ON THE TIP OF THE TONGUE

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Dedicated to Vladimir Igorevich Arnold on the occasion of his 70th Birthday

Abstract. Recently we investigated the family of *double standard maps* of the circle onto itself, given by $f_{a,b}(x) = 2x + a + \frac{b}{\pi}\sin(2\pi x) \pmod 1$. Similarly to the family of Arnold standard maps of the circle, $A_{a,b}(x) = x + a + \frac{b}{2\pi}\sin(2\pi x) \pmod 1$, if $0 < b \le 1$ then any such map has at most one attracting periodic orbit. The values of the parameters for which such orbit exists are grouped into *Arnold tongues.* Here we study the shape of the boundaries of the tongues, especially close to their tips. It turns out that the shape is fairly regular, mainly due to the real analyticity of the maps.

1. Introduction

In the Dynamical Systems Theory, one of the best known families of maps is the family of *standard maps*, called also *Arnold maps.* They are maps of the circle onto itself, given by the formula

$$A_{a,b}(x) = x + a + \frac{b}{2\pi}\sin(2\pi x) \pmod 1 \tag{1}$$

(when we write "mod 1," we mean that both the arguments and the values are taken modulo 1). This family appeared in [1] and its study was useful in the creation of the KAM Theory.

In [7], motivated by [2, 3, 4] and [6], we studied *double standard maps*, which are obtained from the standard maps by replacing rotations of the circle by its doubling:

$$f_{a,b}(x) = 2x + a + \frac{b}{\pi}\sin(2\pi x) \pmod 1. \tag{2}$$

The main feature of the family of standard maps is that the values of the parameters for which there is an attracting periodic orbit are grouped into cusp-like sets, called *Arnold tongues.* For the family of double standard maps the same is true, but with some modifications (see Figure 1). Namely, the tongues do not begin at the level $b = 0$ and are ordered differently. Moreover, we do not know much about their shapes. For the family of standard maps, the tongues are classified in a simple way by the rotation numbers of attracting periodic orbits. For the family of double standard maps it is not so obvious what points to consider as belonging to the same tongue. In [7] to do this we used complexification of the maps. Then it was even not clear whether all tongues are connected. Since in this paper we concentrate on the more local properties of

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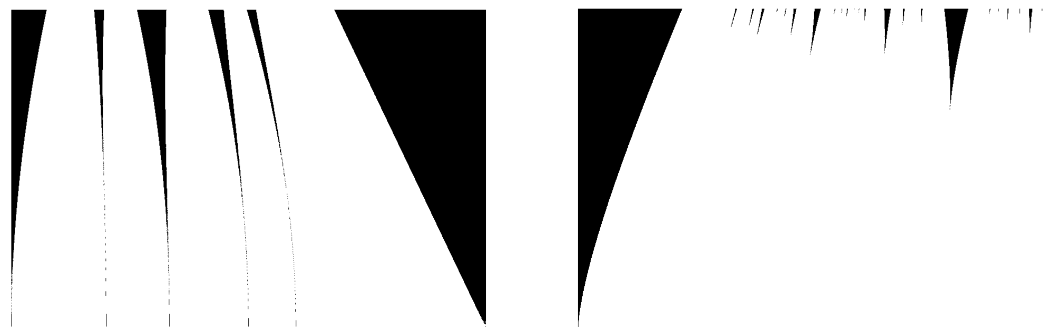

FIGURE 1. Tongues for the families of standard (left) and double standard (right) maps.

tongues, we will consider a tongue to be a connected component of a tongue from [7]. That is, a *tongue* will be a connected component of the set of those pairs (a, b) with $0 \leq b \leq 1$, for which $f_{a,b}$ has an attracting periodic orbit. Clearly, each tongue is open in the cylinder $\mathbb{T} \times [0, 1]$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The aim of this paper is to study the boundaries of the tongues, and especially their tips (loosely speaking, their lowest and perhaps highest points of tongues; for the precise definition see Section 2).

The paper is organized as follows. In Section 2 we recall the basic results proven in [7] which will be used later, and define left and right boundaries and tips of tongues. In Section 3 we prove the main technical result of the paper. In Section 4 we apply the results of the preceding sections to get new information about the tongues for the family of double standard maps. In Section 5 we show that getting more results about the tongue tips for this family would require extracting new information from the specific formula (2).

2. Tongue boundary

Let us fix the terminology concerning periodic orbits. We are working with real analytic increasing maps of the circle $\mathbb{T}$. An orbit $(x, f(x), \dots, f^{n-1}(x))$, where $f^n(x) = x$, of a map f will be called *attracting* if $(f^n)'(x) < 1$, *repelling* if $(f^n)'(x) > 1$, and *neutral* if $(f^n)'(x) = 1$. If a confusion is possible, we will call it *differentiably* attracting, repelling and neutral. If there is $\varepsilon > 0$ such that $f^n(y) < y$ for $y \in (x, x + \varepsilon)$, then x will be called *topologically attracting from the right*. Similarly one defines periodic orbits *topologically attracting from the left*, and *topologically repelling from the right* (*left*). An orbit topologically attracting (repelling) from both sides is just *topologically attracting* (*repelling*). Note that since our map is analytic and not equal to the identity (we can assume this because double standard maps have degree 2), each periodic orbit is either topologically attracting or topologically repelling from each side.

One of the main tools used to study tongues in the family of double standard maps is the following theorem (Theorem 3.5 of [7]).

Theorem 2.1. *If $0 \leq b \leq 1$ then the double standard map $f_{a,b}$, given by (2), has at most one attracting or neutral periodic orbit.*

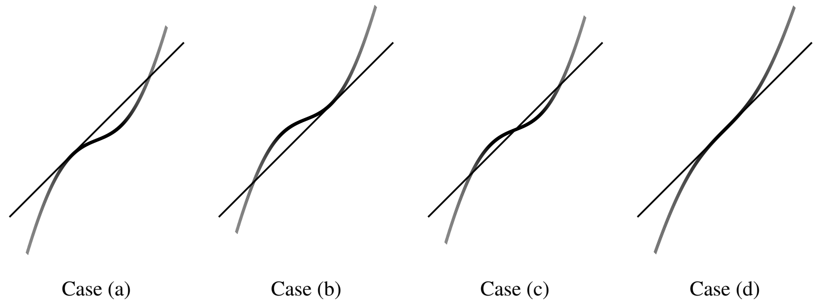

FIGURE 2. Four cases from Lemma 2.3

The proof of this theorem involved conjugacy of $f_{a,b}$ via $e^{2\pi ix}$ with a complex map

$$g_{a,b}(z) = e^{2\pi ia} z^2 e^{b\left(z-\frac{1}{z}\right)}, \tag{3}$$

of the unit circle to itself. For this map any immediate basin of attraction of an attracting or neutral periodic orbit has to contain a critical point, and there is only one pair of critical points, symmetric (in the complex sense) with respect to the unit circle. The map also preserves this symmetry and the theorem follows.

Since we are now interested in the topological types of the periodic orbits more than in the differentiable ones, we will add one more property to the above theorem.

Lemma 2.2. *A neutral periodic orbit mentioned in Theorem 2.1 cannot be topologically attracting from both sides.*

Proof. If a neutral periodic orbit is topologically attracting from both sides, it has at least two immediate basins of attraction that are not symmetric with respect to each other (but each of them is symmetric itself). Thus, such situation is also impossible. □

Another important tool from [7] is the monotone degree one circle map $\varphi_{a,b}$, which semiconjugates $f_{a,b}$ with the doubling map $D : x \mapsto 2x \pmod 1$. It is used in one of the key lemmas about the family of double standard maps (this is Lemma 4.1 of [7]).

Lemma 2.3. *Assume that p is an attracting or neutral periodic point of $f_{a,b}$ of period n. Let J be the set of all points x for which $\varphi_{a,b}(x) = \varphi_{a,b}(p)$. Then J is either a closed interval (modulo 1) or a singleton and $f^n_{a,b}|_J$ is an orientation preserving homeomorphism of J onto itself. The endpoints of J are fixed points of $f^n_{a,b}$, and one of the following four possibilities holds. In the first three cases J is an interval.*

(a) *The left endpoint of J is neutral, topologically attracting from the right and topologically repelling from the left; the right endpoint of J is repelling; there are no other fixed points of $f^n_{a,b}$ in J.*
(b) *The right endpoint of J is neutral, topologically attracting from the left and topologically repelling from the right; the left endpoint of J is repelling; there are no other fixed points of $f^n_{a,b}$ in J.*
(c) *Both endpoints of J are repelling; there is an attracting fixed point of $f^n_{a,b}$ in the interior of J; there are no other fixed points of $f^n_{a,b}$ in J.*
(d) *The set J consists of one neutral fixed point of $f^n_{a,b}$, repelling from both sides.*

Let us look at the four possibilities carefully, distinguishing between differentiable and topological properties. In Cases (a) and (b) one endpoint of J is neutral, so the other one has to be (differentiably) repelling. In Case (c) the fixed point in the interior of J is topologically attracting, so by Lemma 2.2 it is (differentiably) attracting. Then the endpoints have to be (differentiably) repelling. In Case (d) the fixed point is neutral, so "repelling" means here "topologically repelling."

Our parameter space is $\mathbb{T} \times [0, 1]$, and when we speak of the *boundary* of a tongue, we mean the boundary relative to this space. In particular, if a tongue contains a segment of the line $b = 1$, only the endpoints of this segment belong to the boundary of the tongue.

Definition 2.4. We will say that a point (a, b) from a boundary of a tongue of period n belongs to the *left boundary of the tongue* if there is $\varepsilon > 0$ such that $(a+t, b)$ belongs to the tongue for $0 < t < \varepsilon$ and does not belong to the tongue for $-\varepsilon < t < 0$. Similarly, a point (a, b) from a boundary of a tongue of period n belongs to the *right boundary of the tongue* if there is $\varepsilon > 0$ such that $(a + t, b)$ belongs to the tongue for $-\varepsilon < t < 0$ and does not belong to the tongue for $0 < t < \varepsilon$. Moreover, a point (a, b) from a boundary of a tongue of period n is the *tip* of this tongue if there is $\varepsilon > 0$ such that $(a + t, b)$ does not belong to the tongue for $-\varepsilon < t < \varepsilon$.

In the classification of Lemma 2.3 if (c) is satisfied then the point (a, b) belongs to a tongue of period n. If (a, b) belongs to a boundary of a tongue of period n then $f^n_{a,b}$ has a neutral periodic point. Thus, one of the Cases (a), (b) or (d) holds. We will investigate how those cases are related to the notions defined in Definition 2.4. However, first we have to get more information about Case (d).

3. Tongue tips

Consider now a family of maps of an interval or a circle that looks locally more or less like the family of double standard maps. We assume that the maps are real analytic and that they depend on the parameters in a real analytic way. We will show that if the dynamics of a map in this family locally looks like the one observed at the tip of a tongue and the parameters belong to the closure of a tongue, then it is really the tip. We will consider here only one parameter, whose change moves the graph of the map up or down. Note that analyticity of the function plays an essential role in the proof.

Lemma 3.1. *Let U be a neighborhood of the origin in $\mathbb{R}^2$ and let $G : U \to \mathbb{R}$ be a real analytic function. Assume that*

$$G(0, x) < 0 \quad \text{for} \quad x < 0, \quad G(0, 0) = 0, \quad \text{and} \quad G(0, x) > 0 \quad \text{for} \quad x > 0; \tag{4}$$

and that

$$\frac{\partial G}{\partial t}(0, 0) \neq 0, \tag{5}$$

where t is the first variable. Then there are open intervals I, J containing 0 such that $I \times J \subset U$ and for every $t \in I$ is exactly one $x \in J$ such that $G(t, x) = 0$. Moreover, for those t and x, if $t \neq 0$ then $\frac{\partial G}{\partial x}(t, x) > 0$.

Proof. By (5) we can apply the real analytic Implicit Function Theorem. Hence, there exists a real analytic function ψ, defined in an open interval I containing 0, such that $G(\psi(x), x) = 0$ for all $x \in I$. Moreover, these are the only solutions of the equation $G(t, x) = 0$ in a small neighborhood of the origin. If ψ is constant, then, since $G(0, 0) = 0$, it is 0, so $G(0, x) = 0$ for x close to 0. This contradicts (4). Thus, ψ is not constant. Since ψ is analytic, this implies that if I is sufficiently small then

(a) $\psi'(x) \neq 0$ for all $x \in I \setminus \{0\}$.

Set $J = \psi(I)$. We may assume that I is so small that furthermore

(b) $J \times I \subset U$;
(c) in $J \times I$ all solutions to $G(t, x) = 0$ are given by $t = \psi(x)$;
(d) $\frac{\partial G}{\partial t}$ has a constant sign in $J \times I$.

Assume that the sign mentioned in (d) is +. Then, by (4), if $G(t, x) = 0$ then x and t have opposite signs. By (a), in each component of $I \setminus \{0\}$ the function ψ is strictly monotone. Those two facts, together with $\psi(0) = 0$, imply that ψ is strictly decreasing in I (in particular, $\psi'(x) < 0$ for all $x \in I \setminus \{0\}$). Therefore, taking (c) into account, we see that if $t \in I$ then there is exactly one $x \in J$ (namely, $x = \psi^{-1}(t)$) such that $G(t, x) = 0$.

We have

$$0 = \frac{d}{dx} G(\psi(x), x) = \frac{\partial G}{\partial t}(\psi(x), x) \cdot \psi'(x) + \frac{\partial G}{\partial x}(\psi(x), x),$$

so since $\frac{\partial G}{\partial t} > 0$ and $\psi'(x) < 0$, we get $\frac{\partial G}{\partial x}(\psi(x), x) > 0$ for all $x \in I \setminus \{0\}$.

If the sign mentioned in (d) is $-$, the proof is the same, except that if $G(t, x) = 0$ then x and t have the same signs, so ψ is strictly increasing and $\psi'(x) > 0$ for $x \in I \setminus \{0\}$, and thus we reach the same conclusion. □

As a corollary we get the main result of this section.

Theorem 3.2. *Let U be a neighborhood of the origin in $\mathbb{R}^2$ and let $F : U \to \mathbb{R}$ be a real analytic function. Set $f_t(x) = F(t, x)$. Assume that f_0 has a topologically repelling fixed point at $x = 0$ and that*

$$\frac{\partial F}{\partial t}(0, 0) \neq 0. \tag{6}$$

Then there are open intervals I, J containing 0 such that $I \times J \subset U$ and for every $t \in I$ the map f_t has exactly one fixed point $x \in J$. Moreover, if $t \neq 0$ then this fixed point is (differentiably) repelling.

Proof. Set $G(t, x) = F(t, x) - x$ and apply Lemma 3.1. □

The interpretation of the conclusion of this theorem is that the point $(0, 0)$ in the parameter space can be neither in the interior of a tongue nor in its left or right boundary. Thus, if it is in the closure of a tongue, it has to be its tip.

4. Applications of Theorem 3.2 to double standard maps

Now we will apply Theorem 3.2 to the family of double standard maps. To do this, we have to know that the partial derivative of $f^n_{a,b}$ with respect to a is non-zero.

Lemma 4.1. *For any $n \geq 1$, $x \in \mathbb{T}$, we have*

$$\frac{\partial f_{a,b}^n}{\partial a}(x) > 0. \tag{7}$$

Proof. Since $f_{a,b}^n(x) = f_{a,b}^{n-1}(f_{a,b}(x))$, we have

$$\frac{\partial f_{a,b}^n}{\partial a}(x) = \frac{\partial f_{a,b}^{n-1}}{\partial a}(f_{a,b}(x)) + (f_{a,b}^{n-1})'(f_{a,b}(x))\frac{\partial f_{a,b}}{\partial a}(x).$$

By (2), $\frac{\partial f_{a,b}}{\partial a}(y) = 1$ for every y, so by induction we obtain

$$\frac{\partial f_{a,b}^n}{\partial a}(x) = \sum_{k=0}^{n-1}(f_{a,b}^k)'(f_{a,b}^{n-k}(x)).$$

Since $f_{a,b}'$ is nonnegative everywhere, so is $(f_{a,b}^k)'$. Moreover, if $k=0$ then $(f_{a,b}^k)' \equiv 1$. This proves (7). □

Now we can return to the classification of the boundary points of tongues, given by Lemma 2.3.

Theorem 4.2. *In Case (a) of Lemma 2.3 the point (a,b) belongs to the left boundary of a tongue of period n, in Case (b) to the right boundary of a tongue of period n, and in Case (d) it is a tip of a tongue of period n.*

Proof. Assume that (a,b) belongs to the boundary of a tongue of period n. Then $f_{a,b}$ has a neutral periodic orbit of period n. There are only finitely many periodic orbits of $f_{a,b}$ of period n and all of them except the neutral one are repelling. Therefore in order to see whether under a small perturbation an attracting one appears, it is enough to look at a small neighborhood of the neutral one.

In Case (a), since the left endpoint x of J is topologically attracting from the right and topologically repelling from the left, in a small neighborhood of x the graph of $f_{a,b}^n$ is below the diagonal, touching it at x. Thus, in view of Lemma 4.1, a small decrease of a will make this fixed point disappear and a small increase of a will make it bifurcate into an attracting and repelling fixed points of $f_{a,b}^n$ (this is a saddle-node bifurcation). This means that (a,b) belongs to the left boundary of a tongue of period n. Similarly, in Case (b) (a,b) belongs to the right boundary of a tongue of period n. Finally, by Theorem 3.2, in Case (d) (a,b) is a tip of a tongue of period n. □

The above theorem has important consequences. Basically, it yields that the tongues have regular, tongue-like shapes. The first corollary is immediate.

Corollary 4.3. *Every point on the boundary of a tongue either belongs to its left or right boundary or is its tip.*

In particular, degeneracies like a horizontal segment contained in the boundary of a tongue are ruled out.

Remember, that the definition of a tongue we adopted in this paper automatically makes tongues connected.

Corollary 4.4. *The intersection of every tongue with any horizontal line $b =$ constant is connected. In particular, every tongue is simply connected.*

Proof. If the intersection of a tongue of period n with a horizontal line is not connected, then, since the tongue itself is connected, there is c such that the intersection of the tongue with the horizontal line $b = c$ contains a point (a, c) belonging to the closure of two different components of this intersection. In this situation small changes of a in both directions produce an attracting periodic orbit of period n. Therefore (a, c) is not a tip and does not belong to either left or right boundary of the tongue, a contradiction. $\square$

The next corollary follows immediately from Theorem 4.2 and Proposition 4.6 of [7] (which states that whenever a piece of the boundary of a tongue consists of points for which the Case (a) or (b) of Lemma 2.3 holds, it has slope with the absolute value at least π).

Corollary 4.5. *The left and right boundaries of a tongue have slope with the absolute value at least* π.

We can get even more information about the shape of a tongue at its tip.

Theorem 4.6. *At a tip of a tongue the left and right boundaries are tangent to each other.*

Proof. Let (a_0, b_0) be a tip of a tongue of period n, and let x be the corresponding neutral periodic point of period n of f_{a_0,b_0}. By Theorem 4.2, Case (d) of Lemma 2.3 applies to it. Fix this x and consider the function $\psi(a, b) = f^n_{a,b}(x)$. By Lemma 4.1, $\frac{\partial\psi}{\partial a} > 0$, and therefore the gradient vector v of ψ is non-zero. The only vectors in direction of which the derivative of ψ is 0 are orthogonal to v. When moving parameters a, b in the direction of any other vector, starting from (a_0, b_0), we get immediately outside of the tongue by Theorem 3.2. Thus, v must be normal to both left and right boundaries of the tongue at (a_0, b_0). Therefore, those boundaries are tangent to each other. $\square$

5. Questions and examples

There are at least two interesting open questions about the tips of the tongues for the double standard family of maps. The first one is whether all tongue tips are at the bottom of the tongues, or maybe there are tongues with the tips on both sides: bottom and top. Such a tongue after rotation by 90 degrees resembles an eye rather than a tongue (see Figure 3), so we will call it an *eye*. Thus, we are asking about the existence of eyes in the double standard family.

Another question is about the order of contact of the left and right tongue boundaries at the tip. This is the same as asking about the rate with which the width of the tongue decreases to 0 as we approach its tip. By the definition of the order of contact, if this order is r then the rate is x^{r+1} (where x is the distance in the vertical direction). For Arnold tongues in the family of standard maps the order of contact depends on the rotation number of the tongue. If the rotation number is p/q (with p and q coprime) then the order is $q - 1$ (see [1]). For the family of double standard maps the situation is different. The generic order of contact is always $1/2$, since the tip of the tongue can be viewed in the generic case as a cusp bifurcation (see, e.g., [5]). In [7] we checked that this is the order for the tongue of period 1. However, we do not know whether the situation is generic for all tongue tips.

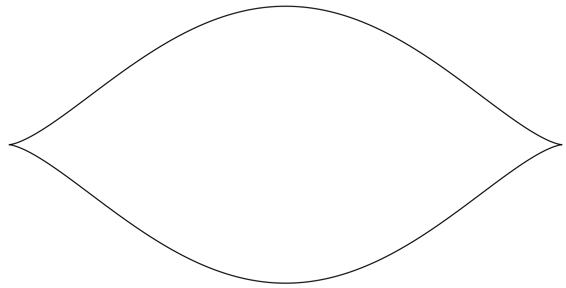

FIGURE 3. An eye, turned by 90 degrees

Answering those two questions would probably require finding new properties of the family of double standard maps, dependent on the specific formula (2). Here we will present simple examples where the families of maps are analytic (even polynomial) and the dependence on a is similar as for double standard family, but the answers are not what we would expect (no eyes and generic tips). We want additionally Theorem 2.1 and Lemma 2.2 to be satisfied locally in a neighborhood of a tongue tip.

Example 5.1. Consider the family of maps given by

$$g_{a,b}(x) = x^3 + x + a + 3xb(b-1).$$

To find the boundary of the period 1 tongue, we have to solve the system of two equations

$$\left\{ \begin{array}{l} g_{a,b}(x) = x \\ g'_{a,b}(x) = 1 \end{array} \right. \tag{8}$$

Eliminating x from this system gives us

$$a = \pm 2\big(b(1-b)\big)^{3/2},$$

which is an eye (see Figure 3).

The tongue tips are at $(0,0)$ and $(0,1)$. Observe that $\frac{\partial^2}{\partial b\, \partial x} g_{0,0}(0) < 0$, while $\frac{\partial^2}{\partial b\, \partial x} g_{0,1}(0) > 0$. This explains why at $(0,0)$ the tongue tip points down, but at at $(0,1)$ it points up.

Now we give examples of families of maps with various orders of contact between the left and right tongue boundaries.

Example 5.2. Consider the family of maps given by

$$g_{a,b}(x) = x^n + x + a - bx^k,$$

with $n \geq 3$ odd and $1 \leq k < n$. If $k \geq 3$ is odd, then for small $b > 0$ the term $-bx^k$ dominates the term x^n as $x \to 0$, so 0 is a neutral fixed point of $g_{a,b}$ topologically attracting from both sides. This violates Lemma 2.2, so we will discard those cases and assume that either $k = 1$ or k is even.

As in Example 5.1, to find the boundaries of the period 1 tongues, we have to solve the system of equations (8). Thus, we solve the system

$$\left\{ \begin{array}{l} x^n + a - bx^k = 0 \\ nx^{n-1} - bkx^{k-1} = 0 \end{array} \right. \tag{9}$$

Now the situation depends on k. Consider first the case $k = 1$. Then the second equation of 9 gives us $x = \pm \sqrt[n-1]{b/n}$. After plugging it into the first equation we obtain

$$a = \pm \frac{n-1}{n^{\frac{n}{n-1}}} b^{\frac{n}{n-1}}.$$

This means that the order of contact of the left and right boundaries of the cusp is $\frac{n}{n-1} - 1 = \frac{1}{n-1}$. If $n = 3$, this is generic $1/2$, but for larger n it is not generic.

Consider now the case of k even. The second equation of (9) has two solutions, $x = 0$ and $x = \sqrt[n-k]{kb/n}$. Thus, we get formulas for the left and right boundaries of the period 1 tongue:

$$a = 0 \quad \text{and} \quad a = \frac{(n-k)k^{\frac{k}{n-k}}}{n^{\frac{n}{n-k}}} b^{\frac{n}{n-k}}.$$

The order of contact is now $\frac{n}{n-k} - 1 = \frac{k}{n-k}$, so it is also not a generic case.

Observe that we have $b = (n/k)x^{n-k}$. If $k = 1$ then $n - k$ is even, so b can be only nonnegative on the boundary of the tongue. However, for k even, $n - k$ is odd, so b can be both positive and negative. This means that instead of one tongue we get two tongues, one pointing up and one pointing down, with a common tip. This is another non-generic phenomenon.

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