Stability of Periodic Points in Piecewise Isometries of Euclidean Spaces

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April 8, 2005

Abstract

In this paper we complete the analysis of the stability of periodic points of piecewise isometric systems which are defined in the entire Euclidean space. The case which we focus on here is when the determinant of the matrix $Id - R_w$ is zero, where R_w is the linear part of the return map generated by the periodic point whose coding is made of a countable repetition of the block w. The case when determinant of $Id - R_w$ is non-zero was studied in Mendes & Nicol [2004] but still the statement is included here for completeness sake.

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1 Introduction

In the context of piecewise isometric systems the existence of a periodic point x_0 has significant implications in the behaviour of the map. Namely, the fact that the map acts locally as an isometry implies that, if we take $d := \mathsf{min}_{k=1,\dots,p}\{\mathsf{dist}\left(T^k.x_0, \mathcal{D}\right)\}$, where p is the period of x_0 and $\mathcal{D}$ is the discontinuity set, then the ball $B_d(x_0)$ follows the orbit of x_0 at all iterations and therefore, the Lebesgue measure supported on the union of the iterates of $B_d(x_0)$ is an invariant measure hence preventing any ergodic measure of being absolutely continuous with respect to Lebesgue measure. This is the basic argument that drives the proof that almost every interval exchange map with flips is nonergodic in Nogueira [1989].

The study of the birth and death of those disks is then intrinsically related to the occurrence of the same phenomena in periodic points. The paper by Goetz [1998] illustrates how these events interact in a system of two rotations in the plane. In a different paper, the same author studies the changes in Lebesgue measure making use of Hausdorff metric and obtains results on the stability of cells and periodic points for a class of irrational piecewise rotations (see Goetz [2001]). This class of maps has been generalised in Mendes & Nicol [2004] where the stability of periodic points such that the linear part of the return map minus identity is invertible was proved. The remaining case is considered here and we prove that the stability of the periodic point depends on the dimension of the space and on the orientation of the linear part of the return map (see classification of isometries according to recurrence).

This paper is organised as follows: in Section 2 we review basic concepts from the Euclidean Group and establish some criteria for the recurrence properties of Euclidean isometries that lead to the classification presented in Section 2.2.3. In Section 3. we present the results on the stability of periodic points preceded by an excursion on the low-dimensional case which serve as motivation for the general case. All proofs are left for the final section.

2 Preliminaries and background

2.1 The structure of the Euclidean group

The Euclidean group $\mathbb{E}(n)$ is the group of all isometries of $\mathbb{R}^n$, that is, the set of all maps $I : \mathbb{R}^n \to \mathbb{R}^n$ preserving Euclidean distance, equipped with the usual composition of maps as the group operation. We may identify an element $I \in \mathbb{E}(n)$ with a pair (A, v) where $A \in \mathbb{O}(n)$ and $v \in \mathbb{R}^n$. Recall that $\mathbb{O}(n) = \{A \in \mathbb{GL}(n,\mathbb{R}) : AA^t = \mathsf{Id}\}$ and that it includes both orientation reversing ($\mathsf{det}(A) = -1$) and orientation preserving ($\mathsf{det}(A) = 1$) transformations. We will always consider $\mathbb{O}(n)$ to have its standard matrix representation acting on $\mathbb{R}^n$ and denote the standard action of a matrix $A \in \mathbb{O}(n)$ on a vector $v \in \mathbb{R}^n$, by $v \mapsto A.v$. The action of I on $x \in \mathbb{R}^n$ is then given by $I.x = A.x + v$.

The composition of maps generates a semi-direct product structure in the Euclidean group, namely, $\mathbb{E}(n)$ is the semi-direct product of $\mathbb{O}(n)$ and $\mathbb{R}^n$. Analogously, the special Euclidean group $\mathbb{SE}(n)$ is the semi-direct product of $\mathbb{SO}(n)$ and $\mathbb{R}^n$.

2.1.1 Canonical Form

For every element A of $\mathbb{O}(n)$ there is an orthonormal basis $\mathcal{B} = \{b_1, \dots, b_n\}$ for which A can be orthogonally diagonalised. If n is even this implies that

$$A = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & B_{n/2} \end{bmatrix} \tag{1}$$

where

$$B_i = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{bmatrix} \text{ with } \theta_i \in [0, 2\pi), \text{ or}$$
$$B_i = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

If n is odd, $\mathcal{B}$ can be chosen so that,

$$A = \begin{bmatrix} \widetilde{A} & 0 \\ 0 & \pm 1 \end{bmatrix} \text{ where } \widetilde{A} \in \mathbb{O}(n-1) . \tag{2}$$

(See Mishina [1965] for more details on the canonical form of matrices in $\mathbb{O}(n)$.)

2.1.2 Haar measure and norm in $\mathbb{E}(n)$

Denote by h the usual left and right invariant Haar measure on $\mathbb{O}(n)(\mathbb{SO}(n))$ and let λ be the n-dimensional Lebesgue measure on $\mathbb{R}^n$. It can easily be checked that $h \times \lambda$ is a right and left invariant Haar measure on $\mathbb{E}(n)(\mathbb{SE}(n))$. When we refer to a full measure set we mean a set whose complement has zero measure, where the measure we refer to will be clear from context. Similarly we say that almost every (a.e.) element $I \in \mathbb{E}(n)(\mathbb{SE}(n))$ satisfies a condition ($\diamondsuit$) if the $h \times \lambda$-measure of elements $g \in \mathbb{E}(n)(\mathbb{SE}(n))$ not satisfying condition ($\diamondsuit$) is zero. This notion can be extended to $\mathbb{E}(n) \times \overset{m}{\dots} \times \mathbb{E}(n)$, $m \in \mathbb{N}$, by considering the product measure $(h \times \lambda)^m := (h \times \lambda) \times \overset{m}{\dots} \times (h \times \lambda)$.

Finally, for a given square matrix $A \in \mathbb{M}(n)$, we will use the norm

$$\|A\|_{\mathbb{M}(n)} = \sqrt{\mathrm{tr}(AA^T)} = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \right)^{\frac{1}{2}} . \tag{3}$$

This norm $\|.\|_{\mathbb{M}(n)}$ actually corresponds to the Euclidean metric on $\mathbb{R}^{n^2}$ if we notice that $\mathbb{M}(n)$ can be identified with $\mathbb{R}^{n^2}$. Note also that for every element $R \in \mathbb{O}(n)$, $\|R\|_{\mathbb{M}(n)} =$

$\sqrt{n}$. An open ball in $\mathbb{O}(n)$ is a set of the form $B_\delta(A) = \{A \in \mathbb{M}(n) : \|A - M\|_{\mathbb{M}(n)} < \delta\} \cap \mathbb{O}(n)$. The topology of $\mathbb{O}(n)$ as Lie group is, of course, equivalent to the induced topology thus defined.

We define the following norm in $\mathbb{E}(n)$. Given $I := R + v \in \mathbb{E}(n)$ let

$$\|I\|_{\mathbb{E}(n)} = \|R\|_{\mathbb{M}(n)} + \|v\|_{\mathbb{R}^n}$$

where $\|v\|_{\mathbb{R}^n}$ denotes the Euclidean norm in $\mathbb{R}^n$. It turns out that the topology of $\mathbb{E}(n)$ generated by the norm $\|.\|_{\mathbb{E}(n)}$ is equivalent to that as a Lie group.

2.2 Recurrence in $\mathbb{E}(n)$

In this context, we will say that a point x is *recurrent* if $\omega(x) \neq \emptyset$. For any map in general, the orbit of such a point may be unbounded and points with bounded orbits may coexist with others with unbounded ones. Due to the rigid structure of isometries, these situations can be disregarded in the context of isometric maps and therefore we can classify elements of $\mathbb{E}(n)$ as being either recurrent or nonrecurrent. In this section, we present some observations concerning isometries in $\mathbb{R}^n$ leading to their classification in terms of recurrence properties.

Given an isometry $I = R + v$ and any $x \in \mathbb{R}^n$, the n^{th} iterate of x under I is

$$I^n.x = R^n.x + R^{n-1}.v + \ldots + v \ . \tag{4}$$

Since $\mathbb{O}(n)$ is a compact group we can draw from (4) the conclusion that the boundedness of the orbit of x depends not on x but on the sum $\Sigma_n := R^{n-1}.v + \ldots + v$. This means that if $\mathsf{limsup}_{n\to\infty} |\Sigma_n| < \infty$ then every point has bounded orbit; whereas if Σ_n diverges then every orbit is unbounded.

Remark 1 *If some $x \in \mathbb{R}^n$ is recurrent under a given isometry I then all orbits are bounded under the same isometry I.*

This leads naturally to the following definition.

Definition 1 *Given an isometry $I \in \mathbb{E}(n)$ we say that I is **recurrent** if there exists a recurrent point under I. Otherwise, I is called a **nonrecurrent** isometry.*

2.2.1 Recurrent Isometries

We now establish some equivalent criteria of recurrence in $\mathbb{E}(n)$.

Lemma 1 *The following statements are equivalent for a given isometry $I = (R, v)$:*

(i) I is recurrent;
(ii) I has at least one fixed point;
(iii) $\langle v, \mathsf{Fix}R\rangle = 0$;
(iv) $I = \tau R \tau^{-1}$ for some translation τ.

Proof. Given a recurrent isometry I we know that there exists a recurrent point x. By Lemma 1 we conclude that $\mathcal{O}(x)$ is a bounded set. Since $\mathcal{O}(x)$ is a I-invariant set it follows by a common property of isometries that the centre of mass of $\mathcal{O}(x)$ is a fixed point for I hence (i) implies (ii).

Suppose there exists $p \in \mathbb{R}^n$ such that $I.p = p$ which implies that $(\mathsf{Id} - R).p = v$. If $\mathsf{det}(\mathsf{Id} - R) \neq 0$ then R has no eigenvalue 1 and therefore $\mathsf{Fix}R = \{0\}$ which trivially implies that $\langle v, \mathsf{Fix}R\rangle = 0$. If $\mathsf{det}(\mathsf{Id} - R) = 0$ then R has an eigenvalue 1 of multiplicity at least one. Let $\{\lambda_1, \dots, \lambda_n\}$ be the set of eigenvalues corresponding to the eigenvectors $\{b_1, \dots, b_n\}$ and suppose $\lambda_{j_i} = 1$ for $i = 1, \dots, k$ and some $k \leq n$. If $\mathcal{B} = \{b_1, \dots, b_n\}$ is the basis in which R is written in canonical form, then $\mathsf{Fix}R$ is the vector space spanned by $b_{j_1}, \dots, b_{j_k}$ and the entries of the matrix $\mathsf{Id} - R$ corresponding to these vectors are null. Therefore, in order that a solution X to the equation $(\mathsf{Id} - R).X = v$ exists, the j_i coordinates of v must be null as well. This is the same as saying that $\langle v, \mathsf{Fix}R\rangle = 0$. This proves that (iii) results from (ii).

Assume that $\langle v, \mathsf{Fix}R\rangle = 0$. Following a similar argument to the one presented previously, we can take u satisfying $(\mathsf{Id} - R).u = v$ and let $\tau.x = x + u$. Therefore,

$$\begin{aligned} \tau R \tau^{-1}.x &= \tau R.(x-u) \\ &= R.(x-u) + u \\ &= R.x + (\mathsf{Id} - R).u \\ &= R.x + v \ . \end{aligned}$$

which proves that (iii) implies (iv).

Finally, suppose that $I = \tau R \tau^{-1}$ for some translation $\tau.x = x + u$. Consequently, $I.u = \tau R \tau^{-1}.u = \tau R.\vec{0} = \tau.\vec{0} = u$ and so u is a fixed point of I and hence recurrent. Therefore, (i) follows from (iv). ■

2.2.2 Nonrecurrent Isometries

In the nonrecurrent case we can also add the following.

Lemma 2 *An isometry $I = R + v$ is nonrecurrent if and only if there exists an affine set V with $\mathsf{dim}(V) \geq 1$ such that $I_{|V}$ is a translation. Furthermore, the following diagram commutes,*

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{I} & \mathbb{R}^n \\ \mathsf{proj}_V \downarrow & & \downarrow \mathsf{proj}_V \\ V & \xrightarrow{\tau} & V \end{array}$$

where proj_V is the natural projection map defined by V and $\tau := I_{|V}$.

Proof. Assertion (iii) in Lemma 1 implies that if I is nonrecurrent then $\mathsf{dim}(\mathsf{Fix}R) \geq 1$. Furthermore, v can be decomposed as $v = \bar{v} + v^{\perp}$ where $\bar{v} \in \mathsf{Fix}R$, $v^{\perp} \in (\mathsf{Fix}R)^{\perp}$. Since $\langle v^{\perp}, \mathsf{Fix}R\rangle = 0$ we can find a vector u such that $(\mathsf{Id} - R)u = v^{\perp}$. Define $V = \mathsf{Fix}R + u$ and take $x \in V$ written as $x = x' + u$ for some $x' \in \mathsf{Fix}R$. Thus,

$$\begin{aligned} I.x &= R.(x' + u) + \bar{v} + v^{\perp} \\ &= R.x' + R.u + \bar{v} + (\mathsf{Id} - R).u \\ &= x' + \bar{v} + u \\ &= x + \bar{v} \ . \end{aligned}$$

Let $\tau.x = x + \bar{v}$ for all $x \in V$ and consider the projection map defined by V, $\mathsf{proj}_V : \mathbb{R}^n \to V$. This map is well defined since V is a closed convex set (see Webster [1994]). Every Euclidean isometry preserves orthogonality, that is, if $\langle v, w\rangle = 0$ then $\langle Iv, Iw\rangle = 0$. Let $y = \mathsf{proj}_V x$. By definition of proj_V we know that $\langle x - y, \tau.y - y\rangle = 0$ since $\tau.y \in V$. Consequently, $\langle I.x - \tau.y, \tau^2.y - \tau.y\rangle = 0$, which implies that $I.x - \tau.y \in V^{\perp}$ and therefore, $\tau.y = \mathsf{proj}_V I.x$. ■

2.2.3 Classification and genericity according to recurrence

Our remarks lead to the following crucial classification:

Class I: $\mathbb{SE}(n)$, n even and $\mathbb{E}(n)\backslash\mathbb{SE}(n)$, n odd. If all $\theta_i \neq 0$ in the canonical representation of R (as shown in (1) and (2)) then $\mathsf{dim}(\mathsf{Fix}R) = 0$ and I is recurrent[1]. Otherwise, $\mathsf{dim}(\mathsf{Fix}R) \geq 2$ and I is recurrent if and only if $\langle v, \mathsf{Fix}R\rangle = 0$.

Class II: $\mathbb{SE}(n)$, n odd or $\mathbb{E}(n)\backslash\mathbb{SE}(n)$, n even. In this case $\mathsf{dim}(\mathsf{Fix}R) \geq 1$ and thus, I is recurrent if and only if $\langle v, \mathsf{Fix}R\rangle = 0$.

Note that when $n = 1$ elements in **Class I** are of the form $I(x) = -x + v$ and therefore, $\mathsf{Fix}(-\mathsf{Id}) = \{0\}$ which implies that $\langle v, \mathsf{Fix}R\rangle = 0$ for every vector $v \in \mathbb{R}$. Consequently, all isometries in **Class I** are recurrent. On the other hand, elements of **Class II** are of the form $I(x) = x + v$, where $v \in \mathbb{R}$, hence the only recurrent isometry occurs for $v = 0$ whereas all other isometries satisfy $\langle v, \mathbb{R}\rangle = v \neq 0$. In general, we have the following situation.

Proposition 3 *Given $n \in \mathbb{N}$, there is an open set of full measure of recurrent isometries in* **Class I** *whereas a set of full measure in* **Class II** *is made of nonrecurrent isometries.*

Proof. According to Lemma 1 we know that a given isometry $I := R + v$ is recurrent if and only if $\langle v, \mathsf{Fix}R\rangle = 0$.

[1] Since $\mathsf{det}A = -1$ and n is odd there must exist an even number of occurrences of the eigenvalue 1 (this number could be zero).

Let ϕ : **Class I** $\to\mathbb{R}$, such that, $\phi(I) = \mathsf{det}(\mathsf{Id}-R)$ and $Z := \phi^{-1}(0)$. Since ϕ is a polynomial map on the entries of R we conclude that Z is a closed algebraic variety which must be of codimension at least 1, hence a zero measure set, because $Z \neq \mathbb{SE}(n)$, n even ($\neq \mathbb{E}(n)\backslash\mathbb{SE}(n)$ n odd, resp.). Hence the complement of Z is an open set of full measure for which $\langle v, \mathsf{Fix}R\rangle = 0$.

Consider now $Z := \{I \in$ **Class II**: $\langle v, \mathsf{Fix}R\rangle = 0\}$. Thus, by definition of product measure,

$$h \times \lambda(Z) = \int_{\widetilde{Z}} \lambda(Z_R) dh$$

where $Z_R = \{v \in \mathbb{R}^n : R + v \in Z\}$ and $\widetilde{Z}$ is either $\mathbb{SO}(n)$ when n is odd or $\mathbb{O}(n)\backslash\mathbb{SO}(n)$ for n even. Obviously, $R + v \in Z$ if and only if $\langle v, \mathsf{Fix}R\rangle = 0$ which is equivalent to $v \in \mathsf{Fix}R^{\perp}$. Since $\mathsf{dim}(\mathsf{Fix}R) \geq 1$ we have that $\mathsf{dim}(\mathsf{Fix}R^{\perp}) \leq n-1$. Hence, $\lambda(Z_R) = 0$ and therefore, $h \times \lambda(Z) = 0$. ∎

2.3 Set-up in the context of Piecewise Isometries

Let $\mathbf{X} = \mathbb{R}^n$ endowed with the standard Euclidean metric $d(\cdot,\cdot)$ and let $\mathcal{P} = \{P_0, \ldots, P_{m-1}\}$ be a finite collection of connected open sets with piecewise smooth boundary such that:

(i) $P_i \cap P_j = \emptyset$, for $i \neq j$;

(ii) $\mathbf{X} = \overline{P}_0 \cup \ldots \cup \overline{P}_{m-1}$;

We will call $\mathcal{P}$ a **partition** of $\mathbf{X}$ (with slight abuse of notation) and each set P_i an **atom** of the partition. We call $\mathcal{D} = \cup_{k=0}^{m-1} \partial P_k$ the **discontinuity set**.

Definition 2 *A* ***Piecewise Isometry*** *on* $\mathbf{X}$ *with partition* $\mathcal{P}$ *is a map* $T : \mathbf{X}\backslash\mathcal{D} \to \mathbf{X}$, *such that,*

$$T(x) = T_i(x), \text{ if } x \in P_i \ ,$$

where T_i *is an isometry defined on* $\mathbf{X}$, *for* $i = 0, \ldots, m-1$.

We do not define T on $\mathcal{D}$ but to simplify notation $T(S)$ will represent $T(S\backslash\mathcal{D})$ for any set $S \subset \mathbf{X}$.

Given a partition $\mathcal{P}$ of an Euclidean space $\mathbf{X}$, we denote by $\mathbf{PWI}(\mathcal{P})$ the set of all piecewise isometries that can be defined in this way on $\mathcal{P}$. Since we restrict our attention to the case when $\mathbf{X} = \mathbb{R}^n$ it turns out that $\mathbf{PWI}(\mathcal{P}) = \mathbb{E}(n)^m$ where m is the number of atoms. Hence $\mathbf{PWI}(\mathcal{P})$ inherits the structure of Lie group associated to $\mathbb{E}(n)^m$ as discussed in Section 2.1.2. Do note that if $\mathbf{X} \neq \mathbb{R}^n$ the structure of $\mathbf{PWI}(\mathcal{P})$ is no longer evident. If we restrict even further to the setting of invertible piecewise isometries then we can have a situation in which $\mathbf{PWI}(\mathcal{P})$ is a discrete set (see Ashwin and Fu [2000]).

The partition $\mathcal{P}$ gives a natural way of coding the trajectories. We define $\mathbf{X}^* \subset \mathbf{X}$ to be the set $\{x \in \mathbf{X} : \forall n \geq 0, I^n(x) \in \mathcal{P}_{i_n}, \text{ for some } i_n\}$. Thus $\mathbf{X}^*$ is the set of points whose forward iterates avoid the discontinuity set.

Let $\mathcal{A}$ be the alphabet $\{0,\ldots,m-1\}$. We define the **coding map** $\chi : \mathbf{X}^* \to \mathcal{A}^{\mathbb{N}}$ according to the following rule:

$$\chi(x) = w_0 \ldots w_n \ldots, \text{ if and only if } I^n(x) \in P_{w_n}, \text{ where } w_n \in \mathcal{A}.$$

If $x \in \mathbf{X}^*$ is such that its coding $\chi(x)$ is eventually periodic that is, if $\chi(x) = uv\ldots v\ldots$ where u and v are finite words on $\mathcal{A}$ then we call x a **rational** point. Otherwise $x \in \mathbf{X}^*$ is called an **irrational** point. Obviously, every periodic coding can be written as an infinite adjacent repetition of a finite block w which we represent by $[w] := ww\ldots$.

The original partition can be refined using the coding map. More precisely, let K_w be the set of all points $x \in \mathbf{X}^*$ with the same coding w. We will call the set K_w a **cell**. If all atoms of the partition are convex, open and w is rational then, K_w is convex and has open interior. Furthermore, if $x, y \in \mathbf{X}^*$ belong to the same cell, then for all $k \in \mathbb{N}$, $d(I^k(x), T^k(y)) = d(x,y)$. Finally, we shall denote by T_w the composition of maps $T_{w_n} \circ \ldots \circ T_{w_0}$ corresponding to a given word $w = w_0 \ldots w_n$.

2.4 Perturbing elements in $\mathbf{PWI}(\mathcal{P})$

In order to consider perturbations of piecewise isometries let us, first of all, remark that, unlike Goetz [2001], we do not consider perturbations of the partition $\mathcal{P}$. The main reason being that our results hold for unbounded spaces whereas results in Goetz [2001] are stated for compact sets. Moreover, if we perturb in the class of Goetz maps, for instance, it turns out that changing the atoms of the partition $\mathcal{P}$ is equivalent to conjugating the induced isometries with a translation. This can be easily visualised since conjugating with a translation is equivalent to moving the centre of rotation.

Finally, we define a metric on $\mathbf{PWI}(\mathcal{P})$ which is a generalisation of the metric defined in Section 2.1.2 to the n-dimensional case.

Definition 3 *Given $T, T' \in \mathbf{PWI}(\mathcal{P})$ where $T = \{(R_i, v_i)\}_{i=0}^{m-1}$ and $T' = \{(R'_i, v'_i)\}_{i=0}^{m-1}$ we define the **distance** between them by,*

$$\mathsf{dist}(T,T') = \sum_{i=0}^{m-1} \left\|R_i - R'_i\right\|_{\mathbb{M}(n)} + \sum_{i=0}^{m-1} \left\|v_i - v'_i\right\|_{\mathbb{R}^n}$$

where $\|.\|_{\mathbb{R}^n}$ denotes the usual Euclidean norm and $\|.\|_{\mathbb{M}(n)}$ is given by (3).

It follows from the definition above and that given in Section 2.1.2 that

$$\mathsf{dist}(T,T') = \sum_{i=0}^{m-1} \mathsf{dist}(T_i, T'_i).$$

For the purpose of developing a perturbation study of a family of dynamical systems one is usually interested in analysing changes under *small perturbations* where this notion may depend upon context. Here, we impose that, given a piecewise isometry $T = (T_i, P_i)_i$ any small perturbation of T cannot reverse orientation of any of the induced isometries. That is, if $T' = (T'_i, P_i)_i$ is a small perturbation of T then $\mathsf{det}(R_i) = \mathsf{det}(R'_i)$ for all $i \in \{0, \ldots, m-1\}$. Note that if we take for instance,

$$R = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \text{ and } R' = \begin{pmatrix} A & 0 \\ 0 & -1 \end{pmatrix}$$

where A is an orthogonal matrix, we have that $\mathsf{dist}(R, R') \geq 2$.

3 Stability of periodic points

3.1 The low-dimensional context

In this section we pay special attention to the one-dimensional and two-dimensional cases. These provide motivation for results in higher dimensions which will be presented in the next section. Let us consider the one-dimensional case first. Given a periodic point $\mathbf{p} \in \mathbb{R}$, with given coding w, of a piecewise isometry T defined in $\mathbb{R}$ there can only be two cases: either the return map is of the form $T_w.x = -x + 2\mathbf{p}$ or $T_w.x = x$. Any small perturbation of the initial piecewise isometry generates a return map with coding w which is of the form, $T'_w.x = -x + 2\mathbf{p} + v'_w$ in the former case, and, $T'_w.x = x + v'_w$ in the latter, where $v'_w \in \mathbb{R}$ is small in both maps. Consequently, in the orientation-reversing case the periodic point is preserved as long as the perturbation is sufficiently small taking into account that the orbit of $\mathbf{p} + \frac{v'_w}{2}$ must be realizable. As to the orientation-preserving case, we conclude that there is no longer a periodic point with coding w, whenever $v'_w \neq 0$. The collection of perturbations for which $v'_w = 0$ forms a zero measure set since $v'_w = 0$ can be seen as an algebraic equation defining an algebraic variety of codimension at least one. In particular, this implies that if the piecewise isometry is either an IET (interval exchange transformation) or an ITM (interval translation mapping) then no periodic points and periodic cells are stable.

As for the two-dimensional case let us assume that the return map is $T_w.x = R_w.x + v$ where $v \in \mathbb{R}^2$, $R_w \in \mathbb{O}(2)$ and its periodic point $\mathbf{p}$ with coding w is such that,

$$v = \mathbf{p} - R_w.\mathbf{p} \ . \tag{5}$$

1. If $\mathsf{Id} \neq R_w \in \mathbb{SO}(2)$ then equation (5) can be written equivalently as

$$\mathbf{p} = (\mathsf{Id} - R_w)^{-1}.v \quad .$$

   Therefore, in a sufficiently small neighbourhood of R_w we must have that $\mathsf{det}(\mathsf{Id} - R') \neq \mathbf{0}$ for all R' in such neighbourhood. By continuity of the composition map, it turns out that small perturbations of the original piecewise isometry will generate return maps whose linear part R'_w satisfies $\mathsf{det}(\mathsf{Id} - R'_w) \neq \mathbf{0}$. This implies that $\mathbf{p}$ is stable.

2. If $R_w = \mathsf{Id}$ then v must be the null vector, for otherwise, no periodic point with coding w would exist at all. Hence, $T_w = \mathsf{Id}$. By slightly perturbing the initial piecewise isometry one will obtain a return map of the form,

$$T'_w \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

where both θ and $\|u\| = \|(u_0, u_1)\|$ are close to zero. It turns out that the fixed point $\mathbf{p} = (p_0, p_1) \in \mathbb{R}^2$ of T'_w is given by,

$$\begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = \begin{bmatrix} 1-\cos\theta & \sin\theta \\ -\sin\theta & 1-\cos\theta \end{bmatrix}^{-1} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

and, therefore, it follows that,

$$\|\mathbf{p}\| = \frac{\|u\|}{\sqrt{2 - 2\cos\theta}} \ .$$

This, in turn, implies that $\|\mathbf{p}\|$ can take any value as $\|u\|, \theta \to 0$ and, consequently, $\mathbf{p}$ might lie anywhere in the plane $\mathbb{R}^2$. Thus, for some perturbations the whole cell may disappear whereas in other cases the periodic point will persist.

3. If $R_w \in \mathbb{O}(2)\backslash\mathbb{SO}(2)$ then R_w is a reflection on some mirror line crossing the origin. If a periodic point exists then it means that $\langle \mathsf{Fix} R_w, v\rangle = 0$. Using the argument in the proof of 3 one can state that the chances of $\langle \mathsf{Fix} R'_w, v\rangle = 0$ are zero according to the Haar measure in $\mathbb{E}(2)^m$. Therefore, almost every small perturbation will not have any periodic point with coding w.

Although these considerations are illustrative of what one can expect in the general case, the latter can be even more complex as Theorem 5 in next section shows.

3.2 Results in higher dimensions

The following result appeared in Mendes and Nicol [2004] and a similar type of result was obtained previously in two-dimensional compact spaces in Goetz [2001] allowing perturbation of partitions. This version holds for unbounded n-dimensional spaces with no perturbation of the original partition $\mathcal{P}$.

Theorem 4 *Suppose $T \in \mathbf{PWI}(\mathcal{P})$ possesses a periodic point $\mathbf{p}$ with coding $[w]$ such that $\mathsf{det}(R_w - \mathsf{Id}) \neq 0$. Then, $\mathbf{p}$ is stable under perturbation of T. More precisely, for every sufficiently small $\varepsilon > 0$ there is a $\delta > 0$ so that for all perturbations T' of T with $\mathsf{dist}(T', T) < \delta$ there is a point $\mathbf{p}'$ such that $\|\mathbf{p} - \mathbf{p}'\| < \varepsilon$ and $\mathbf{p}'$ is periodic under T' with coding $[w]$.*

For the remaining situation, *i.e.*, when $\mathsf{det}(R_w - \mathsf{Id}) = 0$, we shall consider two cases separately: $T_w \in$ **Class I** and $T_w \in$ **Class II**.

Theorem 5 *Suppose $T \in \mathbf{PWI}(\mathcal{P})$ possesses a periodic point $\mathbf{p}$ with coding $[w]$ such that $\mathsf{det}(R_w - \mathsf{Id}) = 0$ and $T_w \in$ **Class I**. Then,*

(i) If $\mathsf{Fix}T_w \subseteq K_{[w]}$ then for every sufficiently small $\varepsilon > 0$ there is $\delta > 0$ such that for an open set of full measure of piecewise isometries $T' \in B_\delta(T)$, $\mathsf{dist}(\mathsf{Fix}T_w, \mathbf{p}') < \varepsilon$, where $\mathbf{p}'$ is the only periodic point for T' with coding $[w]$.

(ii) If $\mathsf{Fix}T_w \nsubseteq K_{[w]}$ then for every $\varepsilon > 0$ there is $\delta > 0$ such that:

1. For an open set $S_\delta \subset B_\delta(T)$, $\mathsf{dist}(\mathsf{Fix}T_w, \mathbf{p}') < \varepsilon$, where $\mathbf{p}'$ is the unique periodic point under T' with coding $[w]$.

2. For a set $U_\delta \subset B_\delta(T)$ with nonempty interior all piecewise isometries have no periodic points with coding $[w]$.

Moreover, $S_\delta \cup U_\delta$ is open and dense in $B_\delta(T)$.

Theorem 6 *Suppose $T \in \mathbf{PWI}(\mathcal{P})$ possesses a periodic point $\mathbf{p}$ with coding w such that $\mathsf{det}(R_w - \mathsf{Id}) = 0$ and $T_w \in$ **Class II**. Then a.e. sufficiently small perturbation of T has no periodic points with coding w.*

This result generalises a previous theorem (see Mendes and Nicol [2004]) which proved instability for periodic points of piecewise rotations in odd dimensional Euclidean spaces.

We summarise our results in the following table.

$T_w := (R_w, v_w)$	$T_w \in$ **Class I**	$T_w \in$ **Class II**
$\mathsf{det}(R_w - \mathsf{Id}) = 0$	*Mixed stability*	*Periodic points are <u>unstable</u>*
$\mathsf{det}(R_w - \mathsf{Id}) \neq 0$	*Periodic points are <u>stable</u>*	—

3.3 Discussion

In this paper we have studied the stability properties of periodic points when the perturbation of piecewise isometries is carried out without perturbing the original partition. In this context we reckon the following points should be raised:

1. If we restrict our attention to convex partitions then, since they must form a partition of the entire $\mathbb{R}^n$ we can conclude that they must indeed, be polytopes, that is, the boundary of each atom must be made of finitely many hyperplanes of codimension 1. This means that we can define an orthogonal space of dimension 1, in any dimension. Therefore, we can measure a perturbation of each of the boundaries of the atoms by considering the angle between the original orthogonal spaces and the resulting ones. Nevertheless, we underline the fact that distinct perturbations of the original partition determine the same angles of perturbation of those orthogonal spaces.

2. We have not considered in detail what might happen to periodic cells when a perturbation (with or without perturbation of the atoms) occurs. When a periodic point is also present then our results offer an answer as to the persistence of that cell. However, nothing can be concluded concerning the changes in shape or measure that those cells might undergo. This offers a possible line of research, though we recognise that in Goetz [2001] the author gives a full description of the perturbation phenomena in terms of Lebesgue measure but only in compact domains of the plane.

3. The fact that we have considered piecewise isometries defined on the entire Euclidean space has enabled us to use all the structure of the Euclidean group as a Lie group. That had enormous implications on our results insofar as they have benefited from the existence of not only a measure space but also of a topological structure. If one restricts to compact domains then it is not so clear what the structure of the set of all possible piecewise isometries defined on that set might be. Even further, if one considers the invertible case then Ashwin & Fu [2002] have proved that one can end up with a finite set of possible maps.

4 Main proofs and auxiliary lemmata

4.1 Proof of Theorem 5

(i) Suppose there exists a periodic point with coding generated by $w = w_1 \dots w_k$ such that $\mathsf{Fix}T_w \subseteq K_{[w]}$. Let us denote by $\sigma(w_1 \dots w_k) = w_2 \dots w_k w_1$ the one-step shift on finite words. Let also, $d := \min_{i=1,\dots,k} \{\mathsf{dist}(T^i(\mathsf{Fix}T_w), \partial P_{w_i})\}$. Since $\mathcal{P}$ is a convex partition of $\mathbb{R}^n$ it turns out that all atoms must be open polytopes hence their boundaries are made of countably many hyperplanes of dimension $n-1$. Therefore, d is well defined and moreover, $d > 0$, taking into account $\mathsf{Fix}T_w$ is an affine space.

Take $\varepsilon \in]0, d[$. By applying Lemma 7 on each of the isometries $T_{\sigma^i(w)}$, $i = 1, \dots, k$, we conclude that there are $\delta_i > 0$ such that for open sets of full measure, $\mathcal{F}_i$ say, of isometries $T' \in B_{\delta_i}(T_{\sigma^i(w)})$, we have that $\mathsf{dist}(\mathsf{Fix}T_{\sigma^i(w)}, \mathbf{p}') < \varepsilon$, where $\mathbf{p}'$ is the unique fixed point of T'.

By Lemma 8 we know that the composition map $\varphi_{\sigma^i(w)}$ is surjective and continuous for each of the finite words $\sigma^i(w)$. Thus, $\mathcal{G}_i := \varphi^{-1}_{\sigma^i(w)}(\mathcal{F}_i)$ is a nonempty open set for each of the open sets $\mathcal{F}_i \subset \mathbb{E}(n)$. Again, by Lemma 8, we know that $\mathcal{N} := \cap_{i=1}^k \varphi^{-1}_{\sigma^i(w)}(B_{\delta_i}(T_{\sigma^i(w)}))$ must be an open neighbourhood (possibly disconnected) of T.

We now choose $\delta > 0$ so that $B_\delta(T) \subseteq \mathcal{N}$ ensuring, in this way, that every $T' \in B_\delta(T)$ is a small perturbation of T in the sense that $T'_w \in$ **Class I** as we have defined in Section 2.4. Moreover, each set,

$$Z_{\sigma^i(w)} := \left\{ T' \in B_\delta(T) : \mathsf{det}(R'_{\sigma^i(w)} - \mathsf{Id}) = 0 \right\}$$

is a closed set of codimension at least 1 in $\mathcal{N}$ by Lemma 9 since each $T_{\sigma^i(w)}$ must be of

Class I. This proves that $\mathcal{G}_i \cap B_\delta(T)$ is an open set of full measure in $\varphi^{-1}_{\sigma^i(w)} B_{\delta_i}(T_{\sigma^i(w)})$. Consequently, $\mathcal{G} := \cap_{i=1}^k \mathcal{G}_i$ must be an open set of full measure in $\mathcal{N}$ and so, T is an accumulation point of $\mathcal{G}$.

The assertion now follows by noting that $\mathcal{G}$ is an open of full measure of isometries in $B_\delta(T)$ whose elements have to satisfy, by construction, the following condition: $\mathsf{dist}(\mathsf{Fix}T_w, \mathbf{p}') < \varepsilon$, where $\mathbf{p}'$ is such that it is the unique fixed point of T'_w. The orbit of $\mathbf{p}'$ is realised by all piecewise isometries in $\mathcal{G}$ because it is the intersection of all neighbourhoods $\mathcal{G}_i$.

(ii) Given $\varepsilon > 0$ and not regarding to an upperbound of ε as in the proof above (hence ignoring any particular partition of $\mathbf{X}$), we use the previous construction to find $\delta > 0$ and an open set of full measure of piecewise isometries $\mathcal{H}$ in $B_\delta(T)$ such that $\mathsf{dist}(\mathsf{Fix}T_w, \mathbf{p}') < \varepsilon$, where $\mathbf{p}'$ is as previously. However, since $\mathsf{Fix}T_w \nsubseteq K_{[w]}$, there exist points $\mathbf{p}$ from $\mathsf{Fix}T_w$ lying outside $K_{[w]}$ and that implies, using Lemma 8 and Lemma 10 combined, that there is a subset, U_δ say, of $\mathcal{H}$ whose elements have their unique periodic point $\mathbf{p}'$ with coding $[w]$ close to $\mathbf{p}$ and thus lying outside $K_{[w]}$. Therefore, those $T' \in U_\delta$ will not have any periodic point with coding $[w]$ according to the given partition $\mathcal{P}$ (in fact, none of the codings $\{\sigma^i(w)\}_i$ is realizable). Since the complement of $K_{[w]}$ has nonempty interior, by continuity in Lemma 8 and in Lemma 10 combined, we conclude that U_δ also has nonempty interior.

If, for other piecewise isometries T', some iterate of $\mathbf{p}'$ lies outside its corresponding atom then $\mathsf{Fix}T_{\sigma^i(w)} \nsubseteq K_{[\sigma^i(w)]}$ for some i and therefore the previous argument is applicable to $T_{\sigma^i(w)}\mathbf{p}'$ instead.

The remaining set of piecewise isometries in $\mathcal{H}$, S_δ say, is such that its unique periodic point $\mathbf{p}'$ and its iterates will lie inside $P_{\sigma^i(w)}$ for all i. It must be a nonempty set because $K_{[\sigma^i(w)]}$ has nonempty interior for every i (note that periodic cells must have nonempty interior). Those piecewise isometries T' will be such that $\mathsf{det}(R'_w - \mathsf{Id}) \neq 0$ and consequently, by Theorem 4 we conclude that $\mathbf{p}'$ is stable hence we can find a neighbourhood of T' which is contained in $\mathcal{H}$ and, thus, S_δ must actually be open.

Finally, the fact that $\mathcal{H} = U_\delta \cup S_\delta$ and that $\mathcal{H}$ is open and has full measure in $B_\delta(T)$ implies that $U_\delta \cup S_\delta$ is also open an set with full measure in $B_\delta(T)$.

4.2 Proof of Theorem 6

Suppose $w = w_1 \dots w_k$ is a word on the alphabet $\{0, \dots, m-1\}$ so that $T_w \in$ **Class II**. Let us denote by $\mathbb{E}[T]$ (resp., $\mathbb{O}[T]$) the connected component of $\mathbb{E}(n)^m$ (resp., $\mathbb{O}(n)^m$)whose i-th coordinate has the same orientation of T_i (resp., R_i), for all $i \in \{0, \dots, m-1\}$. In other words, $\mathbb{E}[T]$ is the set on which small perturbations of T can be performed and $\mathbb{O}[T]$ is the corresponding linear component.

The composition of isometries, T_w, is of the form

$$\begin{aligned} T_w.x &= R_{w_1 \dots w_k}.x + R_{w_1 \dots w_{k-1}}.v_{w_k} + R_{w_1 \dots w_{k-2}}.v_{w_{k-1}} + \dots + v_{w_1} = \\ &= R_w.x + U_0.v_0 + \dots + U_{m-1}.v_{m-1} \end{aligned} \tag{6}$$

where U_j is a sum of words on matrices of $\mathbb{O}(n)$ or $U_j = 0$ depending on whether or not the element T_j appears in the word T_w. The last isometry applied, T_{w_1}, distinguishes U_{w_1} as a sum of the identity matrix and the sum of products of matrices and this is the only matrix U_j which we can guaranty it is nonzero for every word w. Without loss of generality, let us assume that $w_1 = 0$.

Let $\mathfrak{Rec}_w := \{T \in \mathbb{E}[T] : T_w \text{ is recurrent}\}$. We now prove that $(h \times \lambda)^m(\mathfrak{Rec}_w) = 0$. From (6) we can write T_w in the form $(R_w, U_0.v_0 + u)$ where $u \in \mathbb{R}^n$. By the definition of product measure, it follows that,

$$(h \times \lambda)^m(\mathfrak{Rec}_w) = \int_{\mathbb{O}[T]} \int_{(\mathbb{R}^n)^{m-1}} \lambda(\mathfrak{Rec}^*_w) dh^m \times d\lambda^{m-1}, \tag{7}$$

where,

$$\begin{aligned}\mathfrak{Rec}^*_w &= \{v_0 \in \mathbb{R}^n : (L, v_0) \in \mathfrak{Rec}_w, L \in \mathbb{O}[T] \times (\mathbb{R}^n)^{m-1}\} = \\ &= \{v_0 \in \mathbb{R}^n : T_w = (R_w, U_0.v_0 + u) \in \mathfrak{Rec}_w\}.\end{aligned}$$

By Lemma 1.(iii) in Section 2.2 it follows that $(R_w, U_0.v_0 + u) \in \mathfrak{Rec}_w$ if and only if $\langle \mathsf{Fix}R_w, U_0.v_0 + u\rangle = 0$ which is equivalent to saying that $v_0 \in U_0^{-1}((\mathsf{Fix}R_w)^\perp - u)$, where $U_0^{-1}(.)$ denotes the standard preimage notation. Therefore,

$$\mathfrak{Rec}^*_w = U_0^{-1}((\mathsf{Fix}R_w)^\perp - u)\ .$$

If U_0 is invertible then the fact that $T_w \in$ **Class II** implies that $\mathsf{dim}(\mathsf{Fix}R_w) \geq 1$ and so, $\mathsf{dim}(\mathsf{Fix}R_w)^\perp \leq n-1$. Consequently, $\lambda(U_0^{-1}((\mathsf{Fix}R_w)^\perp - u)) = 0$ since U_0^{-1} exists and it preserves dimension.

When U_0 is not invertible, by linearity of U_0, the only situation for which $U_0^{-1}((\mathsf{Fix}R_w)^\perp - u)$ has positive Lebesgue measure is when $((\mathsf{Fix}R_w)^\perp - u) \cap \mathsf{Im}U_0$ has dimension equal to that of $\mathsf{Im}U_0$, that is, $\mathsf{Im}U_0 \subset (\mathsf{Fix}R_w)^\perp - u$. The fact that $\mathsf{Im}U_0$ is a vector subspace of $\mathbb{R}^n$ implies that $\vec{0}$ must belong to $(\mathsf{Fix}R_w)^\perp - u$. Since $(\mathsf{Fix}R_w)^\perp$ also is a vector subspace we must have that $u = \vec{0}$.

Let $\psi_w : \mathbb{O}[T] \times (\mathbb{R}^n)^{m-1} \to \mathbb{R}^n$ be defined as $\psi_w(T') = \sum_{k=1}^{m-1} U_k.v_k$ where each U_k is determined according to formula (6). Since ψ_w is a polynomial map on the entries of elements in $\mathbb{O}[T] \times (\mathbb{R}^n)^{m-1}$ we conclude that $\psi_w^{-1}(\vec{0})$ is an algebraic variety. We now prove that $\psi_w^{-1}(\vec{0}) \neq \mathbb{O}[T] \times (\mathbb{R}^n)^{m-1}$.

Let R be an element of $\mathbb{O}(n) \backslash \mathbb{SO}(n)$ such that $R^2 = \mathsf{Id}$ (any reflection on an hyperplane of $\mathbb{R}^n$) and let $L = (L_i, v_j) \in \mathbb{O}[T] \times (\mathbb{R}^n)^{m-1}$ where $v_j = v$ for all $j \in \{1, \ldots, m-1\}$ and v is a nonzero vector that does not belong to either $\mathsf{Fix}R$ or $(\mathsf{Fix}R)^\perp$. Moreover, let $L_i = R$ or $L_i = \mathsf{Id}$ depending on whether $\mathsf{det}(T_i) = -1$ or $\mathsf{det}(T_i) = -1$ for all $i \in \{1, \ldots, m\}$ (this ensures that $(L_1, \ldots, L_m) \in \mathbb{O}[T]$). It then follows that $\psi_w(L) = \sum_{k=1}^{m-1} R^{p_k}.v$ where ,

$$R^{p_k} = \begin{cases} \mathsf{Id}\text{ , if } p_k \text{ is even} \\ R\text{, if } p_k \text{ is odd}\end{cases}$$

Therefore, $\psi_w(L) = (pR + q\mathsf{Id})\,.v$ for some $p, q \in \mathbb{N}$ such that $p + q = |w|$. Now,

$$\psi_w(L) = \vec{0} \Leftrightarrow (pR + q\mathsf{Id})\,.v = \vec{0} \Leftrightarrow R.v = -\frac{q}{p}.v$$

Since R preserves length one concludes that $\left|-\frac{q}{p}\right| = 1$ and therefore, $R.v = \pm v$. This would imply that either v would belong to $\mathsf{Fix}R$ or $(\mathsf{Fix}R)^{\perp}$, which is false by construction. Consequently, $\psi_w^{-1}(\vec{0})$ is a closed algebraic variety of codimension at least 1.

Finally, we have proved that the integral (7) can be decomposed in two integrals, separating $\mathbb{O}[T] \times (\mathbb{R}^n)^{m-1}$ in two disjoint sets differing on whether the resulting matrix U_0 is invertible or not. If it is then the integrand is zero. Otherwise, the chances of that same integrand taking nonzero values are zero according to $dh^m \times d\lambda^{m-1}$ measure.

The assertion now follows since the set of words with finite length is countable.

4.3 Auxiliary lemmata

Lemma 7 *Suppose $T = (R, v) \in$ **Class I** is such that $\mathsf{det}(R - \mathsf{Id}) = 0$. Then, for every $\varepsilon > 0$ there is $\delta > 0$ such that for an open set of full measure of isometries $T' \in B_\delta(T)$, $\mathsf{dist}(\mathsf{Fix}T, \mathbf{p}') < \varepsilon$, where $\mathbf{p}'$ is the unique fixed point for T.*

Proof. Let $\mathcal{F}$ denote the set of isometries $T = (R, v)$ in **Class I** such that $\mathsf{det}(R-\mathsf{Id}) \neq 0$. This implies, in particular, that $\mathsf{Fix}T \neq \emptyset$. Arguments shown in the proof of Proposition 3 imply that $\mathcal{F}$ is an open set of full measure in **Class I** (see also proof of Lemma 9 for $m = 1$ and $|w| = 1$). An equivalent statement is that the function,

$$\begin{array}{rcl} \psi_T : \mathcal{F} & \rightarrow & \mathbb{R}_0^+ \\ T' & \mapsto & \mathsf{dist}(\mathsf{Fix}T, \mathbf{p}') \end{array}$$

is continuous in $\mathcal{F}$. The assertion follows by noticing that $\psi_T(T') = \|\mathbf{p}' - proj_{\mathsf{Fix}T}\mathbf{p}'\|$ where $\mathbf{p}' = (\mathsf{Id} - R')^{-1}.v'$ and $proj_{\mathsf{Fix}T}\mathbf{p}'$ are both continuous maps, since $\mathsf{det}(R' - \mathsf{Id}) \neq 0$ and $\mathsf{Fix}T$ is a closed convex space. ■

Lemma 8 *For every finite word w, the composition map $\varphi_w : \mathbb{E}(n)^m \to \mathbb{E}(n)$, $\varphi_w(T) = T_w$ is surjective and differentiable.*

Proof. Let $\overline{\varphi}_w : \mathbf{M}(n)^m \to \mathbf{M}(n)$ be the composition map determined by the word w, *i.e.*, $\overline{\varphi}_w(A_1, \ldots, A_m) = A_{w_k} \ldots A_{w_1}$. The differentiability of $\overline{\varphi}_w$ can be easily derived from the fact that $\overline{\varphi}_w$ can be seen as a map from the Euclidean space $\mathbb{R}^{m.n^2}$ onto $\mathbb{R}^{n^2}$ whose coordinates are polynomials. Hence φ_w is also differentiable because it is a restriction of a differentiable map to a smooth manifold. Surjectivity follows from the fact for any word w and any isometry $I \in \mathbb{E}(n)$ we have that $\varphi_w(T) = I$ where $T = (A_1, \ldots, A_m)$ is chosen so that $A_{w_1} \overset{r}{\ldots} A_{w_1} = I$ and $A_i = Id$ for all $i \neq w_1$. Note that such a r^{th} root can always be found simply by manipulating the canonical form of the original isometry. ■

Since $\mathbb{O}(n)$ is made of two connected components then so will $\mathbb{E}(n)$ be. Therefore, the Lie group $\mathbb{E}(n)^m$ will be made of 2^m connected components. Given a finite word w we split $\mathbb{E}(n)^m$ into the following disjoint union of 2^{m-1} connected components of $\mathbb{E}(n)^m$:

$$\begin{array}{rcl} \mathbb{E}(w)_I & = & \{T \in \mathbb{E}(n)^m : T_w \in \textbf{Class I}\} \\ \mathbb{E}(w)_{II} & = & \{T \in \mathbb{E}(n)^m : T_w \in \textbf{Class II}\} \end{array}$$

Obviously, $\mathbb{E}(n)^m = \mathbb{E}(w)_I \cup \mathbb{E}(w)_{II}$. This notation will be used in the following lemma.

Lemma 9 *For every finite word w, the set*

$$Z_w := \{T \in \mathbb{E}(n)^m : \mathsf{det}(R_w - \mathsf{Id}) = 0\}$$

is a closed Zariski variety such that $Z_w \cap \mathbb{E}(w)_I$ has codimension at least 1 in $\mathbb{E}(w)_I$ and the dimension of $Z_w \cap \mathbb{E}(w)_{II}$ equals that of $\mathbb{E}(w)_{II}$.

Proof. Since det is a polynomial map on the entries of matrices it follows immediately that Z is Zariski subvariety of $\mathbb{E}(n)^m$. The subset $Z_w \cap \mathbb{E}(w)_I$ has codimension at least 1 in $\mathbb{E}(w)_I$ since we can choose a matrix $A \in$ **Class I** such that its canonical form has only irrational $\theta's$ and, therefore,

$$\varphi_w\left(\mathsf{Id}, \ldots, \underset{\uparrow\, w_1 \text{ position}}{A}, \ldots, \mathsf{Id}\right) = A^r ,$$

where r is the number of appearances of w_1 in w, implying that $\mathsf{det}(A^r - \mathsf{Id}) \neq 0$. On the other hand, all elements $T \in Z_w \cap \mathbb{E}(w)_{II}$ are such that $T_w \in$ **Class II** and thus, $\mathsf{det}(R_w - \mathsf{Id}) = 0$. ■

Lemma 10 *Suppose $T = (R, v) \in$ **Class I** is such that $\mathsf{det}(R - \mathsf{Id}) = 0$. Let $\mathbf{B}_\delta = \{T' \in B_\delta(T) : \mathsf{det}(T' - \mathsf{Id}) \neq 0\}$. For every $\delta > 0$ and every $\mathbf{p} \in \mathsf{Fix}T$ let us define the function:*

$$\begin{array}{rcl} \tau_{\delta,\mathbf{p}} : \mathbf{B}_\delta & \to & \mathbb{R}_0^+ \\ T' & \mapsto & \left\|\mathbf{p} - \mathbf{p}'\right\| \end{array}$$

where $T'.\mathbf{p}' = \mathbf{p}'$ is the unique fixed point for T'. Then, $\mathbf{B}_\delta$ is an open set of full measure in $B_\delta(T)$ and $\tau_{\delta,\mathbf{p}}$ is surjective and continuous on $\mathbf{B}_\delta$.

Proof. Again, the fact that $T \in$ **Class I** and that $\mathsf{det}(R - \mathsf{Id}) = 0$ imply, in particular, that $\mathsf{Fix}T \neq \emptyset$. Following the argument in the proof of Proposition 3 we conclude that there is an open set of full measure of isometries $T' = (R', v') \in B_\delta(T)$ such that $\mathsf{det}(R' - \mathsf{Id}) \neq 0$. The continuity of $\tau_{\delta,\mathbf{p}}$ follows trivially from the continuity of the Euclidean norm and the continuity of ρ, defined by $\rho(T') := (\mathsf{Id} - R')^{-1} .v'$, since $\tau_{\delta,\mathbf{p}}(T') = \|\mathbf{p} - \rho(T')\|$.

As to the surjectivity of $\tau_{\delta,\mathbf{p}}$ let $r \in \mathbb{R}_0^+$ and take $\mathbf{p}' \in \mathsf{Fix}T$ so that $\|\mathbf{p}-\mathbf{p}'\| = r$ (Note that $\mathsf{dim}(\mathsf{Fix}T) \geq 1$). Let us also define $T' := R'(x-\mathbf{p}')+\mathbf{p}'$, where R' is chosen as follows. First of all, by continuity of the action of $\mathbb{O}(n)$ on $\mathbb{R}^n$, we conclude, that, independently of the vector $\mathbf{p}' \in \mathbb{R}^n$, there is $\varepsilon > 0$ such that,

$$\left\|R - R'\right\|_{\mathbf{M}(n)} < \varepsilon \Rightarrow \left\|(R-R').\mathbf{p}'\right\|_{\mathbb{R}^n} < \delta/2 \ .$$

Let $\bar{\delta} := \mathsf{min}\{\delta/2, \varepsilon\}$ and let us choose a matrix R' so that $\mathsf{det}(R' - \mathsf{Id}) \neq 0$ and $R' \in B_{\bar{\delta}}(R)$. Such matrix R' exists since the set of matrices M satisfying $\mathsf{det}(M - \mathsf{Id}) \neq 0$ is open and has full measure in **Class I**. By doing this, we have arranged an isometry which has only one fixed point $\mathbf{p}'$. Since $\mathbf{p}' \in \mathsf{Fix}T$ we can write $v = \mathbf{p}' - R\mathbf{p}'$. Therefore,

$$\begin{aligned} \left\|v - v'\right\|_{\mathbb{R}^n} &= \left\|\mathbf{p}' - R\mathbf{p} - (\mathbf{p}' - R'\mathbf{p}')\right\|_{\mathbb{R}^n} = \\ &= \left\|(R'-R).\mathbf{p}'\right\|_{\mathbb{R}^n} < \delta/2 \end{aligned}$$

This implies that,

$$\mathsf{dist}(T', T) = \left\|R - R'\right\|_{\mathbf{M}(n)} + \left\|v - v'\right\|_{\mathbb{R}^n} < \bar{\delta} + \delta/2 < \delta.$$

and consequently, $T' \in \mathbf{B}_\delta$ and, moreover, $\tau_{\delta,\mathbf{p}}(T') = r$. ■

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