CONTINUOUS MAPS OF TYPE 2^∞ ARE INFINITELY RENORMALIZABLE

SALETE ESTEVES AND FERNANDO J. MOREIRA

Abstract. We prove that a continuous endomorphisms of the interval whose set of periods is the set of all powers of two, is infinitely renormalizable.

1. Introduction

In this work we will denote by $C^0(I)$ the set of continuous map from de compact interval I into itself. Let $f \in C^0(I)$ be a map with zero topological entropy and $x \in I$. Smítal has proven [7] that if the ω-limit set of x, $\omega(x)$, is infinite, then f is infinitely renormalizable. V. Jiménez López showed [3] that if f is normal (see section 2) with set of periods (of its periodic orbits)

$$P(f) = \{2^i : i \in \mathbb{N}_0\},$$

then there exists a point $x \in I$ such that $\omega(x)$ is infinite. We obtain a slight improvement of the result stated above.

Theorem 1. *Let $f \in C^0(I)$. If $P(f) = \{2^i : i \in \mathbb{N}_0\}$, then f is infinitely renormalizable.*

We remark that if f is infinitely renormalizable, then $P(f) \supseteq \{2^i : i \in \mathbb{N}_0\}$. Combining this claim with Sarkovskii Theorem and Smítal result we have:

Corollary 2. *Let $f \in C^0(I)$ be a map with zero topological entropy and $x \in I$. If $\omega(x)$ is infinite, then $P(f) = \{2^i : i \in \mathbb{N}_0\}$.*

2. Preliminary definitions and results

For any $n \in \mathbb{N}_0$, we define f^n inductively in $\mathbb{N}_0$, by $f^0 = id$ and $f^n = f \circ f^{n-1}$. A point x is a *periodic point* (of period k) if $f^k(x) = x$ and if x, $f(x)$, $f^2(x)$, $\cdots$, $f^{k-1}(x)$ are distinct. The set

$$\mathcal{O}(x) = \{x, f(x), \cdots, f^{k-1}(x)\}$$

is called *periodic orbit* (of period k). If $n = 1$, then x is a *fixed point* of f. We define $\omega(x)$ as the set of accumulation of $\mathcal{O}(x)$. We say that x is a *recurrent point* iff $x \in \omega(x)$. Periodic points are always recurrent points and if x is recurrent, then any iterate $x_n = f^n(x)$ is also a recurrent point.

Consider the Sharkovskii ordering of the positive integers: $3 \lhd 5 \lhd 7 \lhd 9 \lhd \cdots \lhd 2 \cdot 3 \lhd 2 \cdot 5 \lhd 2 \cdot 7 \lhd 2 \cdot 9 \lhd \cdots \lhd 2^2 \cdot 3 \lhd 2^2 \cdot 5 \lhd 2^2 \cdot 7 \lhd 2^2 \cdot 9 \lhd \cdots \lhd 2^n \cdot 3 \lhd 2^n \cdot 5 \lhd 2^n \cdot 7 \lhd 2^n \cdot 9 \lhd \cdots \lhd 2^\infty \lhd \cdots \lhd 16 \lhd 4 \lhd 2 \lhd 1$. Let $f \in C^0(I)$ and $n, k \in \mathbb{N}$. Sarkovskii showed that if $n \in P(f)$ and $n \lhd k$, then $k \in P(f)$ (see, e.g., [8]).

The next Proposition gives a sufficient condition for the existence of periodic orbits of period 3. This result have an elementary proof.

Proposition 3. *Let $f \in C^0(I)$ and let R, S closed subintervals of I. If the set $R \cap S$ doesn't contain fixed points of f and $f(R) \cap f(S) \supseteq R \cup S$, then $3 \in P(f)$.*

For $A \subseteq I$, $\mathrm{Conv}(A)$ denote the convex hull of A. We say that a continous map $f : I \to I$ is *normal* if for any $k \in \mathbb{N}$ and any monotone sequence $(p_n)_{n\in\mathbb{N}}$ of fixed points for f^k, there is $s \in \mathbb{N}$ such that

$$f^k(\mathrm{Conv}\{p_r, p_{r+1}\}) = \mathrm{Conv}\{p_r, p_{r+1}\}.$$

Note that piecewise monotone maps are normal, as well as C^1 maps (see [3]).

A continuous map $f : I \to I$ is called *renormalizable* if there exists an interval $J \subsetneq I$ and $p \geq 2$ satisfying:

(1) the intervals $f^i(J)$, $i \in \{0, 1, \cdots, p-1\}$, have disjoint interiors,
(2) $f^p(J) = J$.

A map $f \in C^0(I)$ is *infinitely renormalizable* if there is an infinite sequence $J_1 \supsetneq J_2 \supsetneq \cdots \supsetneq J_n \supsetneq \cdots$ of nested intervals and a sequence $(a_n)_{n\in\mathbb{N}}$ of integers greater or equal to two such that, for each $n \in \mathbb{N}$,

(1) the intervals J_n, $f(J_n)$, $, \cdots$, $f^{a_1 \cdots a_n - 1}(J_n)$ have disjoint interiors,
(2) $f^{a_1 \cdots a_n}(J_n) = J_n$.

When more precision is required, we shall say that f is $(a_n)_{n\in\mathbb{N}}$-infinitely renormalizable. The sets $f^i(J_n)$, for $0 \leq i \leq a_1 \cdots a_n - 1$ are called the *atoms of generation* n of f.

Let $n \in \mathbb{N}$, $\epsilon > 0$, (X, d) a compact metric space and $f : X \to X$ a continuous map. We say that a set $S \subseteq X$ is (n, ϵ)-separated if

$$x, y \in S,\, x \neq y \Rightarrow \max_{0 \leq j < n} d(f^j(x), f^j(y)) > \epsilon.$$

Set

$$s(n, \epsilon, f) = \max\{\#S : S \subseteq X \text{ is } (n, \epsilon)\text{-separated}\}.$$

Then, the limit

$$h(f) = \lim_{\epsilon \to 0^+} \limsup_{n \to +\infty} \frac{\log s(n, \epsilon, f)}{n}$$

exists and we call $h(f)$ the *topological entropy* of f.

For continuous maps on an interval, the next theorem connect entropy with periodic orbits.

Theorem 4. ([1],[4]) *Let $f \in C^0(I)$. Then $h(f) > 0$ if and only if it has a periodic point whose period is not a power of 2.*

3. Proof of the Theorem 1

We prove the Theorem 1 applying inductively the next lemma.

Lemma 5. *Let $f : I \to I$ be a C^0 interval map with zero entropy. If f has a recurrent point x_0 such that $\#\mathcal{O}(x_0) \geq 3$, then f is 2-renormalizable.*

Proof. We will define the class $\mathcal{A}$ of closed intervals $J \subseteq I$ by

$$J \in \mathcal{A} \Leftrightarrow \begin{cases} f(J) \subseteq J \\ x_0 \in J \end{cases}$$

The closed set $J_0 = \bigcap_{J \in \mathcal{A}} J$ is also in $\mathcal{A}$ relation $\subseteq$ in the class $\mathcal{A}$). Since $x_0 \in f(J_0)$, we see that $f(J_0) \in \mathcal{A}$, hence $f(J_0) = J_0$. We set $J_0 = [a_0, b_0]$.

We claim that f has a fixed point $p_0 \in J_0$, such that, for some $i \in \mathbb{N}$,

$$a_0 \leq x_i < p_0 < x_{i+1} \leq b_0.$$

Since $O(x_0) \subseteq \omega(x_0)$, $(x_n)_{n \geq 0}$ cannot be a monotonic sequence. So there is $i \in \mathbb{N}$ such that $f(x_i) = x_{i+1} > x_i$ and $f(x_{i+1}) = x_{i+2} < x_{i+1}$. Therefore there exists a point $p_0 \in (x_i, x_{i+1})$ for which $f(p_0) = p_0$.

If $f(x) > p_0$ for all $x \in (a_0, p_0)$ and $f(x) < p_0$ for all $x \in (p_0, b_0)$, then the lemma is proved. If is not the case there exists a point $q_0 \in (a_0, b_0)$

such that $q_0 \neq p_0$ and $f(q_0) = f(p_0) = p_0$. We assume that $q_0 \in (p_0, b_0)$. The other possibility is handled similarly.

Set

$$J_1 = \bigcup_{n\geq 0} f^n([p_0, q_0]).$$

J_1 is an f-invariant set, that is to say $f(J_1) \subseteq J_1$. Moreover, since $p_0 \in f^n([p_0, q_0])$, for all $n \in \mathbb{N}_0$, we have J_1 is an interval. It follows by the continuity of f that $J_2 = J_1 \cup \partial J_1$ is an f- invariant set.

The next step is to prove that $J_2 \subseteq [a_0, q_0]$. Otherwise, there would be a point $y \in (p_0, q_0)$ and $n > 0$ such that $f^n(y) > q_0$. We get $f^n([p_0, y]) \cap f^n([y, q_0]) \supseteq [p_0, q_0]$, then f^n has a periodic point of period three, which imply that $h(f) > 0$, a contradiction.

Since $J_2 \subseteq [a_0, q_0] \subsetneq J_0$ and $f(J_2) \subseteq J_2$, the minimality of J_0 ensure us that $x_0 \notin J_2$. Consequently $\mathcal{O}(x_0) \cap J_2 = \emptyset$. Consider the class $\mathcal{B}$ of intervals $J \subseteq I$ such that

$$J \in \mathcal{B} \Leftrightarrow \begin{cases} J_2 \subseteq J \subseteq J_0 \\ f(J) \subseteq J \\ \mathcal{O}(x_0) \cap J = \emptyset \end{cases}.$$

The interval $J_3 = \bigcup\limits_{J\in\mathcal{B}} J$ belongs to $\mathcal{B}$. It is the maximum element in the class $\mathcal{B}$ for the relation $\subseteq$. Since J_0 is closed and f is continuous, it follows that

$$J_2 \subseteq \overline{J_3} \subseteq J_0 \text{ e } f(\overline{J_3}) \subseteq \overline{J_3}.$$

We set $\partial J_3 = \{p_3, q_3\}$ $(p_3 < q_3)$. If $f(p_3) \in (p_3, q_3)$ or $f(q_3) \in (p_3, q_3)$, then there would be $K \in \mathcal{B}$ such that $J_3 \subsetneq K$, which contradicts the definition of J_3. Hence $f(\partial J_3) \subseteq \partial J_3$. Since $\#\mathcal{O}(x_0) \geq 3$, it follows that $\mathcal{O}(x_0) \cap \partial J_3 = \emptyset$ and so we have that $\overline{J_3} \in \mathcal{B}$. Therefore $\overline{J_3} = J_3$.

We remark that, since $x_i \in [a_0, p_3]$ and $x_{i+1} \in [q_3, b_0]$, $f([a_0, p_3]) \cap [q_3, b_0] \neq \emptyset$ and $f([q_3, b_0]) \cap [a_0, p_3] \neq \emptyset$.

We claim that $f(p_3) = q_3$ and $f(q_3) = p_3$. Suppose $f(q_3) = q_3$. By the maximality of J_3, there is $z \in [q_3, b_0]$ such that $f(z) > q_3$ (otherwise there would be a larger interval than J_3, whose image by f is contained in J_3). Since the interval $[q_3, b_0]$ is not f- invariant, there exists $r \in (q_3, b_0)$ such that $f(r) = q_3$. The set L defined by

$$L = \bigcup_{n\geq 0} f^n([q_3, r])$$

is an interval, because q_3 is a fixed point of f. Since $P(f) \subseteq \{2^i : i \in \mathbb{N}_0\}$, we get $L \subseteq [a_0, r] \subsetneq J_0$. Thus, $L_1 = J_3 \cup L$ is an interval f-invariant. Therefore $f(\overline{L_1}) \subseteq \overline{L_1}$ and $\overline{L_1} \subsetneq J_0$, which implies that $\mathcal{O}(x_0) \cap \overline{L_1} = \emptyset$. It follows that $\overline{L_1} \in \mathcal{B}$ and $J_3 \subsetneq \overline{L_1}$, a contradiction. So $f(q_3) \neq q_3$. Since the boundary of J_3 is f-invariant, $f(q_3) = p_3$. A similar argument gives $f(p_3) = q_3$.

We now finish the proof of the lemma by proving that $f([a_0, p_3]) = [q_3, b_0]$ and $f([q_3, b_0]) = [a_0, p_3]$. Suppose $f([q_3, b_0]) \nsubseteq [a_0, p_3]$. It follows that there is $w \in [q_3, b_0]$ such that $f(w) > p_3$. Since $f([q_3, b_0]) \nsubseteq [p_3, b_0]$, there exists $q_3 < s < b_0$ with $f(s) = p_3$. Next, consider

$$L_3 = \bigcup_{n \geq 0} f^n([p_3, s]).$$

Then the set L_3 is an interval, because, for all $n \in \mathbb{N}_0$, $p_3 \in f^n([p_3, s])$ and $L_3 \subseteq [a_0, s]$. If is not the case, since $f([p_3, q_3]) \subseteq [p_3, q_3]$, there would be a point $t \in [q_3, s]$ and $m \in \mathbb{N}$ such that $f^m(t) > s$. Therefore $f^m([q_3, t]) \cap f^m([t, s]) \supseteq [q_3, s]$, which is impossible because $h(f) = 0$. Hence $\overline{L_3}$ is an interval f- invariant and $J_3 \subsetneq \overline{L_3} \subsetneq J_0$. By the maximality of J_0, we have $\mathcal{O}(x_0) \cap \overline{L_3} = \emptyset$, but this contradicts the maximality of J_3. Then $f([q_3, b_0]) \subseteq [a_0, p_3]$. The inclusion $f([a_0, p_3]) \subseteq [q_3, b_0]$ can be proved analogously. Since $f(J_0) = J_0$, $J_0 = J_3 \cup [a_0, p_3] \cup [q_3, b_0]$ e $f(J_3) \subseteq J_3$, we conclude the equality. □

Proof of Theorem 1. Since f has a periodic orbit of period four, by the Lemma 5 we can find intervals K_0, K_1 with $f(K_0) = K_1$ and $f(K_1) = K_0$.

Since $P(f) = \{2^i : i \in \mathbb{N}_0\}$, we can apply Lemma 5 to the map $f^2_{|K_0} : K_0 \to K_0$ in order to obtain two intervals $K_{0,0}$, $K_{0,1}$ of a 2-renormalization of $f^2_{|K_0} : K_0 \to K_0$. For $i = 0, 1$ and $j = 0, 1$

$$K_{i,j} = f^{i+2j}(K_{0,0})$$

are the atoms of the second generation for a $(2, 2)$-renormalization of f.

Applying inductively Lemma 5, we get that f is $(2)_{n\in\mathbb{N}}$-infinitely renormalizable. □

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E-mail address: saleteesteves@ipb.pt

E-mail address: fjm@fc.up.pt