Discontinuous Maps exhibiting Symmetry Lebesgue-almost everywhere*

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Abstract

In this paper we analyse the existence of a certain type of symmetry in the context of discontinuous maps. The classical notions of symmetry cannot be applied due to the existence of discontinuities and a broader version using a measure-theory perspective is introduced. We show that a group structure is also present under the new type of symmetry and derive results which are analogous in nature to results in the theory of continuous maps. Our motivation stems from examples of symmetric patterns arising in simulations with the Goetz map.

Contents

1 Main Statements and Definitions

Neglecting sets of zero measure is a standard technique in Ergodic theory. Proving results for Lebesgue-almost every point in a measure space or defining properties for Lebesgue-almost every point implies disregarding sets of zero measure. In our paper, we make use, once again, of this technique in order to generalise the concept of symmetry for discrete dynamical systems. As far as we are aware, this has never been published nor has this

*Version without figures.

been given a theoretical ground from which one can attempt to generalise the whole theory of continuous symmetric maps. This is a first step in pursuing that goal.

First and foremost, let X denote some Euclidean space $\mathbb{R}^n$ and let λ denote the usual n-dimensional Lebesgue measure. The maps which shall be considered in this paper have the following structure. Let $T : X^* \to X$ be a transformation defined and continuous on a set $X^* \subset X$ whose complement - the *singularity set,* $\mathcal{S} := X \backslash X^*$- is assumed to have zero Lebesgue measure. The case when $X^* = X$ implies that T is continuous everywhere. Since we are primarily interested in the case when $X^* \neq X$, the singularity set is simply the set of points where T is discontinuous. Therefore, by definition of continuity, we conclude that X^* has to be open. Furthermore, its complement has to have empty interior (otherwise $\mathcal{S}$ would not have zero measure) and thus, it has to be dense as well. For our purposes, T must also satisfy the following condition: $\lambda(Z) = 0 \Rightarrow \lambda(T^{-1}(Z)) = 0$ given $Z \subset X$ (*), where $T^{-1}(Z) := \{x \in X : T(x) \in Z\}$.

Since X^* is an open set, it must be made of countably[1] many connected components[2], namely, $X^* = \cup_{k=0}^{n} P_k$, where each P_k is open and connected, and $n \in \mathbb{N} \cup \{\infty\}$. This partition, inherited from the structure of T, induces a *coding map,* χ, according to the following rule: $\chi(x) = \omega_0\omega_1 \dots \omega_n \dots$, with $\omega_i \in \mathbb{N}$, for all i, if and only if $T^n(x) \in P_{\omega_n}$. Sets of points having the same coding will be called *cells* and shall be represented by K_ω if $\chi(x) = \omega$ for every $x \in K_\omega$, where w is a right-infinite word. The collection of all cells is denoted by $\mathcal{K}$. Obviously, the coding map cannot be defined uniquely on points which will eventually fall on the singularity set, namely those that belong to the *exceptional set* $\mathcal{E} := \cup_{i=0}^{\infty} T^{-i}(\mathcal{S})$. Hence, we consider χ not being defined on $\mathcal{E}$ which, in turn, has zero Lebesgue measure as it follows from condition (*).

1.1 Symmetry in an almost-everywhere sense

Let us now define the concept of symmetry bearing in mind that T might possess a non-empty set of discontinuities. All results concerning the issues raised in this section are provided in Section 2.1.

Definition 1 *We say that σ is an* ***a.e.-symmetry*** *of T if the complement of $\{x \in X : T\sigma(x) = \sigma T(x)\}$ has zero Lebesgue measure.*

In order to define a.e.-reversing-symmetries we have to go through some subtleties. First of all, we generalise the notion of invertibility so that it fits our context (see Section 2.1). We leave the details out so that readability is enhanced in this section. Afterwards, one has to carefully deal with the equalities $T\rho(x) = \rho T^{-1}(x)$ and $\rho T(x) = T^{-1}\rho(x)$ which are equivalent in the continuous case (see Lemma 4). Having done so, it is natural to define the following:

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[1]Note that $\mathbb{R}^n$ admits a topology spanned by a countable basis.
[2]Usually called *atoms* in this context.

Definition 2 *We say that ρ is an **a.e.-reversing-symmetry** of T if the complement of either $\{x \in X : T\rho(x) = \rho T^{-1}(x)\}$ or $\{x \in X : \rho T^{-1}(x) = T\rho(x)\}$ has zero Lebesgue measure.*

From the definitions presented previously, it is natural to say that T is *essentially equivariant* if it possesses an a.e.-symmetry and *essentially reversible* when it possesses an a.e.-rev.-symmetry. One can then show (see Proposition 6) the set of a.e.-symmetries and a.e.-reversing-symmetries forms a group.

Propositions 7 and 8 complete our analysis by establishing results concerning the symmetry properties of cells, codings and exceptional sets.

1.2 Examples of a.e.-symmetry from planar piecewise rotations

A *piecewise rotation* T is a piecewise continuous map acting on $\mathbb{R}^n$ in the following way. Let $\mathcal{P} = \{P_0, \dots, P_{m-1}\}$ be a collection of disjoint open polytopes such that $\mathbb{R}^n = \cup_{i=0}^{m-1}\overline{P}_i$. For every $x \in P_i$, let $T(x) := R_i(x) := A_i(x - C_i) + C_i$ where $A_i \in \mathbb{SO}(n)$ is a orientation-preserving matrix and $C_i \in \mathbb{R}^n$ is the centre of rotation.

In this section we are concerned with the special case of $n = 2$ and $m = 2$ which is known as the Goetz map since it was first studied by Arek Goetz in his PhD thesis [Goetz, 1996].

1.2.1 The Goetz map

Here we describe all possible a.e.-(rev.)-symmetries for Goetz maps and the resulting admissible symmetry groups. A Goetz map is defined for every $z \in \mathbb{C}$ (we use $\mathbb{C}$ instead of $\mathbb{R}^2$ for simplicity of calculations) as follows,

$$G(z) := \begin{cases} e^{i\alpha_0}(z - C_0) + C_0, & \text{if } \mathfrak{Re}(z) < 0 \\ e^{i\alpha_1}(z - C_1) + C_1, & \text{if } \mathfrak{Re}(z) > 0 \end{cases}$$

where $\alpha_0, \alpha_1 \in [0, 2\pi[$, $C_0 \neq C_1$[3]$\in \mathbb{C}$, and, as per usual, $\mathfrak{Re}(x + iy) = x$.

Proposition 1 *If G is such that $\alpha_0 = -\alpha_1$ and $C_0 = -\overline{C}_1$ then G is essentially-equivariant for the a.e.-symmetry $\sigma.z = -\overline{z}$ (see Figure **??**-left)). If G is such that $\alpha_0 = \alpha_1$ and $C_0 = -C_1$ then G is essentially-equivariant for the a.e.-symmetry $\sigma.z = -z$ (see Figure **??**-right).*

Furthermore, there are no other cases of essential-equivariance with respect to $\mathbb{D}_n$, for Goetz maps.

Let ρ_a be the reflection on the line passing through both centres of rotation and ρ_b the reflection on the line passing through the origin and perpendicular to the previous one. We refer to Figure **??** (see Appendix) in which we give a geometric construction of quasi-invertible Goetz maps.

[3]The case when $C_0 = C_1$ degenerates into a piecewise rotation on S^1.

Proposition 2 *Every invertible Goetz map is essentially-reversible for ρ_a. Furthermore, ρ_a and ρ_b are the only admissible a.e.-reversing-symmetries (see Figure* **??***).*

Corollary The admissible groups of symmetry of a Goetz map are $\mathbb{Z}_2$ and $\mathbb{D}_2$.

Case (i) includes maps with only one reversing-symmetry and other maps where there is just one symmetry, whereas case (ii), concerns those maps with a group of symmetry generated by one symmetry and one reversing-symmetry. Both of them generate normal subgroups isomorphic to $\mathbb{Z}_2$.

1.2.2 Examples with $\mathbb{Z}_n$-symmetry

From the results obtained for the Goetz map we are led to believe that examples with $\mathbb{Z}_n$-symmetry should be found considering maps with n atoms. However, the existence of symmetries of order m $(= \#\mathcal{P})$ may prove to be quite restricting for a piecewise continuous map T possessing a redundant atom, that is, an atom P_i such that $T_i(P_i) \subset P_j$ for some j.

Suppose σ is a symmetry of order m and P_i is a redundant atom. Thus, we can write $\{i_k\}_{k=0}^{m-1}$ as a sequence of indices such that $i = i_0$ and $\sigma(P_{i_k}) = P_{i_{k+1}}$, for all $k < m-1$. Assume that $T(P_i) \subset P_{i_r}$. Then,

$$T_{i_1}(P_{i_1}) = T_{i_1}\sigma(P_{i_0}) = \sigma T_{i_0}(P_{i_0}) \subset \sigma(P_{i_r}) = P_{i_{r+1}}$$

Applying this successively we conclude that $T_k(P_k) \subset P_{i_{r+k (\mathsf{mod}\ m)}}$ for all k and so T is an trivial example since its coding map produces only one word which is periodic. Therefore, in order to obtain examples with cyclic symmetry we must avoid redundant atoms.

For this purpose we let $\mathcal{P}^n$ denote the collection of partitions of $\mathbb{C}$ into n cones with vertex on the origin and angle at the vertex $\theta := \frac{2\pi}{n}$. Let $z_1 \in \mathbb{C}$ and take $z_k := e^{ik\theta}.z_1$. We then define a piecewise rotation T on $\mathbb{C}$ with partition $\mathcal{P} := \{P_0, \ldots, P_{n-1}\} \in \mathcal{P}^n$ and induced rotations $R_k := e^{i\alpha}(z - z_k) + z_k$ such that $T(z) = R_k(z)$ if and only if $z \in P_k$, where $\alpha \in]0, 2\pi[$ is a common angle of rotation.

It can be easily seen that any piecewise rotation constructed in this way is $\mathbb{Z}_n$-equivariant. Let $\sigma.z := e^{i\theta}.z$ and suppose $z \in P_1 \in \mathcal{P}$ without loss of generality. Thus,

$$\sigma T.z = \sigma R_1.z = \sigma(e^{i\alpha}(z - z_1) + z_1) = e^{i(\theta+\alpha)}.z - e^{i\alpha}.z_2 + z_2$$

and,

$$T\sigma.z = R_2\sigma.z = e^{i\alpha}(\sigma.z - z_2) + z_2) = e^{i(\theta+\alpha)}.z - e^{i\alpha}.z_2 + z_2$$

since $\sigma(P_k) = P_{k+1(\mathsf{mod}\ n)}$ and $\sigma.z_k = z_{k+1(\mathsf{mod}\ n)}$ by construction. Check Figure **??** for examples with $\mathbb{Z}_4$-symmetry [4].

[4] In order to simplify the codings we have considered only four atoms. Other partitions would have increased the complexity of our programming considerably.

2 Proofs of all results

2.1 Properties of a.e.-symmetries

Given an element of $\mathbb{O}(n)$, say $\sigma : \mathbb{R}^n \to \mathbb{R}^n$, let $\Sigma_\sigma^T := \{x \in X : T\sigma(x) = \sigma T(x)\}$ denote the set of *σ-equivariance* of T.

Proposition 3 *If σ is an a.e.-symmetry of T then $\Sigma_\sigma^T = X^* \cap \sigma^{-1}(X^*)$.*

proof: Let $U(x) := T\sigma(x) - \sigma T(x)$. Obviously, U is continuous at every point in $X^* \cap \sigma^{-1}(X^*)$, which is an open set since X^* is open and σ^{-1} is linear. In particular, given $x \in X^* \cap \sigma^{-1}(X^*)$ there is an open ball B such that $x \in B \subseteq X^* \cap \sigma^{-1}(X^*)$ and $U_{|B}$ is continuous. Since $\lambda(X \backslash \Sigma_\sigma^T) = 0$, we conclude that $B \cap \Sigma_\sigma^T$ must be dense on B. By continuity of $U_{|B}$ and connectedness of B it follows that $U_{|B} \equiv 0$ since $U_{|B\cap\Sigma_\sigma^T} \equiv 0$. In particular, $x \in \Sigma_\sigma^T$.

For any point x in the complement of $X^* \cap \sigma^{-1}(X^*)$ either $T(x)$ or $T\sigma(x)$ is not defined hence the inclusion $\Sigma_\sigma^T \subset X^* \cap \sigma^{-1}(X^*)$ follows. $\square$

In particular, we can conclude from Proposition 3 that the definition of a.e.-symmetry coincides with the classical notion of symmetry in the setting of continuous maps since in that case $X^* = X$.

Let us now denote $T_i := T_{|P_i}$. We shall say that T is *quasi-invertible* whenever the following conditions are satisfied:

(C1). T_i is an homeomorphism on its image, for every i;

(C2). $T_i(P_i) \cap T_j(P_j) = \emptyset$, for every i and $j \neq i$;

(C3). $\lambda(X \backslash T(X^*)) = 0$;

(C4). $\lambda(Z) = 0 \Rightarrow \lambda(T(Z)) = 0$, and $Z \subset X$.

Conditions (C1) to (C3) allow us to consider the map $T^{-1} : T(X^*) \to X$, which assigns $T^{-1}(x) := T_i^{-1}(x)$ whenever $x \in T_i(P_i)$ and satisfies the following properties:

1. The singularity set $\mathcal{S}^{-1} := X \backslash T(X^*)$ has zero Lebesgue measure,
2. $T(X^*)$ is open and its connected components are $\{T_i(P_i)\}_{i\in\mathbb{N}}$,
3. $TT^{-1} \equiv Id$ and $T^{-1}T \equiv Id$ at every point in $Y := T(X^*) \cap X^*$, whose complement has zero Lebesgue measure.

Additional condition (C4) ensures that T^{-1} has the same properties as T, as defined in the first paragraph of this section. It also implies that T^{-1} is a quasi-invertible map.

In the remainder, T will be taken as being quasi-invertible. Let $\rho : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isometry and let us define $\Pi_{\rho,1}^T := \{x \in X : T\rho(x) = \rho T^{-1}(x)\}$ and $\Pi_{\rho,2}^T := \{x \in X : \rho T^{-1}(x) = T\rho(x)\}$.

Lemma 4 *Given a quasi-invertible map T, $\lambda(X \backslash \Pi_{\rho,1}^T) = 0$ if and only if $\lambda(X \backslash \Pi_{\rho,2}^T) = 0$, for any linear isometry ρ.*

proof: Assume that $\lambda(X \backslash \Pi_{\rho,1}^T) = 0$. Let $y \in \Pi$, where,

$$\Pi := X^* \cap T^{-1}(\Pi_{\rho,1}^T) \cap \rho^{-1} T(X^*) \cap T^{-1}\rho^{-1}(X^*) \ .$$

Since $y \in X^* \cap T^{-1}(\Pi^T_{\rho,1})$ there must exist a unique $x \in \Pi^T_{\rho,1}$ such that $T(y) = x$. Therefore,

$$T\rho(x) = \rho T^{-1}(x) \Leftrightarrow T\rho T(y) = \rho(y) \ .$$

Moreover, since $y \in T^{-1}\rho^{-1}(X^*)$ we know that $\rho T(y) \in X^*$ which implies that $T^{-1}T\rho T(y) = \rho T(y)$. Finally, we can state that $T^{-1}\rho(y)$ is well defined since $y \in \rho^{-1}T(X^*)$. Consequently,

$$T\rho T(y) = \rho(y) \Leftrightarrow \rho T(y) = T^{-1}\rho(y) \ .$$

Now, it suffices to show that $\lambda(X \backslash \Pi) = 0$ because $\Pi \subseteq \Pi^T_{\rho,2}$ by the previous argument. Obviously,

$$\lambda(X\backslash\Pi) \leq \lambda(X\backslash X^*) + \lambda(X\backslash T^{-1}(\Pi^T_{\rho,1})) + \lambda(X\backslash\rho^{-1}T(X^*)) + \lambda(X\backslash T^{-1}\rho^{-1}(X^*)) \ .$$

By assumption, $\lambda(X\backslash X^*) = 0$. In general, if a map $f : X \to X$ is invertible then $X\backslash f(Z) = f(X\backslash Z)$ for every $Z \subset X$. In our setting, given $Z \subset X$ and a quasi-invertible map T, it is true that $X\backslash T(Z) \cap Y = T(X\backslash Z) \cap Y$, where $Y = X^* \cap T(X^*)$, as defined previously. Therefore, $X\backslash T^{-1}(\Pi^T_{\rho,1}) \cap Y = T^{-1}(X\backslash \Pi^T_{\rho,1}) \cap Y$. Since $\lambda(X\backslash Y) = 0$, it follows that $\lambda(Z \cap Y) = \lambda(Z)$ which, in turn, implies that,

$$\lambda(X\backslash T^{-1}(\Pi^T_{\rho,1})) = \lambda(T^{-1}(X\backslash\Pi^T_{\rho,1})) \ .$$

It turns out, by condition (*), above, that $\lambda(T^{-1}(X\backslash\Pi^T_{\rho,1})) = 0$ since $\lambda(X\backslash\Pi^T_{\rho,1}) = 0$. Similar arguments, using also the fact that both ρ and ρ^{-1} are isometries, show that $\lambda(X\backslash\rho^{-1}T(X^*)) = \lambda(X\backslash T^{-1}\rho^{-1}(X^*)) = 0$. Thus, $\lambda(X\backslash\Pi) = 0$.

Analogously, we can prove that $\lambda(X\backslash\Pi^T_{\rho,2}) = 0$ assuming that $\lambda(X\backslash\Pi^T_{\rho,1}) = 0$. □

If ρ is an a.e.-rev.-symmetry of T then its *ρ-reversibility set* is $\Pi^T_\rho := \Pi^T_{\rho,1} \cap \Pi^T_{\rho,2}$, which is a set whose complement also has zero Lebesgue measure. Analogously to the equivariant case we have the following result.

Proposition 5 *If ρ is an a.e.-rev.-symmetry of T then $\Pi^T_\rho = Y \cap \rho^{-1}(Y)$.*

proof: Let $U_1(x) := T\rho(x) - \rho T^{-1}(x)$ and $U_2(x) := \rho T(x) - T^{-1}\rho(x)$. Simply note that U_1 is continuous on $X^* \cap \rho^{-1}T(X^*)$ and U_2 is continuous on $T(X^*) \cap \rho^{-1}X^*$. All other arguments follow accordingly. □

Again, notice that the definition of a.e.-rev.-symmetry coincides with the classical notion of reversing-symmetry in the setting of continuous maps as it follows from the proposition above.

As in the continuous case, one can still generate a group of a.e.-symmetries.

Proposition 6 *The set of a.e.-(reversing)-symmetries of a piecewise continuous map is a group.*

proof: Let σ be a symmetry of T. Let $x \in \sigma(\Sigma_\sigma) = \sigma(X^*) \cap X^*$ by Proposition 3. Therefore, both $T(x)$ and $T\sigma^{-1}(x)$ are well defined. Moreover, there must exist $y \in \Sigma_\sigma$ such that $x = \sigma(y)$ and $\sigma T(y) = T\sigma(y)$. This implies that,

$$\sigma T\sigma^{-1}(x) = \sigma T(y) = T\sigma(y) = T(x) \ .$$

Hence, $T\sigma^{-1}(x) = \sigma^{-1}T(x)$ .Since the complement of $\sigma(\Sigma_\sigma)$ has zero Lebesgue measure we conclude that σ^{-1} is a symmetry of T. For the reversible case it suffices to consider $x \in T(X^*) \cap \rho(T(X^*))$ given a reversing-symmetry ρ.

Given two symmetries, σ and γ say, we choose $x \in \Sigma_\gamma \cap \gamma^{-1}(\Sigma_\sigma)$. Thus,

$$\begin{aligned} \sigma\gamma T(x) &= \sigma T\gamma(x) \text{ since } x \in \Sigma_\gamma \\ &= T\sigma\gamma(x) \text{ because } \gamma(x) \in \Sigma_\sigma \ . \end{aligned}$$

Once again, $\Sigma_\gamma \cap \gamma^{-1}(\Sigma_\sigma)$ is a set whose complement has zero Lebesgue measure. The remaining cases follow similar arguments. Therefore, the set of all (rev.)-symmetries is a group. □

Given a symmetry of T of order r, that is, $\sigma^r \equiv Id$, we extend the singularity set to $\hat{\mathcal{S}} := \cup_{k=1}^{r}\sigma^k(\mathcal{S})$. Not only the new singularity set is symmetric, *i.e.*, $\sigma(\hat{\mathcal{S}}) = \hat{\mathcal{S}}$, but also the new partition of $\hat{X} := X\backslash\hat{\mathcal{S}} = \cup_{k=0}^{n}\hat{P}_k$ (where $n \in \mathbb{N} \cup \{\infty\}$) has the property that, for every i there must exist a j such that $\sigma(\hat{P}_i) = \hat{P}_j$. Hence, $\Sigma_\sigma^T = \hat{X}$, provided we have extended the singularity set. In what follows, we assume that the singularity set is extended (hence disregarding the hat notation).

Let $\mathcal{E}^{(m)} := \cup_{k=0}^{m}T^{-k}(\mathcal{S})$ denote the exceptional set of order m. We shall also write, by slight abuse of notation, $\sigma(i) := j$ whenever, $\sigma(P_i) \subset P_j$.

Proposition 7 *Let σ be a symmetry of T. Then, the following statements hold:*
(i) $\sigma(\mathcal{E}^{(m)}) = \mathcal{E}^{(m)}$*, for all* $m \in \mathbb{N}$*;*
(ii) $\sigma T^k(x) = T^k\sigma(x)$ *for all* $k \in \mathbb{N}$ *if and only if* $x \notin \mathcal{E}$*;*
(iii) $\chi(x) = \omega_0\omega_1\ldots$ *if and only if* $\chi(\sigma(x)) = \sigma(\omega_0)\sigma(\omega_1)\ldots$ *for every* $x \notin \mathcal{E}$*.*

The proof of this result is a simplified version of the proof of Proposition 8.

Corollary In particular: $\sigma(\mathcal{E}) = \mathcal{E}$, $\sigma(K_\omega) = K_{\sigma(\omega)}$ for every cell $K_\omega \in \mathcal{K}$ and $\sigma(\mathcal{O}_T^+(x)) = \mathcal{O}_T^+(\sigma(x))$[5] if and only if $x \notin \mathcal{E}$.

Let us now consider the case of reversing symmetries. Given a rev.-symmetry of T of order r, that is, $\rho^r \equiv Id$, we extend the singularity set to $\hat{\mathcal{S}} := \cup_{k=1}^{r}\rho^k(\mathcal{S} \cup \mathcal{S}^{-1})$. It then follows that $\rho(\hat{\mathcal{S}}) = \hat{\mathcal{S}}$ and, moreover, $\rho(\hat{P}_i) = \hat{P}_j$ where both $\hat{P}_i$ and $\hat{P}_j$ are connected components of $\hat{X} := X\backslash\hat{\mathcal{S}}$. Consequently, $\Pi_\rho^T = Y$, provided we have extended the singularity set. In what follows, we assume that the singularity set is extended (once again disregarding the hat notation).

[5] $\mathcal{O}_T^+(x)$ denotes the forward orbit of x under the transformation T.

We denote by $\mathcal{E}^{(-m)}$ the exceptional set of order m for the transformation T^{-1}. In general, we assign a -1 exponent whenever we are referring to some object with respect to the transformation T^{-1}.

Proposition 8 *Let ρ be a rev.-symmetry of T. Then, the following statements hold:*
(i) $\rho(\mathcal{E}^{(m)}) = \mathcal{E}^{(-m)}$ and $\rho(\mathcal{E}^{(-m)}) = \mathcal{E}^{(m)}$ for all $m \in \mathbb{N}$.
(ii) $\rho T^k(x) = T^{-k}\rho(x)$ and $\rho T^{-k}(x) = T^k\rho(x)$ for all $k \in \mathbb{N}$ if and only if $x \notin \mathcal{E} \cup \mathcal{E}^{-1}$.
(iii) The coding under T of every $x \notin \mathcal{E}$ is $\omega = \omega_0\omega_1 \ldots$ if and only if the coding under T^{-1} of $\rho(x) \notin \mathcal{E}^{-1}$ is $\rho(\omega) := \rho(\omega_0)\rho(\omega_1)\ldots$. Also, the coding under T^{-1} of every $x \notin \mathcal{E}^{-1}$ is $\omega = \omega_0\omega_1 \ldots$ if and only if the coding under T of $\rho(x) \notin \mathcal{E}$ is $\rho(\omega) := \rho(\omega_0)\rho(\omega_1)\ldots$.

proof: (i) We will proceed by induction. For $m = 0$ the assertion holds by definition of ρ and by extending the singularity set. Suppose $\rho(\mathcal{E}^{(m-1)}) = \mathcal{E}^{(1-m)}$, for some integer $m \geq 2$. Let $x \in \rho(\mathcal{E}^{(m)} \backslash \mathcal{S})$. In particular, $x \notin \mathcal{S}$ and so, $x \in \Pi_\rho^T = \Pi_{\rho^{-1}}^T$ since extending $\mathcal{S}$ implies that $T(X^*) = X^*$ and, therefore, $Y = X^*$. Thus,

$$\begin{aligned} x \in \rho(\mathcal{E}^{(m)} \backslash \mathcal{S}) \Leftrightarrow \rho^{-1}(x) \in \mathcal{E}^{(m)} \backslash \mathcal{S} \Leftrightarrow T\rho^{-1}(x) \in \mathcal{E}^{(m-1)} \\ \Leftrightarrow \rho^{-1}T^{-1}(x) \in \mathcal{E}^{(m-1)} \Leftrightarrow T^{-1}(x) \in \mathcal{E}^{(1-m)} \Leftrightarrow x \in \mathcal{E}^{(-m)} \backslash \mathcal{S} \ . \end{aligned}$$

Consequently, $\rho(\mathcal{E}^{(m)}) = \rho(\mathcal{E}^{(m)} \backslash \mathcal{S}) \cup \rho(\mathcal{S}) = (\mathcal{E}^{(-m)} \backslash \mathcal{S}) \cup \mathcal{S} = \mathcal{E}^{(-m)}$.

(ii) Given $k \geq 1$, take $\Lambda_k := \Lambda_k^{(1)} \cap \Lambda_k^{(2)}$ where $\Lambda_k^{(1)} = \Pi_\rho^T \cap \ldots \cap T^{-k+1}(\Pi_\rho^T)$ and $\Lambda_k^{(2)} = \Pi_\rho^T \cap \ldots \cap T^{k-1}(\Pi_\rho^T)$. Let us define $\Pi_\rho^{(k)} := \{x \in X : \rho T^k(x) = T^{-k}\rho(x)$ and $\rho T^{-k}(x) = T^k\rho(x)\}$. For every $x \in \Lambda_k^{(1)}$, it follows that,

$$\begin{aligned} \rho T^k(x) &= \rho T(T^{k-1}(x)) \\ &= T^{-1}\rho T^{k-1}(x), \text{ since } T^{k-1}(x) \in \Pi_\rho^T \\ &\vdots \\ &= T^{-k}\rho(x), \text{ since } x \in \Pi_\rho^T \ . \end{aligned}$$

Analogously, we prove that for every $x \in \Lambda_k^{(2)}$,

$$\rho T^{-k}(x) = T^k\rho(x) \ .$$

This proves that $\Lambda_k \subset \Pi_\rho^{(k)}$. Also note that $\Lambda_k = \cap_{i=1-k}^{k-1} T^i(X^*)$ which implies that $X \backslash \Lambda_k = \cup_{i=1-k}^{k-1} T^{-i}(\mathcal{S}) \overset{by\ def.}{=} \mathcal{E}^{(k-1)} \cup \mathcal{E}^{(1-k)}$. Let $\Pi_\rho^{(\infty)} := \cap_{n=1}^{\infty} \Pi_\rho^{(n)}$ and $\Lambda_\infty := \cap_{n=1}^{\infty} \Lambda_n$. It then follows that, $X \backslash \Pi_\rho^{(\infty)} \subset X \backslash \Lambda_\infty = X \backslash (\cap_{k=1}^{\infty} \Lambda_k) = \cup_{k=-\infty}^{\infty} \mathcal{E}^{(k)} \overset{by\ def.}{=} \mathcal{E} \cup \mathcal{E}^{-1}$. Conversely, if $x \in \mathcal{E}^{(k)} \cup \mathcal{E}^{(-k)}$ then for some $-k \leq m \leq k$, $T^m(x) \in \mathcal{S}$ and obviously $x \notin \Pi_\rho^{(|m|+1)}$ since either $T^{|m|+1}(x)$ or $T^{-|m|-1}(x)$ is not defined. Consequently, $X \backslash \Pi_\rho^{(\infty)} = \mathcal{E} \cup \mathcal{E}^{-1}$.

(iii) Given a positive integer k, we have that, $T^k(x) \in P_{\omega_k}$ if and only if $\rho T^k(x) \in \rho(P_{\omega_k})$. However, given any point $x \notin \mathcal{E}$ we know from the proof in (ii) that $\rho T^k(x) =$

$T^{-k}\rho(x)$ for all $k \in \mathbb{N}$. Therefore, $T^k(x) \in P_{\omega_k}$ if and only if $T^k\rho(x) \in \rho(P_{\omega_k}) = P_{\rho(\omega_k)}$ by definition of $\rho(\omega_k)$ since $\rho(P_{\omega_k})$ must equal some connected component P_i. The remaining assertion follows analogously. □

Corollary In particular: $\rho(\mathcal{E} \cup \mathcal{E}^{-1}) = \mathcal{E} \cup \mathcal{E}^{-1}$, $\rho(K_w) = K^{-1}_{\rho(w)} \in \mathcal{K}^{-1}$ for every cell $K_w \in \mathcal{K}$ and $\rho(\mathcal{O}^+_T(x)) = \mathcal{O}^+_{T^{-1}}(\rho(x))$ if and only if $x \notin \mathcal{E} \cup \mathcal{E}^{-1}$.

2.2 The symmetry group of the Goetz map

Proposition 1 If G is such that $\alpha_0 = -\alpha_1$ and $C_0 = -\overline{C}_1$ then G is essentially-equivariant for the a.e.-symmetry $\sigma.z = -\overline{z}$. If G is such that $\alpha_0 = \alpha_1$ and $C_0 = -C_1$ then G is essentially-equivariant for the a.e.-symmetry $\sigma.z = -z$. Furthermore, there are no other cases of essential-equivariance with respect to $\mathbb{D}_n$, for Goetz maps.

proof: This proof is divided into two cases: firstly σ is considered to be a reflection and secondly, σ will be a rotation.
Case A: *σ is a reflection in some mirror line L.*

Firstly, we will assume that L is not the vertical axis. For simplicity, we consider that L intersects the origin. Thus, we can find open sets (hence of positive measure) $U \subset P_0$ and $V \subset P_1$ such that $\sigma(U)$ and $\sigma(V)$ belong to the same atom, respectively. Consequently, it has to be true for at least $z \in U$ and $\hat{z} \in V$,

$$R_0\sigma.z = \sigma R_0.z \ ,R_1\sigma.\hat{z} = \sigma R_1.\hat{z} \ .$$

Let γ be the angle that L makes with the real axis, measured counter-clockwise and let $h.z = e^{i\gamma}.z$ which is defined so that $h(L)$ is in fact the real axis. We can then write $\sigma(z) = h^{-1}\varphi h(z)$, where $\varphi(z) = \overline{z}$ and $h^{-1}(z) = e^{-i\gamma}.z$. More precisely, it follows that,

$$\sigma(z) = h^{-1}\varphi(e^{i\gamma}.z) = h^{-1}(e^{-i\gamma}.\overline{z}) = e^{-i\gamma}(e^{-i\gamma}.\overline{z}) = e^{-2i\gamma}.\overline{z} \ .$$

Computing $R_0\sigma$ and σR_0 ,

$$R_0\sigma.z = e^{i\alpha_0}(e^{-2i\gamma}.\overline{z} - C_0) + C_0 = e^{i(\alpha_0-2\gamma)}.\overline{z} + \dots \ ,$$

$$\sigma R_0.z = e^{-2i\gamma}(\varphi(e^{i\alpha_0}\,(z - C_0) + C_0)) = e^{-i(2\gamma+\alpha_0)}.\overline{z} + \dots \ ,$$

we can force the polynomial $p(z) = R_0\sigma.z - \sigma R_0.z = a\overline{z} + b$, in the complex variable z, to be null, which implies that both a and b must be zero.

Calculating a,

$$a = e^{i(\alpha_0-2\gamma)} - e^{-i(2\gamma+\alpha_0)},$$

we conclude that $a = 0$ if and only if $\alpha_0 = 0$ and $\alpha_1 = 0$, by applying the same calculations to $\hat{z}$, which implies that G is, in fact, the identity map and hence commutes with any map. In conclusion, no reflection whose mirror line is not the vertical axis is allowed as a.e.-symmetry of a Goetz map.

When L is the vertical axis, one has that $\sigma(P_0) = P_1$. Moreover, note that σ can be written as $\sigma.z = -\overline{z}$. Given $z \in P_1$, we obtain,

$$R_0\sigma.z = R_0(-\overline{z}) = e^{i\alpha_0}\left(-\overline{z} - C_0\right) + C_0,$$

$$\sigma R_1.z = \sigma(e^{i\alpha_1}\left(z - C_1\right) + C_1) = -e^{-i\alpha_1}\left(\overline{z} - \overline{C}_1\right) - \overline{C}_1 \ ,$$

and so, $T\sigma = \sigma T$, for a given $z \in P_1$, if and only if the polynomial $p(z) = R_0\sigma.z - \sigma R_1.z = a\overline{z} + b$, has null coefficients. Therefore,

$$a = -e^{i\alpha_0} + e^{-i\alpha_1} = 0 \Leftrightarrow \alpha_0 = -\alpha_1 \ ,$$

$$b = -e^{i\alpha_0}C_0 + C_0 - e^{i\alpha_0}\overline{C}_1 + \overline{C}_1 = (1 - e^{i\alpha_0})(C_0 + \overline{C}_1) \ ,$$

which implies that $b = 0$ if and only if $\alpha_0 = 0$ or $C_0 = -\overline{C}_1 = \sigma(C_1)$. The case when $\alpha_0 = 0$ implies $\alpha_1 = 0$ (since $\alpha_0 = -\alpha_1$).

It can be easily verified that when $\alpha_0 = -\alpha_1, C_0 = -\overline{C}_1$ and $\sigma.z = -\overline{z}$ then, for every $z \in P_0 \cup P_1$, $G\sigma(z) = \sigma G(z)$. Therefore, the only possibility of having a reflection as an a.e.-symmetry of a Goetz map is when,

$$\boxed{\alpha_0 = -\alpha_1 \ , \ C_0 = -\overline{C}_1 \text{ and } \sigma.z = -\overline{z} \ .}$$

The dynamical properties in these cases were explained in great detail in [Goetz, 1998].

Case B: σ *is a rotation on some centre* C*.*

We may write σ as $\sigma.z = e^{i\beta}(z - C) + C$. If $\beta \neq 0$ then clearly open sets (hence of positive measure) can be found $U \subset P_0$ and $V \subset P_1$ such that $\sigma(U) \subset P_1$ and $\sigma(V) \subset P_0$ and consequently, in order to prove essential-equivariance, we must verify whether, for $z \in V$ and $\hat{z} \in U$ the following equalities hold:

$$R_0\sigma.z = \sigma R_1.z \ , R_1\sigma.\hat{z} = \sigma R_0.\hat{z} \ .$$

Computing $R_0\sigma$ and σR_1,

$$\begin{aligned} R_0\sigma.z &= R_0(e^{i\beta}(z - C) + C) \\ &= e^{i\alpha_0}(e^{i\beta}(z - C) + C - C_0) + C_0 \\ &= e^{i(\alpha_0+\beta)}(z - C) + e^{i\alpha_0}(C - C_0) + C_0 \ , \end{aligned}$$

$$\begin{aligned} \sigma R_1.z &= e^{i\beta}(e^{i\alpha_1}\left(z - C_1\right) + C_1 - C) + C \\ &= e^{i(\beta+\alpha_1)}\left(z - C_1\right) + e^{i\beta}\left(C_1 - C\right) + C \ . \end{aligned}$$

One may force, once again, the polynomial $p(z) = R_0\sigma.z - \sigma R_1.z = az + b$, to have null coefficients. So,

$$a = e^{i(\beta+\alpha_0)} - e^{i(\beta+\alpha_1)} = 0 \Leftrightarrow \alpha_0 = \alpha_1.$$

Computing b,

$$\begin{aligned} b &= -e^{i(\beta+\alpha_0)}C + e^{i\alpha_0}(C - C_0) + C_0 + e^{i(\beta+\alpha_1)}C_1 - e^{i\beta}\,(C_1 - C) - C \\ &= (e^{i(\beta+\alpha_0)} - e^{i\beta})(C_1 - C) + (e^{i\alpha_0} - 1)(C - C_0) \\ &= (e^{i\alpha_0} - 1)(e^{i\beta}(C_1 - C) + (C - C_0)) \ . \end{aligned}$$

We claim that $b = 0 \Leftrightarrow \alpha_0 = \alpha_1 = 0 \vee e^{i\beta}(C_1 - C) = C_0 - C$. Applying the same arguments to the second equivariance equality (in $\hat{z}$) we conclude that $e^{i\beta}(C_0 - C) = C_1 - C$. Therefore, $e^{i2\beta}(C_1 - C) = C_1 - C$ which implies $\beta = \pi$, as it is not interesting considering the case when $C = C_0 = C_1$ (degenerate case).

Consequently,

$$e^{i\beta}(C_1 - C) = C_0 - C \Leftrightarrow C_0 = -C_1 + 2C. \tag{1}$$

If C is not on the vertical axis, then we can find a disc centred at C and contained in one of the atoms, say P_0. For this disc, one has to prove that,

$$R_0\sigma.z = \sigma R_0.z \ ,$$

for otherwise there would not be essential-equivariance since the disc has positive measure. On account of both R_0 and σ being rotations, we conclude that $C = C_0$, since two rotations commute if and only if their centres are the same, unless both angles are null. Moreover, by the above equality (1), $C_0 = C_1$. However, once more, this leads to the degenerate case. Thus, C must belong to the vertical axis.

It can be easily verified that if $\alpha_0 = \alpha_1, C_0 = -C_1 + 2C$ and $\sigma.z = -\overline{z} + 2C$ then, for every $z \in P_0 \cup P_1$, $G\sigma(z) = \sigma G(z)$. In conclusion, the only possibility of having a rotation as an a.e.-symmetry of a Goetz map, up to translation of the origin along the vertical axis, is when,

$$\boxed{\alpha_0 = \alpha_1,\ C_0 = -C_1 \text{ and } \sigma.z = -z.}$$

□

Let ρ_a be the reflection on the line passing through both centres of rotation and ρ_b the reflection on the line passing through the origin and perpendicular to the previous one.

In Figure **??** we give a geometric construction of quasi-invertible Goetz maps.

Proposition 2 Every invertible Goetz map is essentially-reversible for ρ_a. Furthermore, ρ_a and ρ_b are the only admissible a.e.-reversing-symmetries.

proof: Firstly, notice from Figure **??** that a Goetz map is invertible if and only if both angles of rotation equal some α (or $\pi - \alpha$) and the line that connects both centres of rotation makes an angle with the real axis (imaginary axis, respectively) equal to $\alpha/2$.

We will consider the case when both angles of rotation are equal to α only, for the remaining one is similar. The following sets are now defined:

$$P_j^i = P_j \cap R_i(P_i) \ ; \ i,j = 0,1 \ .$$

It can be easily verified that $\rho_a(P_j^i) = P_i^j$ and that all rotations, R, are reversible with respect to any reflection, ρ, whose mirror line goes through the centre of rotation, *i.e.*, $R\rho = \rho R^{-1}$. Since that G is quasi-invertible, its inverse can be written as follows,

$$G^{-1}(z) = \begin{cases} R_0^{-1}(z) \text{ if } z \in R_0\,(P_0) \\ R_1^{-1}(z) \text{ if } z \in R_1\,(P_1) \end{cases}$$

Therefore, in order to prove essential-reversibility, one has to show that $G\rho_a = \rho_a G^{-1}$. Clearly, the sets $\{P_j^i\}_{i,j=0,1}$ form a partition of the phase space with zero measure complement, or in other words, $z \in P_j^i$ almost surely for some i and j. Let $z \in P_j^i$. Then $G^{-1}.z = R_i^{-1}.z$ and $\rho_a(z) \in P_i^j$, which implies that $G\rho_a(z) = R_i\rho_a(z)$. Consequently,

$$G\rho_a(z) = \rho_a G^{-1}(z) \Leftrightarrow R_i\rho_a(z) = \rho_a R_i^{-1}(z) \ ,$$

which is clearly true for all $z \in \cup_{i,j} P_j^i$.

Suppose γ is another a.e.-reversing-symmetry. So, $\rho_a\gamma$ must be an a.e.-symmetry. As seen previously in Proposition 1, $\rho_a\gamma = -\mathsf{Id}$ since any Goetz map with $C_0 = -\overline{C}_1$ cannot be invertible. This last equality implies that $\gamma = -\rho_a = \rho_b$. $\square$

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References

[1] Goetz [1996], *Dynamics of Piecewise Isometries*, University of Illinois.

[2] Goetz [1998], Dynamics of a Piecewise Rotation; *Disc. and Cont. Dyn. Sys.*, **4** (4), 593-608.

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