The algorithmic potential of continuous transductions

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Abstract

We give a topological characterization of the transductions τ from a monoid M into a monoid N, such that if R is a recognizable subset of N, $\tau^{-1}(R)$ is a recognizable subset of M. It follows that these transductions correspond to continuous mappings of a particular type and so may be called continuous. We impose two conditions on the monoids, which are fullfilled in all cases of practical interest: the monoids must be residually finite and, for every positive integer n, must have only finitely many congruences of index n. We proceed to express as continuous transductions various operators occurring naturally in the study of rational languages. We show that the concept is powerful enough to enclose the whole first-order theory induced by these operators.

1 Introduction

This paper, both summarizing and expanding [16], is a contribution to the mathematical foundations of automata theory that also aims at the development of combinatorial tools for the study of algorithmic properties of rational languages. We are mostly interested in the study of transductions τ from a monoid M into another monoid N such that, for every recognizable subset R of N, $\tau^{-1}(R)$ is a recognizable subset of M. We propose to call such transductions *continuous*, a term introduced in [7] in the case where M is a finitely generated free monoid.

In mathematics, the word "continuous" generally refers to a topology. We find appropriate topologies for which our use of the term *continuous* coincides with its usual topological meaning.

This problem was already solved when τ is a *mapping* from A^* into B^*. In this case, a result which goes back at least to the eighties (see [13]) states that τ is continuous in our sense if and only if it is continuous for the *profinite topology* on A^* and B^*. This result actually extends to mappings from A^* into a residually finite monoid N, thanks to a result of [7].

However, a transduction $\tau : M \to N$ is not a map from M into N, but a map from M into the set of subsets of N, which calls for a more sophisticated

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solution, since it does not suffice to find an appropriate topology on N. Our solution proceeds in two steps. We first show, under fairly general assumptions on M and N, which are fulfilled in all cases of practical interest, that M and N can be equipped with a metric, the Hall metric, for which they become metric monoids whose completion (as metric spaces) is compact. Next we prove that τ can be lifted to a map $\widehat{\tau}$ from M into the monoid $\mathcal{K}(\widehat{N})$ of compact subsets of $\widehat{N}$, the completion of N. The monoid $\mathcal{K}(\widehat{N})$, equipped with the Hausdorff metric, is again a compact monoid. Finally, our main result states that τ is continuous in our sense if and only if $\widehat{\tau}$ is continuous in the topological sense.

We can express as continuous transductions various operators occurring naturally in the study of rational languages, related to operations such as product and star, composition, direct products, intersecting rational languages and many others. We show that the concept is powerful enough to enclose the whole first-order theory induced by these operators, in the sense that we can provide through our study of continuous transductions recognizable solution sets for first-order formulae involving these operators. This generalizes results presented in [19] and [20].

2 Recognizable languages and transductions

Recall that a subset P of a monoid M is *recognizable* if there exists a finite monoid F and a monoid morphism $\varphi : M \to F$ and a subset Q of F such that $P = \varphi^{-1}(Q)$. The set of recognizable subsets of M is denoted by $\mathrm{Rec}(M)$. Recognizable subsets are closed under boolean operations, quotients and inverse morphisms. By Kleene's theorem, a subset of a finitely generated free monoid is recognizable if and only if it is rational.

The description of the recognizable subsets of a product of monoids was given by Mezei (see [5, p. 54] for a proof).

Theorem 2.1 (Mezei) *Let $M_1, \ldots, M_n$ be monoids. A subset of $M_1 \times \cdots \times M_n$ is recognizable if and only if it is a finite union of subsets of the form $R_1 \times \cdots \times R_n$, where $R_i \in \mathrm{Rec}(M_i)$.*

The following result is perhaps less known. See [5, p. 61].

Proposition 2.2 *Let $A_1, \ldots, A_n$ be finite alphabets. Then $\mathrm{Rec}(A_1^* \times A_2^* \times \cdots \times A_n^*)$ is closed under concatenation product .*

Given two monoids M and N, recall that a *transduction* from M into N is a relation on M and N, that we shall also consider as a map from M into the monoid of subsets of N. If X is a subset of M, we set

$$\tau(X) = \bigcup_{x \in X} \tau(x)$$

Observe that "transductions commute with union": if $(X_i)_{i \in I}$ is a family of subsets of M, then

$$\tau(\bigcup_{i \in I} X_i) = \bigcup_{i \in I} \tau(X_i)$$

If $\tau : M \to N$ is a transduction, then the inverse relation $\tau^{-1} : N \to M$ is also a transduction, and if P is a subset of N, the following formula holds:

$$\tau^{-1}(P) = \{x \in M \mid \tau(x) \cap P \neq \emptyset\}$$

A transduction $\tau : M \to N$ *preserves recognizable sets* if, for every set $R \in \mathrm{Rec}(M)$, $\tau(R) \in \mathrm{Rec}(N)$. It is said to be *continuous* if τ^{-1} preserves recognizable sets, that is, if for every set $R \in \mathrm{Rec}(N)$, $\tau^{-1}(R) \in \mathrm{Rec}(M)$.

Continuous transductions were characterized in [7] when M is a finitely generated free monoid. Recall that a transduction $\tau : M \to N$ is *rational* if it is a rational subset of $M \times N$. According to [7], a transduction $\tau : A^* \to N$ is *residually rational* if, for any morphism $\varphi : N \to F$, where F is a finite monoid, the transduction $\varphi \circ \tau : A^* \to F$ is rational. We can now state:

Proposition 2.3 [7] *A transduction $\tau : A^* \to N$ is continuous if and only if it is residually rational.*

3 Topology

Throughout this section, we assume the reader to be acquainted with the basic concepts envolving metric spaces. Some definitions are nevertheless presented to make reading easier.

A *metric* d on a set E is a map from E into the set of nonnegative real numbers satisfying the three following conditions, for all $(x, y, z) \in E^3$:

(1) $d(x, y) = 0$ if and only if $x = y$,

(2) $d(y, x) = d(x, y)$,

(3) $d(x, z) \leqslant d(x, y) + d(y, z)$

A metric is an *ultrametric* if (3) is replaced by the stronger condition

(3′) $d(x, z) \leqslant \max\{d(x, y), d(y, z)\}$

A *metric space* is a set E together with a metric d on E. Given a positive real number ε and an element x in E, the *open ball* of center x and radius ε is the set

$$B(x, \varepsilon) = \{y \in E \mid d(x, y) < \varepsilon\}.$$

A function φ from a metric space (E, d) into another metric space (E', d') is uniformly continuous if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $(x, x') \in E^2$, $d(x, x') < \delta$ implies $d(\varphi(x), \varphi(x')) < \varepsilon$. It is an *isometry* if, for all $(x, x') \in E^2$, $d(\varphi(x), \varphi(x')) = d(x, x')$.

Let M be a monoid. A monoid morphism $\varphi : M \to N$ *separates* two elements u and v of M if $\varphi(u) \neq \varphi(v)$. By extension, we say that a monoid N *separates* two elements of M if there exists a morphism $\varphi : M \to N$ which separates them. A monoid is *residually finite* if any pair of distinct elements of M can be separated by a finite monoid.

Residually finite monoids include finite monoids, free monoids, free groups and many others. They are closed under direct products and thus monoids of the form $A_1^* \times A_2^* \times \cdots \times A_n^*$ are also residually finite.

Any residually finite monoid M can be equipped with the *Hall metric* d, defined as follows. We first set, for all $(u, v) \in M^2$:

$$r(u, v) = \min\bigl\{\mathrm{Card}(N) \bigm| N \text{ separates } u \text{ and } v \bigr\}$$

Then we set $d(u,v) = 2^{-r(u,v)}$, with the usual conventions $\min\emptyset = +\infty$ and $2^{-\infty} = 0$. It is easy to see (see e.g. [16]) that, equipped with this metric, M becomes a *metric monoid*, multiplication being uniformly continuous.

A sequence $(x_n)_{n\geqslant 0}$ of elements of E is converging to a limit $x \in E$ if, for every $\varepsilon > 0$, there exists N such that for all integers $n > N$, $d(x_n, x) < \varepsilon$. It is a *Cauchy sequence* if, for every positive real number $\varepsilon > 0$, there is an integer N such that for all integers $p, q \geqslant N$, $d(x_p, x_q) < \varepsilon$. A metric space E is said to be *complete* if every Cauchy sequence of elements of E converges to a limit.

For any metric space E, one can construct a complete metric space $\widehat{E}$, such that $\widehat{E}$ is the closure of E and satisfies the following universal property: if F is any complete metric space and φ is any uniformly continuous function from E to F, then there exists a unique uniformly continuous function $\widehat{\varphi} : \widehat{E} \to F$ which extends φ. The space $\widehat{E}$ is determined up to isometry by this property, and is called the *completion* of E.

The completion of the metric space (M, d), denoted by $(\widehat{M}, d)$, is called the *Hall completion* of M. Since multiplication on M is uniformly continuous, it extends, in a unique way, into a multiplication onto $\widehat{M}$, which is again uniformly continuous. In particular, $\widehat{M}$ is a metric, complete monoid. Furthermore, any morphism from $(\widehat{M}, d)$ onto a finite discrete monoid is uniformly continuous.

A metric space (E, d) is said to be *compact* if, for each family of open sets covering E, there exists a finite subfamily that still covers E. We now characterize the residually finite monoids M such that $\widehat{M}$ is compact.

Proposition 3.1 *[16, Proposition 3.5] Let M be a residually finite monoid. Then $\widehat{M}$ is compact if and only if, for every positive integer n, there are only finitely many congruences of index n on M.*

An important sufficient condition is given in the following corollary.

Corollary 3.2 *[16, Corollary 3.6] Let M be a residually finite monoid. If M is finitely generated, then $\widehat{M}$ is compact.*

Proposition 3.1 justifies the following terminology. We will say that a monoid M is *Hall-compact* if it is residually finite and if, for every positive integer n, there are only finitely many congruences of index n on M. Proposition 3.1 can now be rephrased as follows:

"A residually finite monoid M is Hall-compact if and only if $\widehat{M}$ is compact."

and Corollary 3.2 states that

"Every residually finite and finitely generated monoid is Hall-compact."

The class of Hall-compact monoids includes most of the examples used in practice: finitely generated free monoids (resp. groups), finitely generated free commutative monoids (resp. groups), finite monoids, trace monoids, finite products of such monoids, etc.

It is shown in [16, Proposition 3.7] that the converse to Corollary 3.2 does not hold.

Recall that a *clopen* subset of a metric space is a subset which is both open and closed. A metric space is *zero-dimensional* if every open subset is a union of clopen subsets.

Proposition 3.3 *[16, Proposition 3.8] Let M be a residually finite monoid. Then (M, d) and $(\widehat{M}, d)$ are zero-dimensional.*

Proposition 3.3 implies that if M is a Hall-compact monoid then $\widehat{M}$ is *profinite* (see [1, 3, 4, 22] for the definition of profinite monoids and several equivalent properties), but we will not use this result in this paper.

We now give three results relating clopen sets and recognizable sets. The first one is due to Hunter [9, Lemma 4], the second one summarizes results due to Numakura [12] (see also [17, 2]). The third result is stated in [3] for free profinite monoids.

Recall that the *syntactic congruence* of a subset P of a monoid M is defined, for all $u, v \in M$, by

$$s \sim t \quad \text{if and only if, for all } (x, y) \in M^2,\ xuy \in P \Leftrightarrow xvy \in P.$$

It is the coarsest congruence of M which saturates P.

Lemma 3.4 (Hunter's Lemma) *In a compact monoid, the syntactic congruence of a clopen set is clopen.*

Proposition 3.5 *In a compact monoid, every clopen subset is recognizable. If M is a residually finite monoid, then every recognizable subset of $\widehat{M}$ is clopen.*

The last result of this subsection is a clone of a standard result on free profinite monoids (see [3] for instance).

Proposition 3.6 *[16, Proposition 3.11] Let M be a Hall-compact monoid, let P be a subset of M and let $\overline{P}$ be its closure in $\widehat{M}$. The following conditions are equivalent:*

(1) *P is recognizable,*

(2) *$P = K \cap M$ for some clopen subset K of $\widehat{M}$,*

(3) *$\overline{P}$ is clopen in $\widehat{M}$ and $P = \overline{P} \cap M$,*

(4) *$\overline{P}$ is recognizable in $\widehat{M}$ and $P = \overline{P} \cap M$.*

Let M be a compact monoid, and let $\mathcal{K}(M)$ be the monoid of compact subsets of M. The *Hausdorff metric* on $\mathcal{K}(M)$ is defined as follows. For $K, K' \in \mathcal{K}(M)$, let

$$\delta(K, K') = \sup_{x \in K} \inf_{x' \in K'} d(x, x')$$

$$h(K, K') = \begin{cases} \max(\delta(K, K'), \delta(K', K)) & \text{if } K \text{ and } K' \text{ are nonempty,} \\ 0 & \text{if } K \text{ and } K' \text{ are empty,} \\ 1 & \text{otherwise.} \end{cases}$$

The last case occurs when one and only one of K or K' is empty. By a standard result of topology, $\mathcal{K}(M)$, equipped with this metric, is compact.

The next result states a property of clopen sets which is crucial in the proof of our main result.

Proposition 3.7 *[16, Proposition 3.12] Let M be a Hall-compact monoid, let C be a clopen subset of $\widehat{M}$ and let $\varphi : \mathcal{K}(\widehat{M}) \to \mathcal{K}(\widehat{M})$ be the map defined by $\varphi(K) = K \cap C$. Then φ is uniformly continuous for the Hausdorff metric.*

4 Transductions

Let M and N be Hall-compact monoids and let $\tau : M \to N$ be a transduction. Then $\mathcal{K}(\widehat{N})$, equipped with the Hausdorff metric, is also a compact monoid. Define a map $\widehat{\tau} : M \to \mathcal{K}(\widehat{N})$ by setting, for each $x \in M$, $\widehat{\tau}(x) = \overline{\tau(x)}$, the closure of $\tau(x)$ in $\widehat{N}$.

Theorem 4.1 *[16, Theorem 4.1] The transduction τ^{-1} preserves the recognizable sets if and only if $\widehat{\tau}$ is uniformly continuous.*

Proof. Suppose that τ^{-1} preserves the recognizable sets. Let $\varepsilon > 0$. Since $\widehat{N}$ is compact, it can be covered by a finite number of open balls of radius $\varepsilon/2$, say

$$\widehat{N} = \bigcup_{1 \leqslant i \leqslant k} B(x_i, \varepsilon/2)$$

Since $\widehat{N}$ is zero-dimensional by Proposition 3.3, every open ball $B(x_i, \varepsilon/2)$ is a union of clopen sets and $\widehat{N}$ is a union of clopen sets each of which is contained in a ball of radius $\varepsilon/2$. By compactness, we may assume that this union is finite. Thus

$$\widehat{N} = \bigcup_{1 \leqslant j \leqslant n} C_j$$

where each C_j is a clopen set contained in, say, $B(x_{i_j}, \varepsilon/2)$. It follows now from Proposition 3.6 that $C_j \cap N$ is a recognizable subset of N. Since τ^{-1} preserves the recognizable sets, the sets $L_j = \tau^{-1}(C_j \cap N)$ are also recognizable. By [16, Proposition 3.4], the syntactic morphism of L_j is uniformly continuous and thus, there exists δ_j such that $d(u, v) < \delta_j$ implies $u \sim_{L_j} v$. Taking $\delta = \min\{\delta_j \mid 1 \leqslant j \leqslant n\}$, we have for all $(u, v) \in M^2$,

$$d(u, v) < \delta \Rightarrow \text{ for all } j \in \{1, \dots, n\},\ u \sim_{L_j} v.$$

We claim that, whenever $d(u, v) < \delta$, we have $h(\overline{\tau(u)}, \overline{\tau(v)}) < \varepsilon$. By definition,

$$L_j = \{x \in M \mid \tau(x) \cap C_j \cap N \neq \emptyset\}$$

Suppose first that $\tau(u) = \emptyset$. Then $u \notin \bigcup_{1 \leqslant j \leqslant n} L_j$. Since $u \sim_{L_j} v$ for every j, it follows that $v \notin \bigcup_{1 \leqslant j \leqslant n} L_j$, so $\tau(v) \cap C_j \cap N \neq \emptyset$ for $1 \leqslant j \leqslant n$. Since $N = \bigcup_{1 \leqslant j \leqslant n} (C_j \cap N)$, it follows that $\tau(v) = \emptyset$. by symmetry, we conclude that $\tau(u) = \emptyset$ if and only if $\tau(v) = \emptyset$.

Thus we may assume that both $\tau(u)$ and $\tau(v)$ are nonempty. Let $y \in \tau(u)$. Then $y \in C_j \cap N$ for some $j \in \{1, \dots, n\}$ and so $u \in L_j$. Since $u \sim_{L_j} v$, it follows that $v \in L_j$ and hence there exists some $z \in \tau(v)$ such that $z \in C_j \cap N$. Since $C_j \subseteq B(x_{i_j}, \varepsilon/2)$, we obtain $d(x_{i_j}, y) < \varepsilon/2$ and $d(x_{i_j}, z) < \varepsilon/2$, whence $d(y, z) < \varepsilon/2$ since d is an ultrametric. Thus $d(y, \overline{\tau(v)}) < \varepsilon/2$. Since $\tau(u)$ is dense in $\overline{\tau(u)}$, it follows that $d(x, \overline{\tau(v)}) \leqslant \varepsilon/2$ for every $x \in \overline{\tau(u)}$ and so

$$\delta(\overline{\tau(u)}, \overline{\tau(v)}) \leqslant \varepsilon/2 < \varepsilon.$$

By symmetry, $\delta(\overline{\tau(v)}, \overline{\tau(u)}) < \varepsilon$ and hence $h(\overline{\tau(u)}, \overline{\tau(v)}) < \varepsilon$ as required.

Next we show that if $\widehat{\tau}$ is uniformly continuous, then τ^{-1} preserves the recognizable sets. First, $\widehat{\tau}$ can be extended to a uniformly continuous mapping

$$\check{\tau} : \widehat{M} \to \mathcal{K}(\widehat{N}).$$

Let L be a recognizable subset of N. By Proposition 3.6, $L = C \cap N$ for some clopen subset C of $\widehat{N}$. Let

$$R = \{K \in \mathcal{K}(\widehat{N}) \mid K \cap C \neq \emptyset\}$$

We show that R is a clopen subset of $\mathcal{K}(\widehat{N})$. Let $\varphi : \mathcal{K}(\widehat{N}) \to \mathcal{K}(\widehat{N})$ be the map defined by $\varphi(K) = K \cap C$. By Proposition 3.7, φ is uniformly continuous and since $R = \varphi^{-1}(\{\emptyset\}^c) = [\varphi^{-1}(\{\emptyset\})]^c$, it suffices that $\{\emptyset\}$ is a clopen subset of $\mathcal{K}(\widehat{N})$. Since $B(\emptyset, 1) = \{\emptyset\}$, $\{\emptyset\}$ is open. Let $K \in \{\emptyset\}^c$. Since $\emptyset \notin B(K, 1)$, we have $B(K, 1) \subseteq \{\emptyset\}^c$ and so $\{\emptyset\}^c$ is also open. Therefore $\{\emptyset\}$ is clopen and so is R. Since $\check{\tau}$ is continuous, $\check{\tau}^{-1}(R)$ is a clopen subset of $\widehat{M}$ and so $M \cap \check{\tau}^{-1}(R)$ is recognizable by Proposition 3.6. Now

$$\begin{aligned} M \cap \check{\tau}^{-1}(R) &= \{u \in M \mid \check{\tau}(u) \in R\} \\ &= \{u \in M \mid \overline{\tau(u)} \in R\} \\ &= \{u \in M \mid \overline{\tau(u)} \cap C \neq \emptyset\} \end{aligned}$$

Since C is open, we have $\overline{\tau(u)} \cap C \neq \emptyset$ if and only if $\tau(u) \cap C \neq \emptyset$, hence

$$\begin{aligned} M \cap \check{\tau}^{-1}(R) &= \{u \in M \mid \tau(u) \cap C \neq \emptyset\} \\ &= \{u \in M \mid \tau(u) \cap L \neq \emptyset\} \\ &= \tau^{-1}(L) \end{aligned}$$

and so $\tau^{-1}(L)$ is a recognizable subset of M. Thus τ^{-1} preserves the recognizable sets. □

Corollary 4.2 *If $\tau : M \to N$ is a function, then τ^{-1} preserves the recognizable sets if and only if τ is continuous (for the Hall topologies).*

Proof. Since $\mathcal{K}(\widehat{N})$ is compact, we may replace "uniformly continuous" by "continuous" in Theorem 4.1. If τ is a function, then $\widehat{\tau}(x) = \tau(x)$. Since N (with the Hall topology) can be viewed as a subspace of $\mathcal{K}(\widehat{N})$, it follows from Theorem 4.1 that τ^{-1} preserves the recognizable sets if and only if τ is continuous. □

5 Examples of continuous transductions

A large number of examples of continuous transductions can be found in the literature [21, 8, 10, 11, 18, 14, 15, 6, 7]. We present a few important cases and show the class is closed for some natural operators.

Proposition 5.1 *[16, Proposition 5.1] Let $L \subseteq N$ and let $\kappa_L : M \to N$ be the transduction defined by $\kappa_L(x) = L$. Then κ_L is continuous.*

Given $A, B \subseteq M$ and $n \in \mathbb{N}$, we define the *bounded shuffle of degree n* of A and B by

$$A \sqcup_n B = \{a_1 b_1 \dots a_n b_n \mid a_i, b_i \in M,\ a_1 \dots a_n \in A,\ b_1 \dots b_n \in B\}.$$

Note that $A \sqcup_1 B = AB$. The *shuffle* of x and y is defined by

$$x \sqcup y = \bigcup_{n \geqslant 0} (x \sqcup_n y).$$

Proposition 5.2 *If $\mathrm{Rec}(M)$ is closed under product, then $\sqcup_n : M \times M \to M$ is continuous for every $n \in N$.*

Proof. Let $R \in \mathrm{Rec}(M)$. Then

$$\begin{aligned}(\sqcup_n)^{-1}(R) &= \{(x, y) \in M \times M \mid x \sqcup_n y \cap R \neq \emptyset\} \\ &= \{(x_1 \dots x_n, y_1 \dots y_n) \in M \times M \mid x_1 y_1 \dots x_n y_n \in R\}.\end{aligned}$$

Let $\sim_R$ be the syntactic congruence of R and let $\mathcal{R}$ be the class of $\sim_R$-classes of R in M. Set

$$E = \{(R_1, \dots, R_{2n}) \mid R_i \in \mathcal{R} \text{ and } R_1 \cdots R_{2n} \cap R \neq \emptyset\}.$$

Since R is saturated by its syntactic congruence, we have

$$R_1 \cdots R_{2n} \cap R \neq \emptyset \text{ if and only if } R_1 \cdots R_{2n} \subseteq R$$

and thus $a_1 a_2 \dots a_{2n} \in R$ if and only if there exists $(R_1, \dots, R_{2n}) \in E$ such that $a_j \in R_j$ for all $1 \leqslant j \leqslant 2n$. It follows that

$$(\sqcup)^{-1}(R) = \bigcup_{(R_1, \dots, R_{2n}) \in E} R_1 R_3 \cdots R_{2n-1} \times R_2 R_4 \cdots R_{2n}.$$

Since $R_1, \dots, R_k$ are recognized by the syntactic morphism of R, they are themselves recognizable. Since $\mathrm{Rec}(M)$ is closed under product, Mezei's Theorem implies that $(\sqcup)^{-1}(R) \in \mathrm{Rec}(M \times M)$ and hence $\sqcup_n$ is continuous. □

This result generalizes [20], where the free monoid case was proved (under different terminology).

Corollary 5.3 *[20, Theorem 5.3] The product $M \times M \to M$ is a continuous transduction.*

Proof. Since $x \sqcup_1 y = xy$ and we do not use the fact that $\mathrm{Rec}(M)$ is closed under product in the proof of the preceding result (for $n = 1$). □

Proposition 5.2 cannot be generalized to the case of shuffle:

Example 5.1 [20, Example 4.1] The shuffle operator $\sqcup : A^* \times A^* \to A^*$ is not continuous.

Proof. Let $A = \{a, b\}$, $R = (ab)^*$. Then

$$\begin{aligned}(\sqcup)^{-1}(R) \cap (a^* \times b^*) &= \{(a^n, b^m) \mid a^n \sqcup b^m \cap (ab)^* \neq \emptyset\} \\ &= \{(a^n, b^n) \mid n \geqslant 0\}.\end{aligned}$$

and it follows easily that $(\sqcup)^{-1}(R) \cap (a^* \times b^*)$ (and consequently $(\sqcup)^{-1}(R)$) are not recognizable. $\square$

However, we can consider shuffle if we fix one of the components:

Proposition 5.4 *Let $L \in \mathrm{Rec}(M)$ and let $\sigma_L : M \to M$ be defined by $\sigma_L(x) = x \sqcup L$. If $\mathrm{Rec}(M)$ is closed under product and star, then σ_L is continuous.*

Proof. Let $R \in \mathrm{Rec}(M)$ and let $\varphi : M \to F$ be a morphism onto a finite monoid recognizing both R and L. Let F^* be the free monoid on F. In order to avoid any confusion with the product in F, we shall denote the elements of F^* as sequences, like $(f_1, \ldots, f_n)$. Finally, set

$$\begin{aligned}Y = \{(f_1, \ldots, f_n) \in F^* \mid \ &\text{there exists } (g_1, \ldots, g_n) \in F^n \text{ such that} \\ &g_1 \cdots g_n \in \varphi(L) \text{ and } f_1 g_1 \cdots f_n g_n \in \varphi(R)\}.\end{aligned}$$

We claim that $\sigma_L^{-1}(R) = \theta(Y)$, where $\theta : F^* \to \mathrm{Rec}(M)$ denotes the morphism defined, for each $f \in F$, by $\theta(f) = \varphi^{-1}(f)$. Indeed

$$\begin{aligned}\sigma_L^{-1}(R) &= \{x \in M \mid (x \sqcup L) \cap R \neq \emptyset\} = \{x \in M \mid \varphi(x \sqcup L) \cap \varphi(R) \neq \emptyset\} \\ &= \{x_1 \cdots x_n \in M \mid \varphi(x_1)\varphi(y_1) \cdots \varphi(x_n)\varphi(y_n) \in \varphi(R) \\ &\qquad \text{for some } (y_1, \ldots, y_n) \in M^n \text{ such that } y_1 \cdots y_n \in L\} \\ &= \{x_1 \cdots x_n \in M \mid \varphi(x_1) g_1 \cdots \varphi(x_n) g_n \in \varphi(R) \\ &\qquad \text{for some } (g_1, \ldots, g_n) \in F^n \text{ such that } g_1 \cdots g_n \in \varphi(L)\} \\ &= \{x_1 \cdots x_n \in M \mid \varphi(x_1) \cdots \varphi(x_n) \in Y\} = \theta(Y).\end{aligned}$$

We now show that Y is a rational subset of F^*. Let $\pi_1 : (F \times F)^* \to F^*$, $\pi_2 : (F \times F)^* \to F$ and $\pi : (F \times F)^* \to F$ denote the morphisms defined respectively, for each $(f, g) \in F \times F$, by $\pi_1(f, g) = (f)$, $\pi_2(f, g) = g$ and $\pi(f, g) = fg$. It is easy to see that

$$Y = \pi_1\big(\pi_2^{-1}(\varphi(L)) \cap \pi^{-1}(\varphi(R))\big).$$

Now, since F is finite, $\pi_2^{-1}(\varphi(L))$ and $\pi^{-1}(\varphi(R))$ are recognizable subsets of $(F \times F)^*$ and so is their intersection. It follows that Y is a rational subset of F^*.

Since $\mathrm{Rec}(M)$ is closed under product and star, a straightforward induction using a rational expression for Y shows that $\theta(Y) \in \mathrm{Rec}(M)$. Thus $\sigma_L^{-1}(R)$ is recognizable, proving that σ_L is continuous. $\square$

Naturally, reversion cannot be defined in general, so we consider just the free monoid case. We denote by $\widetilde{w} = a_n \ldots a_1$ the reversion of the word $w = a_1 \ldots a_n$ ($a_i \in A$).

Proposition 5.5 *[20] The function $\tau : A^* \to A^*$ defined by $\tau(w) = \widetilde{w}$ is continuous.*

Proposition 5.6 *[20, 16] The function $\tau : M \times \mathbb{N} \to M$ defined by $\tau(x, n) = x^n$ is continuous.*

Corollary 5.7 *[16, Corollary 5.5] The transduction $\sigma : M \to M$ defined by $\sigma(x) = x^*$ is continuous.*

The following simple operator will play a crucial role in the following section:

Proposition 5.8 *Let $L \in \mathrm{Rec}(M)$. The function $\eta_L : M \to M$ defined by $\eta_L(x) = \{x\} \cap L$ is continuous.*

Proof. Let $R \in \mathrm{Rec}(M)$. Since $\mathrm{Rec}(M)$ is closed for intersection, we have

$$\eta_L^{-1}(R) = \{x \in M \mid (\{x\} \cap L) \cap R \neq \emptyset\} = L \cap R \in \mathrm{Rec}(M)$$

and η_L is continuous. □

Considered as binary operations, quotients are not continuous:

Example 5.2 [20, Example 4.2] The left (respectively right) quotient operator $\tau : A^* \times A^* \to A^*$ (respectively $\tau' : A^* \times A^* \to A^*$) defined by $\tau(x, y) = x^{-1}y$ (respectively $\tau'(x, y) = xy^{-1}$) is not continuous.

Proof. Let $A = \{a\}$. Then

$$\begin{aligned} \tau^{-1}(1) &= \{(x, y) \in A^* \times A^* \mid x^{-1}y = 1\} \\ &= \{(a^n, a^n) \mid n \geqslant 0\} \notin \mathrm{Rec}(A^* \times A^*) \end{aligned}$$

and so τ is not continuous. The proof for τ' is identical. □

Continuity holds if we consider different versions of quotients as (unary) transductions:

Proposition 5.9 *Let $L \in \mathrm{Rec}(M)$ and let $\sigma_L, \tau_L : M \to M$ be defined by*

$$\sigma_L(x) = xL^{-1}, \qquad \tau_L(x) = L^{-1}x.$$

If $\mathrm{Rec}(M)$ is closed under product, then σ_L and τ_L are continuous.

Proof. Let $R \in \mathrm{Rec}(M)$. Then

$$\begin{aligned} \sigma_L^{-1}(R) &= \{x \in M \mid xL^{-1} \cap R \neq \emptyset\} \\ &= \{x \in M \mid \text{there exists } y \in R \text{ such that } x \in yL\} \\ &= RL. \end{aligned}$$

Since $\mathrm{Rec}(M)$ is closed under product, $RL \in \mathrm{Rec}(M)$ and hence σ_L is continuous. Similarly, τ_L is continuous. □

Proposition 5.10 *Let $L \in \mathrm{Rec}(M)$ and let $\sigma_L, \tau_L : M \to M$ be defined by*

$$\sigma_L(x) = x^{-1}L, \quad \tau_L(x) = Lx^{-1}.$$

Then σ_L and τ_L are continuous.

Proof. Let $R \in \mathrm{Rec}(M)$. Then

$$\begin{aligned}\sigma_L^{-1}(R) &= \{x \in M \mid x^{-1}L \cap R \neq \emptyset\}\\ &= \{x \in M \mid \text{there exists } y \in R \text{ such that } xy \in L\}\\ &= LR^{-1}\end{aligned}$$

which is recognizable since recognizable sets are closed under quotients. Hence σ_L is continuous. Similarly, τ_L is continuous. □

We consider now different versions of insertion.

Proposition 5.11 *Let $L \in \mathrm{Rec}(M)$. The transduction $\sigma_L : M \to M$ defined by $\sigma_L(x) = \{uxv \in M \mid uv \in L\}$ is continuous.*

Proof. Let $R \in \mathrm{Rec}(M)$ and let φ be a morphism from M onto a finite monoid F recognizing both L and R. Setting $E = \{(\ell, r) \in F^2 \mid \ell r \in \varphi(L)\}$, we have

$$\begin{aligned}\sigma_L^{-1}(R) &= \{x \in M \mid \text{there exist } u, v \in M \text{ such that } uxv \in R, uv \in L\}\\ &= \{x \in M \mid \text{there exist } u, v \in M \text{ such that}\\ &\qquad\qquad \varphi(u)\varphi(x)\varphi(v) \in \varphi(R) \text{ and } \varphi(u)\varphi(v) \in \varphi(L)\}\\ &= \bigcup_{(\ell,r)\in E^2} \{x \in M \mid \ell\varphi(x)r \in \varphi(R)\}\\ &= \bigcup_{(\ell,r)\in E^2} \varphi^{-1}(\ell^{-1}\varphi(R)r^{-1}).\end{aligned}$$

Since recognizable sets are closed under quotients, it follows that $\sigma_L^{-1}(R)$ is recognizable and thus σ_L is continuous. □

Proposition 5.12 *Let $L \in \mathrm{Rec}(M)$ and let $\tau_L : M \to M$ be the transduction defined by $\tau_L(x) = \{uyv \in M \mid uv = x$ and $y \in L\}$. If $\mathrm{Rec}(M)$ is closed under product, then τ_L is continuous.*

Proof. Let $R \in \mathrm{Rec}(M)$ and let φ be a morphism from M onto a finite monoid F recognizing both L and R. Setting $E = \{(\ell, r) \in F^2 \mid \ell\varphi(L)r \cap \varphi(R) \neq \emptyset\}$, we have

$$\begin{aligned}\tau_L^{-1}(R) &= \{x \in M \mid \text{there exist } u, v \in M,\ y \in L \text{ such that}\\ &\qquad\qquad uv = x \text{ and } \varphi(u)\varphi(y)\varphi(v) \in \varphi(R)\}\\ &= \bigcup_{(\ell,r)\in E} \{x \in M \mid x = uv \text{ for some } u \in \varphi^{-1}(\ell) \text{ and } v \in \varphi^{-1}(r)\}\\ &= \bigcup_{(\ell,r)\in E} \varphi^{-1}(\ell)\varphi^{-1}(r)\end{aligned}$$

Since $\mathrm{Rec}(M)$ is closed under product, it follows that $\tau_L^{-1}(R)$ is recognizable and thus τ_L is continuous. □

Similar results hold for the analogous versions of deletion.

Proposition 5.13 *Let $L \in \mathrm{Rec}(M)$. The transduction $\sigma'_L : M \to M$ defined by $\sigma'_L(x) = \{uv \in M \mid uxv \in L\}$ is continuous.*

Proof. Let $R \in \mathrm{Rec}(M)$. Then ${\sigma'_L}^{-1}(R) = \sigma_R^{-1}(L)$, where σ_L denotes the continuous transduction of Proposition 5.11. It follows that σ'_L is continuous. $\square$

Proposition 5.14 *Let $L \in \mathrm{Rec}(M)$ and let $\tau'_L : M \to M$ be the transduction defined by $\tau'_L(x) = \{uv \in M \mid x \in uLv\}$. If $\mathrm{Rec}(M)$ is closed under product, then τ'_L is continuous.*

Proof. Let $R \in \mathrm{Rec}(M)$. Let $\varphi : M \to F$ be a morphism from M onto a finite monoid recognizing both L and R. Setting $E = \{(\ell, r) \in F^2 \mid \ell r \in \varphi(R)\}$, we have

$$\begin{aligned}
{\tau'_L}^{-1}(R) &= \{x \in M \mid \text{there exist } u, v \in M \text{ such that } uv \in R \text{ and } x \in uLv\} \\
&= \{x \in M \mid \text{there exist } u, v \in M \text{ such that } \varphi(u)\varphi(v) \in \varphi(R) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and } x \in uLv\} \\
&= \bigcup_{(\ell,r)\in E} \{x \in M \mid x \in uLv \text{ for some } u \in \varphi^{-1}(\ell) \text{ and } v \in \varphi^{-1}(r)\} \\
&= \bigcup_{(\ell,r)\in E} \varphi^{-1}(\ell)L\varphi^{-1}(r).
\end{aligned}$$

Since $\mathrm{Rec}(M)$ is closed under product, it follows that ${\tau'_L}^{-1}(R)$ is recognizable and thus τ'_L is continuous. $\square$

We show now how the class of continuous transductions is closed for a certain number of operators.

Theorem 5.15 *[16, Theorem 5.2] The composition of two continuous transductions is a continuous transduction.*

Continuous transductions are also closed under product, in the following sense:

Proposition 5.16 *[16, Proposition 5.3] Let $\tau_1 : M \to N_1$ and $\tau_2 : M \to N_2$ be continuous transductions. Then the transduction $\tau : M \to N_1 \times N_2$ defined by $\tau(x) = \tau_1(x) \times \tau_2(x)$ is continuous.*

We can extend the domain of a continuous transduction by means of a direct product in the natural way:

Proposition 5.17 *Let $\tau : M \to N$ be a continuous transduction. Then the transduction $\tau' : M \times M' \to N$ defined by $\tau'(x, y) = \tau(x)$ is continuous.*

Proof. By Proposition 5.15, it is enough to show that $\sigma : M \times M' \to M$ defined by $\sigma(x, y) = x$ is continuous. For every $L \in \mathrm{Rec}(M)$, we have

$$\sigma^{-1}(L) = \{(x, y) \in M \times M' \mid x \in L\} = L \times M' \in \mathrm{Rec}(M \times M').$$

$\square$

Proposition 5.18 *Let $\tau : M \to N$ be a continuous transduction. Then the transduction $\tau_{neg} : M \to N$ defined by*

$$\tau_{neg}(x) = \begin{cases} \emptyset & \text{if } \tau(x) \neq \emptyset \\ N & \text{otherwise} \end{cases}$$

is continuous.

Proof. Let $L \in \mathrm{Rec}(N)$. We may assume that $L \neq \emptyset$. Then we have

$$\begin{aligned} \tau_{neg}^{-1}(L) &= \{x \in M \mid \tau_{neg}(x) \cap L \neq \emptyset\} = \{x \in M \mid \tau_{neg}(x) = N\} \\ &= \{x \in M \mid \tau(x) = \emptyset\} = M \setminus \tau^{-1}(N) \end{aligned}$$

Since τ is continuous and $\mathrm{Rec}(M)$ is closed for complement, it follows that $\tau_{neg}^{-1}(L) \in \mathrm{Rec}(M)$. Thus τ_{neg} is continuous. □

Proposition 5.19 *Let $\sigma, \tau : M \to N$ be continuous transductions. Then the transduction $\sigma \cup \tau : M \to N$ defined by $(\sigma \cup \tau)(x) = \sigma(x) \cup \tau(x)$ is continuous.*

Proof. Let $L \in \mathrm{Rec}(N)$. Since $\mathrm{Rec}(M)$ is closed for union, we have

$$\begin{aligned} (\sigma \cup \tau)^{-1}(L) &= \{x \in M \mid (\sigma \cup \tau)(x) \cap L \neq \emptyset\} \\ &= \{x \in M \mid \sigma(x) \cup \tau(x) \cap L \neq \emptyset\} \\ &= \{x \in M \mid \sigma(x) \cap L \neq \emptyset\} \cup \{x \in M \mid \tau(x) \cap L \neq \emptyset\} \\ &= \sigma^{-1}(L) \ \cup \ \tau^{-1}(L) \in \mathrm{Rec}(M) \end{aligned}$$

and so $\sigma \cup \tau$ is continuous. □

The analogous result for intersection (and therefore for complement) does not hold, as we show in the next example:

Example 5.3 There exist continuous transductions $\sigma, \tau : A^* \to A^*$ such that $\sigma \cap \tau : A^* \to A^*$ is not continuous.

Proof. Let $A = \{a, b\}$. We define morphisms $\sigma, \tau : A^* \to A^*$ by $\sigma(a) = \tau(b) = a$ and $\sigma(b) = \tau(a) = 1$. Clearly, σ and τ are continuous transductions.

Now $\sigma \cap \tau$ is not continuous since

$$\begin{aligned} (\sigma \cap \tau)^{-1}(A^*) &= \{x \in M \mid \sigma(x) \cap \tau(x) \neq \emptyset\} \\ &= \{x \in A^* \mid \sigma(x) = \tau(x)\} \end{aligned}$$

consists of all words having the same number of occurrences of a and b, and this is a well-known non-recognizable language. □

6 First-order logic

In [19] and [20], the second author developped a variation of a first-order language aimed at solving certain types of equations involving recognizable languages. We show in this section that the concept of continuous transduction is powerful enough to cover all the operations, predicates, first-order connectives and quantifiers considered in that approach, and much more. For the latter, it is of course enough to consider negation, conjunction and the existential quantifier.

As far as operations go, bounded shuffle, reversion and powers (concatenation is bounded shuffle of degree 1) were the operations considered in [20]. Propositions 5.2, 5.5 and 5.6 are their counterparts in terms of continuous transductions.

The unique predicates considered involved intersecting a given recognizable language, and this situations is expressed through Proposition 5.8.

Assume now that $\mathcal{C}$ is a set of continuous transductions from M^n to M ($n \geqslant 1$) or from $M \times \mathbb{N}$ to M, where M is a monoid. Considering the direct product with $\mathbb{N}$ enables us to deal with powers. We define a first-order language (without equality) $\mathcal{L}(\mathcal{C})$ consisting of

- an operational symbol $\widehat{\tau}$ with appropriate arity for each $\tau \in \mathcal{C}$,
- a predicate symbol $\cap_L$ for each $L \in \mathrm{Rec}(M)$.

Let X be a set of variables for elements of M and let P be a set of variables for naturals. An interpretation is a mapping

$$\theta : X \cup P \to M \cup \mathbb{N}$$

satisfying $\theta(X) \subseteq M$ and $\theta(P) \subseteq \mathbb{N}$. We denote by $\mathcal{I}$ the set of all interpretations. If we restrict ourselves to some formula $\varphi(x_1, \ldots, x_n, p_1, \ldots, p_m)$ (where $x_1, \ldots, x_n \in X$ and $p_1, \ldots, p_m \in P$ are the variables occurring in φ), the corresponding set of interpretations $\mathcal{I}(\varphi)$ can viewed as the direct product $M^n \times \mathbb{N}^m$.

Denote by $\mathcal{T}$ the set of all terms of $\mathcal{L}(\mathcal{C})$ and by $\mathcal{T}' = \mathcal{T} \setminus \mathcal{P}$ the set of nonnumerical terms. Any interpretation θ can be inductively extended from variables to terms by setting, in the usual way:

(t) $\theta(\widehat{\tau}(t_1, \ldots, t_n)) = \tau(\theta(t_1), \ldots, \theta(t_n))$ for all $\tau \in \mathcal{C}$ and $t_1, \ldots, t_n \in \mathcal{T}$.

We define an atomic formula to be a formula of the form $\cap_L(t)$, where $t \in \mathcal{T}'$.

Given two interpretations θ and θ' and a variable v, we write $\theta \sim_v \theta'$ if θ and θ' coincide in every element of their domain with the possible exception of v. We define the *solution set* $\mathrm{Sol}(\varphi)$ of a first-order formula φ of $\mathcal{L}(\mathcal{C})$ according to the following inductive rules:

(f1) $\mathrm{Sol}(\cap_L(t)) = \{\theta \in \mathcal{I} \mid \theta(t) \cap L \neq \emptyset\}$ for all $L \in \mathrm{Rec}(M)$ and $t \in \mathcal{T}'$;

(f2) $\mathrm{Sol}(\neg\varphi) = \mathcal{I} \setminus \mathrm{Sol}(\varphi)$ for every formula φ;

(f3) $\mathrm{Sol}(\varphi \vee \psi) = \mathrm{Sol}(\varphi) \cup \mathrm{Sol}(\psi)$ for all formulae φ and ψ;

(f4) $\mathrm{Sol}(\exists v\varphi) = \{\theta \in \mathcal{I} \mid \theta \sim_v \theta' \text{ for some } \theta' \in \mathrm{Sol}(\varphi)\}$ for every formula φ and every variable v.

The set $\mathrm{Sol}(\varphi)$ encodes all the possible values that can assigned to both types of variables to obtain a true statement in the monoid M. We can of course view $\mathrm{Sol}(\varphi)$ as a subset of $\mathcal{I}(\varphi)$.

Theorem 6.1 *Given a first-order formula φ of $\mathcal{L}(\mathcal{C})$, it is possible to construct a continuous transduction $\sigma : \mathcal{I}(\varphi) \to M$ such that $Sol(\varphi) = \sigma^{-1}(M)$. In particular, $Sol(\varphi)$ is a recognizable set effectively constructible from φ.*

Proof. By induction on the set of formulae, starting from the atomic case.

Let $\varphi = \cap_L(t)$, with $t \in \mathcal{T}'$. Let $\sigma : M^n \times \mathbb{N}^m \to M$ denote the transduction defined by replacing each $\widehat{\tau}$ in t by τ. By Propositions 5.17, 5.16 and 5.15, σ is continuous. Moreover, it follows easily from (t) that

$$\theta(t) = \sigma\theta(X \cup P).$$

Thus

$$\begin{aligned}\mathrm{Sol}(\varphi) &= \{\theta \in \mathcal{I}(\varphi) \mid \theta(t) \cap L \neq \emptyset\} = \{\theta \in \mathcal{I}(\varphi) \mid \eta_L\theta(t) \neq \emptyset\} \\ &= \{\theta \in \mathcal{I}(\varphi) \mid \eta_L\sigma\theta(X \cup P) \neq \emptyset\} = \{\theta \in \mathcal{I}(\varphi) \mid \eta_L\sigma\theta(X \cup P) \cap M \neq \emptyset\}.\end{aligned}$$

Viewing $\mathcal{I}(\varphi)$ as the direct product $M^n \times \mathbb{N}^m$, we obtain

$$\mathrm{Sol}(\varphi) = \{\theta \in \mathcal{I}(\varphi) \mid \eta_L\sigma\theta(X \cup P) \cap M \neq \emptyset\} = (\eta_L\sigma)^{-1}(M).$$

Since $\eta_L\sigma$ is continuous by Propositions 5.8 and 5.15, the theorem holds for atomic formulae.

Assume now that $\varphi \equiv \neg\psi$, and $\mathrm{Sol}(\psi) = \tau^{-1}(M)$ with σ effectively constructible from ψ. By Proposition 5.18, τ_{neg} is continuous. We have

$$\begin{aligned}\mathrm{Sol}(\varphi) &= \mathcal{I}(\varphi) \setminus \mathrm{Sol}(\psi) = \mathcal{I}(\varphi) \setminus \tau^{-1}(M) \\ &= \{x \in \mathcal{I}(\varphi) \mid \tau(x) \cap M = \emptyset\} \\ &= \{x \in \mathcal{I}(\psi) \mid \tau_{neg}(x) \cap M \neq \emptyset\} = \tau_{neg}^{-1}(M)\end{aligned}$$

and the theorem holds for φ.

Assume next that $\varphi = \psi \vee \psi'$ and $\mathrm{Sol}(\psi) = \sigma^{-1}(M)$ with σ effectively constructible from ψ, $\mathrm{Sol}(\psi') = \sigma'^{-1}(M)$ with σ' effectively constructible from ψ'. We may assume that $\mathrm{dom}(\sigma) = \mathrm{dom}(\sigma')$ by Proposition 5.17. Let $\tau = \sigma \cup \sigma'$. By Proposition 5.19, τ is continuous. We have

$$\begin{aligned}\mathrm{Sol}(\varphi) &= \mathrm{Sol}(\psi) \cup \mathrm{Sol}(\psi') = \sigma^{-1}(M) \cup \sigma'^{-1}(M) \\ &= \{x \in \mathrm{dom}(\sigma) \mid \sigma(x) \cap M \neq \emptyset\} \cup \{x \in \mathrm{dom}(\sigma') \mid \sigma'(x) \cap M \neq \emptyset\} \\ &= \{x \in \mathrm{dom}(\tau) \mid \tau(x) \cap M \neq \emptyset\} = \tau^{-1}(M)\end{aligned}$$

and the theorem holds for φ.

Finally, we assume that $\varphi = \exists v\psi$, and $\mathrm{Sol}(\psi) = \sigma^{-1}(M)$ with $\sigma : \mathcal{I}(\psi) \to M$ effectively constructible from ψ. Assuming that v corresponds to the first component in $\mathcal{I}(\varphi) = \mathcal{I}(\psi)$, we define $\tau : \mathcal{I}(\varphi) \to M$ by

$$\tau = \sigma(\kappa_M \times id \times \ldots \times id).$$

By Propositions 5.1, 5.17, 5.16 and 5.15, τ is continuous. Since

$$\kappa_M(x) = \{x' \in \mathcal{I}(\varphi) \mid x' \sim_v x\},$$

we obtain

$$\begin{aligned}\mathrm{Sol}(\varphi) &= \{x \in \mathcal{I}(\varphi) \mid x \sim_v x' \text{ for some } x' \in \mathrm{Sol}(\psi)\}\\ &= \{x \in \mathcal{I}(\varphi) \mid x \sim_v x' \text{ for some } x' \text{ such that } \sigma(x') \cap M \neq \emptyset\}\\ &= \{x \in \mathcal{I}(\varphi) \mid \sigma\kappa_M(x) \cap M \neq \emptyset\}\\ &= \tau^{-1}(M)\end{aligned}$$

and the theorem holds. $\square$

This shows that the main results of [20, 16] can now be derived from all our results on continuous transductions.

Example 6.1 Let us show that the set of all overlapping factors of a recognizable language admits a recognizable parametric description.

Proof. By definition, the set of overlapping factors of a language L is the language

$$F = \{(uv)^n u \mid u \in A^+, v \in A^*, n > 1, x(uv)^n uy \in L \text{ for some } x, y \in A^*\}.$$

It follows from Corollary 5.3, Proposition 5.6 and Theorems 5.15 and 6.1 that the set

$$\{(u, v, n) \in A^* \times A^* \times \mathbb{N} \mid \exists x \in A^* \, \exists y \in A^* \; x(uv)^n uy \in L\}$$

is an effectively constructible recognizable language. Since the set $A^+ \times A^* \times (\mathbb{N} \setminus \{0, 1\})$ is recognizable, the set

$$\{(u, v, n) \in A^+ \times A^* \times (\mathbb{N} \setminus \{0, 1\}) \mid \exists x \in A^* \, \exists y \in A^* \; x(uv)^n uy \in L\}$$

is an effectively constructible recognizable language, which provides a parametric description of F. $\square$

7 Conclusion

We gave some topological arguments to call *continuous* transductions whose inverse preserve recognizable sets. It remains to see whether this approach can be pushed forward to use purely topological arguments, like fixpoint theorems, to obtain new results on transductions and recognizable sets. We also provided enough algorithmic results that together are more powerful than other previous approaches to solve equations involving recognizable languages.

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