On random topological Markov chains with big images and preimages

Manuel Stadlbauer

Departamento de Matemática Pura, Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal. E-mail: manuel.stadlbauer@fc.up.pt

April 28, 2009

Abstract. We introduce a relative notion of the 'big images and preimages'-property for random topological Markov chains. This then implies that a relative version of the Ruelle-Perron-Frobenius theorem holds with respect to summable and locally Hölder continuous potentials.

1 Introduction

In this note we give a further contribution to the extension of thermodynamic formalism for topological Markov chains to random transformations and, in particular, obtain a sufficient condition for the existence of random conformal measures and random eigenfunctions of the Ruelle operator which applies e.g. to a random full shift with countably many states. In particular, we obtain an extension of the results for random subshifts of finite type obtained by Bogenschütz, Gundlach and Kifer ([1, 7, 8]) to random shift spaces with countably many states. For illustration, we also give applications to countable random matrices, that is we deduce a Perron-Frobenius theorem and a sufficient condition for the existence of a stationary distribution for a countable-state Markov chain with random transition probabilities.

For deterministic dynamical systems the following results are known. Recall that it was shown by Sarig ([12]) that the Ruelle-Perron-Frobenius theorem extends to deterministic topological Markov chains with countably many states and locally Hölder continuous potentials if and only if the system is positive recurrent. If the potential is summable, results in this direction are obtained by imposing topological mixing conditions, 'finite irreducibility' or 'finite primitivity', on the shift space (see [9, 13]). Furthermore, if the topological

1991 Mathematics Subject Classification. Primary: 37D35; Secondary: 37H99.

Key words and phrases. random countable Markov shift, random bundle transformation, Ruelle-Perron-Frobenius theorem, Markov chains with random transitions, finite primitivity, big images and preimages.

Acknowledgement. The author acknowledges support by FCT through grant SFRH/BPD/39195/2007 and the Centro de Matemática da Universidade do Porto.

Markov chain is topologically mixing then these conditions coincide with the 'big images and preimages'-property introduced in [11] where it is shown that this condition is equivalent to positive recurrence for summable potentials (see [11]). Note that these results are advantageous in many applications since they can be, in contrast to positive recurrence, verified easily.

The goal of this paper is to obtain an extension of these results to random bundle transformations, that is we consider a commuting diagram (or fibered system)

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \pi \downarrow & & \downarrow \pi \\ \Omega & \xrightarrow{\theta} & \Omega, \end{array}$$

where θ is an ergodic automorphism of the abstract probability space (Ω, P) and π is onto and measurable. With X_ω referring to $\pi^{-1}(\{\omega\})$ the restriction $T_\omega : X_\omega \to X_{\theta\omega}$ of T to fibers then has a natural interpretation as a random transformation in a random environment. In here, we consider the class of random topological Markov chains, that is X is a subset of $\mathbb{N}^{\mathbb{N}} \times \Omega$ such that each fiber X_ω has a random Markov structure (for details, see Section 2).

For the extension of the notion of big images and preimages (b.i.p.) to this setting, we only require that a corresponding property holds for returns to subsets Ω_{bi} and Ω_{bp} of positive measure in the base Ω. That is, for $\omega \in \Omega_{\text{bi}}$, there exists a finite union of cylinders $F_{\theta\omega} \subset X_{\theta\omega}$ such that $T_\omega([a]) \cap F_{\theta\omega} \neq \emptyset$ for all cylinders $[a] \subset X_\omega$ (*big images*) and, for $\omega \in \Omega_{\text{bp}}$, there exists a finite union of cylinders $F'_{\theta^{-1}\omega} \subset X_{\theta^{-1}\omega}$ such that $T_{\theta^{-1}\omega}(F'_{\theta^{-1}\omega}) = X_\omega$ (*big preimages*), respectively. Note that this property is a purely topological property with respect to the fibers.

We then consider topologically mixing systems equipped with a potential ϕ which is locally Hölder continuous in the fibers. Our further analysis relies on the divergence at the radius of convergence of a random power series whose coefficients are given by random partition functions. Systems with this property will be called of divergence type. As a first result we obtain in Theorem 3.4 that a system with summable potential and the b.i.p.-property is of divergence type.

For systems of divergence type with summable potential, it then follows that a random conformal measure exists (Theorem 4.3). That is, there there exists a family of probability measures $\{\mu_\omega\}$ and a positive random variable $\lambda : \Omega \to \mathbb{R}$ such that, for $x \in X_\omega$,

$$d\mu_{\theta\omega} \circ T_\omega / d\mu_\omega(x) = \lambda(\omega) e^{P_G(\phi) - \phi(x)},$$

where $P_G(\phi)$ refers to the relative Gurevič pressure as introduced in [3]. The proof of this statement relies on the construction of λ as the limit of the quotient of random power series and the application of Crauel's random Prohorov theorem (see [2]) to a family of random measures. Note that the construction of this family of random measures is an adaption of the construction in [3]. However, it turns out that the summability assumption significantly simplifies the tightness argument compared to the proof in there.

In particular, this result gives that $\lambda e^{P_G(\phi)}$ is the spectral radius of the dual of the random Ruelle operator. For systems with the b.i.p.-property, the identification of λ as quotient of random power series then gives rise to application of results in [3], that is the system is positive recurrent and a relative version of the Ruelle-Perron-Frobenius theorem holds (Corollary 4.6 and Theorem 4.7). As immediate consequences of these results, we obtain a Perron-Frobenius theorem for random matrices (Corollary 4.9) and an application to random stochastic matrices (Theorem 4.10).

2 Preliminaries

Let θ be an automorphism (i.e. bimeasurable, invertible and probability preserving) of the probability space $(\Omega, \mathscr{F}, P)$, $\ell = \ell_\omega > 1$ be a $\mathbb{N} \cup \{\infty\}$-valued random variable and for a.e. $\omega \in \Omega$, let $A_\omega = \left(\alpha_{ij}(\omega), i < \ell_\omega, j < \ell_{\theta\omega}\right)$ be a matrix with entries $\alpha_{ij}(\omega) \in \{0,1\}$ such that $\omega \mapsto A_\omega$ is measurable and $\sum_{j<\ell_{\theta\omega}} a_{ij}(\omega) > 0$ for all $i < \ell_\omega$. For the random shift spaces

$$X_\omega = \{x = (x_0, x_1, ...) : \alpha_{x_i x_{i+1}}(\theta^i \omega) = 1 \; \forall i = 0, 1, ...\},$$

the (random) shift map $T_\omega : X_\omega \to X_{\theta\omega}$ is defined by $T_\omega : (x_0, x_1, x_2 ...) = (x_1, x_2, ...)$. This gives rise to a globally defined map T of X, where $X := \{(\omega, x) : x \in X_\omega\}$, and $T(\omega, x) = (\theta\omega, T_\omega x)$. In this situation, the pair (X, T) is referred to as a *random countable topological Markov chain*. For $n \in \mathbb{N}$, set $T_\omega^n = T_{\theta^{n-1}\omega} \circ \cdots \circ T_{\theta\omega} \circ T_\omega$, and note that $T^n(\omega, x) = (\theta^n \omega, T_\omega^n x)$.

A finite word $a = (x_0, x_1, \dots, x_{n-1}) \in \mathbb{N}^n$ of length n is called ω-*admissible*, if $x_i < \ell_{\theta^i \omega}$ and $\alpha_{x_i x_{i+1}}(\theta^i \omega) = 1$, for $i = 0, \dots, n-1$. In here, $\mathscr{W}_\omega^n$ denotes the set of ω-admissible words of length n (in particular, $\mathscr{W}_\omega^1 = \{a : a < \ell_\omega\}$) and, for $a = (a_0, a_1, \dots, a_{n-1}) \in \mathbb{N}^n$,

$$[a]_\omega = [a_0, a_1, ..., a_{n-1}]_\omega := \{x \in X_\omega : x_i = a_i, \, i = 0, 1, ..., n-1\}$$

is called *cylinder set*. The set of those $\omega \in \Omega$ where the cylinder is nonempty will be denoted by Ω_a, that is

$$\Omega_a = \{\omega : \; [a]_\omega \neq \emptyset\} = \{\omega : a \in \mathscr{W}_\omega^n\}.$$

Finally, $\mathscr{W}^n$ refers to the set of words of length n defined by $P(\Omega_a) > 0$ for all $a \in \mathscr{W}^n$. In this paper, we exclusively consider *topologically mixing* random topological Markov chains. That is, for $a, b \in \mathscr{W}^1$, there exists a $\mathbb{N}$-valued random variable $N_{ab} = N_{ab}(\omega)$ such that for $n \geq N_{ab}(\omega)$, $a \leq \mathscr{W}_\omega^1$ and $\theta^n \omega \in \Omega_b$ it follows that $[a]_\omega \cap (T_\omega^n)^{-1}[b]_{\theta^n \omega} \neq \emptyset$.

As mentioned above we are interested in thermodynamic aspects of random topological Markov chains with respect to locally Hölder continuous potentials. Therefore, recall that, for a function $\phi : X \to \mathbb{R}$, $(\omega, x) \mapsto \phi^\omega(x)$, the n-th variation is defined by

$$V_n^\omega(\phi) = \sup\{|\phi^\omega(x) - \phi^\omega(y)| : x_i = y_i, \, i = 0, 1, \dots, n-1\}.$$

The function ϕ is referred to as a *locally fiber Hölder continuous function with index* $k \in \mathbb{N}$ if there exists a random variable $\kappa = \kappa(\omega) \geq 1$ such that $\int \log \kappa dP < \infty$ and for all $n \geq$

k, $V_n^\omega(\phi) \leq \kappa(\omega) r^n$. For abbreviation, such a function will be referred to as a *k-Hölder continous* function. This then leads to the following elementary but useful estimate. For $n \leq m$, $x, y \in [a]_\omega$ for some $a \in \mathscr{W}_\omega^m$, and a $(m-n+1)$-Hölder continous function ϕ,

$$\begin{aligned}|\phi_n^\omega(x) - \phi_n^\omega(y)| &\leq \sum_{k=0}^{n-1} |\phi^{\theta^k\omega}(T_\omega^k(x)) - \phi^{\theta^k\omega}(T_\omega^k(y))| \leq \sum_{k=0}^{n-1} V_{m-k}^{\theta^k\omega}(\phi) \\ &\leq \sum_{k=0}^{n-1} \kappa(\theta^k\omega) r^{m-k} \leq \sum_{k>m-n} \kappa(\theta^{m-k}\omega) r^k = r^{m-n} \sum_{k=1}^{\infty} \kappa(\theta^{n-k}\omega) r^k.\end{aligned}$$

Since $\log\kappa \in L^1(P)$, we obtain $(\log\kappa)/n \to 0$ as a consequence of the ergodic theorem. So the radius of convergence of the series on the right hand side of the above estimate is equal to 1 and, in particular, the right hand side is finite. Hence, for a locally fiber Hölder continuous function with index less than or equal to $(m-n+1)$, we obtain

$$(B_{\theta^n\omega})^{-1} \leq (B_{\theta^n\omega})^{-r^{m-n}} \leq e^{\phi_n^\omega(x) - \phi_n^\omega(y)} \leq (B_{\theta^n\omega})^{r^{m-n}} \leq B_{\theta^n\omega}, \tag{1}$$

where $B_\omega := \exp\sum_{k=1}^\infty \kappa(\theta^{-k}\omega) r^k$. Note that this definition differs from the one in [3] by the choice of the element in the base - in here we replaced ω by $\theta^n\omega$. A further basic notion is the *(random) Ruelle operator* L_ϕ associated to a potential (function) $\phi = (\phi^\omega) : X \to \mathbb{R}$, which is defined by, for a function $f : X \to \mathbb{R}$,

$$L_\phi^\omega f(\theta\omega, x) = \sum_{y \in X_\omega, T_\omega y = x} e^{\phi^\omega(y)} f(\omega, y).$$

In here, we consider potentials ϕ satisfying some of the following additional assumptions.

(H1) The potential ϕ is 1-Hölder continuous, and $\int \log B_\omega dP(\omega) < \infty$.

(H2) The potential ϕ is 2-Hölder continuous, and $\int \log B_\omega dP(\omega) < \infty$.

(S1) $\int \log M_\omega dP(\omega) < \infty$, where $M_\omega := \sup\{L_\phi^\omega(1)(x) : x \in X_{\theta\omega}\}$.

(S2) $\int \log m_\omega dP(\omega) > -\infty$, where $m_\omega := \inf\{L_\phi^\omega(1)(x) : x \in X_{\theta\omega}\}$.

These assumptions might be seen as randomised versions of Hölder continuous (H1-2) and summable potentials (S1-2), respectively. Recall that, if ϕ is locally fiber Hölder continuous and κ is integrable, then (H1) holds (see [3]). Also note that (S1-2) is equivalent to $\|\log L_\phi^\omega(1)\| \in L^1(P)$. Below, after introducing big images and preimages, we will give a further Hölder condition (H*) for which 1-Hölder continuity is only required on a subset of Ω and 2-Hölder continuity else.

3 Partition functions and big images and preimages

In this section, we introduce the notion of big images and preimages for random topological Markov chains. Moreover, we discuss immediate consequences of this notion in terms of estimates for the random version of the Gurevič partition functions. These estimates will then be used to prove that the preimage function diverges at its radius of convergence (Theorem 3.4).

In order to define the relevant objects, we now introduce the following notation. For $a,b \in \mathscr{W}^1$, $\omega \in \Omega_a$ and $n \in \mathbb{N}$, set

$$\mathscr{W}_\omega^n(a,b) := \{(w_0,\dots,w_{n-1}) \in \mathscr{W}_\omega^n :\ w_0 = a, w_{n-1}b \in \mathscr{W}^2_{\theta^{n-1}\omega}\}.$$

Moreover, for $w \in \mathscr{W}_\omega^n$, and $m \le n$, set $\exp(\phi_m^\omega([w])) := \sup\{\exp(\phi_m^\omega(x)) :\ x \in [w]_\omega\}$. As an immediate consequence of (1) we have, for a k-Hölder continuous potential ϕ,

$$0 < \inf\{\exp(\phi^\omega_{n-k+1}(x)) :\ x \in [w]_\omega\} \le \exp(\phi^\omega_{n-k+1}([w])) < \infty \text{ a.s.} \tag{2}$$

We now consider a fixed topologically mixing random Markov chain (X,T), a potential ϕ satisfying (H2) and $a \in \mathscr{W}^1$. For $\omega \in \Omega_a$ and $n \in \mathbb{N}$, the *n-th (random) Gurevič partition function* is defined by

$$Z_n^\omega(a) := \sum_{w \in \mathscr{W}_\omega^n(a,a)} e^{\phi_n^\omega([wa])},$$

where we use the convention that $Z_n^\omega(a) = 0$ if $\mathscr{W}_\omega^n(a,a) = \emptyset$. Note that this definition differs from the one in [3]. In here, $\phi_n^\omega([w])$ is replaced by $\phi_n^\omega([wa])$ in order to obtain a partition function applicable to 2-Hölder continuous potentials. Since (X,T) is topologically mixing, it follows that $Z_n^\omega(a) > 0$ for all $n \ge N_{aa}(\omega)$ with $\theta^n\omega \in \Omega_a$. Furthermore, given a measurable family $\{\xi_\omega \in [a]_\omega : \omega \in \Omega\}$, the *n-th local preimage function* is defined by

$$\mathscr{Z}_n^\omega(a) := \sum_{w \in \mathscr{W}_\omega^n(a,a)} e^{\phi_n^\omega(\tau_w(\xi_{\theta^n\omega}))} = L_\phi^{\omega,n}(1_{[a]})(\xi_{\theta^n\omega}),$$

where τ_w refers to the inverse branch $T_\omega^n([w]_\omega) \to [w]_\omega$. In particular, if ϕ is 2-Hölder, then (1) implies that $Z_n^\omega(a) \ge \mathscr{Z}_n^\omega(a) \ge Z_n^\omega(a) B^{-1}_{\theta^n\omega}$. Moreover, the *n-th preimage function* is defined by, for $\omega \in \Omega$,

$$\mathscr{Z}_n^\omega := \sum_{w \in \mathscr{W}_\omega^n} e^{\phi_n^\omega(\tau_w(\xi_{\theta^n\omega}))} = L_\phi^{\omega,n}(1)(\xi_{\theta^n\omega}).$$

As a consequence of (S1), we have $\mathscr{Z}_n^\omega \le M_\omega \cdots M_{\theta^{n-1}\omega} < \infty$. Finally, set

$$A_n^\omega := \sum_{w \in \mathscr{W}_\omega^n} e^{\phi_n^\omega([w])},$$

and note that $0 < A_n^\omega \le \infty$. We now introduce the the *relative Gurevič pressure* $P_G(\phi)$ adapted to the situation under consideration. For $\Omega' \subset \Omega$ and $\omega \in \Omega$, set $J_\omega(\Omega') := \{n \in \mathbb{N} :\ \theta^n\omega \in \Omega'\}$, and choose $N \in \mathbb{N}$ such that $\Omega^* := \{\omega \in \Omega_a :\ N_{aa}(\omega) \le N\}$ is a set of positive measure. The following proposition is a slight generalization of Theorem 3.2 in [3] to 2-Hölder continuous (H2) and summable (S1) potentials.

Proposition 3.1. *For a mixing system (X,T) and a potential satisfying (H2) and (S1), the limits*

$$P_G(\phi) := \lim_{\substack{n\to\infty,\\ n\in J_\omega(\Omega^*)}} \frac{1}{n}\log Z_n^\omega(a) = \lim_{\substack{n\to\infty,\\ n\in J_\omega(\Omega^*)}} \frac{1}{n}\log \mathscr{Z}_n^\omega(a) \geq -\infty$$

exist, are a.s. constant with respect to ω and independent of the choice of a and N.

Proof. Since most of the arguments can be found in [3] we only give a sketch of proof. For a.e. $\omega\in\Omega^*$ and $m,n\geq N$ with $\theta^m\omega,\theta^{m+n}\omega\in\Omega^*$ it follows from (1) that

$$\mathscr{Z}_m^\omega(a)\mathscr{Z}_n^{\theta^m\omega}(a) \leq B_{\theta^m\omega}\mathscr{Z}_{m+n}^\omega(a). \tag{3}$$

It is well known that the induced transformation $\hat{\theta}:\Omega^*\to\Omega^*$ given by

$$\begin{aligned}&\eta:\Omega'\to\mathbb{N},\quad \omega\to\eta(\omega):=\min\{n\in\mathbb{N}:\ \theta^n\omega\in\Omega'\}\\ &\hat{\theta}:\Omega'\to\Omega',\quad \omega\to\theta^{\eta(\omega)}\omega.\end{aligned}$$

is an invertible, measure preserving, conservative and ergodic transformation with respect to P restricted to Ω^*. Set $\eta_k(\omega):=\sum_{l=0}^{k-1}\eta(\hat{\theta}^l\omega)$. It then follows from (3), for $M\geq N$ and $k,l\in\mathbb{N}$, that

$$-\log\mathscr{Z}^\omega_{\eta_{(k+l)M}(\omega)}(a)+\log\mathscr{Z}^\omega_{\eta_{kM}(\omega)}(a)+\log\mathscr{Z}^{\hat{\theta}^{kM}\omega}_{\eta_{lM}(\hat{\theta}^{kM}\omega)}(a)\leq\log B_{\hat{\theta}^{kM}\omega}.$$

Since $\mathscr{Z}_n^\omega(a)\leq\mathscr{Z}_n^\omega\leq M_\omega M_{\theta\omega}\cdots M_{\theta^{n-1}\omega}$, it follows from (H2), (S1) and Kac's theorem that the almost subadditive ergodic theorem as stated in [5] is applicable to $-\log\mathscr{Z}_\cdot^\omega(a)$ with respect to the measure preserving transformation $\hat{\theta}^M$. Since η_k/k converges by Birkhoff's ergodic theorem it follows that

$$f(\omega):=\lim_{k\to\infty}\frac{1}{\eta_{kM}(\omega)}\log\mathscr{Z}^\omega_{\eta_{kM}(\omega)}(a)$$

exists a.s., and is $\hat{\theta}^M$-invariant. It is now easy to see that this limit is independent from the choice of $M\geq N$ and hence the limit is a constant function. In order to show that the limit does exist along $J_\omega(\Omega^*)$ we now use a different argument as in [3]). For $k>3N$ set $a_k:=N+(k \mod N)$. Then $k-a_k$ is a multiple of N, and $2N>a_k\geq N$. In particular, $\mathscr{Z}^\omega_{\eta_{a_k}(\omega)}(a),\mathscr{Z}^{\hat{\theta}^{a_k}(\omega)}_{\eta_{k-a_k}(\omega)}(a)>0$. Hence, by (3),

$$\frac{1}{\eta_k(\omega)}\log\left(\mathscr{Z}^\omega_{\eta_{a_k}(\omega)}(a)\mathscr{Z}^{\hat{\theta}^{a_k}(\omega)}_{\eta_{k-a_k}(\omega)}(a)\right)\leq\frac{1}{\eta_k(\omega)}\log\left(B_{\hat{\theta}^{a_k}(\omega)}\mathscr{Z}^\omega_{\eta_k}(a)\right).$$

By passing to the limit, we obtain that $\inf\{(\log\mathscr{Z}_n^\omega(a)):\ n\in J_\omega(\Omega^*)\}\geq f(\omega)$ a.s. The other direction then follows by the same argument with $b_k=2N-(k \mod N)$ and using

$$\mathscr{Z}^{\hat{\theta}^{-b_k}(\omega)}_{\eta_{b_k}(\hat{\theta}^{-b_k}(\omega))}(a)\mathscr{Z}^\omega_{\eta_{k-b_k}(\omega)}(a)\leq B_\omega\mathscr{Z}^{\hat{\theta}^{-b_k}(\omega)}_{\eta_k}(a).$$

The remaining assertions now follow from $Z_n^\omega(a)\geq\mathscr{Z}_n^\omega(a)\geq Z_n^\omega(a)B^{-1}_{\theta^n\omega}$ and Step 2 in the proof of Theorem 3.2 in [3]. ☐

We now introduce the notion of big images and preimages. In here, we will write $\#B$ for the cardinality of a set B. So assume that there exists $\Omega_{\mathrm{bi}} \subset \Omega$ of positive measure, and a family $\{\mathscr{I}_{\mathrm{bi}}^{\omega} \subset \mathscr{W}_{\omega}^{1} : \omega \in \Omega_{\mathrm{bi}}\}$ such that

(i) $\#\mathscr{I}_{\mathrm{bi}}^{\omega} < \infty$,

(ii) for each $a \in \mathscr{W}_{\theta^{-1}\omega}^{1}$, there exists $b \in \mathscr{I}_{\mathrm{bi}}^{\omega}$ such that $ab \in \mathscr{W}_{\theta^{-1}\omega}^{2}$.

We then say that (X,T) has the *big image property*. By choosing a subset of Ω_{bi}, one may assume without loss of generality that there exists a finite set $\mathscr{I}_{\mathrm{bi}}$ such that $\mathscr{I}_{\mathrm{bi}}^{\omega} \subset \mathscr{I}_{\mathrm{bi}}$ for each $\omega \in \Omega_{\mathrm{bi}}$.

Moreover, if there exists $\Omega_{\mathrm{bp}} \subset \Omega$ of positive measure, and a family $\{\mathscr{I}_{\mathrm{bp}}^{\omega} \subset \mathscr{W}_{\theta^{-1}\omega}^{1} : \omega \in \Omega_{\mathrm{bp}}\}$ such that

(i) $\#\mathscr{I}_{\mathrm{bp}}^{\omega} < \infty$,

(ii) for each $a \in \mathscr{W}_{\omega}^{1}$, there exists $b \in \mathscr{I}_{\mathrm{bp}}^{\omega}$ such that $ba \in \mathscr{W}_{\theta^{-1}\omega}^{2}$,

then (X,T) is said to have the *big preimage property*. As above, one may assume without loss of generality, that each $\mathscr{I}_{\mathrm{bp}}^{\omega}$ is a subset of a globally defined finite set $\mathscr{I}_{\mathrm{bp}}$. If (X,T) is topologically mixing and has the big image and big preimage property, then (X,T) is said to have the *(relative) b.i.p.-property*.

Lemma 3.2. *If (X,T) has the b.i.p.-property, then for $a \in \mathscr{W}^{1}$ and almost every $\omega \in \Omega_a$, there exist $\alpha_\omega, \beta_\omega \in \mathbb{N}$ such that*

(i) $\mathscr{W}_{\omega}^{n}(a,b) \neq \emptyset$*, for all* $n \geq \alpha_\omega$*, and* $b \in \mathscr{W}_{\theta^{n}\omega}^{1}$*,*

(ii) $\mathscr{W}_{\theta^{-n}\omega}^{n}(b,a) \neq \emptyset$*, for all* $n \geq \beta_\omega$*, and* $b \in \mathscr{W}_{\theta^{-n}\omega}^{1}$*.*

Proof. In order to show the first assertion, set $N_\omega := \max\{N_{ac}(\omega) : \ c \in \mathscr{I}_{\mathrm{bp}}\}$, and $\alpha_\omega := \min\{n \geq N_\omega : \theta^{n} \in \Omega_{\mathrm{bp}}\}$. The second assertion follows by a similar construction. $\square$

The following Lemma now shows that the above partition and preimage functions are proportional to each other along subsequences. In the proof of the result, we only will need that ϕ^{ω} is 1-Hölder for $\omega \in \theta^{-1}(\Omega_{\mathrm{bi}} \cup \Omega_{\mathrm{bp}})$. The precise condition is as follows.

(H*) The potential ϕ has property (H2) and for a.e. $\omega \in \theta^{-1}(\Omega_{\mathrm{bi}} \cup \Omega_{\mathrm{bp}})$, we have $V_1^{\omega}(\phi) < \infty$.

Lemma 3.3. *For (X,T,ϕ) with b.i.p.-property, (H*) and (S1-2), the following holds.*

(i) For a.e. $\omega \in \Omega_a$, and $k,n \in \mathbb{N}$ with $k \geq \alpha_\omega$, $\theta^{k}\omega \in \Omega_{bp}$ and $\theta^{k+n}\omega \in \Omega_a$, there exists $1 \leq C_\omega(a,k) < \infty$ such that

$$\mathscr{Z}_n^{\theta^k\omega} \leq C_\omega(a,k)\ \mathscr{Z}_{k+n}^{\omega}(a).$$

(ii) For a.e. $\omega \in \Omega$, and $n,k \in \mathbb{N}$ with $\theta^n\omega \in \Omega_{bi}$, $\theta^{k+n}\omega \in \Omega_a$ and $k \geq \beta_{\theta^n\omega}$, there exists $1 \leq D_{\theta^n\omega}(a,k) < \infty$ such that

$$A_n^\omega \leq B_{\theta^n\omega} D_{\theta^n\omega}(a,k)^{-1} \; \mathscr{Z}_{n+k}^\omega.$$

Moreover, $P_G(\phi)$ is finite and $C_\omega(a,k)$ and $D_\omega(a,k)$ are measurable.

Proof. In order to show the first assertion, note that the big preimage property combined with Lemma 3.2 implies the existence of $\{v_j \in \mathscr{W}_\omega^k : \; j = 1, \ldots, \#\mathscr{I}_{\text{bp}}^{\theta^k\omega}\}$ such that for each $b \in \mathscr{W}_{\theta^k\omega}^1$, there exists $j \in \{1, \ldots, \#\mathscr{I}_{\text{bp}}^{\theta^k\omega}\}$ with $v_j b \in \mathscr{W}_\omega^{k+1}$. This then gives that

$$\begin{aligned}
\mathscr{Z}_{k+n}^\omega(a) &\geq \sum_{j=1}^{\#\mathscr{I}_{\text{bp}}^{\theta^k\omega}} \sum_{w:\, v_j w \in \mathscr{W}_\omega^{k+n}} e^{\phi_{n+k}^\omega(\tau_{v_j w}(\xi_{\theta^{k+n}\omega}))} \\
&\geq \sum_{j=1}^{\#\mathscr{I}_{\text{bp}}^{\theta^k\omega}} \inf\left\{ e^{\phi_k^\omega(x)} : x \in [v_j]_\omega \right\} \sum_{w:\, v_j w \in \mathscr{W}_\omega^{k+n}} e^{\phi_n^{\theta^k\omega}(\tau_w(\xi_{\theta^{k+n}\omega}))} \\
&\geq \inf\left\{ e^{\phi_k^\omega(x)} : x \in [v_j]_\omega, j = 1, \ldots, \#\mathscr{I}_{\text{bp}}^\omega \right\} \; \mathscr{Z}_n^\omega =: (C_\omega(a,k))^{-1} \; \mathscr{Z}_n^\omega.
\end{aligned}$$

Observe that $C_\omega(a,k) > 0$ which follows from (H*) and (2). Moreover, by choosing e.g. $v_1, \ldots v_{\#\mathscr{I}_{\text{bp}}^\omega}$ to be minimal with respect to the lexicographic ordering, it follows that $\omega \to C_\omega(a,k)$ is measurable. Assertion (ii) follows by a similar argument, that is by

$$\begin{aligned}
\mathscr{Z}_{n+k}^\omega &\geq \sum_{j=1}^{\#\mathscr{I}_{\text{bi}}^{\theta^n\omega}} \sum_{w:\, w v_j \in \mathscr{W}_\omega^{n+k}} e^{\phi_{n+k}^\omega(\tau_{w,v_j}(\xi_{\theta^{n+k}\omega}))} \\
&\geq A_n^\omega B_{\theta^n\omega}^{-1} \exp(-V_1^{\theta^{n-1}\omega}(\phi)) \sum_{j=1}^{\#\mathscr{I}_{\text{bi}}^{\theta^n\omega}} \inf\left\{ e^{\phi_k^{\theta^n\omega}(x)} : x \in [v_j]_{\theta^n\omega} \right\},
\end{aligned}$$

where $\{v_j \in \mathscr{W}_{\theta^n\omega}^k : \; j = 1, \ldots, \#\mathscr{I}_{\text{bi}}^{\theta^n\omega}\}$ are constructed from the big image property. For the proof of $|P_G(\phi)| < \infty$, note that

$$\frac{1}{n}\sum_{k=0}^{n-1} \log m_{\theta^k\omega} \xrightarrow{n\to\infty} \int \log m_\omega dP \;\text{ and }\; \frac{1}{n}\sum_{k=0}^{n-1} \log M_{\theta^k\omega} \xrightarrow{n\to\infty} \int \log M_\omega dP$$

by the ergodic theorem. It hence follows from $\mathscr{Z}_n^\omega(a) \leq M_\omega \cdots M_{\theta^{n-1}\omega}$ that $P_G(\phi) < \infty$. Furthermore, from assertions (i) and (ii) combined with $\log A_n^\omega \geq \sum_{k=0}^{n-1} \log m_{\theta^k\omega}$ and the convergence in Proposition 3.1, we obtain that $P_G(\phi) \geq \int \log m_\omega dP(\omega)$. □

Using these estimates, we are now in position to prove the main result of this section. In the statement of the theorem, Ω^* refers to the subset of Ω_a in the definition of the relative Gurevič pressure.

Theorem 3.4. *Assume that (X,T) has the b.i.p.-property, and (H^*) and (S1-S2) are satisfied. Then $P_G(\phi)$ is finite and, for a.e. $\omega \in \Omega$,*

$$\sum_{n\in J_\omega(\Omega^*)} s^n \mathscr{Z}_n^\omega \quad \begin{cases} < \infty & s < e^{-P_G(\phi)}, \\ \infty & s = e^{-P_G(\phi)}. \end{cases}$$

Proof. Note that $P_G(\phi)$ is finite by Lemma 3.3. By replacing ϕ by $\phi - \log\phi$ we now assume without loss of generality, that $P_G(\phi) = 0$. By Lemma 3.3 (i), it then follows, for a.e. $\omega \in \Omega_{\text{bp}}$, that

$$\lim_{n\in J_\omega(\Omega^*)} \frac{1}{n} \log \mathscr{Z}_n^\omega = 0.$$

We will show that $\sum A_n^\omega < \infty$ leads to a contradiction of $P_G(\phi) = 0$. So assume that, for a.e. $\omega \in \Omega_{\text{bi}}$, $\sum_{n\in J_\omega(\Omega_{\text{bi}})} A_n^\omega < \infty$. Hence, for $\varepsilon > 0$, there exist $\Omega' \subset \Omega_{\text{bi}}$ and $N \in \mathbb{N}$ such that $A_n^\omega < \varepsilon$ for all $\omega \in \Omega'$ and $n \geq N$. Now consider the jump transformation $\theta^* : \Omega' \to \Omega'$ given by

$$\eta : \Omega' \to \mathbb{N}, \quad \omega \to \eta^*(\omega) := \min\{n \in \mathbb{N} : \ n \geq N, \theta^n\omega \in \Omega'\}$$
$$\theta^* : \Omega' \to \Omega', \quad \omega \to \theta^{\eta^*(\omega)}\omega.$$

Note that θ^* is invertible, and that $P|_{\Omega'}$ is a finite θ^*-invariant measure (see e.g. [14]). In particular, it follows that $\theta^*(\Omega') = \Omega' \mod P$. Set

$$\eta_k^*(\omega) := \sum_{i=0}^{k-1} \eta^*((\theta^*)^i\omega).$$

Since the sequence $(\log A_n^\omega)$ is subadditive, it follows that $A^\omega_{\eta_k^*(\omega)} \leq \varepsilon^k$. Furthermore, by the ergodic theorem, $\eta_k^*(\omega)/k$ converges to an invariant function which is bigger than or equal to N. In particular,

$$\lim_{k\to\infty} \frac{1}{\eta_k^*(\omega)} \log A^\omega_{\eta_k^*(\omega)} \leq \frac{k}{\eta_k^*(\omega)} \log \varepsilon \leq \frac{\log \varepsilon}{N} \text{ a.s.}$$

Using $A_n^\omega \geq \mathscr{Z}_n^\omega$ we obtain that $\lim_{n\in J_\omega(\widetilde{\Omega})}(\log \mathscr{Z}_n^\omega)/n < 0$ for a.e. $\omega \in \Omega_{\text{bi}}$, and a suitable subset $\widetilde{\Omega} \subset \Omega^*$ of positive measure. Since this is a contradiction to $P_G(\phi) = 0$, it follows that

$$\sum_{n\in J_\omega(\Omega_{\text{bi}})} A_n^\omega = \infty$$

for a.e. $\omega \in \Omega_{\text{bi}}$, and by subadditivity, for a.e. $\omega \in \Omega$. By Lemma 3.3 (ii), the assertion follows. $\square$

4 Random eigenvalues and conformal measures

The first step of this section is to construct random eigenvalues and conformal measures for random topological Markov chains for which the sum of the preimage function diverges. As a corollary, we obtain that the random eigenvalue can be identified with the quotient of two random power series. In particular, this then gives in analogy to deterministic topological Markov chains (see [11]) that the b.i.p.-property implies positive recurrence. Throughout this section we assume that (X,T) is topologically mixing, ϕ satisfies (H2) and (S1-2) and $P_G(\phi)$ is finite. In particular, without loss of generality, $P_G(\phi)=0$. Now fix $a \in \mathscr{W}^1$, and for $\widetilde{\Omega} \subset \Omega_a$, $\omega \in \Omega$ and $0 < s \leq 1$, set

$$P_\omega(s) := \sum_{n \in J_\omega(\widetilde{\Omega})} s^n \mathscr{Z}_n^\omega.$$

If there exists $\widetilde{\Omega} \subset \Omega_a$ such that $P_\omega(1) = \infty$, and $P_\omega(s) < \infty$ for $0 < s < 1$, we say that (X,T,ϕ) is of *divergence type*. In particular, observe that for a system of divergence type, we have $\lim_{n \in J_\omega(\widetilde{\Omega})} (\log \mathscr{Z}_n^\omega)/n = 0 = P_G(\phi)$ by Hadamard's formula for the radius of convergence. Also note that systems with the b.i.p.-property are in this class as a consequence of Theorem 3.4.

Lemma 4.1. *There exists a sequence* $(s_n : n \in \mathbb{N})$ *with* $s_n \nearrow 1$ *and* $\lambda^* : \omega \to \mathbb{R}$ *with* $\log \lambda^* \in L^1(P)$ *such that*

$$\int g(\omega) \log(\lambda^*(\omega)) dP(\omega) = \lim_{n\to\infty} \int g(\omega) \log(P_\omega(s_n)/P_{\theta\omega(s_n)}) dP(\omega)$$

for all $g \in L^\infty(P)$. *Furthermore, we have* $\int \log \lambda^* dP = 0$ *and* $m_\omega \leq \lambda^*(\omega) \leq M_\omega$, *for P-a.e.* $\omega \in \Omega$.

Proof. Observe that

$$\begin{aligned}
P_\omega(s) &= \sum_{n \in J_\omega(\widetilde{\Omega})} s^n \sum_{x \in X_\omega : T_\omega^n(x) = \xi_{\theta^n\omega}} e^{\phi^\omega(x)} e^{\phi_{n-1}^{\theta\omega}(T_\omega(x))} \\
&= sL_\phi^\omega(1)(\xi_{\theta\omega}) + \sum_{n \in J_\omega(\widetilde{\Omega}), n \geq 2} s^n \sum_{y \in X_{\theta\omega} : T_{\theta\omega}^{n-1}(y) = \xi_{\theta^n\omega}} L_\phi^\omega(1)(y) e^{\phi_{n-1}^{\theta\omega}(y)} \\
&\leq s \cdot M_\omega(1 + P_{\theta\omega}(s)).
\end{aligned}$$

By applying the same argument and using $\left(1 + (P_{\theta\omega}(s))^{-1}\right) \geq 1$, we arrive at

$$sm_\omega \leq \frac{P_\omega(s)}{P_{\theta\omega}(s)} \leq sM_\omega \left(1 + (P_{\theta\omega}(s))^{-1}\right). \tag{4}$$

Since $\log \|L_\phi^\omega(1)\| \in L^1(P)$, the set $\{\log(P_\omega(s)/P_{\theta\omega}(s)) : s < 1\}$ is uniformly integrable. This shows the existence of $\log \lambda^* \in L^1(P)$ as a weak limit. By applying the ergodic theorem, it then follows that $\int \log \lambda^* dP = 0$. The remaining assertion can be proved by combining $\lim_{s \to 1+} P_{\theta\omega}(s) = \infty$ with the above inequalities. □

In order to obtain pointwise convergence of $P_\omega(s)/P_{\theta\omega}(s)$ as $s\to 1$ we construct a random conformal measure using a randomized version of the construction in [4]. As a consequence of (S1-2), the construction and the proof of relative tightness will turn out to be significantly easier than in [3]. For $s<1$ and $\omega\in\Omega$, set

$$\mu_{\omega,s} := \frac{1}{P_\omega(s)} \sum_{n\in J_\omega(\widetilde{\Omega})} s^n \sum_{x:T_\omega^n(x)=\xi_{\theta^n\omega}} e^{\phi_n^\omega(x)}\delta_x,$$

where δ_x refers to the Dirac measure at $x\in X_\omega$. For $A\in\mathscr{B}_\omega$, it hence follows that

$$\mu_{\omega,s}(A) := \frac{1}{P_\omega(s)} \sum_{n\in J_\omega(\widetilde{\Omega})} s^n L_\phi^{\omega,n}(1_A)(\xi_{\theta^n\omega}).$$

In order to show that a reasonable limit of this family of measures exists (for $s\nearrow 1$), we will employ Crauel's random Prohorov theorem (see [2]). So recall that $\{\mu_{\omega,s}:\omega\in\Omega, s\geq s_0\}$ is relatively tight if for all $\varepsilon>0$ there exists a set $K\subset X$ such that $K\cap X_\omega$ is compact for a.e. $\omega\in\Omega$ and $\int\mu_s(K)dP>1-\varepsilon$ for all $s>s_0$.

Lemma 4.2. *The family $\{\mu_{\omega,s}:\omega\in\Omega, n\in\mathbb{N}\}$ is relatively tight.*

Proof. For the proof, for $k\in\mathbb{N}$, $\omega\in\Omega$, and $b\in\mathbb{N}$, set

$$A_\omega^{k,b} := \{(x_0,x_1,\dots)\in X_\omega:\ x_k=b\} = T_\omega^{-k}([b]_{\theta^k\omega}),$$
$$E_n^\omega := T_\omega^{-n}(\{\xi_{\theta^n\omega}\}),\quad E_n^\omega(b,k) := E_n^\omega\cap T_\omega^{-k}([b]_{\theta^k\omega}).$$

By construction, it then follows that

$$\begin{aligned}
&\mu_{\omega,s}(A_\omega^{k,b})\\
=&\frac{1}{P_\omega(s)}\left(\sum_{\substack{n\in J_\omega(\widetilde{\Omega}),n\leq k,\\ x\in E_n^\omega(b,k)}} s^n e^{\phi_n^\omega(x)} + \sum_{x\in E_{k+1}^\omega(b,k)} s^{k+1}e^{\phi_n^\omega(x)} + \sum_{\substack{n\in J_\omega(\widetilde{\Omega}),n\geq k+2,\\ x\in E_n^\omega(b,k)}} s^n e^{\phi_n^\omega(x)}\right)\\
=:&\frac{1}{P_\omega(s)}\left(\Sigma_1^\omega(b)+\Sigma_2^\omega(b)+\Sigma_3^\omega(b)\right).
\end{aligned}$$

For the third summand, one immediately obtains that

$$\begin{aligned}
\frac{\Sigma_3^\omega(b)}{P_\omega(s)} &\leq \frac{s^{k+1}}{P_\omega(s)} \sum_{n\in J_\omega(\widetilde{\Omega}),n\geq k+2} s^{n-(k+1)}\\
&\quad\cdot \sum_{w\in\mathscr{W}_\omega^k:\ wb\in\mathscr{W}_\omega^{k+1}} e^{\phi_k^\omega([wb])} \sum_{x\in E_{n-(k+1)}^{\theta^{k+1}\omega}\cap T_{\theta^k\omega}([b]_{\theta^k\omega})} e^{\phi_{n-(k+1)}^{\theta^{k+1}\omega}(x)}\\
&\leq \frac{s^{k+1}P_{\theta^{k+1}\omega}(s)}{P_\omega(s)}\left(\sum_{w\in\mathscr{W}_\omega^k:\ wb\in\mathscr{W}_\omega^{k+1}} e^{\phi_k^\omega([w])}\right) e^{\phi^{\theta^k\omega}([b])}\mu_{\theta^{k+1}\omega,s}(T_{\theta^k\omega}([b]_{\theta^k\omega}))\\
&\leq \frac{s^{k+1}P_{\theta^{k+1}\omega}(s)}{P_\omega(s)}\left(\prod_{l=0}^{k-1} M_{\theta^l\omega}\|\right) e^{\phi^{\theta^k\omega}([b])} \leq \left(\prod_{l=0}^{k-1}\frac{M_{\theta^l\omega}}{m_{\theta^l\omega}}\right) e^{\phi^{\theta^k\omega}([b])},
\end{aligned}$$

where the last inequality follows from (S2) and (4). By the same arguments, it follows that

$$\Sigma_2^\omega(b) \leq \left(\prod_{l=0}^{k-1} M_{\theta^l\omega}\right) e^{\phi^{\theta^k\omega}([b])}.$$

Finally, for $n = 1, \dots k$, note that the set $E_n^\omega \cap T_\omega^{-k}([b]_{\theta^k\omega})$ is nonempty for at most one $b \in \mathscr{W}^1_{\theta^k\omega}$. Hence, there exists $c_{\omega,k} \leq \infty$ with $\sum_{b>c_{\omega,k}} \Sigma_1^\omega(b) = 0$.

For a given $\varepsilon > 0$, choose a triple (C, s_0, Ω') with $C > 0$, $s_0 \in (0,1)$ and $\Omega' \subset \Omega$ such that $P(\Omega') > 1-\varepsilon$ and $P_\omega(s) \geq C$ for all $s \geq s_0$, $\omega \in \Omega'$. For $\omega \in \Omega'$ and $c \geq c_{\omega,k}$, we hence have that

$$\begin{aligned}
\sum_{b\geq c} \mu_{\omega,s}(A_\omega^{k,b}) &= \frac{1}{P_\omega(s)} \sum_{b\geq c} (\Sigma_2^\omega(b) + \Sigma_3^\omega(b)) \\
&\leq \left(C^{-1}\prod_{l=0}^{k-1} M_{\theta^l\omega} + \prod_{l=0}^{k-1} \tfrac{M_{\theta^l\omega}}{m_{\theta^l\omega}}\right) \sum_{b\geq c} e^{\phi^{\theta^k\omega}([b])} && (5) \\
&\leq B_{\theta^{k+1}\omega} \left(C^{-1}\prod_{l=0}^{k-1} M_{\theta^l\omega} + \prod_{l=0}^{k-1} \tfrac{M_{\theta^l\omega}}{m_{\theta^l\omega}}\right) M_{\theta^k\omega} < \infty && (6)
\end{aligned}$$

Combining the summability in (6) with the independence of the estimate from s in (5) then gives rise to the existence of $c^*_{\omega,k} \geq c_{\omega,k}$, for $\omega \in \Omega'$ and $k \in \mathbb{N}$, such that $c^*_{\omega,k} < \infty$ and

$$\sum_{b\geq c^*_{\omega,k}} \mu_{\omega,s}(A_\omega^{k,b}) \leq \frac{\varepsilon}{2^k}.$$

Observe that $\omega \to c^*_{\omega,k}$ might chosen to be measurable, which can be seen e.g. by construction of $c^*_{\omega,k}$ as a maximum. For $K := \{(\omega, (x_0, x_1, \dots)) :\ \omega \in \Omega', x_k < c^*_{\omega,k}\}$ it then follows that

$$\begin{aligned}
\int \mu_{\omega,s}(K^c) dP &\leq \varepsilon + \int_{\Omega'} \mu_{\omega,s}(K^c) dP \\
&= \varepsilon + \int_{\Omega'} \mu_{\omega,s}(\{(x_0, \dots) :\ \exists k \text{ s.t. } x_k \geq c^*_{\omega,k}\}) dP \\
&\leq \varepsilon + \int_{\Omega'} \sum_{\substack{k=1,\dots,\infty \\ b\geq c^*_{\omega,k}}} \mu_{\omega,s}(A_\omega^{k,b}) dP \leq \varepsilon + P(\Omega') \sum_{k=1}^\infty \frac{\varepsilon}{2^k} \leq 2\varepsilon.
\end{aligned}$$

Since $K \cap X_\omega$ is compact for a.e. $\omega \in \Omega'$, the assertion follows. $\square$

As an immediate consequence of Crauel's relative Prohorov theorem we hence obtain that there exists a sequence (s_n) with $s_n \nearrow 1$ and a random probability measure $\{\mu_\omega\}$ such that

$$\lim_{n\to\infty} \int f d\mu_{\omega,s_n} dP(\omega) = \int f d\mu_\omega dP(\omega)$$

for all $f \in \mathscr{L}_1^C(P)$, where $\mathscr{L}_1^C(P) := \{f : X \to \mathbb{R} :\ f|_{X_\omega} \in C(X_\omega), \int \|f_{|X_\omega}\|_\infty dP(\omega) < \infty\}$ and $C(X_\omega)$ denotes the set of continuous functions defined on X_ω. The following theorem is now stated without the assumption that $P_G(\phi) = 0$.

Theorem 4.3. *Let (X,T,ϕ) be a topologically mixing system of divergence type with (H2), S(1-2) and $P_G(\phi) > -\infty$. Then there exists a sequence (s_n) with $s_n \nearrow \exp(-P_G(\phi))$ such that*

$$\lambda(\omega) := \lim_{n\to\infty} P_\omega(s_n)/P_{\theta\omega}(s_n)$$

exists almost surely, with $|\log\lambda| \in L^1(P)$ and $\int \log\lambda dP = 0$. Furthermore, there exists a random probability measure $\{\mu_\omega\}$ as a weak limit of the sequence $\{\mu_{\omega,s_n}\}$, such that, for $x \in X_\omega$,

$$\frac{d\mu_{\theta\omega}\circ T_\omega}{d\mu_\omega}(x) = \lambda(\omega)e^{P_G(\phi)-\phi^\omega(x)}.$$

Proof. By replacing ϕ with $\phi - P_G(\phi)$, assume without loss of generality that $P_G(\phi) = 0$. Let $(t_k : k \in \mathbb{N})$ be a sequence given by Lemma 4.1. By Lemma 4.2, there exists a subsequence $(s_n : n \in \mathbb{N})$ of (t_k) and a random probability measure $\{\mu_\omega : \omega \in \Omega\}$ which is the weak limit of $\{\mu_{\omega,s_n}\}$. For $n \in \mathbb{N}$, $a \in \mathscr{W}^n\omega$ and $A \subset [a]_\omega$ with $\mu_\omega(A) > 0$, it follows that

$$\begin{aligned}\mu_{\omega,s}(A) = &\frac{1}{P_\omega(s)} \sum_{\substack{k\in J_\omega(\tilde{\Omega}),k\le n\\ x\in E^k_\omega\cap A}} s^k e^{\phi^k_\omega(x)}\\ &+ \frac{P_{\theta^k\omega}(s)}{P_\omega(s)}\frac{1}{P_{\theta^k\omega}(s)} \sum_{\substack{n\in J_\omega(\tilde{\Omega}),k>n,\\ x\in E^{k-n}_{\theta^n\omega}\cap T^n_\omega(A)}} s^k e^{\phi^\omega_n(\tau_a(x))} e^{\phi^{\theta^n\omega}_{k-n}(x)}\end{aligned}$$

When passing to the limit, the first summand tends to zero and as a consequence of estimate (4), we obtain that $\mu_{\theta^n\omega}$ and $T^n_\omega \circ \mu_\omega$ are absolutely continuous for a.e. $\omega \in \Omega$. Hence, $d\mu_{\theta^n\omega}\circ T^n_\omega/\mu_\omega$ exists a.e. and

$$\frac{d\mu_{\theta^n\omega}\circ T^n_\omega}{\mu_\omega}(x) = e^{-\phi^\omega_n(x)} \lim_{k\to\infty}\frac{P_\omega(s_k)}{P_{\theta^k\omega}(s_k)}.$$

In particular, $\lim_{k\to\infty} P_\omega(s_k)/P_{\theta^k\omega}(s_k)$ exists a.e. and by Lemma 4.1 we have $|\log\lambda| \in L^1(P)$ and $\int \log\lambda dP = 0$.

If $P_G(\phi) \neq 0$ then the radius of convergence of $P_\omega(s)$ is equal to $\exp(-P_G(\phi))$. With $P^*_\omega(s)$ referring to the random power series associated with the potential $\phi^* = \phi - P_G(\phi)$, we have $P^*_\omega(s) = P_\omega(s\cdot\exp(-P_G(\phi)))$. The remaining assertions follow from this. □

Remark 4.4. For $a \in \mathscr{W}_\omega$ and $A \subset [a]_\omega$, the above result implies that

$$\mu_{\theta\omega}(T_\omega(A)) = \lambda(\omega)\int_A e^{P_G(\phi)-\phi^\omega} d\mu_\omega$$

for a.e. $\omega \in \Omega$. Hence $\{\mu_\omega\}$ is a $(\lambda\exp(P_G(\phi)-\phi))$-random conformal measure. Furthermore, this leads to the characterisation of $\{\mu_\omega\}$ in terms of the dual $(L^\phi_\omega)^*$ acting on the space of Radon measures (see e.g. [3]), that is

$$(L^\phi_\omega)^*(\mu_{\theta\omega}) = \lambda(\omega)e^{P_G(\phi)}\mu_\omega.$$

Remark 4.5. Now assumme that ϕ has property (H1). For $a = (a_0, \dots, a_{n-1}) \in \mathscr{W}_\omega^n$ we then immediately obtain an estimate for $\mu_\omega([a]_\omega)$ in terms of the measure of $T_{\theta^{n-1}\omega}([a_{n-1}]_\omega)$. Set $\Lambda_n(\omega) := \lambda(\omega) \cdot \lambda(\theta\omega) \cdots \lambda(\theta^{n-1}\omega)$. We then have

$$\frac{1}{B_{\theta^n\omega}} \mu_{\theta^n\omega}(T_{\theta^{n-1}\omega}([a_{n-1}]_\omega)) \leq \Lambda_n(\omega) \frac{\mu_\omega([a]_\omega)}{e^{\phi_\omega^n(x) - nP_G(\phi)}} \leq B_{\theta^n\omega} \mu_{\theta^n\omega}(T_{\theta^{n-1}\omega}([a_{n-1}]_\omega)),$$

for all $x \in [a]_\omega$. If (X,T) has the big image property, then

$$D_\omega := \inf\{\mu_\omega(T_{\theta^{-1}\omega}([b]_{\theta^{-1}\omega})) : \; b \in \mathscr{W}^1_{\theta^{-1}\omega}\} > 0$$

for all $\omega \in \Omega_{\text{bi}}$. Hence, for a.e. $\omega \in \Omega$ and n with $\theta^n\omega \in \Omega_{\text{bi}}$, we have

$$(B_{\theta^n\omega})^{-1} D_{\theta^n\omega} \leq \Lambda_n(\omega) \frac{\mu_\omega([a]_\omega)}{e^{\phi_\omega^n(x) - nP_G(\phi)}} \leq B_{\theta^n\omega},$$

which is a natural analogue of the Gibbs property for random topological Markov chains.

We proceed with applications of the above theorem to systems with the b.i.p.-property. In this case, by Theorem 3.4, we have that the system is of divergence type, and hence the above theorem is applicable. The representation of λ as a quotient then gives rise to a an estimate for the asymptotic behaviour of the Gurevič partition functions.

Corollary 4.6. *If (X,T,ϕ) has the b.i.p.-property and (H^*) and (S1-2) hold then there exist positive measurable functions $K, K^* : \Omega_a \to \mathbb{R}$, $\mathscr{N} : \Omega_a \to \mathbb{N}$ such that for all $\omega \in \Omega_a$ and $n \geq \mathscr{N}(\omega)$ with $\omega \in \Omega_a \cap \theta^n\Omega_a$,*

$$K(\omega)K(\theta^{-n}\omega) \leq \frac{Z_n^{\theta^{-n}\omega}(a)}{\Lambda_n(\theta^{-n}\omega)e^{nP_G(\phi)}} \leq K^*(\omega). \tag{7}$$

Proof. Assume without loss of generality that $P_G(\phi) = 0$. We divide the proof into two steps. We first show that $Z_n^{\theta^{-n}\omega}(a)\mathscr{Z}_m^\omega \gg \mathscr{Z}_{n+m}^{\theta^{-n}\omega}(a)$, and then use this estimate prove the assertion.

Choose $k, l \in \mathbb{N}$ such that $\theta^{-l+1}\omega \in \Omega_{\text{bi}}$, $\theta^k\omega \in \Omega_{\text{bp}}$, and $N_{ba}^{\theta^{-l}\omega} < l$ for all $b \in \mathscr{I}_{\text{bi}}^{\theta^{-l}\omega}$. Set

$$\mathscr{M}_\omega := \sup \left\{ \frac{\displaystyle\sum_{w \in \mathscr{W}^{k+l}_{\theta^{-l}\omega}(b_1,b_2)} e^{\phi_{k+l}^{\theta^{-l}\omega}([w])}}{\displaystyle\sum_{\substack{u \in \mathscr{W}^l_{\theta^{-l}\omega}(b_1,a), \\ v \in \mathscr{W}^k_{\theta^m\omega}(a,b_2)}} e^{\phi_l^{\theta^{-l}\omega}([ua])} e^{\phi_k^\omega([v])}} : \; b_1 \in \mathscr{W}^1_{\theta^{-l}\omega}, b_2 \in \mathscr{W}^1_{\theta^k\omega} \right\}$$

It then follows from (1) and the b.i.p.-property that $\mathscr{M}_\omega < \infty$. Hence, for all $n, m \in \mathbb{N}$ with $\theta^{-n}\omega \in \Omega_a, \theta^m\omega \in \Omega_a$ and $n \geq l$, $m \geq k + N_{ba}^{\theta^k\omega}$ for all $b \in \mathscr{I}_{\text{bp}}^{\theta^k\omega}$, we have

$$\mathscr{Z}_{m+n}^{\theta^{-n}\omega}(a) \leq \mathscr{M}_\omega B_{\theta^{-l}\omega} Z_n^{\theta^{-n}\omega}(a) B_\omega \mathscr{Z}_m^\omega(a) B_{\theta^k\omega} =: \mathscr{M}'_\omega Z_n^{\theta^{-n}\omega}(a) \mathscr{Z}_m^\omega(a).$$

As a consequence of Theorems 3.4 and 4.3 there exists a sequence (s_j) with $s_j \nearrow 1$ such that the limit $\lambda(\omega) = \lim_{j\to\infty} P_\omega(s_j)/P_{\theta\omega}(s_j)$ exists for a.e. $\omega \in \Omega$. Hence, for n as above, we have

$$\frac{Z_n^{\theta^{-n}\omega}(a)}{\Lambda_n(\theta^{-n}\omega)} = \lim_{j\to\infty} \frac{Z_n^{\theta^{-n}\omega}(a)P_\omega(s_j)}{P_{\theta^{-n}\omega}(s_j)} \geq \lim_{j\to\infty} \frac{Z_n^{\theta^{-n}\omega}(a)\sum_{i\in J_\omega(\widetilde{\Omega})} s_j^i \mathscr{Z}_i^\omega(a)}{\sum_{i\in J_{\theta^{-n}\omega}(\widetilde{\Omega})} s_j^i \mathscr{Z}_i^{\theta^{-n}\omega}}$$

$$\geq \mathscr{M}'_\omega \lim_{j\to\infty} \frac{\sum_{i\in J_\omega(\widetilde{\Omega})} s_j^i \mathscr{Z}_{n+i}^{\theta^{-n}\omega}(a)}{\sum_{i\in J_{\theta^{-n}\omega}(\widetilde{\Omega})} s_j^i \mathscr{Z}_i^{\theta^{-n}\omega}}$$

With $k = k(\theta^{-n}\omega) := \min\{l \geq \alpha_{\theta^{-n}\omega} : \ \theta^{k-l}\omega \in \Omega_{\mathrm{bp}}\}$, it follows from Lemma 3.3 (i) that

$$\frac{Z_n^{\theta^{-n}\omega}(a)}{\Lambda_n(\theta^{-n}\omega)} \geq \frac{\mathscr{M}'_\omega C_{\theta^{-n}\omega}(a,k)}{\Lambda_k(\theta^{-n}\omega)}.$$

Hence the left hand side of (7) holds for $n \geq \mathscr{N}(\omega)$, where

$$\mathscr{N}(\omega) := \min\{l \in \mathbb{N} : \ \theta^{-l}\omega \in \Omega_{\mathrm{bi}},\ N_{ba}^{\theta^{-l}\omega} < l \text{ for all } b \in \mathscr{I}_{\mathrm{bi}}^{\theta^{-l}\omega}\},$$
$$K(\omega) := \min\{\mathscr{M}'_\omega, C_\omega(a,k(\omega))/\Lambda_{k(\omega)}(\omega)\}.$$

The remaining assertion follows by similar agruments by using the estimate $\mathscr{Z}_i^{\theta^{-n}\omega} \geq Z_n^{\theta^{-n}\omega}(a)\mathscr{Z}_{i-n}^\omega(a)B_\omega^{-1}$ and Lemma 3.3 (i). □

By choosing a subset Ω_r of Ω_a for which $K(\omega)$ is uniformly bounded, it immediately follows that there exists $\widetilde{K} : \Omega_r \to \mathbb{R}$, $\widetilde{K} > 0$ with

$$\widetilde{K}^{-1}(\omega) \leq \frac{Z_n^{\theta^{-n}\omega}(a)}{\Lambda(\theta^{-n}\omega)} \leq \widetilde{K}(\omega),$$

for a.e. $\omega \in \Omega_r$ and $n \geq \mathscr{N}(\omega)$ with $\theta^{-n}\omega \in \Omega_r$. This in particular shows that (X,T,ϕ) is positive recurrent as introduced in [3]. For the definition of relative exactness in the statement of the Theorem below, we refer to [6, 3].

Theorem 4.7. *Assume that (X,T,ϕ) has the b.i.p.-property and (H*) and S(1-2) hold. Then there exists a measurable family of functions $(h^\omega : \omega \in \Omega)$ such that, for μ and λ given by Theorem 4.3, the following holds.*

(i) For a.e. $\omega \in \Omega$, $h^\omega : X_\omega \to \mathbb{R}$ is a positive, 1-Hölder continuous function which is bounded from above and below on cylinders.

(ii) For a.e. $\omega \in \Omega$, we have $L_\phi^\omega h^\omega = \lambda(\omega)e^{P_G(\phi)}h^{\theta\omega}$, $\int h^\omega \mu_\omega = 1$.

(iii) The random topological Markov chain is relatively exact with respect to (μ_ω). In particular, for $\{f^\omega : \omega \in \Omega'\}$ with $f^\omega \in L^1(\mu_\omega)$ for a.e. $\omega \in \Omega$, we have

$$\lim_{n\to\infty} \left\| \frac{L_\phi^{\omega,n} f^\omega}{\Lambda_n(\omega)e^{nP_G(\phi)}} - h^{\theta^n\omega} \int f^\omega d\mu_\omega \right\|_{L^1(\mu_{\theta^n\omega})} = 0.$$

(iv) The probability measure given by $h^\omega d\mu_\omega dP$ is T-invariant and ergodic.

Proof. These are immediate consequences of Theorem 5.3, Proposition 7.3 and Proposition 7.4 in [3]. □

Remark 4.8. Recall e.g. from [1], that a random subshift of finite type is a random topological Markov chain with $\ell_\omega < \infty$. Now assume that a random subshift (X,T) of finite type is topologically mixing and has properties (H2) and (S1-2). Clearly, (X,T) has the b.i.p.-property. Moreover, it easily can be seen that $V_1^\omega(\phi) < \infty$. Hence above Theorem is applicable and hence is an extension of Ruelle's theorem in [1].

Furthermore, by considering a potential which is constant on cylinders of length two, we obtain a Perron-Frobenius-theorem for the following class of random matrices. So let $A = \{A_\omega : \omega \in \Omega\}$ with $A_\omega = \left(p_{ij}^\omega, i < \ell_\omega, j < \ell_{\theta\omega}\right)$ and $p_{ij} \geq 0$ a.s. be a measurable family of random matrices. We refer to A as a summable random matrix with the b.i.p.-property if

(i) the signum of A gives rise to a random topological Markov chain with the b.i.p.-property,

(ii) For a.e. $\omega \in \theta^{-1}(\Omega_{\text{bi}} \cup \Omega_{\text{bp}})$, we have

$$\sup\left\{\frac{p_{ij}^\omega}{p_{ik}^\omega} : i < \ell_\omega, j,k < \ell_{\theta\omega}, p_{ik}^\omega \neq 0\right\} < \infty.$$

(iii) there exist positive random variables $\omega \mapsto m_\omega$ and $\omega \mapsto M_\omega$ with $\log m, \log M \in L^1(P)$ such that, for a.e. $\omega \in \Omega$,

$$m_\omega \leq \inf_{j<\ell_{\theta\omega}} \sum_{i<\ell_\omega} p_{ij}^\omega \leq \sup_{j<\ell_{\theta\omega}} \sum_{i<\ell_\omega} p_{ij}^\omega \leq M_\omega.$$

By viewing A as a locally constant potential we arrive at the following random Perron-Frobenius theorem. Below, $\mathbb{R}^{\infty-1}$ stands for $\mathbb{R}^{\mathbb{N}}$, and $(B)_{ij}$ for the coefficient of the matrix B with coordinates (i,j).

Corollary 4.9. *For a summable random matrix A with the b.i.p.-property, there exist a positive random variable $\lambda : \Omega \to \mathbb{R}$ and strictly positive random vectors $h = \{h^\omega \in \mathbb{R}^{\ell_\omega - 1} : \omega \in \Omega\}$ and $\mu = \{\mu^\omega \in \mathbb{R}^{\ell_{\theta\omega}-1} : \omega \in \Omega\}$ such that, for a.e. $\omega \in \Omega$,*

$$(h^\omega)^t A^\omega = \lambda(\omega) h^{\theta\omega}, \quad A^\omega \mu^{\theta\omega} = \lambda(\omega)\mu^\omega, \quad (h^\omega)^t \mu^\omega = 1.$$

Furthermore, for a.e. $\omega \in \Omega$ and $i < \ell_\omega$, we have

$$\lim_{n\to\infty} \sum_{j<\ell_{\theta^n\omega}} \left| \frac{(A^\omega \cdot A^{\theta\omega} \cdots A^{\theta^{n-1}\omega})_{ij}}{\Lambda_n(\omega)} - \mu_i^\omega h_j^{\theta^n\omega} \right| \mu_j^{\theta^n\omega} = 0.$$

Proof. Let (X,T) be the random topological Markov chain given by the signum of A and, for $x \in [a_0a_1]_\omega$, set $\phi^\omega(x) := \log p^\omega_{a_0a_1}$. Then ϕ is 2-Hölder continuous and, by condition (ii), is 1-Hölder continuous for $\omega \in \theta^{-1}(\Omega_{\text{bi}} \cup \Omega_{\text{bp}})$. As a consequence of the summability assumption (iii) it then follows that Theorem 4.7 is applicable to (X,T,ϕ). So let λ', h' and μ' be given by this result. The random variable λ is then defined by $\lambda := \lambda' e^{P_G(\phi)}$. Furthermore, since L_ϕ acts on functions which are constant on cylinders, it follows by the construction of the eigenfunction in Proposition 7.3 in [3] that h' is constant on cylinders of length 1. Hence, with h given by $h^\omega_a := h'|_{[a]_\omega}$, we have that, for a.e. $\omega \in \Omega$ and $x \in [b]_{\theta\omega}$,

$$((h^\omega)^t A^\omega)_b = L^\omega_\phi(h')(x) = \lambda(\omega)h'(x) = \lambda(\omega)h^{\theta\omega}_b.$$

Furthermore, for μ given by $\mu^\omega_a := \mu'_\omega([a]_\omega)$, the identity $A^\omega \mu^{\theta\omega} = \lambda(\omega)\mu^\omega$ follows by similar arguments. The remaining assertion is an application of Theorem 4.7 (iii) to the indicator function $1_{[a]_\omega}$. $\square$

As a concluding remark, we give an application of our results to the existence of a stationary vector (or stationary distribution) for a stationary Markov chain with countably many states in a stationary environment. Recall that such a Markov chain is given by a *random stochastic matrix* $A = \{(p^\omega_{ij} : i < \ell_\omega, j < \ell_{\theta\omega}) : \omega \in \Omega\}$, that is $\sum_{j<\ell_{\theta\omega}} p^\omega_{ij} = 1$ for every $i < \ell_{\theta\omega}$ and a.e. $\omega \in \Omega$, where p^ω_{ij} stands for the random transition probability from state i to j. Furthermore, a random vector $\pi = \{(\pi^\omega_i : i < \ell_\omega) : \omega \in \Omega\}$ is called *random stationary distribution* if $\pi^\omega A^\omega = \pi^{\theta\omega}$ and $\sum_i \pi^\omega_i = 1$ for a.e. $\omega \in \Omega$. The following result answers a question of Orey in [10].

Theorem 4.10. *Assume that A is a random stochastic matrix such that*

(i) the signum of the transpose A^t of A defines a random top. mixing topological Markov chain with the big preimages property,

(ii) for a.e. $\omega \in \theta(\Omega_{bi})$, we have

$$\sup\left\{ \frac{p^\omega_{ji}}{p^\omega_{ki}} : i < \ell_\omega, j,k < \ell_{\theta\omega}, p^\omega_{ki} \neq 0 \right\} < \infty.$$

Then a random stationary distribution π exists, and for $i < \ell_\omega$,

$$\lim_{n\to\infty} \sum_{j<\ell_{\theta^{-n}\omega}} \left| (A^{\theta^{-n+1}\omega} \cdots A^{\theta^{-1}\omega} \cdot A^\omega)_{ji} - \pi^\omega_i \right| \pi^{\theta^{-n}\omega}_j = 0.$$

Proof. By assumption, the signum of A^t defines a topological mixing system $((X,T),(\Omega,\theta^{-1})$ with the big preimages property. Moreover, as a consequence of (ii), the potential defined by $\phi^\omega|_{[ij]} := \log p^\omega_{ji}$ is 1-Hölder continuous for $\omega \in \theta(\Omega_{\text{bp}})$.

Since A is a stochastic matrix, it follows that the constant function 1 is an eigenfunction of L_ϕ, and hence $\mathscr{Z}^\omega_n = 1$ for all $n \in \mathbb{N}$ and a.e. $\omega \in \Omega$. For $a \in \mathscr{W}^1$, it follows from Lemma 3.3 (i) that there exists $C_\omega > 0$ such that $\mathscr{Z}^\omega_n(a) \cdot C_\omega \geq 1$ for all $n \in J_\omega(\Omega_a)$ sufficiently large. Hence $P_G(\phi) = 0$, (X,T,ϕ) is of divergence type and positive recurrent. The assertion now follows from Theorem 4.3 above and Theorem 5.3 in [3]. $\square$

References

[1] T. Bogenschütz and V. M. Gundlach, Ruelle's transfer operator for random subshifts of finite type, *Ergod. Th. Dynam. Sys.* **15** (1995) 413–447.

[2] H. Crauel, *Random probability measures on Polish spaces*, *Stochastics Monographs*, Vol. 11 (Taylor & Francis, London, 2002).

[3] M. Denker, Y. Kifer and M. Stadlbauer, Thermodynamic formalism for random countable markov shifts, *Discrete Contin. Dyn. Syst.* **22** (2008) 131–164.

[4] M. Denker and M. Urbański, On the existence of conformal measures, *Trans. Am. Math. Soc.* **328** (1991) 563–587.

[5] Y. Derriennic, Un théorème ergodique presque sous-additif, *Ann. Probab.* **11** (1983) 669–677.

[6] Y. Guivarc'h, Propriétés ergodiques, en mesure infinie, de certains systèmes dynamiques fibrés, *Ergod. Th. Dynam. Sys.* **9** (1989) 433–453.

[7] Yu. Kifer, Perron-frobenius theorem, large deviations, and random perturbations in random environments, *Math. Z.* **222** (1996) 677–698.

[8] Yu. Kifer, Thermodynamic formalism for random transformations revisited, *Stoch. Dyn.* **8** (2008) 77–102.

[9] R. D. Mauldin M. and Urbański, Gibbs states on the symbolic space over an infinite alphabet, *Israel J. Math.* **125** (2001) 93–130.

[10] S. Orey, Markov chains with stochastically stationary transition probabilities, *Ann. Probab.* **19** (1991) 907–928.

[11] O. Sarig, Existence of Gibbs measures for countable markov shifts, *Proc. Amer. Math. Soc.* **131** (2003) 1751–1758.

[12] O. Sarig, Thermodynamic formalism for countable markov shifts, *Ergod. Th. Dynam. Sys.* **19** (1999) 1565–1593.

[13] B. O. Stratmann and M. Urbański, Pseudo-Markov systems and infinitely generated Schottky groups, *Amer. J. Math.* **129** (2007) 1019–1062.

[14] M. Thaler, Transformations on $[0,1]$ with infinite invariant measure, *Israel J. Math.* **46** (1983) 67–96.