but by the definition, they are all conjugate). It is well known that in this way we get all braid types that correspond to the finite order maps in the Thurston's classification (it follows, for instance, from Theorem 12.5 of [8]).

The most natural $1/n$-twist braid is the ppc braid with the permutation τ for which $\tau(i) = i+1$ for $i = 1, 2, \dots, n-1$ and $\tau(n) = 1$ (look at the second orbit at Figure 2). We will denote this braid by $B_{1/n}$. Using the standard generators $\sigma_1, \dots, \sigma_{n-1}$ of the n-th braid group, we can write $B_{1/n} = \sigma_{n-1} \dots \sigma_2\sigma_1$.

Composing disk maps sharing the same finite set as a periodic orbit (although the map on this orbit may be different) corresponds to the multiplication of braids. Thus, if m and n are coprime, the braid $B_{m/n} = B_{1/n}^m$ is an m/n-twist braid. If $2m < n$ (we always assume that m is positive), then it is easy to see that it looks as follows. First $n-m$ strands go from i to $i+m$ ($i=1,\dots,n-m$) as straight segments. The last m strands go from i to $i+m-n$, but are twisted together by the full positive rotation (by 360 degrees; we do not want to use the word "twist" here, since we are using it with a slightly different meaning).

Let us conjugate this braid via the positive half-rotation (by 180 degrees) on the last m strands. That is, we first apply the negative half-rotation on the last m strands, then $B_{m/n}$, and then the positive half-rotation on the last m strands. The first negative half-rotation will untwist partially the last m strands, so there will be only one crossing between each pair of them. However, the positive half-rotation will twist the strands from $n-2m+1$ through $n-m$, so again there will be one crossing between each pair of them. Thus, we can think of three bands of strands. The first one consists of $n-2m$ strands and it goes in a monotone way (without crossings) by m to the right (remember that the strands are going up). The second band consists of m strands, the whole band goes to the right by m, but it is half-twisted. The third band consists also of m strands and is half-twisted; the whole band goes by $n-m$ to the left. We denote this braid by $C_{m/n}$; it is a ppc braid. In Figure 4 we show the braids $C_{3/7}$ and $C_{1/7}$, corresponding to the twist orbits from Figure 2. Since each pair of strands crosses at most at one point, it is convenient to draw the strands as segments of straight lines (as we already did when defining extensions).

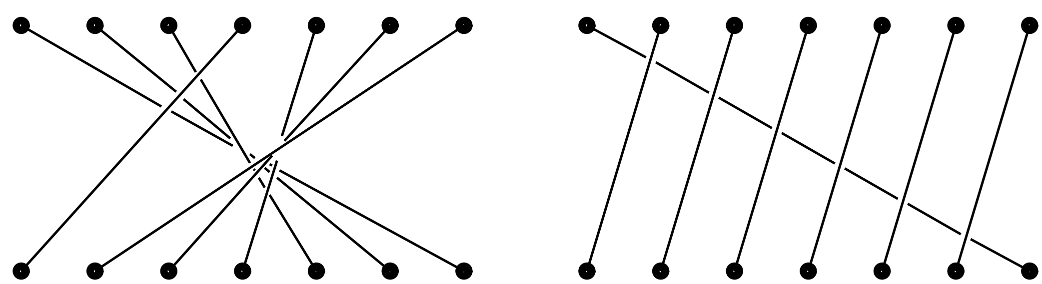

FIGURE 4. Braids $C_{3/7}$ and $C_{1/7}$.

Let us compute the turning numbers of $B_{m/n}$. By Theorem 2.5, the turning numbers will be the same for all other m/n-twist braids.

Theorem 5.1. *All turning numbers (from the first to $(n-1)$-st) of an m/n-twist braid are equal to m.*

Proof. The structure of $B_{1/n}$ is so simple, that the verification of $\mathrm{TN}_k(B_{1/n})=1$ for $k=1,\dots,n-1$ is straightforward. Thus, by Theorem 2.6, we get $\mathrm{TN}_k(B_{m/n})=m$. By the definition, each m/n-twist braid is conjugate to $B_{m/n}$, so by Theorem 2.5 its turning numbers are equal to m. $\square$

Now we see that if $2m>n$ then by Theorems 5.1 and 2.4 $\mathrm{es}(B_{m/n})\ge n(n-1)/2$, so no m/n-twist braid can be a ppc braid.

Lemma 5.2. *If a ppc braid with n strands has the first turning number 1, then its exponent sum is $n-1$.*

Proof. Let B be a ppc braid with n strands and with the first turning number 1. Let τ be its permutation. Then, by Lemma 4.1, there are only two numbers $i \in \{1, \dots, n\}$ such that $\tau^{-1}(i) - i$ and $i - \tau(i)$ have opposite signs. Those i's have to be 1 and n. Therefore, there is $k < n$ such that

$$1 < \tau(1) < \tau^2(1) < \cdots < \tau^k(1) = n \text{ and } n > \tau(n) > \tau^2(n) > \cdots > \tau^{n-k}(n) = 1.$$

This means that in a picture like Figure 4, over each gap between j-th and $(j+1)$-st points we see one strand going to the right and one going to the left. This proves that there are $n - 1$ crossings, so $\mathrm{es}(B) = n - 1$. □

Observe that since we are assuming here that the braid is cyclic, there must be a crossing in each gap as above, or, in other words, if we write a braid as a word in generators σ_i, each generator has to appear in this word.

Now we will use the results of the paper [10]. Since not everything that is proven there is stated as a theorem (some things important to us are hidden in the proofs of Theorems 2, 3 and 5), we will restate those results in a form useful to us. By Δ_n we denote the positive half-rotation on all n strands. In order to obtain a knot from a braid we join the top and the bottom of each strand, this is, we "close" the braid [6].

Theorem 5.3 ([10]). *Let B be a ppc braid with n strands. Then*

(a) *If B closes to the unknot then* $\mathrm{es}(B) = n - 1$*; if B closes to a trefoil, then* $\mathrm{es}(B) = n + 1$.
(b) *If* $\mathrm{es}(B) = n - 1$ *then B is conjugate to $\sigma_1\sigma_2 \dots \sigma_{n-1}$.*
(c) *If* $\mathrm{es}(B) = n + 1$ *then B is conjugate to $\sigma_1^3\sigma_2 \dots \sigma_{n-1}$.*
(d) *If* $\mathrm{es}(B) = n(n-1)/2 - \lfloor (n-1)/2 \rfloor$ *then B is conjugate to the braid $E_n = \Delta_n\sigma_1^{-1}\sigma_2^{-1} \dots \sigma_{\lfloor (n-1)/2 \rfloor}^{-1}$ (see Figure 5).*

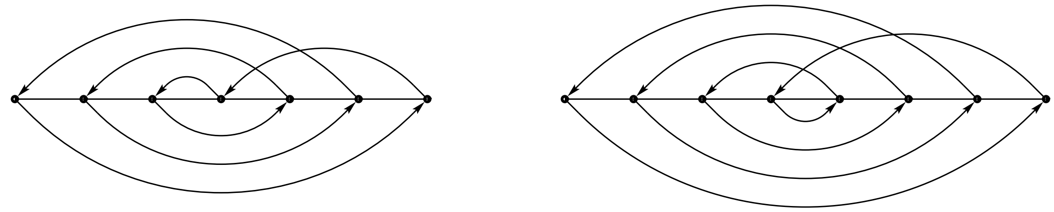

FIGURE 5. Braids E_7 and E_8.

Remark 5.4. In [10], it is stated without proof that the number from (d) above is the maximal possible exponent sum of a ppc braid. To see that a ppc braid with this exponent sum exists, one can just compute the exponent sum of E_n (which we will do later).

However, one has to check that a larger exponent sum cannot occur. If n is odd, this follows immediate from our Theorems 2.4 and 4.2. To give a general proof, note that a ppc braid can be written as $\Delta_n\sigma_{i_1}^{-1}\sigma_{i_2}^{-1} \dots \sigma_{i_r}^{-1}$ (see [7]). The permutation of Δ_n consists of $\lfloor (n+1)/2 \rfloor$ cycles. Each $\sigma_{i_j}^{-1}$ can reduce the number of cycles at most by 1. Thus, $r \geq \lfloor (n+1)/2 \rfloor - 1 = \lfloor (n-1)/2 \rfloor$, and therefore the exponential sum of our braid is at most $n(n-1)/2 - \lfloor (n-1)/2 \rfloor$.

Now we can strengthen statements (b)-(d) of Theorem 5.3.

Theorem 5.5. *Let B be a ppc braid with n strands. Then the following conditions are equivalent:*

(a) $\mathrm{es}(B) = n-1$,
(b) $\mathrm{TN}_1(B) = 1$,
(c) $\mathrm{TN}_1(B) = \mathrm{TN}_2(B) = \cdots = \mathrm{TN}_{n-1}(B) = 1$,
(d) *B is a $1/n$-twist braid.*

Proof. (a)⇒(d). By Theorem 5.3 (b), B is conjugate to $\sigma_1\sigma_2\dots\sigma_{n-1}$. By Theorems 5.1 and 2.4, $B_{1/n}$ has exponent sum $n-1$, so by Theorem 5.3 (b) it is also conjugate to $\sigma_1\sigma_2\dots\sigma_{n-1}$. Thus, B is conjugate to $B_{1/n}$, so it is a $1/n$-twist braid.

(d)⇒(c). Follows from Theorem 5.1.

(c)⇒(b). Obvious.

(b)⇒(a). This is Lemma 5.2. □

Theorem 5.6. *Let B be a ppc braid with n strands. Then the following conditions are equivalent:*

(a) $\mathrm{es}(B) = n+1$,
(b) $\mathrm{TN}_1(B) = \mathrm{TN}_{n-1}(B) = 2$ *and* $\mathrm{TN}_2(B) = \cdots = \mathrm{TN}_{n-2}(B) = 1$.
(c) *B is conjugate to $\sigma_1^3\sigma_2\dots\sigma_{n-1}$.*

Proof. (a)⇒(c). This is Theorem 5.3 (c).

(c)⇒(b). By Theorem 5.1, $\mathrm{TN}_1(B_{1/n}) = \mathrm{TN}_2(B_{1/n}) = \cdots = \mathrm{TN}_{n-1}(B_{1/n}) = 1$. Compared to $B_{1/n}$, we have in $\sigma_1^3\sigma_2\dots\sigma_{n-1}$ two more crossings, between strands number 2 and $1 = \tau(2)$ (where τ is the permutation of this braid). This adds 1 to the first and $(n-1)$-st turning numbers.

(b)⇒(a). Follows from Theorem 2.4. □

Theorem 5.7. *Let B be a ppc braid with n strands. If n is odd then the following conditions are equivalent:*

(a) $\mathrm{es}(B) = (n-1)^2/2$,
(b) $\mathrm{TN}_1(B) = \mathrm{TN}_2(B) = \cdots = \mathrm{TN}_{n-1}(B) = (n-1)/2$,
(c) *B is a $\frac{n-1}{2}/n$-twist braid.*

If n is even then the following conditions are equivalent:

(a') $\mathrm{es}(B) = ((n-1)^2+1)/2$,
(b') *if $1 \le k \le n-1$ then $\mathrm{TN}_k(B) = n/2$ for k odd and $\mathrm{TN}_k(B) = n/2-1$ for k even.*
(c') *B is an extension of $B_{1/2}$ by $B_{(n/2-1)/(n/2)}$.*

Proof. Note first that if n is odd then $n(n-1)/2 - \lfloor (n-1)/2 \rfloor = (n-1)^2/2$, and if n is even then $n(n-1)/2 - \lfloor (n-1)/2 \rfloor = ((n-1)^2+1)/2$.

(a)⇒(c). By Theorem 5.3 (d), B is conjugate to E_n. By Theorems 5.1 and 2.4, $B_{\frac{n-1}{2}/n}$ has exponent sum $(n-1)^2/2$, so by Theorem 5.3 (d) it is also conjugate to E_n. Thus, B is conjugate to $B_{\frac{n-1}{2}/n}$, so it is a $\frac{n-1}{2}/n$-twist braid.

(c)⇒(b). Follows from Theorem 5.1.

(b)⇒(a). Follows from Theorem 2.4.

Now we consider the case of n even. Let us start with showing that E_n is an extension of $B_{1/2}$ by $B_{(n/2-1)/(n/2)}$. Since Δ_n is an extension of $B_{1/2}$ by $\Delta^2_{n/2}$, the braid

E_n is an extension of $B_{1/2}$ by the braid $D = \Delta_{n/2}^2\sigma_1^{-1}\sigma_2^{-1}\dots\sigma_{n/2-1}^{-1} = \Delta_{n/2}^2 B_{1/(n/2)}^{-1}$ (cf. Figure 5). Since $\Delta_{n/2}^2 = B_{1/(n/2)}^{n/2}$, we get $D = B_{1/(n/2)}^{n/2-1} = B_{(n/2-1)/(n/2)}$. Thus, E_n is an extension of $B_{1/2}$ by $B_{(n/2-1)/(n/2)}$.

(a')⇒(c'). In view of what we proved above, this is Theorem 5.3 (d).

(c')⇒(b'). Follows from Theorems 5.1, 2.4 and 3.1.

(b')⇒(a'). Follows from Theorem 2.4. □

Motivated by Theorems 5.5 and 5.7, we state the following conjecture.

Conjecture 5.8. Let m, n be positive coprime integers with $2m < n$ and let B be a ppc braid with n strands and with $\mathrm{TN}_1(B) = \mathrm{TN}_2(B) = \cdots = \mathrm{TN}_{n-1}(B) = m$. Then B is an m/n-twist braid.

6. First and second turning numbers

In this section we will explain how to compute easily the first and second turning numbers for the ppc braids and how to compare them. Let B be a ppc braid with $n > 2$ strands and permutation τ. We can think of τ as walking on $\{1, 2, \dots, n\}$, sometimes to the right, sometimes to the left, and with varying step length. We compare two first steps we make from $i \in \{1, 2, \dots, n\}$. If we move in the same direction, that is $\tau(i) - i$ and $\tau^2(i) - \tau(i)$ have the same signs, we assign to i the symbol o. If we move in opposite directions, that is $\tau(i) - i$ and $\tau^2(i) - \tau(i)$ have opposite signs, we compare the lengths of those two steps. If the second step is shorter than the first one, that is, $|\tau^2(i) - \tau(i)| < |\tau(i) - i|$, then we assign to i the symbol s. If the second step is longer than the first one, that is, $|\tau^2(i) - \tau(i)| > |\tau(i) - i|$, then we assign to i the symbol ℓ. Now, when we follow our τ-orbit (that is, consider $\tau^k(1)$, $k = 0, 1, \dots$, where k is taken modulo n), we get a circular sequence of symbols o, s, ℓ of length n. We call this sequence the *code* of τ (or of B).

Theorem 6.1. *The number of symbols s and ℓ (together) in the code of a ppc braid B is equal to $2\,\mathrm{TN}_1(B)$.*

Proof. Symbols s and ℓ are assigned exactly to those i's for which there is a crossing between the i-th and $\tau(i)$-th strand. □

Theorem 6.2. *The number of blocks $s\ell$ in the code of a ppc braid B is equal to $\mathrm{TN}_1(B) - TN_2(B)$.*

Proof. Let us find out for which pairs of symbols assigned to i and $\tau(i)$ the numbers $\tau^3(i) - \tau(i)$ and $\tau^2(i) - i$ have opposite signs. According to Lemma 4.1, those are the i's for which the strands number i and $\tau^2(i)$ cross each other, so the number of those i's is equal to $2TN_2(B)$. A direct inspection (see Figure 6) shows that this happens for the pairs (blocks of length 2) $o\ell, so, ss, \ell\ell$.

In other words, in order to compute $2TN_2(B)$ we go around the code and count how many times we encounter blocks $o\ell, so, ss, \ell\ell$. Since these are all pairs whose first element is s or the second element is ℓ, except the pair $s\ell$, we have to add the number of s's and ℓ's in the code and subtract twice the number of the blocks $s\ell$ in the code. Thus, in view of Theorem 6.1, $2TN_2(B)$ is equal to $2\,\mathrm{TN}_1(B)$ minus twice the number of the blocks $s\ell$ in the code. □

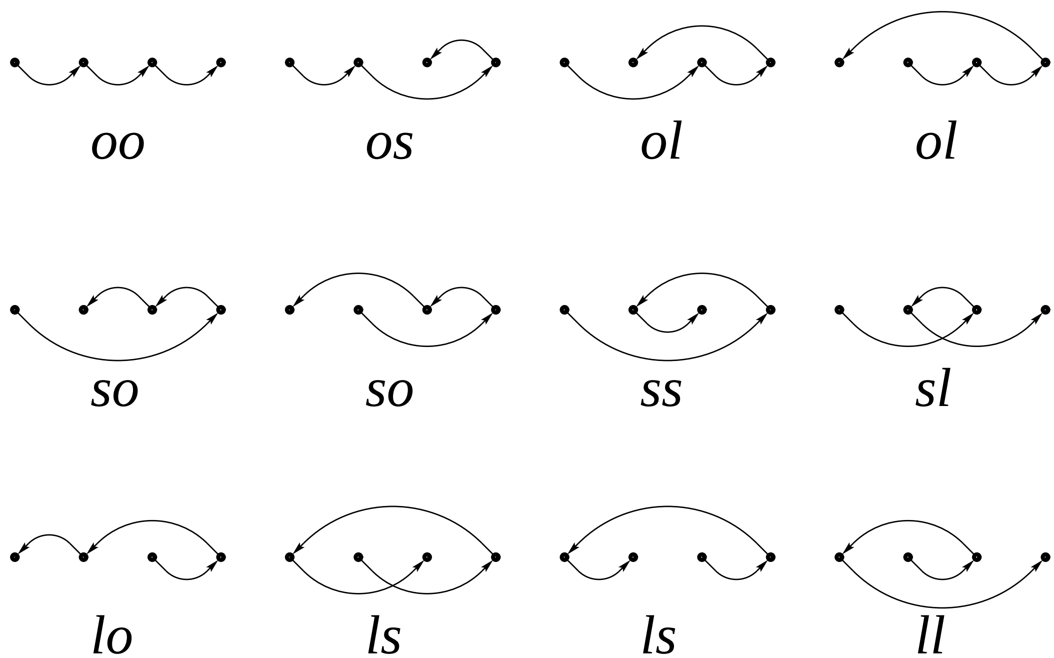

FIGURE 6. Various blocks of length 2 in the code of B.

Corollary 6.3. *If B is a ppc braid then $TN_2(B) \le TN_1(B)$. Moreover, the equality holds if and only if there are no blocks $s\ell$ in the code of B.*

7. Experimental results

Computations show that for ppc braids with up to 7 strands, two braids with the same turning numbers are always conjugate. Unfortunately, there is a counterexample to this statement for braids with 8 strands. Both braids corresponding to the periodic orbits from Figure 7 have the same turning numbers: $3, 1, 2, 1, 2, 1, 3$, but they are not conjugate.

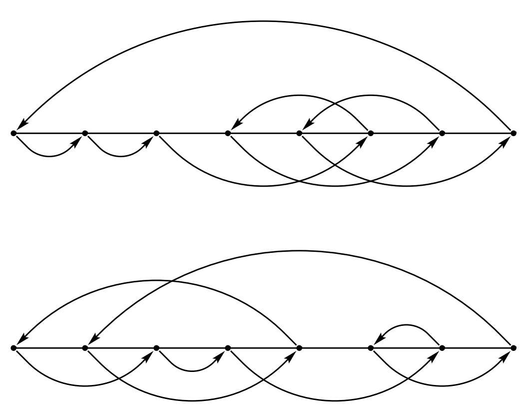

FIGURE 7. Two orbits of period 8 with the same turning numbers but different braid types.

To see that the braids are not conjugate, one can find their pseudo-Anosov representations, using the methods of [1] or [8] (see also [9]). Corresponding Markov partitions give transition matrices that allow us to estimate their entropies.

The first matrix is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the second matrix is

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

The cube of the second matrix has the minimal row sum 21, so the topological entropy of any orientation preserving disk homeomorphism exhibiting the second orbit is at least $\frac{1}{3}\log 21$. On the other hand, the cube of the first matrix has the maximal row sum 15, so there is an orientation preserving disk homeomorphism exhibiting the second orbit with topological entropy less than or equal to $\frac{1}{3}\log 15$.

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