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Iteration functions for pth roots of complex numbers

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Abstract A novel way of generating higher–order iteration functions for the computation of pth roots of complex numbers is the main contribution of the present work. The behavior of some of these iteration functions will be analyzed and the conditions on the starting values that guarantee the convergence will be stated. The illustration of the basins of attractions of the pth roots will be carried out by some computer generated plots. In order to compare the performance of the iterations some numerical examples will be considered.

Keywords basins of attraction · high order convergence · iteration function · pth root, residual

1 Introduction

Let $p \geqslant 2$ be a positive integer. Given a complex number w which do not belong to the closed negative real axis, there is a unique complex z such that $z^p = w$ and $-\pi/p < \arg(z) < \pi/p$, where $\arg(z)$ denotes the argument of z. This unique z is called the *principal* pth root of the complex w and will be denoted by $w^{1/p}$. All the pth roots of w can be related with $w^{1/p}$ by

$$\sqrt[p]{w} = e^{\frac{2\ell\pi i}{p}} w^{1/p},$$

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with $\ell = 0, 1, \ldots, p-1$. The pth roots of w are exactly the real or complex roots of the polynomial equation

$$z^p - w = 0. \tag{1}$$

Recently, the iterative methods for computing the principal pth root of a complex number have received particular interest. This has been motivated, in part, by the relevance of this topic in the computation of pth roots of matrices [9, 10, 12, 13, 16], which involves the pth roots of their complex or real eigenvalues. Moreover, the scalar iteration functions can be extended to matrices in a natural way.

The Newton's and Halley's iterative methods are the ones most widely used for solving nonlinear equations and, in particular, polynomial equations, whose orders of local convergence are, respectively, 2 and 3. For more details we refer the reader to [6, 23, 4, 19, 1] and the references therein. The Newton iteration function associated to the polynomial equation (1) is given by

$$N(z) = \frac{(p-1)z + wz^{1-p}}{p} \tag{2}$$

whereas the Halley iteration function is

$$H(z) = z\frac{(p-1)z^p + (p+1)w}{(p+1)z^p + (p-1)w}. \tag{3}$$

Since both iteration functions are rational, their dynamical behavior can be analyzed using results from the classical theory developed by Fatou and Julia [8, 14, 3].

Let λ be a fixed point of a rational iteration function $\psi(z)$, that is, $\psi(\lambda) = \lambda$. The set of initial values for which the sequence generated by $z_{k+1} = \psi(z_k)$ converge to λ is called the *basin of attraction* of λ, while the connected component of this set that contains λ is the *immediate basin of attraction*. It is well known that the boundaries of the basins of attraction are Julia sets and so exhibit a fractal behavior. In general, when the initial guess is close to such sets, the iterative methods are particularly sensitive, which means that the sequence $\{z_k\}_{k=0}^{\infty}$ can converge to a complex number distinct from λ or not converge at all.

A fixed point λ is called *attracting*, *repelling* or *indifferent* whether $|\psi'(\lambda)|$ is less than, greater than or equal to 1, respectively. If $\psi'(\lambda) = 0$ then λ is called *superattracting*. In the case of Newton iteration $N(z)$ the only existing fixed points are the pth roots of w, which are superattracting. For Halley iteration $H(z)$, the pth roots of w are also superattracting fixed points, but it has the extra fixed point $\lambda = 0$, which is repelling.

A rational iteration function $\psi(z)$ converges to λ with order of convergence j if there exist a complex constant $c > 0$ and an integer $k > 0$ sufficiently large such that $|z_{k+1} - \lambda| \leq c|z_k - \lambda|^j$, with $z_{k+1} = \psi(z_k)$. A classical convergence result for fixed point iterations is stated below.

Theorem 1 [11, 22] *Let $\psi(z)$ be an analytic function in a neighborhood of λ such that*

$$\psi(\lambda) = \lambda,\ \psi'(\lambda) = \ldots = \psi^{(j-1)}(\lambda) = 0,\ \psi^{(j)}(\lambda) \neq 0.$$

Then there is z_0 sufficiently close to λ such that the sequence $\{z_k\}_{k=0}^{\infty}$ generated by $z_{k+1} = \psi(z_k)$ converges to λ with order j.

In this paper we present a novel way of generating infinitely many iteration functions for computing pth roots of complex numbers, with prefixed orders of convergence. Many known iteration functions, including Newton and Halley as well as the Schröder iteration functions, will appear as particular cases. As pointed out in [23], higher-order iteration functions may have additional fixed points that are not pth roots of w. Although in general these extraneous fixed points are repelling, they may complicate the structure of the basins of attractions of the pth roots by inserting new sets of “petals”. This phenomenon will be analyzed for some higher-order iterations and will be illustrated by computer generated plots of the basins of attractions of the pth roots.

One important issue of our work is the computation of the principal pth root of a complex number by an iterative method. It is widely known that the success of these kind of methods depends on the choice of the initial guess. In [12,13,9,10] the authors derived practical conditions for choosing an appropriate initial guess that guarantees the convergence of Newton's and Halley's method to the principal pth root. We extend their results to other iteration functions, by proving that if the initial approximation satisfy a certain condition then the sequence generated by the iterations converges toward the right pth root with the expected order.

This paper is organized as follows. In Section 2 an important result (Theorem 2), which can be used to generate several families of iterative methods with higher–order convergence, is derived and a definition for the residual of the terms of a sequence converging to a pth root is proposed. We also give examples of some families of iteration functions and find connections with families already existing in the literature. Sections 3, 4 and 5 are devoted to the study of particular families of iterations, where some convergence results are stated and, in particular, conditions for choosing an initial guess in terms of the residuals are derived. The results we bring into discussion are illustrated by computer generated basins of attraction. Numerical examples that permit to compare the performance of several iteration functions are given in Section 6. Finally, in the last section, after presenting a brief conclusion, we point out some issues that need further research.

Unless otherwise stated, throughout the text we will systematically consider $p \geqslant 2$ and $j \geqslant 1$ as integer numbers and w as a complex number not belonging to the closed negative real axis.

2 Generation of iteration functions

The following theorem allows the generation of infinitely many families of higher–order iteration functions that converge to the pth roots of a complex number.

Theorem 2 *Consider the complex function f defined by $f(z) = [\alpha(z)]^{1/p}$, where $\alpha(\cdot)$ represents an analytic, one-to-one complex-valued function defined in an open set containing zero. If $T_j(z)$ denotes the Taylor polynomial of degree j of $f(z)$ at zero, then the principal pth root $w^{1/p}$ of w is a superattracting fixed point of the function*

$$F_j(z) := z\,T_{j-1}\left(\alpha^{-1}(wz^{-p})\right),$$

with $\alpha^{-1}(\cdot)$ representing the inverse function of α: $\left(\alpha^{-1}\circ\alpha\right)(t) = t$. Moreover, the iteration functions $F_j(z)$ are of orders at least j.

Proof. Consider $R(z) := \alpha^{-1}(wz^{-p})$. The k-th order derivative of the composite function $\Big(f\circ R\Big)(z)$ at the point $z = w^{1/p}$ can be easily computed and is given by

$$\frac{d^k}{dz^k}\Big(f\circ R\Big)(z)\Big|_{z=w^{1/p}} = \frac{d^n}{dz^n}\Big[w\,z^{-p}\Big]^{1/p}\Big|_{z=w^{1/p}} = \frac{(-1)^k\,k!}{w^{k/p}} \quad , \quad k = 1,2,3,\ldots,j-1,\ldots \tag{4}$$

As we are dealing with derivatives of the composite function, this latter may also be computed based on the *Faà di Bruno's formula* [7,5, pp. 137-138], and, in this case we have:

$$\frac{d^k}{dz^k}\Big(f\circ R\Big)(z)\Big|_{z=w^{1/p}} = \sum_{\mu=1}^{k}\frac{d^\mu f(t)}{dt^\mu}\Big|_{t=0} B_{k,\mu}(r_1,r_2,\ldots,r_{k-\mu+1}) \quad , \quad k = 1,2,3,\ldots,j-1,\ldots \tag{5}$$

where $B_{k,\mu}(r_1,r_2,\ldots,r_{k-\mu+1})$ denotes the *Bell* polynomials [5, pp. 133-136], [18, Chapter 5] and

$$r_\sigma = \frac{d^\sigma R(z)}{dz^\sigma}\Big|_{z=w^{1/p}} := \frac{d^\sigma}{dz^\sigma}\Big(\alpha^{-1}(wz^{-p})\Big)\Big|_{z=w^{1/p}} \quad , \quad \text{for } \sigma = 1,\ldots,k-\mu+1 \text{ and } k = 1,2,\ldots \ . \tag{6}$$

Likewise, the formula of *Faà di Bruno* permits to obtain an expression for the k-th order derivative of the composite function $\Big(T_{j-1}\circ R\Big)(z)$ at the point $z=w^{1/p}$ and we have:

$$\frac{d^k}{dz^k}\Big(T_{j-1}\circ R\Big)(z)\bigg|_{z=w^{1/p}}=\sum_{\mu=1}^{k}\frac{d^\mu T_{j-1}(t)}{dt^\mu}\bigg|_{t=0}B_{k,\mu}(r_1,r_2,\dots,r_{k-\mu+1})\quad,\quad k=1,2,3,\dots,j-1,\dots \tag{7}$$

where r_σ (with $\sigma=1,\dots,k-\mu+1$) are given by (6). Since

$$T_{j-1}(z)=\sum_{\nu=0}^{j-1}\left(\frac{d^\nu f(t)}{dt^\nu}\bigg|_{t=0}\right)\frac{z^\nu}{\nu!}$$

we then have

$$\frac{d^\mu T_{j-1}(t)}{dt^\mu}\bigg|_{t=0}=\begin{cases}\dfrac{d^\mu f(t)}{dt^\mu}\bigg|_{t=0} & ,\ 0\leqslant\mu\leqslant j-1\\[2ex] 0 & ,\ \mu\geqslant j\end{cases};$$

consequently (7) becomes

$$\frac{d^k}{dz^k}\Big(T_{j-1}\circ R\Big)(z)\bigg|_{z=w^{1/p}}=\sum_{\mu=1}^{\min(k,j-1)}\frac{d^\mu f(t)}{dt^\mu}\bigg|_{t=0}B_{k,\mu}(r_1,r_2,\dots,r_{k-\mu+1})\quad,\quad k=1,2,3,\dots,j-1,\dots. \tag{8}$$

The comparison between (5) and (8) ensures the equality

$$\frac{d^k}{dz^k}\Big(T_{j-1}\circ R\Big)(z)\bigg|_{z=w^{1/p}}=\frac{d^k}{dz^k}\Big(f\circ R\Big)(z)\bigg|_{z=w^{1/p}}\quad\text{for any}\quad k=1,2,\dots,j-1$$

which, after (4), provides

$$\frac{d^k}{dz^k}\Big(T_{j-1}\circ R\Big)(z)\bigg|_{z=w^{1/p}}=\frac{(-1)^k\,k!}{w^{k/p}}\quad\text{for any}\quad k=1,2,\dots,j-1$$

implying

$$\frac{d^k}{dz^k}F_j(z)\bigg|_{z=w^{1/p}}=0\quad\text{for any}\quad k=1,2,\dots,j-1,$$

inasmuch as, according to the Leibniz formula for the product derivation, it holds

$$\frac{d^k}{dz^k}F_j(z)\bigg|_{z=w^{1/p}}=\left(z\,\frac{d^k\big(T_{j-1}\circ R\big)}{dz^k}(z)+k\,\frac{d^{k-1}\big(T_{j-1}\circ R\big)}{dz^{k-1}}(z)\right)\bigg|_{z=w^{1/p}}.$$

Finally, Theorem 1 allows to complete the proof. ■

Using similar arguments, it is straightforward that Theorem 2 can be extended to the non principal pth roots of w. One important consequence of the previous result is that for any function $\alpha(z)$ satisfying the required conditions, the sequence $\{z_k\}_{k=0}^{\infty}$ defined by

$$z_{k+1}=z_k\,T_{j-1}\left[\alpha^{-1}(wz_k^{-p})\right],$$

converges, with order at least j, to a pth root of w, provided that an initial guess z_0 sufficiently close to that root is taken. Note that there are an infinity of possibilities for choosing $\alpha(z)$. For instance, for each $a,b,c\in\mathbb{C}$, the Möbius transformation

$$\alpha(z)=\frac{az+b}{cz+b}$$

(with $ab-bc\neq 0$) provides an important source of iteration functions. Some particular Möbius transformations will be analyzed throughout the text. A list of illustrating examples of iteration functions thus generated is discussed below, but first it shall be recalled that for any two complex numbers a and x the function $(a+z)^{-x}$ has a formal (binomial) series [5, p.37]:

$$(a+z)^{-x}=\sum_{n\geqslant 0}(-1)^n\,(x)_n\,\frac{a^{-x-n}\,z^n}{n!}\,, \tag{9}$$

where the symbol $(x)_n$ denotes the *Pochhammer symbol* (also known as *rising factorial*) which is defined by $(x)_0=1$ and $(x)_n:=\prod_{\nu=0}^{n-1}(x+\nu)=x(x+1)\dots(x+n-1)$ for any integer $n\geqslant 1$.

Example 1 Let $f(z)=(1-z)^{1/p}$ and $T_j(z)$ be the Taylor polynomial of order j associated to f at $z=0$. Since $\alpha(z)=1-z$ satisfies the conditions required in Theorem 2, one has the following family of iteration functions of order j:

$$\begin{aligned} N_j(z) &= zT_{j-1}(1-wz^{-p}) \\ &= z\left(a_0+a_1(1-wz^{-p})+\cdots+a_{j-1}(1-wz^{-p})^{j-1}\right), \end{aligned} \tag{10}$$

where $j=2,3,\dots$ and $a_n=\frac{1}{n!}\left(-\frac{1}{p}\right)_n$, for any $n=0,\dots,j-1$. For the particular cases $j=2,3,$

$$N_2(z)=z\left(1-\frac{1}{p}(1-wz^{-p})\right)$$

and

$$N_3(z)=z\left(1-\frac{1}{p}(1-wz^{-p})+\frac{1-p}{2!p^2}(1-wz^{-p})^2\right),$$

which have orders 2 and 3, respectively. It is to check that $N_2(z)\equiv N(z)$ given in (2), that is, N_2 is exactly the Newton iteration for the pth root of w. Moreover, as it will be seen in Section 3, N_j coincides with the Schröder iteration of order j applied to the polynomial equation (1). ■

Example 2 If $f(z)=1/(1-z)^{1/p}$ then

$$\begin{aligned} L_j(z) &= zT_{j-1}(1-w^{-1}z^p) \\ &= z\left(a_0+a_1(1-w^{-1}z^p)+\cdots+a_{j-1}(1-w^{-1}z^p)^{j-1}\right), \end{aligned} \tag{11}$$

where $a_n=\frac{1}{n!}\left(\frac{1}{p}\right)_n$, for $n=0,1,\dots,j-1$. This iteration function is similar to the one addressed firstly in [21], and later on in [15] and [9]. The only difference is that our sequence $z_{k+1}=L_j(z_k)$ converges toward a pth root of w instead a pth root of w^{-1}. ■

Example 3 We can also find non rational iteration functions for computing the principal pth root $w^{1/p}$. One example is $F(z)=z\left(1+\frac{1}{p}\log(wz^{-p})\right)$, where log stands for the principal logarithm, which is generated by $f(z)=(e^z)^{1/p}=e^{z/p}$ with $j=2$. This iteration has the drawback of involving the computation of logarithms. For this reason, it does not provide an effective method for pth roots. ■

For a given $\alpha(z)$ as in Theorem 2, the complex valued function

$$R(z):=\alpha^{-1}(wz^{-p})$$

will play an important role in our analysis. When dealing with the sequence defined by

$$z_{k+1} = z_k T_{j-1}\left(\alpha^{-1}(wz_k^{-p})\right),$$

$R(z_k)$ can be seen as a residual, in the sense that it controls the error of the approximation

$$wz_k^{-p} \approx 1.$$

Indeed, assuming that, for a sufficiently large k, $R(z_k) = \alpha^{-1}(wz_k^{-p}) \approx 0$, the continuity of α^{-1} ensures that

$$wz_k^{-p} \approx \alpha(0) = 1.$$

Residuals of iteration functions for pth roots have already appeared in the literature; see, for instance, [21], [15] and [10]. Nevertheless, they have been treated separately as being associated to a particular iteration function. Our approach here is new in the sense that we can give an unified definition that includes those residuals as particular cases. The same holds for the recurrence relationship stated in the next lemma.

Lemma 1 *Under the same assumptions and notations of Theorem 2 and for each $k \in \mathbb{N}$, the residual $R(z_k) = \alpha^{-1}(wz_k^{-p})$ fulfills*

$$R(z_{k+1}) = \alpha^{-1}\left(\alpha(R(z_k))\,[T_{j-1}(R(z_k))]^{-p}\right).$$

Proof. The result follows from the identities

$$\begin{aligned} R(z_{k+1}) &= \alpha^{-1}(wz_{k+1}^{-p}) \\ &= \alpha^{-1}\left(w\left[z_k T_{j-1}\left(\alpha^{-1}(wz_k^{-p})\right)\right]^{-p}\right) \\ &= \alpha^{-1}\left(wz_k^{-p}\left[T_{j-1}\left(\alpha^{-1}(wz_k^{-p})\right)\right]^{-p}\right) \\ &= \alpha^{-1}\left(\alpha(R(z_k))\,[T_{j-1}(R(z_k))]^{-p}\right). \blacksquare \end{aligned}$$

3 The iteration functions N_j

In this section we focus our attention on the iteration functions N_j defined in (10). We start by proving that N_j is a Schröder iteration function associated to the polynomial equation $z^p - w = 0$ (Lemma 2) and for $j = 2, 3$ we define conditions over the initial guesses that guarantee the convergence to the right pth root.

In [20] Schröder constructed a family of prescribed order iteration functions for finding the simple roots of a nonlinear equation $g(z) = 0$. These functions are defined by the expression

$$S_j(z) = z + \sum_{n=1}^{j-1} c_n(z)[-g(z)]^n,$$

where

$$c_n(z) = \frac{1}{n!}\left[\frac{1}{g'(z)}\frac{d}{dz}\right]^{n-1}\frac{1}{g'(z)} \tag{12}$$

and $\left[\frac{1}{g'(z)}\frac{d}{dz}\right]^k := \frac{1}{g'(z)}\frac{d}{dz}\left(\left[\frac{1}{g'(z)}\frac{d}{dz}\right]^{k-1}\right)$ for any $k = 1, 2, \ldots$.

Lemma 2 *Let N_j be defined as in (10) and let S_j be the Schröder iteration function associated to the polynomial equation (1). Then*

$$N_j(z) = S_j(z),$$

for all $z \in \mathbb{C} \setminus \{0\}$.

Proof. Let $R(z)$ denote the residual associated to $N_j(z)$, that is,

$$R(z) = \alpha^{-1}\left(wz^{-p}\right) = 1 - wz^{-p}.$$

If $g(z) = z^p - w$, then $R(z) = g(z)/z^p$. In this case $N_j(z)$ becomes

$$N_j(z) := z\,T_{j-1}(R(z)) = z\left(a_0 z + \frac{a_1 z}{z^p} g(z) + \frac{a_2 z}{(z^p)^2}[g(z)]^2 + \cdots + \frac{a_{j-1} z}{(z^p)^{j-1}}[g(z)]^{j-1}\right) \tag{13}$$

with $a_n = \frac{1}{n!}\left(-\frac{1}{p}\right)_n$, for any $n = 0, \ldots, j-1$. On the other hand, proceeding by finite induction, one has

$$c_n(z) = \frac{(-1)^n}{n!}\left(-\frac{1}{p}\right)_n \frac{z}{(z^p)^n} \quad , \quad n = 0, 1, \ldots, j-1.$$

The comparison between (13) and the expression of S_j yields the equalities

$$c_n(z) = (-1)^n \frac{a_n z}{(z^p)^n} \quad , \quad n = 0, 1, \ldots, j-1\ ,$$

whence the result. ■

Theorem 2 (and also Lemma 2) confirms that, for each j, the iteration function $N_j(z)$ converges locally to a root of the polynomial equation (1) with order j. Now, we analyze the residuals of the sequence generated by N_j,

$$z_{k+1} = N_j(z_k). \tag{14}$$

For each $k = 0, 1, 2, \ldots$, the residuals of (14) are given by

$$R(z_k) = 1 - wz_k^{-p}.$$

By Lemma 1,

$$R(z_{k+1}) = 1 - (1 - R(z_k))\,[T_{j-1}(R(z_k))]^{-p}\,.$$

In order to find a condition on the residual $R(z_0)$ that ensures the convergence of (14) to a pth root of w, we seek an upper bound of $|R(z_{k+1})|$ in terms of $|R(z_k)|$, which allows us to relate the absolute values of the residuals of the successive terms of the sequence. This has been already done for the particular case of Newton's method ($j = 2$). We recall the result in the next theorem.

Theorem 3 [10, 16] *Consider the Newton sequence $z_{k+1} = N_2(z_k)$ and the corresponding residuals $R(z_k)$. If*

$$|R(z_0)| = |1 - wz_0^{-p}| < 1, \tag{15}$$

then, for each k,

$$|R(z_{k+1})| \le |R(z_k)|^2 \le |R(z_0)|^{2^k}\,.$$

Our goal is to extend this result for $j \geqslant 3$. A proof for the case $j = 3$ is based on Theorem 4 and will be presented in Corollary 1, however unfortunately we did not succeed to reach a proof for $j \geqslant 4$. The main reason for this is connected with the requirement of the computation of the coefficients of the Taylor series expansion for the residual function

$$R_j(z;p) := 1-(1-z)\,[T_{j-1}(z)]^{-p}\,,$$

whose expressions become more complicate as j increases. For $j = 2$ there is a nice and simple expression for the Taylor coefficients of R_2 (see [10,16]), while the case $j = 3$ requires more laborious arguments, as it can be followed in Theorem 4.

Consider the 2-degree Taylor polynomial

$$T_2(z) = 1-\frac{1}{p}\,z-\frac{p-1}{2p^2}\,z^2 \tag{16}$$

associated to the function $f(z) = (1-z)^{1/p}$ and the residual function

$$R_3(z;p) = 1-(1-z)\Big(T_2(z)\Big)^{-p}. \tag{17}$$

Let us consider the power series expansion of the two rational functions $(T_2(\cdot))^{-p}$ and $R_3(\cdot;p)$, denoted by

$$(T_2(z))^{-p} = \sum_{\nu=0}^{\infty} c_\nu^{(p)}\,z^\nu \qquad , \qquad R_3(z;p) = \sum_{\nu=0}^{\infty} d_\nu^{(p)}\,z^\nu.$$

The roots of $T_2(z;p)$ are

$$z_1 = -\frac{p\left(1+\sqrt{2p-1}\right)}{p-1} \qquad , \qquad z_2 = -\frac{p\left(1-\sqrt{2p-1}\right)}{p-1}$$

which are real and lie outside the unit circle. Consequently, the two rational functions $(T_2(z))^{-p}$ and $R_3(z;p)$ are analytic inside the unit circle (i.e., for any z such that $|z| < 1$).

Theorem 4 *The following statements hold:*

(i) The coefficients $d_\nu^{(p)} = 0$ *whenever* $\nu = 0,1,2$, *and* $d_\nu^{(p)} > 0$ *for any* $\nu = 3,4,\ldots$.

(ii) The series $\sum_{\nu=0}^{\infty} c_\nu^{(p)}$ *is convergent.*

(iii) $\sum_{\nu=0}^{\infty} d_\nu^{(p)}$ *is convergent and* $\sum_{\nu=0}^{\infty} d_\nu^{(p)} = 1$.

(iv) The series $\sum_{\nu=0}^{\infty} c_\nu^{(p)} z^\nu$ *is absolutely convergent for any* z *such that* $|z| < 1$.

(v) The series $\sum_{\nu=0}^{\infty} d_\nu^{(p)} z^\nu$ *is absolutely convergent for any* z *such that* $|z| < 1$.

Proof. We begin by deriving explicit expressions for $c_\nu^{(p)}$ and $d_\nu^{(p)}$, $\nu \in \mathbb{N}$, in order to prove statements (i) and (ii). At last we will show that (ii) provides (iii). The statements (iv) and (v) are a direct consequence of (ii) and (iii), on the basis of Abel's convergence theorem for formal power series.

From its definition, the polynomial $T_2(z;p)$ may be written as $T_2(z;p) = T_1(z;p) + b_2\, z^2$ with $b_2 = \frac{-(p-1)}{2\,p^2}$, and on account of (9), we successively have

$$\begin{aligned}\Big(T_2(z)\Big)^{-p} &= \Big(T_1(z) + b_2 z^2\Big)^{-p}\\ &= \sum_{n\geqslant 0} \frac{(-1)^n (p)_n}{n!} \Big(T_1(z)\Big)^{-p-n} \Big(b_2 z^2\Big)^n\\ &= \sum_{n\geqslant 0} \frac{(p-1)^n (p)_n}{n! 2^n\, p^{2n}}\, z^{2n} \Big(T_1(z)\Big)^{-p-n}.\end{aligned}$$

According to (9), it follows

$$\Big(T_1(z)\Big)^{-p-n} = \Big(1 - \frac{1}{p} z\Big)^{-p-n} = \sum_{\nu\geqslant 0} \frac{(-1)^\nu (p+n)_\nu}{\nu!} (-1/p)^\nu z^\nu = \sum_{\nu\geqslant 0} \frac{(p+n)_\nu}{\nu!\; p^\nu}\, z^\nu$$

whence

$$\Big(T_2(z)\Big)^{-p} = \sum_{n\geqslant 0} \frac{(p-1)^n (p)_n}{n! 2^n\, p^{2n}} \sum_{\nu\geqslant 0} \frac{(p+n)_\nu}{\nu!\; p^\nu}\, z^{\nu+2n}. \tag{18}$$

Now, based on the equality,

$$\sum_{n\geqslant 0} \sum_{\nu\geqslant 0} A(n,\nu) = \sum_{\nu\geqslant 0} \sum_{n\geqslant 0} A(n,\nu) = \sum_{\nu\geqslant 0} \sum_{n=0}^{\lfloor \nu/2 \rfloor} A(n, \nu - 2n),$$

with $\lfloor x \rfloor$ denoting the floor function of x, that is, the highest integer lower than or equal to x, the relation (18) becomes like

$$\Big(T_2(z)\Big)^{-p} = \sum_{\nu\geqslant 0} \sum_{n=0}^{\lfloor \nu/2 \rfloor} \frac{(p)_n\, (p+n)_{\nu-2n}\, (p-1)^n}{n!(\nu-2n)!\; p^\nu\, 2^n}\, z^\nu = \sum_{\nu\geqslant 0} \sum_{n=0}^{\lfloor \nu/2 \rfloor} \frac{(p)_{\nu-n}\, (p-1)^n}{n!(\nu-2n)!\; p^\nu\, 2^n}\, z^\nu,$$

yielding

$$c_\nu^{(p)} = \sum_{n=0}^{\lfloor \nu/2 \rfloor} \frac{(p)_{\nu-n}\, (p-1)^n}{n!(\nu-2n)!\; p^\nu\, 2^n}. \tag{19}$$

Clearly, we have $c_0^{(p)} = c_1^{(p)} = c_2^{(p)} = 1$. Inasmuch as $d_0^{(p)} = 0$ and

$$d_{\nu+1}^{(p)} = c_\nu^{(p)} - c_{\nu+1}^{(p)} \quad , \quad \nu \geqslant 2 \tag{20}$$

it subsequently follows that $d_1^{(p)} = d_2^{(p)} = 0$. With a few more computations, we derive the expression for $d_\nu^{(p)}$ whenever $\nu \geqslant 3$:

$$\begin{aligned}d_{\nu+1}^{(p)} &= \sum_{n=0}^{\lfloor \frac{\nu}{2} \rfloor} \frac{(p)_{\nu-n}\, (p-1)^n}{n!(\nu-2n)!\; p^\nu\, 2^n} - \sum_{n=0}^{\lfloor \frac{\nu+1}{2} \rfloor} \frac{(p)_{\nu+1-n}\, (p-1)^n}{n!(\nu+1-2n)!\; p^{\nu+1}\, 2^n}\\ &= \sum_{n=0}^{\lfloor \frac{\nu+1}{2} \rfloor} \frac{(p)_{\nu+1-n}\, (p-1)^n}{n!(\nu+1-2n)!\; p^{\nu+1}\, 2^n} \frac{((\nu-2n+1)p - (p+\nu-n))}{(p+\nu-n)}\end{aligned}$$

whence

$$d_{\nu+1}^{(p)} = \sum_{n=0}^{\lfloor \frac{\nu+1}{2} \rfloor} \frac{(p)_{\nu+1-n}\, (p-1)^n}{n!(\nu+1-2n)!\; p^{\nu+1}\, 2^n} \frac{\Big((\nu-2n)(p-1) - n\Big)}{(p+\nu-n)} \quad , \quad \nu \geqslant 2\,. \tag{21}$$

From (21) and on account of the fact that $p \geqslant 2$, we successively have

$$\begin{aligned} d_\nu^{(p)} &\geqslant \frac{1}{p^\nu} \sum_{n=0}^{\lfloor \frac{\nu}{2} \rfloor} \frac{(1)_{\nu-n}}{n!\,(\nu-2n)!\,2^n} \frac{\nu-1-3n}{p+\nu+1-n} \\ &= \frac{1}{p^\nu} \sum_{n=0}^{\lfloor \frac{\nu}{2} \rfloor} \binom{\nu-n}{n} \frac{(\nu-1-3n)}{2^n(p+\nu+1-n)} \\ &\geqslant \frac{1}{p^\nu\,(p+\nu+1)} \sum_{n=0}^{\lfloor \frac{\nu}{2} \rfloor} \frac{(\nu-1-3n)}{2^n} \\ &= \frac{1}{p^\nu\,(p+\nu+1)} \left[2\nu-8+2^{-k}(-\nu+3k+7)\right]_{k=\lfloor \frac{\nu}{2} \rfloor} \end{aligned}$$

whence (ii) holds.

On the other hand, and again based on the fact that $p \geqslant 2$, we are able to establish the inequality

$$\frac{(p)_{\nu-n}\,(p-1)^n}{p^\nu} = \left(\prod_{\mu=0}^{\nu-n-1} \frac{p+\mu}{p}\right)\left(\frac{p-1}{p}\right)^n \leqslant \prod_{\mu=0}^{\nu-n-1}\left(1+\frac{\mu}{2}\right) = \frac{(\nu+1-n)!}{2^{\nu-n}}$$

and from the definition of (19), it follows

$$0 < c_\nu^{(p)} \leqslant \frac{(\nu+2)(\nu+4)}{2^{\nu+3}}$$

because

$$c_\nu^{(p)} \leqslant \sum_{n=0}^{\lfloor \nu/2 \rfloor} \frac{(\nu+1-n)!}{n!(\nu-2n)!\,2^\nu} = \frac{1}{2^\nu} \sum_{n=0}^{\lfloor \nu/2 \rfloor} \binom{\nu+1-n}{n+1}(n+1) \leqslant \frac{1}{2^\nu} \sum_{n=0}^{\lfloor \nu/2 \rfloor} (n+1)\,.$$

Consequently, the series of nonnegative terms $\sum_{\nu=0}^{\infty} c_\nu^{(p)}$ is convergent because of the convergence of the series $\sum_{\nu=0}^{\infty} \frac{(\nu+2)(\nu+4)}{2^{\nu+3}}$.

The statement (ii) provides statement (iii), as we show below. Indeed from (20), the sequence of the partial sums of the series $\sum_{\nu=3}^{\infty} d_\nu^{(p)}$ is given by

$$D_N^{(p)} = \sum_{\nu=3}^{N} d_\nu^{(p)} = c_2^{(p)} - c_N^{(p)} = 1 - c_N^{(p)}\,.$$

Since $\sum_{\nu=0}^{\infty} c_\nu^{(p)}$ is a convergent series, necessarily, we have

$$\lim_{N\to\infty} c_N^{(p)} = 0\,,$$

yielding (iii).

On account of the well known Abel's theorem on power series - if the series $\sum_{n=0}^{\infty} a_n$ converges, then the power series $\sum_{n=0}^{\infty} a_n\, z^n$ converges absolutely for $|z| < 1$ - the statements (iv) and (v) come as consequence of statements (ii) and (iii), respectively. ■

Corollary 1 *Consider the residual function $R(z) = 1 - wz^{-p}$ and let z_k be defined by*

$$z_{k+1} = N_3(z_k).$$

If $|R(z_0)| = |1 - wz_0^{-p}| < 1$, then

$$|R(z_{k+1})| \leq |R(z_k)|^3 \leq |R(z_0)|^{3^k},$$

for all $k = 1, 2, \ldots$.

Proof. From Theorem 4 we have

$$\begin{aligned}
|R(z_1)| &= \left|1 - (1 - R(z_0))\Big(T_2(R(z_0))\Big)^{-p}\right| \\
&= \left|\sum_{\nu=0}^{\infty} d_\nu^{(p)}\ (R(z_0))^\nu\right| \\
&\leq |R(z_0)|^3 \sum_{\nu=0}^{\infty} d_{\nu+3}^{(p)}\ |R(z_0)|^\nu \\
&\leq |R(z_0)|^3 \sum_{\nu=0}^{\infty} d_{\nu+3}^{(p)} \\
&= |R(z_0)|^3.
\end{aligned}$$

Now the result follows by induction. ■

From Theorem 2 we know that if z_0 is sufficiently close to a pth root of w, then the sequence (14) converges to that pth root with order j. The following result states a stronger convergence result for $j = 2, 3$, because it gives a specific condition on the initial guess.

Lemma 3 *Let $R(z) = 1 - wz^{-p}$. If z_0 is such that $|R(z_0)| = |1 - wz_0^{-p}| < 1$, then, for $j = 2, 3$, the sequence (14) converges to a pth root of w with order of convergence j.*

Proof. Let s be a pth root of w and assume that the sequence (14) converges to s. We know that $R(z_k) - R(s) = R(z_k)$ because $R(s) = 0$. The derivative of the residual at s can be written as

$$\mu := R'(s) = \lim_{k\to\infty} \frac{R(z_k) - R(s)}{z_k - s}.$$

Thus for each $\varepsilon > 0$ there exists a positive integer k_0 such that for any $k \geq k_0$,

$$\left|\frac{R(z_k) - R(s)}{z_k - s} - \mu\right| < \varepsilon$$

and then

$$|R(z_k) - R(s)| - |\mu(z_k - s)| \leq |R(z_k) - R(s) - \mu(z_k - s)| < \varepsilon|z_k - s|.$$

Hence the following inequality holds:

$$|R(z_k) - R(s)| < (\varepsilon + |\mu|)|z_k - s|,$$

or equivalently

$$|R(z_k) - R(s)| < \eta_1|z_k - s|, \tag{22}$$

with $\eta_1 = \varepsilon + |\mu|$. Since

$$\frac{1}{R'(s)} = \lim_{k\to\infty} \frac{z_k - s}{R(z_k) - R(s)},$$

a similar argument lead us to conclude that for $k \geq k_1$, for some k_1, there is $\eta_2 > 0$ such that

$$\eta_2 |z_k - s| < |R(z_k) - R(s)|. \tag{23}$$

From (22), (23) and Corollary 1 it follows that

$$\begin{aligned}
|z_{k+1} - s| &\leq \frac{1}{\eta_2} |R(z_{k+1}) - R(s)| \\
&= \frac{1}{\eta_2} |R(z_{k+1})| \\
&\leq \frac{1}{\eta_2} |R(z_k)|^j \\
&\leq \frac{1}{\eta_2} (\eta_1 |z_k - s|)^j \\
&= \frac{\eta_1^j}{\eta_2} |z_k - s|^j
\end{aligned}$$

which shows that (14), with $j = 2, 3$ is of order j. ■

Lemma 4 *Let $\theta = \arg(w) \in]-\pi, \pi[$ and let $\mathscr{S} = \{z \in \mathbb{C} : \ |1 - wz^{-p}| < 1\}$. Then*

$$\mathscr{S} = \mathscr{S}_0 \cup \mathscr{S}_1 \cup \cdots \cup \mathscr{S}_{p-1} \tag{24}$$

where, for each $n = 0, 1, \ldots, p-1$,

$$\mathscr{S}_n = \left\{ z \in \mathscr{S} : \ \frac{(2n-1)\pi + \theta}{p} < \arg(z) < \frac{(2n+1)\pi + \theta}{p} \right\}$$

is a connected set and $\mathscr{S}_n \cap \mathscr{S}_\ell = \phi$, for $\ell \neq n$.

Proof. Firstly we prove that, for each n, the points of the ray emerging from the origin

$$\mathscr{R}_n = \left\{ z \in \mathbb{C} : \ \arg(z) = \frac{(2n-1)\pi + \theta}{p} \right\} \tag{25}$$

are not in $\mathscr{S}$. If $z \in \mathscr{R}_n$ then

$$z = \rho_1 \, e^{\frac{(2n-1)\pi+\theta}{p} i},$$

for some nonnegative ρ_1. Let $w = \rho e^{\theta i}$ be the polar decomposition of w. Since

$$\begin{aligned}
|1 - wz^{-p}| &= \left| 1 - \rho e^{\theta i} \left(\rho_1 e^{\frac{(2n-1)\pi+\theta}{p} i} \right)^{-p} \right| \\
&= |1 + \rho \rho_1^{-p}| \\
&\geq 1,
\end{aligned}$$

it follows that $z \notin \mathscr{S}$, which shows that (24) holds. The sets $\mathscr{S}_n$ are disjoint by definition. Now we show that each $\mathscr{S}_n$ is a connected set. Let $P_n(z)$ be the complex valued function which assigns to each $z \in \mathbb{C} \setminus \mathbb{R}_0^-$ the unique pth root of z that satisfies

$$\frac{(2n-1)\pi}{p} < \arg(z) < \frac{(2n+1)\pi}{p}.$$

By definition, we have $[P_n(z)]^p = z$. Consider the function

$$g_n(z) = \frac{w^{1/p}}{P_n(z)}$$

which is continuous in $\mathbb{C} \setminus \mathbb{R}_0^-$. If

$$C = \{z \in \mathbb{C} : \ |1 - z| < 1\},$$

it is not hard to show that, for each n, $g_n(C \setminus \{0\}) = \mathscr{S}_n$. Therefore $\mathscr{S}_n$ is a connected set because it is the image of the connected set $C \setminus \{0\}$ by a continuous function. ■

Theorem 5 *Let $\mathscr{S}_n$ be as in Lemma 4. If $z_0 \in \mathscr{S}_n$ then for $j = 2,3$ the sequence (14) converges with order j to the unique pth root of w which lies on the wedge*

$$\mathscr{W}_n = \left\{ z \in \mathbb{C} : \ \frac{(2n-1)\pi + \theta}{p} < \arg(z) < \frac{(2n+1)\pi + \theta}{p} \right\} \tag{26}$$

Proof. Let $j = 2,3$. By Theorem 3 and Corollary 1, N_j takes $\mathscr{S}$ into $\mathscr{S}$. Moreover $N_j(\mathscr{S}_n) \subset \mathscr{S}_\ell$, for some ℓ, because all the sets in the decomposition (24) are disjoint and each $\mathscr{S}_n$ is connected. Let w_n be the pth root of w which belongs to $\mathscr{W}_n$. It is easy to see that $w_n \in \mathscr{S}_n$. Since $N_j(w_n) = w_n \in \mathscr{S}_n$, the set $N_j(\mathscr{S}_n)$ must lie entirely in $\mathscr{S}_n$, for all n, and consequently, for $z_0 \in \mathscr{S}_n$, $z_k \in \mathscr{S}_n$ for all n. By Corollary 1, z_k converges to w_n. Thanks to Theorem 3 and Lemma 3, the convergence is of order j. ■

Corollary 2 *Let $\mathscr{S}_0$ be the connected component of $\mathscr{S}$ as in (24). If $z_0 \in \mathscr{S}_0$ for $j = 2,3$ the sequence (14) converges to the principal pth root of w, $w^{1/p}$, with order j.*

Proof. According to Theorem 5, one needs to show that $w^{1/p}$ lies in $\mathscr{S}_0$. We know that that $|1 - w(w^{1/p})^{-p}| = 0 < 1$ and that the argument of the principal pth root of w is θ/p which satisfies

$$-\frac{\pi}{p} < \frac{\theta}{p} < \frac{\pi}{p}.$$

From these inequalities we conclude that

$$\frac{-\pi + \theta}{p} < \frac{-\pi + \theta}{2p} < \frac{\theta}{p} < \frac{\pi + \theta}{2p} < \frac{\pi + \theta}{p},$$

which shows that $w^{1/p} \in \mathscr{S}_0$. ■

If one is interested in taking real numbers as starting values, next corollary states the conditions that need to be satisfied.

Corollary 3 *Assume that w is such that $\Re(w) > 0$, where $\Re(w)$ denotes the real part of w and let z_0 be a positive real number such that*

$$z_0 > \left(\frac{|w|^2}{2\Re(w)} \right)^{1/p}.$$

Then for $j = 2,3$ the sequence (14) converges with order j to the principal pth root of w.

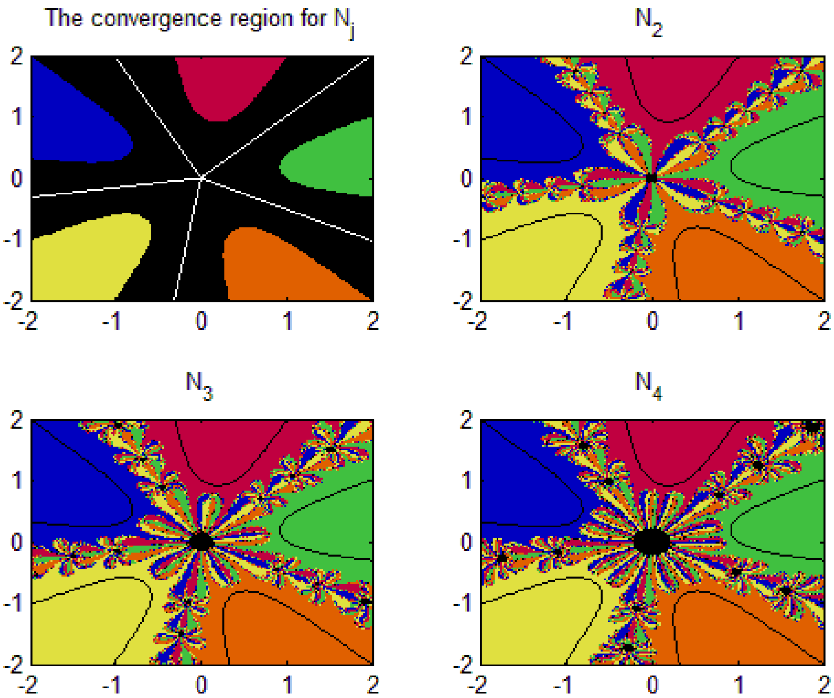

Fig. 1 Regions of convergence defined by the residuals of N_j (top left) and basins of attraction of the 5-roots of $w = 1+i$ for N_2, N_3 and N_4.

Proof. The condition $|1 - wz_0^{-p}| < 1$ can be rewritten in the form

$$|z_0^p - w| < z_0^p.$$

This means that the positive real number z_0^p must lie on the open half plane that contains w and which is defined by the perpendicular bisector of the segment joining w with the origin. Since $\Re(w) > 0$, some calculations lead us to conclude that this half plane contains the part of real axis corresponding to real numbers greater than $\frac{|w|^2}{2\Re(w)}$. Thus

$$z_0^p > \frac{|w|^2}{2\Re(w)},$$

and so the set

$$\left\{ z_0 \in \mathrm{IR} : \; z_0 > \left(\frac{|w|^2}{2\Re(w)} \right)^{1/p} \right\}$$

is entirely contained in $\mathscr{S}_0$ defined in (24). ■

In Figure 1, the top-left plot displays the sets $\mathscr{S}_0, \ldots, \mathscr{S}_4$ with different colors together with the rays $\mathscr{R}_n$ defined in (25), for $w = 1+i$ and $p = 5$. For each n, $\mathscr{S}_n$ corresponds to a specific 5th root of w and the previous results guarantee that, at least for $j = 2,3$, it is contained in the basin of attraction of that pth root. This latter fact is illustrated in the remaining plots, where the basins of attraction of each 5th root of w were determined experimentally in Matlab. More precisely, a point z_0 in the rectangle is marked with the same color of the 5th root $\sqrt[5]{w}$ of w, whenever $|z_{50} - \sqrt[5]{w}| < 10^{-4}$. We have also overlapped the boundaries of the regions of convergence on the basins of attraction for N_2, N_3 and N_4. Black color is assigned to points that are not in $\mathscr{S}_n$ (top-left) or that do not belong to the basins of attraction of the pth roots of w (remaining plots).

While the only fixed points of N_2 are the pth roots of w, the iterations N_3 and N_4 have fixed points other than the pth roots of w. Since these extra fixed points are in general repelling, they do belong to the Julia sets of N_j and so contribute to change the shape of the boundaries. In the plots for N_3 and N_4, we can see new sets of petals which are typical in iteration functions of orders greater than 2. We refer the reader to [23] for more details about the Julia sets of N_j.

The plot corresponding to N_4 (bottom right) suggests that the condition

$$|R(z_k)| = |1 - wz_0^{-p}| < 1$$

also guarantees the convergence of the sequence

$$z_{k+1} = N_4(z_k)$$

to a pth root of w. So assuming that results similar to the ones in Theorem 3 and Corollary 1 are stated for $j \geq 4$, it is easy to conclude that lemmas 3 and 4, Theorem 5 and corollaries 2 and 3 also hold for $j \geq 4$.

4 The iteration functions L_j

The analysis of the iterations functions L_j defined in (11) is the topic of this section. We proceed as in the previous section. For the particular cases $j = 2, 3, 4$, L_j is defined by

$$\begin{aligned}
L_2(z) &= z\left(1 + \frac{1}{p}(1 - w^{-1}z^p)\right) \\
L_3(z) &= z\left(1 + \frac{1}{p}(1 - w^{-1}z^p) + \frac{1+p}{2!p^2}(1 - w^{-1}z^p)^2\right) \\
L_4(z) &= z\left(1 + \frac{1}{p}(1 - w^{-1}z^p) + \frac{1+p}{2!p^2}(1 - w^{-1}z^p)^2 + \frac{(1+p)(1+2p)}{3!p^3}(1 - w^{-1}z^p)^3\right).
\end{aligned}$$

The residuals associated to the sequence

$$z_{k+1} = L_j(z_k) \tag{27}$$

are given by $R(z_k) = 1 - w^{-1}z_k^p$. By Lemma 1 these residuals satisfy the following relationship:

$$R(z_{k+1}) = 1 - (1 - R(z_k))\,[T_{j-1}(R(z_k))]^p\,,$$

where T_{j-1} is given as in Example 2. An important convergence result involving the sequence (27) and its residuals is stated in the following theorem. The proof can be found in [21,15].

Lemma 5 *Let $R(z_k)$ denotes the residual of z_k defined in (27). If $|R(z_0)| = |1 - w^{-1}z_0^p| < 1$, then*

$$|R(z_{k+1})| \leq |R(z_k)|^j \leq |R(z_0)|^{j^k},$$

for all k and j. Moreover, the sequence (27) converges to a pth root of w.

There are two important issues not addressed in [21,15]. The first one is to show that (27) has order of convergence j (according to the definition stated in the Introduction) and the second one is to determine for which pth root the sequence converges. These and other issues will be analyzed in the following results. The proofs will be omitted because they are similar to the ones derived in Section 3 for N_j.

Lemma 6 *If z_0 is such that $|R(z_0)| = |1 - w^{-1} z_0^p| < 1$, then the sequence (27) converges to a pth root of w with order of convergence j.*

Lemma 7 *Let $\theta = \arg(w) \in]-\pi, \pi[$ and let $\mathscr{T} = \{z \in \mathbb{C} : \ |1 - w^{-1} z^p| < 1\}$. Then*

$$\mathscr{T} = \mathscr{T}_0 \cup \mathscr{T}_1 \cup \cdots \cup \mathscr{T}_{p-1} \tag{28}$$

where, for each $n = 0, 1, \ldots, p-1$,

$$\mathscr{T}_n = \left\{ z \in \mathscr{T} : \ \frac{(2n-1)\pi + \theta}{p} < \arg(z) < \frac{(2n+1)\pi + \theta}{p} \right\}$$

is a connected set and $\mathscr{T}_n \cap \mathscr{T}_\ell = \phi$, for $\ell \neq n$.

Theorem 6 *Let $\mathscr{T}_n$ be as in Lemma 7. If $z_0 \in \mathscr{T}_n$ then the sequence (27) converges with order j to the unique pth root of w which lies on the wedge*

$$\mathscr{W}_n = \left\{ z \in \mathbb{C} : \ \frac{(2n-1)\pi + \theta}{p} < \arg(z) < \frac{(2n+1)\pi + \theta}{p} \right\}.$$

Corollary 4 *Let $\mathscr{T}_0$ be the connected component of $\mathscr{T}$ as in (28). If $z_0 \in \mathscr{T}_0$ then z_k converges with order j to the principal pth root of w, $w^{1/p}$.*

Corollary 5 *Assume that w is such that $\Re(w) > 0$ and let z_0 be a positive real number such that*

$$z_0 < (2\Re(w))^{1/p}.$$

Then z_k converges with order j to the principal pth root of w.

Figure 2 displays Matlab generated plots to compare the domains of convergence defined by the sets $\mathscr{T}_n$ in (28) with the basins of attractions of the 5th roots of $w = 1 + i$. The top-left plot shows the sets $\mathscr{T}_n$ together with the rays $\mathscr{R}_n$ defined in (25). The remaining plots include the basins of attraction of the 5th roots of w with respect to the iteration functions L_2, L_3 and L_4 combined with the boundaries of the sets $\mathscr{T}_n$. We can observe that the iteration functions L_j have much smaller domains of convergence than N_j. Although this can be seen as a disadvantage, it turns out that iterations L_j have been implemented successfully for computing matrix pth roots [15,9]. The main reason lies on the fact that each iterate of (27) does not involve the computation of the inverse of z_k. When working with matrices this contributes to reduce the computational cost of the method and eventually to improve the accuracy.

Similarly to the iteration functions N_j, it happens that for $j \geq 3$, L_j has extra fixed points that are not pth roots of w. In general these fixed points are repelling and are responsible for the complicated structure of the Julia sets of L_j, by inserting new sets of petals.

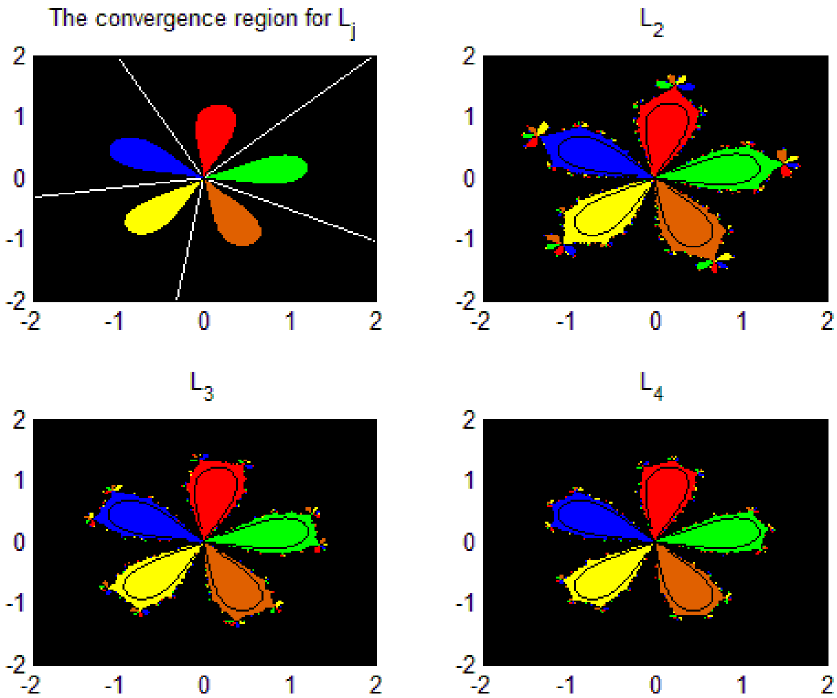

Fig. 2 Regions of convergence defined by the residuals of L_j (top left) and basins of attraction of the 5th roots of $w = 1+i$ for L_2, L_3 and L_4.

5 More iteration functions

In this section we consider more iteration functions generated by the functions of the form

$$f(z) = \left(\frac{az+b}{cz+b}\right)^{1/p}, \tag{29}$$

with $ab - bc \neq 0$. Let T_j denotes the j-degree Taylor polynomial of f. For each j the iteration function

$$F_j(z) = zT_{j-1}\left(\frac{b(wz^{-p}-1)}{a-cwz^{-p}}\right)$$

has order j. The 2-degree Taylor polynomial of f is

$$T_2(z) = 1 + \frac{a-c}{bp}z + \frac{(a-c)^2 + p(c^2-a^2)}{2b^2p^2}z^2.$$

If a and c are related by

$$\frac{a}{c} = \frac{1+p}{1-p}, \tag{30}$$

and $b \neq 0$, then the coefficient of z^2 in T_2 vanishes. This means that we have an iteration of order 3 defined upon a 1-degree Taylor polynomial, which has a simpler expression. Thus assuming that $a = 1+p$, $b = 1$ and $c = 1-p$, the function

$$h(z) = [(1+(p+1)z)/(1+(1-p)z)]^{1/p},$$

gives rise to the following 3-order iteration function

$$\begin{aligned} H_3(z) &= zT_1\left(\frac{w-z^p}{(p+1)z^p+(p-1)w}\right) \\ &= z\frac{(p-1)z^p+(p+1)w}{(p+1)z^p+(p-1)w} \end{aligned} \tag{31}$$

which is exactly the Halley iteration $H(z)$ mentioned in (3). We note that other choices for a, b and c satisfying (30) lead to the same iteration function. This optimal property of Halley iteration is not shared by other iterations generated by functions of the form (29), because the coefficient of z^2 in the corresponding Taylor polynomial does not vanish. However, several tests we carried out showed that the corresponding iteration functions perform better as the numerator of the coefficient of z^2 becomes close to zero, that is, when

$$\frac{a}{c} \approx \frac{1+p}{1-p}. \tag{32}$$

Another function of the form (29) that will be analyzed is

$$m(z) = \left(\frac{1+z}{1-z}\right)^{1/p},$$

which satisfies (32) for p sufficiently large. The corresponding iteration functions are

$$M_j(z) = zT_{j-1}\left(\frac{w-z^p}{w+z^p}\right) \tag{33}$$

For $j=2$, we have

$$M_2(z) = z\left(\frac{(p-2)z^p+(p+2)w}{pz^p+pw}\right).$$

Consider also the iteration functions generated by h,

$$H_j(z) = zT_{j-1}\left(\frac{p(w-z^p)}{(p+1)z^p+(p-1)w}\right). \tag{34}$$

and the sequences generated by the recursions

$$m_{k+1} = M_j(m_k), \tag{35}$$

and

$$h_{k+1} = H_j(h_k). \tag{36}$$

According to Lemma 1 the following residual recurrences hold:

$$R(m_{k+1}) = \frac{(1+R(m_k))-(1-R(m_k))\,[T_{j-1}(R(m_k))]^p}{(1+R(m_k))+(1-R(m_k))\,[T_{j-1}(R(m_k))]^p} \tag{37}$$

and

$$R(h_{k+1}) = \frac{\Big(1+(p+1)R(h_k)\Big)-\Big((1-p)R(h_k)+1\Big)\,[T_{j-1}(R(h_k))]^p}{(p-1)\Big(1+(p+1)R(h_k)\Big)+(p+1)\Big(1+(1-p)R(h_k)\Big)\,[T_{j-1}(R(h_k))]^p}. \tag{38}$$

Consider the following regions defined by the residuals of the sequences (35) and (36):

$$\mathscr{M} = \left\{m_0 \in \mathbb{C} : |R(m_0)| = \left|\frac{w-m_0^p}{w+m_0^p}\right| < 1\right\}$$

and

$$\mathscr{H} = \left\{ h_0 \in \mathbb{C} : |R(h_0)| = \left| \frac{w - h_0^p}{(p-1)w + (p+1)h_0^p} \right| < 1 \right\}.$$

A question that arises now is to know if $\mathscr{M}$ (resp. $\mathscr{H}$) defines a region of convergence for (35) (resp. (36)), that is, if $m_0 \in \mathscr{M}$ (resp., $h_0 \in \mathscr{H}$) do the sequence (35) (resp. (36)) converge to a pth root of w with order j? This seems to be a very hard issue because of the complicated expressions of the residuals recurrences (37) and (38). This means that we cannot expect to state nice convergence results like the ones in Corollary 1 and Lemma 5. However, we proved the following convergence result which concerns to the weaker case when (35) is restricted to real numbers and $j = 2$.

Theorem 7 *Consider the real version of the iterative formula (35), with $j = 2$,*

$$m_{k+1} = \frac{m_k}{p} \left(\frac{(p-2)m_k^p + (p+2)a}{m_k^p + a} \right),$$

where a is a positive real number. If m_0 is a positive real number then the sequence $\{m_k\}_{k=0}^{\infty}$ converges to $a^{1/p}$ with order of convergence at least 2.

Proof. Firstly, we define the real function of real variable (see (37))

$$g(x) = \frac{(1+x) - (1-x)(1+2/p)^p}{(1+x) + (1-x)(1+2/p)^p}$$

and show that

$$|g(x)| < x^2,$$

for all $0 < x < 1$, which is equivalent to

$$g(x) < x^2 \tag{39}$$

and

$$-x^2 < g(x). \tag{40}$$

Let $0 < x < 1$. Since

$$(1+2/p)^p = a_0 + a_1 x + a_2 x^2 + \cdots + a_p x^p,$$

where $a_n = \binom{p}{p-n} \left(\frac{2}{p}\right)^n$, performing some calculations it is straightforward to conclude that the following equivalences hold:

$$\begin{aligned} g(x) < x^2 &\Leftrightarrow (1+x)^2 - (1+x^2)(1+2/p)^p < 0 \\ &\Leftrightarrow -a_2 x^2 - \left(\sum_{n=3}^{p} (a_n + a_{n-2}) x^n \right) - a_{p-1} x^{p+1} - a_p x^{p+2} < 0. \end{aligned} \tag{41}$$

We can easily observe that (41) holds and so (39) follows. To prove (40), we note that

$$\begin{aligned} -x^2 < g(x) &\Leftrightarrow (1+x^2) - (1-x)^2(1+2/p)^p > 0 \\ &\Leftrightarrow (2a_1 - a_2)x^2 + (-a_3 + 2a_2 - a_1)x^3 + \left(\sum_{n=4}^{p} (-a_n + 2a_{n-1} - a_{n-2}) x^n \right) + \\ &\quad + (2a_p - a_{p-1}) x^{p+1} - a_p x^{p+2} > 0. \end{aligned} \tag{42}$$

Let b_n denote the coefficient of x^n ($n=2,3,\ldots,p+2$) in (42). It is easy to show that b_2 and b_3 are nonnegative, that is, $b_2 = 2a_1 - a_2 \geq 0$ and $b_3 = -a_3 + 2a_2 - a_1 \geq 0$. Next we show that the remaining coefficients are negative. Indeed, for $n = 4,\ldots,p$, we have that

$$\begin{aligned}
b_n &= -a_n + 2a_{n-1} - a_{n-2} \\
&= \left(\frac{2}{p}\right)^{n-2} \frac{(p-1)\ldots(p-n+3)}{pn(n-1)(n-2)!} \times \\
&\quad \times \left((5n-n^2-4)p^2 + (-4n^2+16n-12)p + (-4n^2+12n-8)\right) < 0, \\
b_{p+1} &= 2a_p - a_{p-1} < 0 \\
b_{p+2} &= -a_p < 0.
\end{aligned}$$

Since

$$b_2 + b_3 + \cdots + b_{p+1} + b_{p+2} = a_1 > 0,$$

it follows that

$$b_2 + b_3 > -b_4 - \ldots - b_{p+1} - b_{p+2}.$$

Thus

$$\begin{aligned}
b_2x^2 + b_3x^3 &> b_2x^3 + b_3x^3 \\
&> -b_4x^3 - \ldots - b_{p+1}x^3 - b_{p+2}x^3 \\
&> -b_4x^4 - \ldots - b_{p+1}x^{p+1} - b_{p+2}x^{p+2}
\end{aligned}$$

and so

$$b_2x^2 + b_3x^3 + b_4x^4 + \ldots + b_{p+1}x^{p+1} + b_{p+2}x^{p+2} > 0,$$

which proves (42) and consequently (40). Since $|R(m_0)| = |(a - m_0^p)/(a + m_0^p)| < 1$ for each positive m_0, proceeding as in the proof of Corollary 1 the result follows. ■

We have compared experimentally the region $\mathscr{M}$ (resp. $\mathscr{H}$) with the basins of attraction of the 5th roots of $w = 1 + i$ corresponding to the iteration function M_2 (resp., H_3). The pictures are displayed in Figure 3. Comparing top-left and top-right plots we can observe that the region $\mathscr{M}$ apparently defines a convergence region for M_2, where starting values belonging to a given sector lead to convergence for the 5th root lying exactly on that sector. However, this does not seem to happen with the region $\mathscr{H}$ for H_3, which suggests that the residuals considered for H_j may not be the most appropriate. We recall that in [10] (and also in [17]) a different residual was proposed to study the convergence of Halley's method.

Another interesting fact we can see in Figure 3 is the similarity between the patterns exhibited by the basins of attraction for the iterations M_2 and H_3. Contrarily to the basins of attraction for N_j and L_j, there are no black regions and so it may happen that both M_2 and H_3 always converge to a root of w, for any nonzero starting value. We note that both iteration functions have 0 and the pth roots of w as fixed points, though 0 is a repelling fixed point.

6 Numerical examples

We have implemented the iterations functions N_2, L_2, M_2, with order of convergence 2, and N_3, L_3 and H_3 with order of convergence 3 to illustrate the computation of the pth roots of some particular complex numbers. For any k, the error $|z_k - w^{1/p}|$ is displayed in each entry of the tables. All the tests were performed in Matlab using

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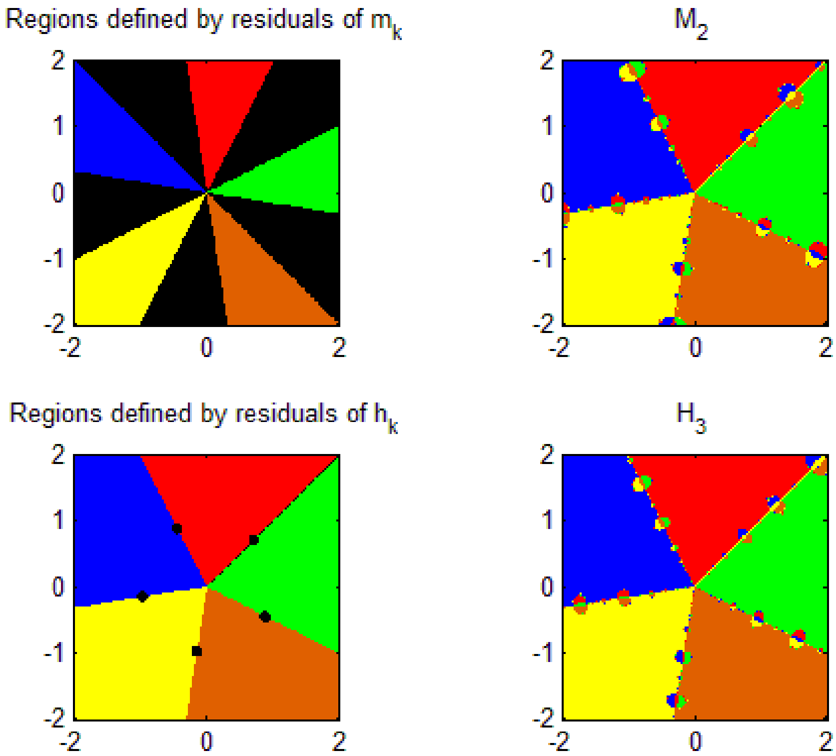

Fig. 3 Region $\mathscr{M}$ (top-left), basins of attractions of the 5th roots of $w = 1+i$ for M_2 (top-right), region $\mathscr{H}$ (bottom-left) and basins of attractions of the 5th roots of w for H_3 (bottom-right).

variable precision arithmetic with 1000 significant decimal digits. Table 1 reports on the principal 5th root of $w = 1+i$, taking $z_0 = 1$ as starting value. The residuals of z_0 are

$$|1 - wz_0^{-5}| = 1$$

for iterations N_2 and N_3 and

$$|1 - w^{-1}z_0^5| \approx 0.7$$

for L_2 and L_3. Although the residuals of N_2 and N_3 do not satisfy the requirements of Corollary 2, the convergence to the principal 5-th root of w occurs. Residuals of the starting value for M_2 and H_3 are respectively

$$\left|\frac{w - z_0^5}{w + z_0^5}\right| \approx 0.4$$

and

$$\left|\frac{5(w - z_0^5)}{6z_0^5 + 4w}\right| \approx 0.5.$$

Table 2 illustrates the computation of a non principal 8th root of $w = -1+2i$. We have taken $z_0 = -1+0.4i$ as the initial guess and computed the 8th root of w, $-0.9535\ldots + 0.5600\ldots i$, that lie on the wedge $\mathscr{W}_4$ defined in (26). The residuals of z_0 have absolute values less than 1 for the iterations M_2 and H_2, and between 1 and 2 for the other iterations. Finally, Table 3 shows the application of the iterations to the particular case of the real pth root of a positive real number, with $p = 11$, $w = 29$ and $z_0 = 1.3$. All the residuals of z_0 have absolute values less than 1.

k	error N_2	error L_2	error M_2	error N_3	error L_3	error H_3
1	$0.669... \times 10^{-1}$	$0.793... \times 10^{-1}$	$0.227... \times 10^{-1}$	$0.387... \times 10^{-1}$	$0.421... \times 10^{-1}$	$0.114... \times 10^{-1}$
2	$0.923... \times 10^{-2}$	$0.182... \times 10^{-1}$	$0.221... \times 10^{-3}$	$0.272... \times 10^{-3}$	$0.800... \times 10^{-3}$	$0.255... \times 10^{-5}$
3	$0.158... \times 10^{-3}$	$0.952... \times 10^{-3}$	$0.229... \times 10^{-7}$	$0.106... \times 10^{-9}$	$0.491... \times 10^{-8}$	$0.290... \times 10^{-16}$
4	$0.469... \times 10^{-7}$	$0.253... \times 10^{-5}$	$0.245... \times 10^{-15}$	$0.628... \times 10^{-29}$	$0.113... \times 10^{-23}$	$0.426... \times 10^{-49}$
5	$0.411... \times 10^{-14}$	$0.180... \times 10^{-10}$	$0.280... \times 10^{-31}$	$0.129... \times 10^{-86}$	$0.140... \times 10^{-70}$	$0.134... \times 10^{-147}$

Table 1 Values of the error $|z_k - (1+i)^{1/5}|$, with $z_0 = 1$

k	error N_2	error L_2	error M_2	error N_3	error L_3	error H_3
1	$0.220... \times 10^{0}$	$0.240... \times 10^{0}$	$0.144... \times 10^{0}$	$0.250... \times 10^{0}$	$0.260... \times 10^{0}$	$0.133... \times 10^{0}$
2	$0.306... \times 10^{0}$	$0.309... \times 10^{0}$	$0.646... \times 10^{-2}$	$0.885... \times 10^{0}$	$0.129... \times 10^{1}$	$0.100... \times 10^{-1}$
3	$0.171... \times 10^{0}$	$0.650... \times 10^{0}$	$0.199... \times 10^{-4}$	$0.531... \times 10^{0}$	$0.729... \times 10^{2}$	$0.434... \times 10^{-5}$
4	$0.755... \times 10^{-1}$	$0.236... \times 10^{1}$	$0.180... \times 10^{-9}$	$0.254... \times 10^{0}$	$0.676... \times 10^{46}$	$0.353... \times 10^{-15}$
5	$0.196... \times 10^{-1}$	$0.184... \times 10^{3}$	$0.147... \times 10^{-19}$	$0.684... \times 10^{-1}$	`inf`	$0.189... \times 10^{-45}$

Table 2 Values of the error $|z_k - (-0.9535... + 0.5600...i)|$, with $p = 8$, $w = 1-2i$ and $z_0 = -1+0.4i$

k	error N_2	error L_2	error M_2	error N_3	error L_3	error H_3
1	$0.149... \times 10^{-1}$	$0.129... \times 10^{-1}$	$0.233... \times 10^{-2}$	$0.561... \times 10^{-2}$	$0.358... \times 10^{-2}$	$0.111... \times 10^{-2}$
2	$0.784... \times 10^{-3}$	$0.722... \times 10^{-3}$	$0.207... \times 10^{-5}$	$0.348... \times 10^{-5}$	$0.112... \times 10^{-5}$	$0.745... \times 10^{-8}$
3	$0.225... \times 10^{-5}$	$0.230... \times 10^{-5}$	$0.159... \times 10^{-11}$	$0.803... \times 10^{-15}$	$0.359... \times 10^{-16}$	$0.224... \times 10^{-23}$
4	$0.187... \times 10^{-10}$	$0.234... \times 10^{-10}$	$0.933... \times 10^{-24}$	$0.985... \times 10^{-44}$	$0.115... \times 10^{-47}$	$0.616... \times 10^{-70}$
5	$0.129... \times 10^{-20}$	$0.242... \times 10^{-20}$	$0.320... \times 10^{-48}$	$0.181... \times 10^{-130}$	$0.384... \times 10^{-142}$	$0.126... \times 10^{-209}$

Table 3 Values of the error $|z_k - 29^{1/11}|$, with $z_0 = 1.3$

One important conclusion we can extract from the tables is that, among the iteration functions of order 2, M_2 performs much better than N_2 and L_2. It is less sensitive to the choice of the starting values and the convergence is faster. Our tests also confirm that Halley's method is a very good choice for computing pth roots.

7 Conclusion

We have presented a new method to generate higher–order iteration functions for computing the pth roots of a complex number. Among the infinitely many iteration functions that can be derived by this method, we can find well known iteration functions such as Newton, Halley and Schröder iterations. Some iteration functions not addressed previously in the literature can also be derived. Important convergence results that shed light on the problem of choosing an appropriated initial guess were proved as well. Our work raises a considerable number of questions that still need to be answered. For example, one needs to investigate if Corollary 1 can be extended to any j and to study the convergence behavior of the sequences (35). The complexity of the associated calculations compelled us to leave such study for a future work.

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