STATISTICAL STABILITY OF EQUILIBRIUM STATES FOR INTERVAL MAPS

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Abstract. We consider families of multimodal interval maps with polynomial growth of the derivative along the critical orbits. For these maps Bruin and Todd have shown the existence and uniqueness of equilibrium states for the potential $\varphi_t : x \mapsto -t \log |Df(x)|$, for t close to 1. We show that these equilibrium states vary continuously in the weak* topology within such families. Moreover, in the case $t = 1$, when the equilibrium states are absolutely continuous with respect to Lebesgue, we show that the densities vary continuously within these families.

1. Introduction

One of the main goals in the study of Dynamical Systems is to understand how the behaviour changes when we perturb the underlying dynamics. In this paper, we examine the persistence of statistical properties of a multimodal interval map (I, f). In particular we are interested in the behaviour of the Cesaro means $\frac{1}{n}\sum_{k=0}^{n-1} \varphi \circ f^k(x)$ for a potential $\varphi : I \to \mathbb{R}$ for 'some' points x, as $n \to \infty$. If the system possesses an invariant *physical measure* μ, then part of this statistical information is described by μ since, by definition of physical measure, there is a positive Lebesgue measure set of points $x \in I$ such that

$$\overline{\varphi}(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^k(x) = \int \varphi \, d\mu.$$

If for nearby dynamics these measures are proven to be close, then the Cesaro means do not change much under small deterministic perturbations. This motivated Alves and Viana [AV] to propose the notion of *statistical stability*, which expresses the persistence of statistical properties in terms of the continuity of the physical measures. A precise definition will be given in Section 1.1

However, the study of Cesaro means is not confined to the analysis of these measures. We can consider the encoding of these statistical properties by 'multifractal decomposition', see [P] for a general introduction. Given $\alpha \in \mathbb{R}$, we define the sets

$$B_\alpha := \{x \in I : \overline{\varphi}(x) = \alpha\}, \; B' := \{x \in I : \overline{\varphi}(x) \text{ does not exist}\}.$$

2000 *Mathematics Subject Classification.* 37D35, 37C75, 37E05, 37D25.

Key words and phrases. Equilibrium states, statistical stability, thermodynamic formalism, interval maps.

JMF is partially supported by POCI/MAT/61237/2004 and MT is supported by FCT grant SFRH/BPD/26521/2006. Both authors are supported by FCT through CMUP.

Then the multifractal decomposition in this case is

$$I = B' \cup \left(\bigcup_{\alpha} B_\alpha \right).$$

Understanding the nature of this decomposition gives us information about the statistical properties of the system. This can be studied via 'equilibrium states'. See [PW] for a fuller account of these ideas, where the theory is applied to subshifts of finite type.

To define equilibrium states, given a potential $\varphi : I \to \mathbb{R}$, we define the *pressure* of φ to be

$$P(\varphi) := \sup \left\{ h_\mu + \int \varphi \, d\mu \right\},$$

where this supremum is taken over all invariant ergodic probability measures. Here h_μ denotes the metric entropy of the system (I, f, μ). Any measure μ which 'achieves the pressure', i.e. $h_\mu + \int \varphi \, d\mu = P(\varphi)$, is called an *equilibrium state* for (I, f, φ).

In this paper for a given map f, we are interested in the equilibrium state μ_t of the 'natural' potential $\varphi_t : x \mapsto -t \log |Df|$ for different values of t. This theory was developed by Ledrappier [L], Bruin and Keller [BK] and then latterly by Pesin and Senti [PSe] and Bruin and the second author [BT3]. For $t = 1$ there is an equilibrium state μ_1 which is a physical measure. In this setting, we also refer to this measure as an absolutely continuous invariant measure (acip) since it is absolutely continuous with respect to Lebesgue. This measure is supported on $B_{\lambda(\mu_1)}$ where $\lambda(\mu_1)$ is the Lyapunov exponent of the acip. For a given value of α, close to $\lambda(\mu_1)$, there is an equilibrium state μ_t supported on B_α, for some t close to 1. Therefore, to understand the statistics of the system with potential φ_1, it is useful to study the properties of the relevant equilibrium states. Note also that these ideas extend to equilibrium states of other potentials, see [BT3] for example. We would also like to point out the theory presented in this paper extends to Manneville-Pommeau maps, see Remark 6.2.

Our main goal is to show that the equilibrium states vary continuously with f.

1.1. Statement of results.

Here we establish our setting and make our statements more precise. Let Crit denote the set of critical points. We say that $c \in \text{Crit}$ is a *non-flat* critical point of f if there exists a diffeomorphism $g_c : \mathbb{R} \to \mathbb{R}$ with $g_c(0) = 0$ and $1 < \ell_c < \infty$ such that for x close to c, $f(x) = f(c) \pm |\varphi_c(x - c)|^{\ell_c}$. The value of ℓ_c is known as the *critical order* of c. Throughout, $\mathcal{H}_\ell$ will be the collection of C^2 interval maps which have finitely many branches, negative Schwarzian (that is, $1/\sqrt{|Df|}$ is convex away from critical points), only non-flat critical points all with fixed order ℓ. Moreover, for simplicity, we assume that maps in this class have no points of inflection and are transitive.

We will consider families of maps in $\mathcal{H}_\ell$ which satisfy the following conditions. The first one is the Collet-Eckmann condition:

(1) There exist $C, \alpha > 0$ such that $|Df^n(f(c))| \geqslant Ce^{\alpha n}$ for all $c \in \text{Crit}$.

Secondly we consider maps satisfying a polynomial growth condition.

(2) There exist $C, \beta > 0$ such that $|Df^n(f(c))| \geqslant Cn^\beta$ for all $c \in \mathrm{Crit}$.

We will take a map $f_0 \in \mathcal{F}$ where we suppose that all maps in $\mathcal{F}$ satisfy either (1) or (2). We will sometimes denote these families as $\mathcal{F}_e(C, \alpha)$ or $\mathcal{F}_p(C, \beta)$ respectively in order to clarify the constants involved.

We will consider equilibrium states for maps in these families. Suppose first that maps in $\mathcal{F}$ satisfy (1). Then by [BT2], there exists an open interval $U_{\mathcal{F}} \subset \mathbb{R}$ containing 1 and depending on α and ℓ so that for $f \in \mathcal{F}$ and $t \in U_{\mathcal{F}}$ the potential $\varphi_{f,t} : x \mapsto -t \log |Df(x)|$ has a unique equilibrium state $\mu = \mu_f$. If instead we assume that maps in $\mathcal{F}$ satisfy (2) then we have the same result but instead $U_{\mathcal{F}}$ is of the form $(t_{\mathcal{F}}, 1]$ where $t_{\mathcal{F}}$ depends on β and ℓ.

We choose our family $\mathcal{F}$ and fix $t \in U_{\mathcal{F}}$ and denote $\varphi_{f,t}$ by φ_f. We fix $f_0 \in \mathcal{F}$ and suppose that $\{f_n\}_n \subset \mathcal{F}$ has $\|f_n - f_0\|_{C^2} \to 0$ as $n \to \infty$. We let $\mu_n = \mu_{f_n,t}$ denote the corresponding equilibrium state for each n with respect to the potential φ_{f_n}. We say that μ_0 is *statistically stable* if a weak* limit μ_∞ of $\{\mu_n\}_n$ is equal to μ_0.

Theorem A. *Let $\mathcal{F} \subset \mathcal{H}_\ell$ be a family satisfying* (1) *or* (2) *with potentials $\varphi_{f,t}$ as above. Then, for every fixed $t \in U_{\mathcal{F}}$ and $f \in \mathcal{F}$, the equilibrium state $\mu_{f,t}$ as above is statistically stable within the family $\mathcal{F}$, i.e., the map $\mathcal{F} \ni f \to \mu_{f,t}$ is continuous in the weak* topology.*

Although the definition of statistical stability involves convergence of measures in the weak* topology, when we are dealing with acips, it makes sense to consider a stronger type of stability due to Alves and Viana [AV]: we say that f is *strongly statistically stable* in the family $\mathcal{F}$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that for

$$\|f - g\|_{C^2} < \delta \text{ implies } \int \left| \frac{d\mu_f}{dm} - \frac{d\mu_g}{dm} \right| \, dm < \epsilon, \tag{3}$$

where m denotes Lebesgue measure, $g \in \mathcal{F}$, and μ_f and μ_g denote the acips for f and g respectively. As a byproduct of the proof of Theorem A we also obtain:

Theorem B. *Let $\mathcal{F} \subset \mathcal{H}_\ell$ be a family satisfying* (1) *or* (2)*. Then, for every $f \in \mathcal{F}$, the acip μ_f is statistically stable in the strong sense, see* (3).

This result generalises the one in [F], where strong statistical stability was proved for Benedicks-Carleson quadratic maps, which are unimodal and satisfy condition (1). The issue of continuous variation of physical measures for unimodal maps was previously addressed in [Ts, RS, Th]. Statistical stability of physical measures (in the strong sense) was also obtained in [A, AV] for non-uniformly expanding maps. This was also considered, under some additional robustness assumptions, in [Ar]. In the non-hyperbolic invertible setting, statistical stability was also studied in [V] for diffeomorphisms with dominated splitting and in [ACF] for Hénon maps of the Benedicks-Carlseon type.

1.2. **Structure of the paper.** In Section 2, we build inducing schemes for each $f \in \mathcal{F}$ and show that the construction can be shadowed for nearby dynamics. Although other methods could be used to build the inducing schemes, we used Hofbauer towers which are explained in more detail in Appendix A. In Section 3, we introduce some thermodynamic formalism, discuss the existence and uniqueness of equilibrium states and study their properties, especially their Gibbs property. In Section 4 we show that the weak* limit of Gibbs measures is also a Gibbs measure. Section 5 is devoted to showing that the induced measures on the tower vary continuously with $f \in \mathcal{F}$. Finally, in Section 6 we show that the continuity of the measures survives the saturation of the induced measures into the original equilibrium states, completing the proof of Theorem A. We finish that section by showing that the choice of inducing schemes and the uniformity properties of the family $\mathcal{F}$ proved along the way allow us to use the results of [AV] to obtain Theorem B.

The main new step in this paper is to notice that the invariant measures on our inducing schemes are Gibbs states. This allows us to pass information from the limiting inducing scheme to other nearby inducing schemes. In this way we can avoid the techniques of [AV] which used convergence in the sense of (3). Those techniques can not be applied in this setting since, unless $\varphi = -\log|Df|$, we are not considering acips, and thus Lebesgue measure has no relevance.

In this paper we write $x = B^{\pm}y$ to mean $\frac{1}{B} \leqslant \frac{x}{y} \leqslant B$. Also for $\lambda \in (0,\infty)$ and an interval J, we let λJ be the interval sharing the same centre as J and having size $\lambda|J|$. For an interval J and a sequence of intervals $\{J_n\}_n$, we write $J_n \to J$ as $n \to \infty$ if the convergence is in the Hausdorff metric.

Acknowledgements: We would like to thank Henk Bruin for his comments on an early version of this paper.

2. Choice of inducing schemes

We will denote the fixed map f in Theorems A and B by f_0 and take a sequence $\{f_n\}_n$ such that $\|f_n - f_0\|_{C^2} \to 0$. An attractor for f_0 is a transitive union of cyclically permuted intervals. For simplicity we will assume that this attractor is unique.

In this section we use the theory of *Hofbauer towers* developed by Hofbauer and Keller [H, HK, K] to produce inducing schemes as described in [B]. We also show how the inducing schemes move with n. Note that we could also have used other methods to make these inducing schemes, see [BLS] for example.

We let $\mathcal{Q}_{n,k}$ be the natural partition into maximal closed intervals on which f_n^k is diffeomorphic. We will denote members of $\mathcal{Q}_{n,k}$ by $\mathrm{X}_{n,k}$. Note that for $\mathrm{X}_{n,k}, \mathrm{X}'_{n,k} \in \mathcal{Q}_{n,k}$ with $\mathrm{X}_{n,k} \neq \mathrm{X}'_{n,k}$ then $\mathrm{X}_{n,k} \cap \mathrm{X}'_{n,k}$ consists of at most one point.

$\mathcal{Q}_{n,1}$ consists of $\#\mathrm{Crit}_n + 1$ intervals. For n large enough this is equal to $\#\mathrm{Crit}_0 + 1$. Thus for all n large enough we can label the intervals in $\mathcal{Q}_{n,1}$ from left to right as $\mathrm{X}^1_{n,1}, \ldots, \mathrm{X}^{\#\mathrm{Crit}_0+1}_{n,1}$. So here $\mathrm{X}^i_{n,1} \to \mathrm{X}^i_{0,1}$ as $n \to \infty$. We let the *k-itinerary of a point* $x \in I \setminus \bigcup_{c\in\mathrm{Crit}} \bigcup_{0\leqslant j\leqslant k} f^{-j}(c)$ to be a sequence $(x_0, \ldots, x_k)$ where $x_j = i \in$

$\{1, \ldots, \mathrm{Crit}_0 + 1\}$ whenever $f^j(x) \in \mathrm{X}^i_{0,1}$. For $k' \geqslant k$, we define the *k-itinerary of a k'-cylinder* $\mathrm{X}_{n,k'}$ to be the k-itinerary of the midpoint of $\mathrm{X}_{n,k'}$.

Given $k \geqslant 1$, we fix a numbering $\mathrm{X}^i_{0,k}$. For $i, k \geqslant 1$ and n large enough there is a corresponding cylinder $\mathrm{X}^{i'}_{n,k}$ which has same k-itinerary as $\mathrm{X}^i_{0,k}$. Indeed for n large enough this cylinder will be unique. So we change i' to i. By Lemma 2.2, $\mathrm{X}^i_{n,k} \to \mathrm{X}^i_{0,k}$ as $n \to \infty$. Then we can directly compare properties of $\mathrm{X}^i_{0,k}$ and $\mathrm{X}^i_{n,k}$ in a canonical way.

We next define the Hofbauer tower. We let

$$\hat{I}_n := \bigsqcup_{k \geqslant 1} \bigsqcup_{\mathrm{X}_{n,k} \in \mathcal{Q}_{n,k}} f^k_n(\mathrm{X}_{n,k}) / \sim$$

where $f^k_n(\mathrm{X}_{n,k}) \sim f^{k'}_n(\mathrm{X}_{n,k'})$ if $f^k_n(\mathrm{X}_{n,k}) = f^{k'}_n(\mathrm{X}_{n,k'})$. Let $\mathcal{D}_n$ be the collection of domains of $\hat{I}_n$ and $\pi_n : \hat{I}_n \to I$ be the inclusion map. A point $\hat{x} \in \hat{I}_n$ can be represented by (x, D) where $\hat{x} \in D$ for $D \in \mathcal{D}_n$ and $x = \pi_n(\hat{x})$.

The map $\hat{f}_n : \hat{I} \to \hat{I}$ is defined as

$$\hat{f}(\hat{x}) = \hat{f}(x, D) = (f(x), D')$$

if there are cylinder sets $\mathrm{X}_{n,k} \supset \mathrm{X}_{n,k+1}$ such that $x \in f^k_n(\mathrm{X}_{n,k+1}) \subset f^k_n(\mathrm{X}_{k,n}) = D$ and $D' = f^{k+1}_n(\mathrm{X}_{n,k+1})$. In this case, we write $D \to D'$, giving $(\mathcal{D}_n, \to)$ the structure of a directed graph. It is easy to check that there is a one-to-one correspondence between cylinder sets $\mathrm{X}_{n,k} \in \mathcal{Q}_{n,k}$ and k-paths $D_0 \to \cdots \to D_n$ starting at the base of the Hofbauer tower. For each $R \in \mathbb{N}$, let $\hat{I}^R_n$ be the compact part of the Hofbauer tower defined by

$$\hat{I}^R_n = \sqcup\{D \in \mathcal{D}_n : \text{ there exists a path } D_0 \to \cdots \to D \text{ of length } r \leqslant R\}$$

The map π_n acts as a semiconjugacy between $\hat{f}_n$ and f_n: $\pi_n \circ \hat{f}_n = f_n \circ \pi_n$.

We will use the method of [B] we next show how Hofbauer towers allow us to define inducing schemes. We fix $\delta > 0$. Then for $A \subset I$ we let $A' = (1 + \delta)A$. We define

$$\hat{A} = \hat{A}(\delta) = \sqcup\{D \cap \pi^{-1}(A) : D \in \mathcal{D}_n, \pi(D) \supset A'\}.$$

The following lemma is left to the reader.

Lemma 2.1. *For $\mathrm{X}^i_{0,k} \in \mathcal{Q}_{0,k}$ as above there exists $N \geqslant 1$ such that for $n \geqslant N$, there exists $\mathrm{X}^i_n \in \mathcal{Q}_{n,k}$ so that $\mathrm{X}^i_{n,k} \to \mathrm{X}^i_{0,k}$. Moreover, $\hat{\mathrm{X}}^i_{n,k} \to \hat{\mathrm{X}}^i_{0,k}$, in the sense that for any R, $\hat{\mathrm{X}}^i_{n,k} \cap I^R_n \to \hat{\mathrm{X}}^i_{0,k} \cap I^R_0$ in the Hausdorff metric.*

As in [B], we consider the first return map $F_{\hat{\mathrm{X}}^i_{n,k}} : \bigcup_j \hat{\mathrm{R}}^j \to \hat{\mathrm{X}}^i_{n,k}$ where $F_{\hat{\mathrm{X}}^i_{n,k}} = \hat{f}^{r_{\hat{\mathrm{X}}^i_{n,k}}}$ for a return time $r_{\hat{\mathrm{X}}^i_{n,k}}$ which is constant on each $\hat{\mathrm{R}}^j$. This gives an inducing scheme $F_{\mathrm{X}^i_{n,k}} : \bigcup \mathrm{R}^j \to \mathrm{X}^i_{n,k}$ with inducing time $\tau_{\mathrm{X}^i_{n,k}}$ where on each R^j, $\tau_{\mathrm{X}^i_{n,k}}$ is a constant $\tau^j_{\mathrm{X}^i_{n,k}} \geqslant 1$. As in [B], for $x \in \bigcup \mathrm{R}^j$, $\tau_{\mathrm{X}^i_{n,k}}(x)$ is given by $r_{\hat{\mathrm{X}}^i_{n,k}}(\hat{x})$ for any $\hat{x} \in \hat{\mathrm{X}}^i_{n,k}$ such that $\pi_n(\hat{x}) = x$. Let $(\mathrm{X}^i_{n,k})^\infty$ denote the set of points for which $F_{\mathrm{X}^i_{n,k}}$ is defined for all time.

Lemma 2.2. *For all $x \in I$ there exists $n_j \to \infty$ and a cylinder $\mathrm{X}^i_{0,k}$ containing x so that $\overline{(\mathrm{X}^i_{n_j,k})^\infty} = \mathrm{X}^i_{n_j,k}$, and for all $y \in (\mathrm{X}^i_{0,k})^\infty$, $F_{\mathrm{X}^i_{n_j,k}}(y) \to F_{\mathrm{X}^i_{0,k}}(y)$ as $j \to \infty$.*

We postpone the proof of this lemma to the appendix. From here on we will replace the sequence $\{f_n\}_n$ with the one given in the lemma.

Remark 2.3. *It is common to suppose that an 'inducing interval' to be chosen to be 'nice' in the sense of Martens: an interval A is nice in this sense if $f^n(\partial A) \cap \overset{\circ}{A} = \emptyset$ for all $n \geqslant 1$. The key property that the inducing scheme then has is as follows. If $x \in A$ has a neighbourhood A_x such that $f^n : A_x \to A$ is a homeomorphism, then $A_x \subset A$. The theory of inducing schemes can then be applied. Our intervals $\mathrm{X}_{n,k}$ are not nice in this sense, but the key property still holds. This can be proved using the structure of cylinder sets.*

3. Pressure

If (X, T) is a dynamical system with potential $\Phi : X \to \mathbb{R}$, then the measure m is Φ-*conformal* if

$$m(T(A)) = \int_A e^{-\Phi(x)}\ dm(x)$$

whenever $T : A \to T(A)$ is one-to-one. In other words, $dm \circ T(x) = e^{-\Phi(x)} dm(x)$. We define the transfer operator for the potential Φ as

$$\mathcal{L}_\Phi g(y) := \sum_{T(y)=x} e^{\Phi(y)} g(y).$$

Assume that $\mathcal{S}_1 = \{\mathrm{C}^i_1\}_i$ is a countable Markov partition of X such that $T : \mathrm{C}^i_1 \to X$ is injective for each $\mathrm{C}^i_1 \in \mathcal{S}_1$.

Suppose that (X, T, Φ) is topologically mixing. For every $\mathrm{C}^i_1 \in \mathcal{S}_1$ and $n \geqslant 1$ let

$$Z_n(\Phi, \mathrm{C}^i_1) := \sum_{T^n x = x} e^{\Phi_n(x)} 1_{\mathrm{C}^i_1}(x),$$

where $\Phi_n(x) = \sum_{j=0}^{n-1} \Phi \circ F^j(x)$. We define

$$V_n(\Phi) := \sup_{\mathrm{C}_n \in \mathcal{S}_n} \sup_{x,y \in \mathcal{S}_n} |\Phi(x) - \Phi(y)|, \tag{4}$$

where $\mathcal{S}_n = \bigvee_{j=0}^{n-1} T^{-j}(\mathcal{S}_1)$ is the n-join of the Markov partition $\mathcal{S}_1$. We say that Φ has summable variations if $\sum_{n \geqslant 1} V_n(\Psi) < \infty$. Under this condition, we set $B_k := \exp\left(\sum_{n \geqslant k+1} V_n(\Psi)\right)$. As in [Sa1], we define the *Gurevich pressure* of Φ as

$$P_G(\Phi) := \lim_{n\to\infty} \frac{1}{n} \log Z_n(\Phi, \mathrm{C}^i_1).$$

This limit exists since $\log Z_n(\Psi, \mathrm{C}^i_1)$ is almost superadditive:

$$\log Z_n(\Psi, \mathrm{C}^i_1) + \log Z_k(\Psi, \mathrm{C}^i_1) \leqslant \log Z_{n+k}(\Psi, \mathrm{C}^i_1) + B_1.$$

Therefore, $P_G(\Psi) = \sup_n \frac{1}{n} \log Z_n(\Psi, \mathrm{C}^i_1) > -\infty$. By the mixing condition, $P_G(\Psi)$ is independent of the choice of C^i_1. To simplify the notation, we will often suppress the dependence of $Z_n(\Phi, \mathrm{C}_1^i)$ on C_1^i. Furthermore, if $\|\mathcal{L}_\Phi 1\|_\infty < \infty$ then $P_G(\Phi) < \infty$, see Chapter 2 of [Sa1].

Assume now that $T : X \to X$ is the full shift. That is $T : \mathrm{C}_1^i \to X$ is bijective for all i.

We say that μ is a *Gibbs measure* if there exists $K < \infty$ such that for all $\mathrm{C}_k \in \mathcal{S}_k$,

$$\frac{1}{K} \leqslant \frac{\mu(\mathrm{C}_k)}{e^{\Phi_{F,k}(x) - kP_G(\Phi)}} \leqslant K$$

for any $x \in \mathrm{C}_n$. Here $\Phi_{F,k}(x) := \Phi(F^{k-1}(x)) + \cdots + \Phi(x)$.

Theorem 3.1 ([Sa3])**.** *If (X, T, Φ) is the full shift, $\sum_{n \geqslant 1} V_n(\Phi) < \infty$ and $P_G(\Phi) < \infty$ then Φ has an invariant Gibbs measure. Moreover the Gibbs measure μ_Φ has the following properties*

(a) *If $h_{\mu_\Phi}(T) < \infty$ or $-\int \Phi d\mu_\Phi < \infty$ then μ_Φ is the unique equilibrium state (in particular, $P(\Phi) = h_{\mu_\Phi}(T) + \int_X \Phi \, d\mu_\Phi$);*
(b) *The Variational Principle holds, i.e., $P_G(\Phi) = P(\Phi)$ $(= h_{\mu_\Phi}(T) + \int_X \Phi \, d\mu_\Phi)$;*
(c) *μ_Φ is finite and $\mu_\Phi = \rho_\Phi \, dm_\Phi$ where $\mathcal{L}_\Phi \rho_\Phi = \lambda \rho_\Phi$ and $\mathcal{L}_\Phi^* m_\Phi = \lambda m_\Phi$ for $\lambda = e^{P_G(\Phi)}$, i.e., $m_\Phi(TA) = \int_A e^{\Phi - \log \lambda} \, dm_\Phi$;*
(d) *This ρ_Φ is unique and m_Φ is the unique $(\Phi - \log \lambda)$-conformal probability measure.*

Note that because μ_Φ is a Gibbs measure, $\mu_\Phi(\mathrm{C}_n^i) > 0$ for every cylinder set $\mathrm{C}_n^i \in \mathcal{S}_n$, $n \in \mathbb{N}$.

From Lemma 2.2, we have inducing schemes $(\mathrm{C}_{n,0}, F_n, \Phi_{F_n})$ for $\mathrm{C}_{n,0} = \mathrm{X}_{n,k}^i$ and $F_n = F_{\mathrm{X}_{n,k}^i}$. As in [BT2] we set $\psi_n = \varphi_{f_n} - P(\varphi_{f_n})$ and get inducing schemes $(\mathrm{C}_{n,0}, F_n, \Psi_{F_n})$. Moreover we get $P_G(\Psi_n) = 0$ and equilibrium states $\mu_{F_n} = \mu_{\Psi_n}$ and $\mu_n = \mu_\psi$ for $(\mathrm{C}_{n,0}, F_n, \Psi_n)$ and (I, f_n, ψ_n) respectively. Note that an equilibrium state for ψ_n is also an equilibrium state for φ_{f_n}.

We denote a k-cylinder of F_n by $\mathrm{C}_{n,k}$, and the collection of these cylinders by $\mathcal{P}_{n,k}$. We denote $\Psi_n := \Psi_{F_n}$ and $\Psi_{n,k}(x) := \Psi_n(F_n^{k-1}(x)) + \cdots + \Psi_n(x)$. We will write Ψ_n^i to mean $\Psi_{F_n}(x)$ for an arbitrarily chosen $x \in \mathrm{C}_{n,1}^i$. The variation $V_{n,k}(\Psi_n)$ is defined as in (4).

Remark 3.2. *We define the distortion constants $B_{n,k}$ as B_k above. By* [BT2, Lemma 7] *there exist $0 < \lambda(\delta, t) < 1$ and $C(\delta) > 0$ so that $e^{V_k(\Psi_n)} \leqslant C(\delta)\lambda(\delta, t)^k$. Then there exist $C'(\delta) > 0$ and $\lambda'(\delta, t)$ so that $B_k = B_k(\delta, t) \leqslant C'(\delta)\lambda'(\delta, t)^k$. Therefore B_k is independent of n.*

For use later, we define $Z_0(\Psi_n) := \sum_i e^{\sup \Psi_n^i}$.

Following Sarig in [Sa3], any constant H_n with $H_n \geqslant (\sup \rho_n)^2$ where ρ_n is as in Theorem 3.1(c) has the following property. For any $\mathrm{C}_{n,k} \in \mathcal{P}_{n,k}$,

$$\frac{1}{H_n B_0} \leqslant \frac{\mu_{F_n}(\mathrm{C}_{n,k})}{e^{\Psi_{n,k}(x) - kP_G(\Psi)}} \leqslant H_n B_0$$

for any $x \in C_{n,k}$. Moreover by [BT2], $P_G(\Psi) = 0$. We are allowed to take a uniform distortion constant B_0 for all of our maps F_n by our choice of $C_{n,0}$. It is important here to replace H_n with a uniform constant H. We consider how H_n was obtained. For this lemma and its proof we fix $f = f_n$, so dropping any extra notation. Note that the bound used in [Sa3, p1754] is not sufficient for us since it depends on the measure of a cylinder, which can be different for different n.

Lemma 3.3. *$V_0(\log \rho_\Psi) \leqslant 2 \log B_0$ and the Gibbs constant can be chosen to be $H_f = B_0^4$.*

Proof. According to [Sa1, (3.12)], $V_1(\log \rho_\Psi) < \log B_1$. We use this to show $V_0(\log \rho_\Psi)$ is uniformly bounded above. Take $x_1, x_2 \in C_0$. Let $y_{1,i}, y_{2,i}$ be the unique points in C_1^i such that $F(y_{1,i}) = x_1$ and $F(y_{2,i}) = x_2$. Then since $\mathcal{L}_\Psi \rho_\Psi = \rho_\Psi$,

$$\left|\frac{\rho_\Psi(x_1)}{\rho_\Psi(x_2)}\right| = \left|\frac{\sum_{Fy_1=x_1} e^{\Psi(y_1)}\rho_\Psi(y_1)}{\sum_{Fy_2=x_2} e^{\Psi(y_2)}\rho_\Psi(y_2)}\right| \leqslant \left|\frac{\sum_i e^{\Psi^i} \sup_{x\in C_1^i} \rho_\Psi(x)}{\sum_i e^{\inf_{x\in C_1^i} \Psi(x)} \inf_{x\in C_1^i} \rho_\Psi(x)}\right| \leqslant B_0 B_1.$$

Therefore the first part of the lemma is finished. There must exist $x_1, x_2 \in C_0$ with $\rho_\Psi(x_1) \leqslant 1$ and $\rho_\Psi(x_2) \geqslant 1$: otherwise in the first case $\mu_F(C_0) > 1$, and in the second case $\mu_F(C_0) < 1$. So setting $H_f := B_0^4$ we have $H_f \geqslant (\sup \rho_\Psi)^2$, so we are finished. $\square$

For use later, we let $H_{\mathcal{F}} := B_0^4$.

4. Gibbs property for the weak* limit of Gibbs measures

By passing to a subsequence if necessary, we may assume that μ_{F_∞} is a weak* limit of $\{\mu_{F_n}\}_n$. From the previous section and a uniqueness argument from [MU], we know that if we prove that μ_{F_∞} satisfies the Gibbs property and is invariant, then $\mu_{F_\infty} = \mu_{F_0}$. This section is devoted to proving that m_{F_∞} has the Gibbs property which will allow us to conclude that μ_{F_∞} has the Gibbs property also.

Lemma 4.1. *For a fixed family $\mathcal{F} = \mathcal{F}_e(C,\alpha)$ or $\mathcal{F} = \mathcal{F}_p(C,\beta)$ satisfying (1) or (2) respectively, there exists $C' > 0$ so that for all $N \geqslant 1$,*

$$\mu_{F_n}\{\tau_n > N\} \leqslant C' e^{-N\alpha} \text{ or } \mu_{F_n}\{\tau_n > N\} \leqslant C' N^{-\beta} \text{ respectively .}$$

Proof. In the proof of Proposition 2 of [BT2], the correspondence between our inducing scheme and the one considered in [BLS] is given, which allows to conclude that the estimates for the tail of our inducing scheme are given by the ones in [BLS]. A careful reading of [BLS] reveals that the constants involved in the tail estimates depend only on constants which are uniform on the respective families $\mathcal{F}_e(C,\alpha)$ and $\mathcal{F}_p(C,\beta)$. $\square$

As a consequence of this lemma, for a given family $\mathcal{F}$ we can choose $\kappa = \kappa_{\mathcal{F}} : \mathbb{N} \to [0,1]$ to be the function so that $\mu_{F_n}\{\tau_n > s_0\} \leqslant \kappa(s_0)$ for any $F_n \in \mathcal{F}$ and $\kappa(s_0) \to 0$ as $s_0 \to \infty$.

We next make conditions on our inducing schemes, so that only some of those in Lemma 2.2 will be appropriate choices. We select our inducing schemes so that the 1-cylinders accumulate on each other. In particular so that a cylinder with a small inducing time is accumulated by 1-cylinders with larger and larger inducing times.

By construction, $f_n^j(\partial \mathrm{X}_{n,k}) \cap \overset{\circ}{\mathrm{X}}_{n,k} = \emptyset$ for all $1 \leqslant j \leqslant k$. However, we will fix an i so that for each n large enough, the cylinder $\mathrm{X}_{n,k}^i$ has $f_n^j(\partial \mathrm{X}_{n,k}^i) \cap \partial \mathrm{X}_{n,k}^i = \emptyset$ for all $1 \leqslant j \leqslant k$ also. It is easy to show that this property can be satisfied for our class of maps. We denote $\mathrm{C}_{n,0}$ to be the cylinder $\mathrm{X}_{n,k}^i$, which is fixed for the rest of this paper up to the appendix. The maps $F_n = F_{\mathrm{X}_{n,k}^i}$ are defined as above. Recall that we $\mathcal{P}_{n,0} := \{\mathrm{C}_{n,0}\}$, and define $\mathcal{P}_{n,k}$ to be the set of k-cylinders for the inducing scheme F_n. This construction means that $\mathrm{C}_{n,1}^{i_1} \cap \mathrm{C}_{n,1}^{i_2} = \emptyset$ for all $i_1 \neq i_2$ for all large n. We exploit this property in Remark 4.2. We may assume that this property actually holds for all n.

Let $\tau_{n,k}^i$ be the kth inducing time on a cylinder $\mathrm{C}_{n,k}^i$, i.e. $f^{\tau_{n,k}^i}(\mathrm{C}_{n,k}^i) = \mathrm{C}_{n,0}$. As in Section 2, after possibly relabelling, for $\mathrm{C}_{0,k}^i$ there exists $N = N(k,i) \geqslant 1$ so that for $n \geqslant N$ there is a cylinder $\mathrm{C}_{n,k}^i$ so that $\tau_{0,k}^i = \tau_{n,k}^i$ and these two intervals have the same itinerary up to time $\tau_{0,k}^i$ by f_0 and f_n respectively. We say that for $n \geqslant N$, $\mathrm{C}_{0,k}^i$ is *matched*; or similarly that $\mathrm{C}_{n,k}^i$ is *matched*. In this case, $\mathrm{C}_{n,k}^i \to \mathrm{C}_{0,k}^i$ as $n \to \infty$.

Remark 4.2. *Given $i \geqslant 1$, for all $M \geqslant 1$ there exists $\eta > 0$ and $N \geqslant 1$ so that for all $n \geqslant N$, $(1+\eta)\mathrm{C}_{n,1}^i \setminus \mathrm{C}_{n,1}^i$ only intersects 1-cylinders with $\tau_n > M$. To show this, we start by choosing N so large that $\{\mathrm{C}_{n,1}^j : \tau_n^j \leqslant M\}$ are matched for all $n \geqslant N$. Now let $\eta := \frac{1}{2}\min_{j \neq i,\ \tau_0^j \leqslant M} d(\mathrm{C}_{0,1}^i, \mathrm{C}_{0,1}^j)$. By the setup, $\eta > 0$. Now we may increase N so that $n \geqslant N$ implies $\mathrm{C}_{n,1}^j \cap \left(1+\frac{\eta}{2}\right)\mathrm{C}_{0,1}^j = \mathrm{C}_{n,1}^j$ for all j with $\tau_0^j \leqslant M$. This means that η has the property required.*

Lemma 4.3. *For all $\varepsilon > 0$ there exists $i_0 \geqslant 1$ and $N \geqslant 1$ such that $\mathrm{C}_{0,1}^i$ is matched for all $1 \leqslant i \leqslant i_0$ for all $n \geqslant N$, and furthermore $n \geqslant N$ implies $\mu_{F_n}\left(\bigcup_{i > i_0} \mathrm{C}_{n,1}^i\right) < \varepsilon$.*

Proof. Let s_0 be so that $\kappa(s_0) < \varepsilon$. So s_0 depends only on ε and $\mathcal{F}$ as in Lemma 4.1. We choose i_0 so that $\tau_0^i > s_0$ for all $i > i_0$. Similarly to Remark 4.2, we can choose N so large that $\mathrm{C}_{n,1}^i$ are matched for all $1 \leqslant i \leqslant i_0$ and that $\tau_n^i > s_0$ for all $i > i_0$ and all $n \geqslant N$. It then follows that $\mu_{F_n}\left(\bigcup_{i > i_0} \mathrm{C}_{n,1}^i\right) < \varepsilon$ as required. □

In the following lemmas we repeatedly use the conformal property of m_{F_n}. This allows us to compare behaviour at small scales with that at large scale.

Lemma 4.4. *For all $\varepsilon > 0$ for all $i_0 \geqslant 1$ there exists $\eta > 0$, such that for all $k \geqslant 1$, any $\mathrm{C}_{0,k}^j \in \mathcal{P}_{0,k}$ with $F_0^{k-1}(\mathrm{C}_{0,k}^j) = \mathrm{C}_{0,1}^i$ and $1 \leqslant i \leqslant i_0$ has*

$$\frac{m_{F_0}\left((1+\eta)\mathrm{C}_{0,k}^j\right)}{m_{F_0}(\mathrm{C}_{0,k}^j)} \leqslant B_0\left(1+\frac{\varepsilon}{4}\right) \text{ and } \frac{m_{F_0}\left(\left(\frac{1}{1+\eta}\right)\mathrm{C}_{0,k}^j\right)}{m_{F_0}(\mathrm{C}_{0,k}^j)} \geqslant \frac{1}{B_0\left(1+\frac{\varepsilon}{4}\right)}.$$

Proof. Let $s_0 \geqslant 1$ be such that
$$\kappa(s_0) \leqslant \frac{\varepsilon}{8}\left(\min_{1\leqslant i\leqslant i_0} m_{F_0}(\mathrm{C}^i_{0,1})\right).$$
For the upper bound, let $\eta'>0$ be such that the set $\bigcup_{1\leqslant i\leqslant i_0}(1+B_0\eta')\mathrm{C}^i_{0,1}\setminus \mathrm{C}^i_{0,1}$ contains only cylinders $\mathrm{C}^i_{0,1}$ with $\tau^i_0\geqslant s_0$. Then $m_{F_0}((1+B_0\eta')\mathrm{C}^i_{0,1})\leqslant \left(1+\frac{\varepsilon}{4}\right)m_{F_0}(\mathrm{C}^i_{0,1})$ for $1\leqslant i\leqslant i_0$.

For $k>1$, we use distortion and conformality to reduce the problem to the 1-cylinders' case just considered. Assume for $k\geqslant 1$ that $F_0^{k-1}(\mathrm{C}^j_{0,k})=\mathrm{C}^i_{0,1}$. Since $(1+\eta)\,\mathrm{C}^j_{0,k}$ is in the same $k-1$-cylinder as $\mathrm{C}^j_{0,k}$, for η' is sufficiently small, bounded distortion implies that
$$F_0^{k-1}\left((1+\eta')\,\mathrm{C}^j_{0,k}\right)\subset(1+B_0\eta')\mathrm{C}^i_{0,1}.$$
Using the conformal property of m_{F_0} and bounded distortion we have
$$\frac{m_{F_0}\left((1+B_0\eta')\,\mathrm{C}^i_{0,1}\right)}{m_{F_0}\left(\mathrm{C}^i_{0,1}\right)}\geqslant\frac{\int_{(1+\eta')\mathrm{C}^j_{0,k}}e^{-\Psi_{0,k-1}}\,dm_{F_0}}{\int_{\mathrm{C}^j_{0,k}}e^{-\Psi_{0,k-1}}\,dm_{F_0}}\geqslant\frac{1}{B_0}\left(\frac{m_{F_0}\left((1+\eta')\,\mathrm{C}^j_{0,k}\right)}{m_{F_0}(\mathrm{C}^j_{0,k})}\right).$$
Hence, by the choice of η' above, we have
$$\frac{m_{F_0}\left((1+\eta')\,\mathrm{C}^j_{0,k}\right)}{m_{F_0}(\mathrm{C}^j_{0,k})}\leqslant B_0\left(1+\frac{\varepsilon}{4}\right).$$

For the lower bound, let $s_1\geqslant 1$ be such that $\kappa(s_1)<\frac{\varepsilon}{8}$. Then we choose $0<\eta\leqslant\eta'$ so that the set $\mathrm{C}_{0,0}\setminus\frac{\mathrm{C}_{0,0}}{1+B_0\eta}$ only contains 1-cylinders $\mathrm{C}^i_{0,1}$ with $\tau^i_0\geqslant s_1$. This implies $m_{F_0}\left(\left(\frac{1}{1+B_0\eta}\right)\mathrm{C}_{0,0}\right)\geqslant 1-\frac{\varepsilon}{8}>\frac{1}{\left(1+\frac{\varepsilon}{4}\right)}$.

For $k>1$ we use the a distortion argument similar to the one above. Bounded distortion implies that
$$F_0^k\left(\left(\frac{1}{1+\eta}\right)\mathrm{C}^j_{0,k}\right)\supset\left(\frac{1}{1+B_0\eta}\right)\mathrm{C}_{0,0}.$$
Using the conformal property of m_{F_0} and bounded distortion we have
$$\frac{m_{F_0}\left(\left(\frac{1}{1+B_0\eta}\right)\mathrm{C}_{0,0}\right)}{m_{F_0}\left(\mathrm{C}_{0,0}\right)}\leqslant\frac{\int_{\left(\frac{1}{1+\eta}\right)\mathrm{C}^j_{0,k}}e^{-\Psi_{0,k}}\,dm_{F_0}}{\int_{\mathrm{C}^j_{0,k}}e^{-\Psi_{0,k}}\,dm_{F_0}}\leqslant B_0\left(\frac{m_{F_0}\left(\left(\frac{1}{1+\eta}\right)\mathrm{C}^j_{0,k}\right)}{m_{F_0}(\mathrm{C}^j_{0,k})}\right).$$
Hence, by the choice of η above, we have
$$\frac{m_{F_0}\left(\left(\frac{1}{1+\eta}\right)\mathrm{C}^j_{0,k}\right)}{m_{F_0}(\mathrm{C}^j_{0,k})}\geqslant\frac{1}{B_0\left(1+\frac{\varepsilon}{4}\right)}.$$

$\square$

Notice that the above proof can be used to show in particular that for the 1-cylinders considered above,

$$\frac{m_{F_0}\left(\mathrm{C}_{0,1}^j \setminus \left(\frac{1}{1+\eta}\right)\mathrm{C}_{0,1}^j\right)}{m_{F_0}(\mathrm{C}_{0,1}^j)} \leqslant \frac{B_0}{1+\frac{\varepsilon}{4}}.$$

Proposition 4.5. *For all $\varepsilon > 0$, $\lambda \in (0,1)$, $k_0 \geqslant 1$ and sequences $(i_1, \ldots, i_{k_0}) \in \mathbb{N}^{k_0}$ there exists $N_0 \geqslant 1$ such that for all $n \geqslant N_0$, $1 \leqslant k \leqslant k_0$ and $1 \leqslant i \leqslant i_k$, we have*

$$\frac{1}{B_0^2(1+\varepsilon)} \leqslant \frac{m_{F_n}(\mathrm{C}_{0,k}^i)}{e^{\Psi_{0,k}(x)}} \leqslant B_0^2(1+\varepsilon)$$

for all $x \in \lambda \mathrm{C}_{0,k}^j$.

Proof. The following claim is left to the reader.

Claim 1. *For all $\varepsilon > 0$, $k_0 \geqslant 1$ and sequences $(i_1, \ldots, i_{k_0}) \in \mathbb{N}^{k_0}$ there exists $N_0 \geqslant 1$ such that for all $n \geqslant N_0$, $1 \leqslant k \leqslant k_0$ and $1 \leqslant i \leqslant i_k$, $\mathrm{C}_{n,k}^i$ is matched. Moreover, for these cylinders, each set $F_n^{k-1}(\mathrm{C}_{n,k}^i)$ is matched.*

We next make the following claim.

Claim 2. *For all $\varepsilon > 0$, $\lambda \in (0,1)$, $k_0 \geqslant 1$ and sequences $(i_1, \ldots, i_{k_0}) \in \mathbb{N}^{k_0}$ there exists $N_1 \geqslant N_0$ such that for all $n \geqslant N_1$, $1 \leqslant k \leqslant k_0$ and $1 \leqslant i \leqslant i_k$,*

$$\frac{m_{F_n}((1+\eta)\mathrm{C}_{n,k}^i)}{m_{F_n}(\mathrm{C}_{n,k}^i)} \leqslant B_0\left(1+\frac{\varepsilon}{4}\right) \text{ and } \frac{m_{F_n}\left(\left(\frac{1}{1+\eta}\right)\mathrm{C}_{n,k}^i\right)}{m_{F_n}(\mathrm{C}_{n,k}^i)} \geqslant \frac{1}{B_0\left(1+\frac{\varepsilon}{4}\right)}.$$

Proof. The proof of the claim is the same as for Lemma 4.4 except that we need to take F_n sufficiently close to F_0 so that the cylinders $\mathrm{C}_{n,k}^i$ considered in Lemma 4.4 have almost exactly the same properties as those $\mathrm{C}_{0,k}^i$ considered here. □

A simple consequence of these claims is that for all $\varepsilon > 0$, $\lambda \in (0,1)$, $k_0 \geqslant 1$ and sequences $(i_1, \ldots, i_{k_0}) \in \mathbb{N}^{k_0}$ there exists $N_2 \geqslant N_1$ such that for all $n \geqslant N_2$, $1 \leqslant k \leqslant k_0$ and $1 \leqslant i \leqslant i_k$, $\mathrm{C}_{n,k}^i \in \mathcal{P}_{n,k}$ is matched and

$$\frac{1}{B_0\left(1+\frac{\varepsilon}{4}\right)} \leqslant \frac{m_{F_n}(\mathrm{C}_{0,k}^i)}{m_{F_n}(\mathrm{C}_{n,k}^i)} \leqslant B_0\left(1+\frac{\varepsilon}{4}\right).$$

Here we take some $N_2 \geqslant N_1$ so that $\mathrm{C}_{0,k}^j \subset (1+\eta)\mathrm{C}_{n,k}^j$ and $\mathrm{C}_{n,k}^j \subset (1+\eta)\mathrm{C}_{0,k}^j$ for all $\mathrm{C}_{n,k}^j$ as in the statement of the proposition.

The Gibbs property for m_{F_n}, which follows directly from conformality, means that $m_{F_n}(\mathrm{C}_{n,k}^i) = B_0^{\pm} e^{\Psi_{n,k}(x)}$ for all $x \in \mathrm{C}_{n,k}^i$. Now we can take N_2 so large that

$$\frac{1}{\left(1+\frac{\varepsilon}{4}\right)} \leqslant e^{\Psi_{n,k}(x)-\Psi_{0,k}(x)} \leqslant 1+\frac{\varepsilon}{4}$$

for all $x \in \mathrm{C}_{n,k}^i \cap \mathrm{C}_{0,k}^i$ for the cylinders $\mathrm{C}_{n,k}^i$ under consideration. To complete the proof of the lemma, we possibly increase N_2 again to ensure that $\mathrm{C}_{n,k}^i \cap \mathrm{C}_{0,k}^i \subset \lambda \mathrm{C}_{0,k}^i$ for all the cylinders we consider. $\square$

Combining Lemma 3.3 and Proposition 4.5 we have that μ_{F_∞} must have the Gibbs property with uniform constant. That is:

Corollary 4.6. *For all k and all $\mathrm{C}_{0,k} \in \mathcal{P}_{0,k}$,*

$$\frac{1}{H_{\mathcal{F}} B_0^2 (1+\varepsilon)} \leqslant \frac{\mu_{F_\infty}(\mathrm{C}_{0,k}^i)}{e^{\Psi_{0,k}(x)}} \leqslant H_{\mathcal{F}} B_0^2 (1+\varepsilon),$$

for all $x \in \mathrm{C}_{0,k}^i$.

We will need the following lemma later.

Lemma 4.7. *For all $\varepsilon > 0$ and $i_0 \geqslant 1$ there exists $N = N(\varepsilon) \geqslant 1$ such that $n \geqslant N$ implies*

$$\mu_{F_n}\left(\bigcup_{i=0}^{i_0} \left(\mathrm{C}_{n,1}^i \triangle \mathrm{C}_{0,1}^i\right)\right) \leqslant \varepsilon.$$

Proof. Combining the arguments in the proof of Lemma 4.4, the paragraph following it and Claim 2 in the proof of Proposition 4.5 we have $\eta > 0$, $i_0 \geqslant 1$ and $N' \geqslant 1$ such that for $n \geqslant N'$,

$$m_{F_n}\left((1+\eta)\mathrm{C}_{n,k}^i \setminus \mathrm{C}_{n,k}^i\right), \; m_{F_n}\left(\mathrm{C}_{n,k}^i \setminus \frac{\mathrm{C}_{n,k}^i}{(1+\eta)}\right) < \frac{\varepsilon}{i_0 H_{\mathcal{F}}}$$

for all $1 \leqslant i \leqslant i_0$. Recall that $H_{\mathcal{F}}$ is the constant from Lemma 3.3. Moreover, there exists $N \geqslant N'$ such that $n \geqslant N$ implies

$$\mathrm{C}_{n,1}^i \triangle \mathrm{C}_{0,1}^i \subset (1+\eta)\mathrm{C}_{n,k}^i \setminus \frac{\mathrm{C}_{n,k}^i}{(1+\eta)}$$

for all $1 \leqslant i \leqslant i_0$. Therefore, $n \geqslant N$ implies

$$m_{F_n}\left(\bigcup_{i=0}^{i_0} \left(\mathrm{C}_{n,1}^i \triangle \mathrm{C}_{0,1}^i\right)\right) \leqslant \frac{\varepsilon}{H_{\mathcal{F}}}.$$

The lemma follows from Lemma 3.3, substituting $\mu_{F_n}(= h_{F_n} m_{F_n})$ for m_{F_n} in the above equation. $\square$

5. Invariance of the weak* limit

We may assume, as in the beginning of Section 4, that μ_{F_∞} is the weak* limit of the sequence $\{\mu_{F_n}\}_n$. In the previous section we saw that μ_{F_∞} is Gibbs. The purpose of this section is to show that μ_{F_∞} is F_0-invariant. Before that, we prove the following technical lemma that will be useful in the remaining arguments.

Lemma 5.1. *For all $i \in \mathbb{N}$ and every continuous $g : C^i_{0,1} \to \mathbb{R}$ we have*
$$\int g.\mathbf{1}_{C^i_{0,1}}\ d\mu_{F_n} \to \int g.\mathbf{1}_{C^i_{0,1}}\ d\mu_{F_\infty}.$$

Proof. We can extend g continuously to $\partial C^i_{0,1}$, and for every $x \in I \setminus \overline{C^i_{0,1}}$, define $\mathrm{b}^i(x)$ as the point of $\partial C^i_{0,1}$ closest to x.

Observing that $g = g^+ - g^-$, where $g^+(x) = \max\{0, g(x)\} \geqslant 0$ and $g^-(x) = \max\{0, -g(x)\} \geqslant 0$, we may assume without loss of generality that $g \geqslant 0$. Also, since, by Corollary 4.6, μ_{F_∞} is a Gibbs measure, we have $\mu_{F_\infty}(\partial C^i_{0,1}) = 0$, which implies that $\int_{\overline{C^i_{0,1}} \setminus \partial C^i_{0,1}} g\ d\mu_{F_\infty} = \int_{\overline{C^i_{0,1}}} g d\mu_{F_\infty} = \int_{C^i_{0,1}} g\ d\mu_{F_\infty}$.

Let $U_k = \{x \in I : \mathrm{dist}(x, \overline{C^i_{0,1}}) < 1/k\}$. Clearly U_k is an open neighbourhood of $\overline{C^i_{0,1}}$ and by the regularity of μ_{F_∞} it follows that $\mu_{F_\infty}(U_k \setminus \overline{C^i_{0,1}}) = \epsilon(k) \to 0$ as $k \to \infty$. Define $h : I \to \mathbb{R}$ as
$$h(x) = \begin{cases} 0 & \text{if } x \notin U_k \\ g(\mathrm{b}^i(x)) \frac{d(x, I\setminus U_k)}{d(x, I\setminus U_k) + d(x, \overline{C^i_{0,1}})} & \text{if } x \in U_k \setminus \overline{C^i_{0,1}} \\ g(x) & \text{if } x \in \overline{C^i_{0,1}} \end{cases}.$$
Notice that h is continuous and, for every $x \in I$, we have $g(x)\mathbf{1}_{C^i_{0,1}}(x) \leqslant h(x) \leqslant \max_{x \in \overline{C^i_{0,1}}} g(x)$ and $h(x) - g(x)\mathbf{1}_{C^i_{0,1}}(x) > 0$ only if $x \in U_k \setminus \overline{C^i_{0,1}}$. Consequently, using the weak* convergence of μ_{F_n} to μ_{F_∞}, it follows
$$\int g\mathbf{1}_{C^i_{0,1}} d\mu_{F_n} \leqslant \int h\ d\mu_{F_n} \xrightarrow[n\to\infty]{} \int h\ d\mu_{F_\infty} \leqslant \int g\mathbf{1}_{\overline{C^i_{0,1}}}\ d\mu_{F_\infty} + \epsilon(k) \max_{x \in \overline{C^i_{0,1}}} g(x).$$
Letting $k \to \infty$ we get $\int g\mathbf{1}_{C^i_{0,1}} d\mu_{F_n} \leqslant \int g\mathbf{1}_{C^i_{0,1}} d\mu_{F_\infty}$. The opposite inequality follows similarly. $\square$

Lemma 5.2. *μ_{F_∞} is F_0-invariant.*

Proof. The F_0-invariance of μ_{F_∞} is equivalent to
$$\int \varphi \circ F_0\ d\mu_{F_\infty} = \int \varphi\ d\mu_{F_\infty}$$
for every continuous $\varphi \colon I \to \mathbb{R}$. Given any $\varphi \colon I \to \mathbb{R}$ continuous we have by hypothesis
$$\int \varphi\ d\mu_{F_n} \to \int \varphi\ d\mu_{F_\infty} \quad \text{as} \quad n \to \infty.$$
On the other hand, since μ_{F_n} is an F_n-invariant probability measure, we have
$$\int \varphi\ d\mu_{F_n} = \int (\varphi \circ F_n)\ d\mu_{F_n} \quad \text{for every } n \geqslant 0.$$
So, it suffices to prove that
$$\int (\varphi \circ F_n)\ d\mu_{F_n} \to \int (\varphi \circ F_0)\ d\mu_{F_\infty} \quad \text{as} \quad n \to \infty. \tag{5}$$

We have

$$\left|\int (\varphi \circ F_n)\ d\mu_{F_n} - \int (\varphi \circ F_0)\ d\mu_{F_\infty}\right| \leqslant$$
$$\left|\int (\varphi \circ F_n)\ d\mu_{F_n} - \int (\varphi \circ F_0)\ d\mu_{F_n}\right| + \left|\int (\varphi \circ F_0)\ d\mu_{F_n} - \int (\varphi \circ F_0)\ d\mu_{F_\infty}\right|.$$

Observing that $\varphi \circ F_0$ is continuous on each $C^i_{0,1}$, we easily deduce from Lemma 5.1 and Lemma 4.3 that the second term in the sum above is close to zero for large n.

The only thing we are left to prove is that the first term in the sum above converges to 0 when n tends to ∞. That term is bounded above by

$$\int |\varphi \circ F_n - \varphi \circ F_0|\ d\mu_{F_n}. \tag{6}$$

Take any $\varepsilon > 0$. Using Lemma 4.1, take $N \geqslant 1$ such that

$$\sum_{\tau^i_n > N} \mu_{F_n}(C^i_{n,1}) < \varepsilon.$$

We write the integral in (6) as

$$\sum_{\tau^i_n > N} \int_{C^i_{n,1}} |\varphi \circ F_n - \varphi \circ F_0|\ d\mu_{F_n} + \sum_{\tau^i_n \leqslant N} \int_{C^i_{n,1}} |\varphi \circ F_n - \varphi \circ F_0|\ d\mu_{F_n}. \tag{7}$$

The first sum in (7) is bounded by $2\varepsilon\|\varphi\|_\infty$. Let us now estimate the second sum in (7).

Using Lemma 4.3, we take n_1 sufficiently large so that for all $n \geqslant n_1$ and every cylinder $C^i_{n,1}$ with $\tau^i_n \leqslant N$ there is a matching cylinder $C^i_{0,1}$ with $\tau^i_n = \tau^i_0$. Moreover, using Lemma 4.7, we may assume that n_1 is large enough so that $n \geqslant n_1$ implies

$$\sum_{\tau^i_n \leqslant N} \mu_{F_n}(C^i_{n,1} \triangle C^i_{0,1}) < \varepsilon.$$

For every i such that $\tau^i_n \leqslant N$ we have

$$\begin{aligned}\int_{C^i_{n,1}} |\varphi \circ F_n - \varphi \circ F_0|\ d\mu_{F_n} \leqslant & \int_{C^i_{n,1} \cap C^i_{0,1}} |\varphi \circ f_n^{\tau^i_0} - \varphi \circ f_0^{\tau^i_0}|\ d\mu_{F_n} \\ & + \int_{C^i_{n,1} \setminus C^i_{0,1}} |\varphi \circ F_n - \varphi \circ F_0|\ d\mu_{F_n}.\end{aligned}$$

Since $f_n \to f_0$ in the C^k topology, there is $n_2 \in \mathbb{N}$ such that for $n \geqslant n_2$

$$\sum_{\tau^i_n \leqslant N} \int_{C^i_{n,1} \cap C^i_{0,1}} |\varphi \circ f_n^{\tau^i_n} - \varphi \circ f_0^{\tau^i_n}|\ d\mu_{F_n} < \varepsilon.$$

On the other hand, for $n \geqslant n_1$

$$\sum_{\tau^i_n \leqslant N} \int_{C^i_{n,1} \triangle C^i_{0,1}} |\varphi \circ F_n - \varphi \circ F_0|\ d\mu_{F_n} \leqslant 2\varepsilon\|\varphi\|_\infty.$$

Thus we have for $n \geqslant \max\{n_1, n_2\}$

$$\int |\varphi \circ F_n - \varphi \circ F_0|\ d\mu_{F_n} \leqslant \varepsilon(4\|\varphi\|_\infty + 1).$$

This proves the result since $\varepsilon > 0$ was arbitrary. $\square$

Since μ_{F_∞} is an invariant Gibbs measure, uniqueness of such measures, [MU, Theorem 3.2], implies $\mu_{F_\infty} \equiv \mu_{F_0}$.

Remark 5.3. *Observe that the whole sequence μ_{F_n} converges in the weak* topology to μ_{F_0}. This is because any subsequence $\left\{\mu_{F_{n_i}}\right\}_i$ admits a convergent subsequence $\left\{\mu_{F_{n_{i_j}}}\right\}_j$, whose weak* limit, μ_{F_∞}, is Gibbs and F_0-invariant, by Corollary 4.6 and Lemma 5.2. Hence, by uniqueness, $\mu_{F_{n_{i_j}}} \to \mu_{F_0}$, in the weak* topology, which clearly implies the statement.*

6. Continuous variation of equilibrium states

So far, we managed to prove that if $f_n \to f_0$, then the induced Gibbs measures converge in the weak* topology, ie, $\mu_{F_n} \to \mu_{F_0}$. We define the *saturation* of μ_F by

$$\mu_f^* = \sum_{i=1}^{\infty} \sum_{k=0}^{\tau_i - 1} f_*^k \left(\mu_F | C_1^i\right) \tag{8}$$

Observe that, for some fixed $t \in U_{\mathcal{F}}$, the unique equilibrium state of f_n for the potential $-t \log |Df_n|$ is such that $\mu_n = \mu^*_{f_n} / \mu^*_{f_n}(I)$, for every $n \geqslant 0$. Consequently, the proof of Theorem A will be complete once we prove:

Proposition 6.1. *For every continuous $g : I \to I$,*

$$\int g \; d\mu^*_{f_n} \xrightarrow[n\to\infty]{} \int g \; d\mu^*_{f_0}.$$

Proof. First observe that as I is compact, g is uniformly continuous and $\|g\|_\infty < \infty$.

Let ε be given. We look for $n_0 \in \mathbb{N}$ sufficiently large so that for every $n > n_0$

$$\left| \int g \; d\mu^*_{f_n} - \int g \; d\mu^*_{f_0} \right| < \varepsilon$$

Recalling (8) we may write for any integer N

$$\mu^*_{f_n} = \sum_{\tau_n^i \leqslant N} \sum_{k=0}^{\tau_n^i - 1} (f_n^k)_* (\mu_{F_n} | C_{n,1}^i) + \eta_{f_n} \text{ and } \mu^*_{f_0} = \sum_{\tau_0^i \leqslant N} \sum_{k=0}^{\tau_0^i - 1} (f_0^k)_* (\mu_{F_0} | C_{0,1}^i) + \eta_{f_0}$$

where $\eta_{f_n} = \sum_{\tau_n^i > N} \sum_{k=0}^{\tau_n^i - 1} (f_n^k)_* (\mu_{F_n} | C_{n,1}^i)$ and $\eta_{f_0} = \sum_{\tau_0^i > N} \sum_{k=0}^{\tau_0^i - 1} (f_0^k)_* (\mu_{F_0} | C_{0,1}^i)$. Using Lemma 4.1 we pick N large enough so that $n \geqslant N$ implies

$$\eta_{f_n}(I) + \eta_{f_0}(I) < \varepsilon/2.$$

Using Lemma 4.3, we take n_1 sufficiently large so that for all $n \geqslant n_1$ and every cylinder $C_{n,1}^i$ with $\tau_n^i \leqslant N$ there is a matching cylinder $C_{0,1}^i$ with $\tau_n^i = \tau_0^i$. Let S_N denote the number of 1-cylinders such that $\tau_n^i \leqslant N$. To complete the proof of the proposition, for every i such that $\tau_n^i \leqslant N$ and $k < \tau_n^i$, we must find a sufficiently large n_2 so that for every $n \geqslant n_2$

$$E := \left| \int (g \circ f_n^k) \mathbf{1}_{C_{n,1}^i} \; d\mu_{F_n} - \int (g \circ f_0^k) \mathbf{1}_{C_{0,1}^i} \; d\mu_{F_0} \right| < \frac{\varepsilon}{2 S_N}.$$

We split E into E_1, E_2 and E_3 presented in respective order:

$$
\begin{aligned}
E \leqslant & \left|\int \left[(g \circ f_n^k) - (g \circ f_0^k)\right] \mathbf{1}_{C_{n,1}^i} d\mu_{F_n}\right| \\
& + \left|\int (g \circ f_0^k) \left[\mathbf{1}_{C_{n,1}^i} - \mathbf{1}_{C_{0,1}^i}\right] d\mu_{F_n}\right| \\
& + \left|\int (g \circ f_0^k)\mathbf{1}_{C_{0,1}^i} d\mu_{F_n} - \int (g \circ f_0^k)\mathbf{1}_{C_{0,1}^i} d\mu_{F_0}\right|.
\end{aligned}
$$

Since

$$E_1 \leqslant \int \left|(g \circ f_n^k) - (g \circ f_0^k)\right| \; d\mu_{F_n},$$

we choose n_2 large enough so that for every $n > n_2$ we have $\left|(g \circ f_n^k) - (g \circ f_0^k)\right| \leqslant \frac{\varepsilon}{6S_N}$ in order to obtain $E_1 \leqslant \frac{\varepsilon}{6S_N}$.

Now,

$$E_2 \leqslant \|g\|_\infty \mu_{F_n}(C_{n,1}^i \triangle C_{0,1}^i).$$

Using Lemma 4.7, we take n_2 large enough so that for all $n > n_2$ we have $E_2 \leqslant \frac{\varepsilon}{6S_N}$.

Regarding the last term, Lemma 5.1 allows us to conclude that if n_2 is sufficiently large then for all $n > n_2$ we have $E_3 \leqslant \frac{\varepsilon}{6S_N}$. □

Proof of Theorem B. Alves and Viana, in [AV], give some abstract conditions for statistical stability of physical measures in the strong sense, that is, convergence of densities in the sense of (3). Our inducing schemes and their properties put us trivially in the setting of Alves and Viana, meaning that both $\mathcal{F}_e(C, \alpha)$ and $\mathcal{F}_p(C, \beta)$ meet the requirements of the family $\mathcal{U}$ of [AV]. Moreover, Lemma 4.7 and Lemma 4.1 imply that conditions U_1 and U_2 of [AV] are satisfied, respectively. Since the constants involved in all estimates are taken uniformly on $\mathcal{F}_e(C, \alpha)$ or $\mathcal{F}_p(C, \beta)$, then condition U_3 also holds. Consequently, by [AV, Therorem A], we have that the map

$$\mathcal{F} \ni f \mapsto \frac{d\mu_f}{dm}$$

is continuous as in (3), where $\mathcal{F}$ stands for either $\mathcal{F}_e(C, \alpha)$ or $\mathcal{F}_p(C, \beta)$ and m denotes Lebesgue measure. □

Remark 6.2. *Note that the theory presented here extends to Manneville-Pommeau maps $f : x \mapsto x + x^{1+\alpha}$ (mod 1) for $\alpha \in (0, 1)$. Given such a map, and a potential $\varphi_t := -t \log |Df|$, it is straightforward to prove an equivalent of* [BT2, Theorem 1], *yielding an equilibrium state μ_t for $t \in [\delta, 1]$ for some $\delta < 0$. One main difference in proving statistical stability for these measures is that in the proofs of Proposition 4.5 and Lemma 4.7 for example, to estimate the measure of sets $\mathrm{C}_{0,k}^i \triangle \mathrm{C}_{n,k}^i$ we can no longer assume that no two cylinders for the inducing schemes are adjacent. Above, this property enabled us to estimate $\mathrm{C}_{0,k}^i \triangle \mathrm{C}_{n,k}^i$ using the measure of 1-cylinders. However, when, as in the Manneville-Pommeau case, we do not have this property, we can use the measure of k-cylinders to give us the required estimates instead.*

Appendix A. Hofbauer Towers

In this appendix we prove Lemma 2.2. We first need some more theory about Hofbauer towers.

For the moment fix n and so we suppress it in the notation. We let $\hat{I}_{\mathcal{T}}$ denote the transitive component of $\hat{I}$. More precisely, $\hat{I}_{\mathcal{T}}$ consists of a union of elements of $\mathcal{D}$ and there is a point $\hat{x} \in \hat{I}_{\mathcal{T}}$ so that $\overline{\bigcup_k \hat{f}^k(\hat{x})} = \hat{I}_{\mathcal{T}}$. The existence of this (maximal) component is implicit in works of Hofbauer and Raith, see also [BT2] for a self contained proof and references.

We next consider how to 'lift' measures to the towers. Let $\iota := \pi|_{D_0}^{-1}$ where D_0 is the lowest level in $\hat{I}$, so $\iota : I \to D_0$ is an inclusion map. Given a probability measure m, let $\hat{m}^0 = m \circ \iota^{-1}$ be a probability measure on D_0. Let

$$\hat{m}^k := \frac{1}{k}\sum_{j=0}^{k-1} \hat{m}^0 \circ \hat{f}^{-j}.$$

We let $\hat{m}$ be a vague limit of this sequence. This is a generalisation of weak* limit for non-compact sets: for details, see [K]. In general it is important to ensure that $\hat{m} \not\equiv 0$. It is known, see for example [BT2] that if m is an ergodic invariant measure with positive Lyapunov exponent then $\hat{m} \circ \pi^{-1} = m$.

We say that x *reaches η-large scale at time j* if there is a neighbourhood $U \ni x$ such that $f^j : U \to (x-\eta, x+\eta)$ is a diffeomorphism (note that we do not require bounded distortion here). As in [MeS], if $f \in \mathcal{H}_\ell$ then $d_k := \sup_{x\in I} |\mathrm{X}_k[x]|$ has $d_k \to 0$ as $k \to \infty$. The following lemma uses the idea of [BT1, Lemma 9], where it was done for complex maps.

Lemma A.1. *Let $f \in \mathcal{H}$ and m be a probability measure. If $\eta, \theta \in (0,1]$ are such that m-a.e. point goes to η-large scale with frequency θ then there exists R depending only on η, θ and how d_k decreases with k, such that $\hat{m}(\hat{I}^R) \geqslant \theta$.*

Proof. Let $R \in \mathbb{N}$ be such that for all $x \in I$, we have $|\mathrm{X}_R[x]| < \eta$. This implies that if $d(\hat{x}, \partial D_{\hat{x}}) > \eta$ then $\pi^{-1}(\mathrm{X}_R[\pi(\hat{x})]) \cap D_{\hat{x}} \Subset D_{\hat{x}}$. Therefore, if $x \in I$ reaches η-large scale at time j, then for $\hat{x} := \iota(x)$, $\pi^{-1}(\mathrm{X}_R(f^j(x))) \cap D_{\hat{f}^j(\hat{x})} \Subset D_{\hat{f}^j(\hat{x})}$. From the way the Hofbauer tower is constructed, this implies that $\hat{f}^{j+R}(\hat{x}) \in \hat{I}^R$. Fix $\varepsilon > 0$. Then there exists $k_0 = k_0(x, \varepsilon) \in \mathbb{N}$ so that $k \geqslant k_0$ implies

$$\frac{1}{k}\#\left\{0 \leqslant j < k : \hat{f}^j(\hat{x}) \in \hat{I}^R\right\} > \frac{\theta}{1+\varepsilon}.$$

Let N be so large that $m\{x \in I : k_0(x,\varepsilon) \leqslant N\} \geqslant 1 - \varepsilon$. Then

$$\hat{m}^k(\hat{I}^R) = \frac{1}{k}\sum_{j=0}^{k-1} \hat{m}^0 \circ \hat{f}^{-j}(\hat{I}^R) \geqslant \theta\left(\frac{p-1}{p}\right)\left(\frac{1-\varepsilon}{1+\varepsilon}\right)$$

for all $k \geqslant pN$. Since $\varepsilon > 0$ was arbitrary, we have $\hat{m}(\hat{I}^R) \geqslant \theta$. $\square$

Proof of Lemma 2.2. Following the ideas of [BT2, Theorem 3] we can show that if $\hat{A}_n$ is chosen in $\hat{I}_{n,\mathcal{T}}$ then there is a first return map to $\hat{A}_n$ which has domains dense in $\hat{A}_n$. For our purposes here, we need to show that we can choose a domain $\hat{A}_n \subset \hat{I}_{n,\mathcal{T}}$ with $x \in \pi_n(\hat{A}_n)$ so that there is a subsequence $\{n_j\}_j$ with $\hat{A}_{n_j} \to \hat{A}_0$ as $j \to \infty$. So the essential thing to prove is that we can choose a uniform R so that $\hat{I}_{n,\mathcal{T}} \cap \hat{I}_n^R \neq \emptyset$. From there, a compactness argument implies that we can pick such a $\hat{A}_{n_j}$. Once we can prove that there is such $\hat{A}_{n_j}$ we can choose $\mathrm{X}^i_{n_j}$ so that $(1+\delta)\mathrm{X}^i_{n_j} \subset \pi_n(\hat{A}_{n_j})$. Hence $\hat{\mathrm{X}}^i_{n,k}$ is always in the transitive part of its Hofbauer tower. It is easy to see that this results in inducing schemes with $F_{\mathrm{X}^i_{n_j,k}}(x) \to F_{\mathrm{X}^i_{0,k}}(x)$ as $j \to \infty$ for all $x \in (\mathrm{X}^i_{0,k})^\infty$. The fact that $\overline{(\mathrm{X}^i_{n_j,k})^\infty} = \mathrm{X}^i_{n_j,k}$ follows as in [BT2, Theorem 3].

Let ν_n be the acip found in [BLS]. By Lemma A.1, there exists $\varepsilon > 0$ and $R \in \mathbb{N}$ so that $\hat{\nu}_n(\hat{I}_n^R) > \varepsilon$ for all n. Here ε and R depend only what scale is reached with what frequency, and the way the cylinders shrink. We claim that every map in $\mathcal{F}$ goes to uniform large scale with uniform frequency. Adding this to the fact that for n large enough, the cylinders shrink sufficiently similarly for us to be able to apply Lemma A.1, the lemma is proved.

The claim can be proved using [BLS]. There, inducing schemes $G_n : \bigcup \Omega_n^i \to \Omega_n$ are constructed for some Ω_n. Here $G_n = f^{r_n}$ for an inducing time r_n. We can take $\eta_n = \frac{|\Omega_n|}{2}$. This is uniformly bounded below. To check this fact we refer to [BLS, Lemma 4.2] where the sets Ω_n are constructed. Then observe that once a map f_0 is fixed the construction of the corresponding Ω_0 involves a finite number of iterations and constants that can be taken uniformly within the families considered. This means that one can mimic the construction for a neighbouring map f_n and hence obtain an interval uniformly close to the original Ω_0. By the Ergodic Theorem, the frequency

$$\lim_{k\to\infty} \frac{1}{k} \# \left\{ 0 \leqslant j < k : \exists U \ni x \text{ s.t. } f^j : U \to \Omega_n \text{ is a diffeomorphism} \right\}$$

for a ν_n-typical point x is bounded below by $\frac{1}{\int r_n \, d\nu_{G_n}}$ where ν_{G_n} is the measure for the inducing scheme. Hence we need only to show that $\int r_n \, d\nu_{G_n}$ is uniformly bounded above for all $f_n \in \mathcal{F}$, which follows from Lemma 4.1. $\square$

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