Infinite periodic points of endomorphisms over special confluent rewriting systems

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December 12, 2007

2000 Mathematics Subject Classification: 68R15, 37B10, 20M35, 68Q70

ABSTRACT

We consider endomorphisms of a monoid defined by a special confluent rewriting system that admit a continuous extension to the completion given by reduced infinite words, and study from a dynamical viewpoint the nature of their infinite periodic points. For prefix-convergent endomorphisms and expanding endomorphisms, we determine the structure of the set of all infinite periodic points in terms of adherence values, bound the periods and show that all regular periodic points are attractors.

1 Introduction

The dynamical study of the automorphisms of a free group and their space of ends is a well established subject in discrete Dynamical Systems [3, 4, 9, 10, 11, 12]. This paper constitutes an effort to study these problems in a more general setting, by considering monoids defined by certain types of rewriting systems instead of just free groups, and endomorphisms instead of automorphisms. The idea is to use combinatorics on words and automata theory to obtain results that have a marked geometric, topological or dynamical nature.

The authors initiated this project in [7], where the foundations of the whole approach were established. In view of the possibilities offered to language theory by the study of free groups [17, 18] and more general structures such as PR-monoids [19], it seemed a natural idea to extend some of the theory on infinite words to the more general setting of monoids defined by finite special confluent rewriting systems. We recall that a rewriting system $\{(r_1, s_1), \ldots, (r_n, s_n)\}$ is said to be *special* if $s_1 = \ldots = s_n = 1$.

Monoids defined through finite special confluent rewriting systems allow normal forms consisting of irreducible elements, hence they can be viewed as proper subsets of a free monoid with a particular binary operation (concatenation followed by total reduction, such as in the free group case). This approach can, up to some extent, be generalized to infinite words that are endowed with algebraic and topological structures that constitute natural generalizations of their free monoid counterparts. The fact that we can view infinite words as the space of ends of the undirected Cayley graph of the original monoid gives geometric significance to this topology.

We should note that infinite iteration of a (finite) word can no longer be assumed in every case due to the existence of periodic elements, thus our approach involves a partial version of the usual concept of ω-monoid [16].

The paper [7] was essentially devoted to the basic problem of endomorphism extensions: under which circumstances can an endomorphism φ of the monoid of finite words be extended to an endomorphism (continuous map, weak endomorphism) of the partial ω-monoid of infinite words? Characterization theorems leading to positive decidability results were obtained in most cases.

In this paper we use the characterization of the uniformly continuous endomorphisms (those that admit a continuous extension to the space of infinite words – that may be viewed as the natural topological completion or as the space of ends originating from the geodesic metric of the Cayley graph) to study the infinite periodic points of these extended endomorphisms. The main results are obtained for *prefix-convergent* endomorphisms and *expanding* endomorphisms, when we succeed in determining the structure of the set of infinite periodic points, bounding the periods and proving that all regular periodic points are attractors.

The paper is organized as follows: Section 2 is devoted to preliminaries. In Section 3 we establish the dynamical concepts relevant to our project. Note that most of these concepts are usually restricted to invertible mappings. In Section 4 we discuss the periodic points for prefix-convergent endomorphisms. This is a natural property to consider from a topological point of view but does not appear to be decidable in general. In Section 5 we accomplish a similar study in the case of expanding endomorphisms, which is proved to be a decidable property for a given endomorphism. As one should expect, the two properties are independent from each other. In Section 6 we develop the particular case of the free monoid, generalizing Konig's Lemma [16] in the spirit of the preceding sections.

2 Preliminaries

For basic concepts and results on language theory (respectively topology), the reader is referred to [2] (respectively [8]).

Let A denote a finite alphabet. Given $u, v \in A^*$, we write $u \leq v$ if u is a prefix of v. A (finite) *rewriting system* over A is a (finite) subset R of $A^* \times A^*$. Given $u, v \in A^*$, we write $u \longrightarrow_R v$ if

$$u = xry, \quad v = xsy$$

for some $x, y \in A^*$ and $(r, s) \in R$. We denote by $\xrightarrow{*}$ the reflexive and transitive closure of the relation $\longrightarrow$. The subscript R will be usually omitted. The *congruence* on A^* generated by R will be denoted by $R^\sharp$. Note that $R^\sharp = \xrightarrow{*}_{R \cup R^{-1}}$. The quotient $M = A^*/R^\sharp$ is said to be the monoid defined by the rewriting system R.

A rewriting system R over A is said to be

- *special* if $R \subseteq A^+ \times \{1\}$;
- *confluent* if, whenever $u \xrightarrow{*} v$ and $u \xrightarrow{*} w$, there exists $z \in A^*$ such that $v \xrightarrow{*} z$ and $w \xrightarrow{*} z$:

$$\begin{tikzcd} u \arrow[r, "*"] \arrow[d, "*"] & v \arrow[d, dashed, "*"] \\ w \arrow[r, dashed, "*"] & z \end{tikzcd}$$

Let R be a special confluent rewriting system over A. We say that $w \in A^*$ is *irreducible* (with respect to R) if $w \notin \cup_{(r,1) \in R} A^* r A^*$. For every $u \in A^*$, there is exactly one irreducible $v \in A^*$ such that $u \xrightarrow{*} v$: existence follows from any reduction sequence being length-reducing, and uniqueness from confluence. We denote this unique irreducible word by $\overline{u}$. It is well known (see [6]) that the equivalence

$$u R^\sharp v \Leftrightarrow \overline{u} = \overline{v}$$

holds for all $u, v \in A^*$, hence $\overline{A^*}$ constitutes a set of normal forms for the monoid $M = A^*/R^\sharp$. Moreover,

$$M \cong (\overline{A^*}, \cdot),$$

where $\cdot$ denotes the binary operation on $\overline{A^*}$ defined by $u \cdot v = \overline{uv}$. We denote the monoid $(\overline{A^*}, \cdot)$ by A_R^*. We shall often abuse notation and identify A_R^* with $\overline{A^*}$. We write also $A_R^+ = \overline{A^*} \setminus \{1\}$.

We denote by A^ω the set of all infinite words of the form $a_1 a_2 a_3 \ldots$, with $a_n \in A$ for every $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. Write

$$A^\infty = A^* \cup A^\omega.$$

Given $\alpha \in A^\infty$ and $n \in \mathbb{N}$, we denote by $\alpha^{(n)}$ the n-th letter of α (if $\alpha \in A^*$ and $n > |\alpha|$, we set $\alpha^{(n)} = 1$). We write also

$$\alpha^{[n]} = \alpha^{(1)} \alpha^{(2)} \ldots \alpha^{(n)}.$$

An infinite word $\alpha \in A^\omega$ is said to be *irreducible* (with respect to R) if $\alpha^{[n]}$ is irreducible for every $n \in \mathbb{N}$. We denote the set of all irreducible infinite words (with respect to R) by A_R^ω and we write

$$A_R^\infty = A_R^* \cup A_R^\omega.$$

For all $\alpha, \beta \in A^\infty$, we define

$$r(\alpha, \beta) = \begin{cases} \min\{n \in \mathbb{N} \mid \alpha^{(n)} \neq \beta^{(n)}\} & \text{if } \alpha \neq \beta \\ \infty & \text{if } \alpha = \beta \end{cases}$$

and we write

$$d(\alpha, \beta) = 2^{-r(\alpha,\beta)},$$

using the convention $2^{-\infty} = 0$. It follows easily from the definition that d is an ultrametric on A^∞, satisfying in particular the axiom

$$d(\alpha, \beta) \leq \max\{\mathrm{d}(\alpha, \gamma), \mathrm{d}(\gamma, \beta)\}.$$

We shall identify A^∞ with the metric space (A^∞, d). It is well known that the metric space A^∞ is compact (and therefore complete) [16, Chapter III]. Note that $\lim_{n\to\infty} \alpha_n = \alpha$ if and only if

$$\forall k \in \mathbb{N}\, \exists m \in \mathbb{N}\, \forall n \in \mathbb{N}\, (n \geq m \Rightarrow \alpha_n^{[k]} = \alpha^{[k]}).$$

Furthermore, since A^∞ is complete, a sequence $u_1, u_2, \ldots \in A^*$ converges if and only if it is a Cauchy sequence, i.e., if the condition

$$\forall k \in \mathbb{N}\, \exists m \in \mathbb{N}\, \forall n, n' \in \mathbb{N}\, (n, n' \geq m \Rightarrow u_n^{[k]} = u_{n'}^{[k]})$$

holds. By [7, Corollary 2.3], (A_R^∞, d) is compact (and therefore complete) whenever R is special confluent. We remark that, since $\alpha = \lim_{n\to\infty} \alpha^{[n]}$ for every $\alpha \in A^\infty$, (A^∞, d) (respectively (A_R^∞, d)) is the completion of (A^*, d) (respectively (A_R^*, d)). Note also that d induces the discrete topology on A^* since the open ball $B_{2^{-(n+1)}}(u) = \{u\}$ for every $u \in A^n$.

Referring to [7], we can mention an interesting geometric viewpoint on the nature of (A_R^ω, d). Let Γ denote the Cayley graph of the monoid A_R^* relative to the generating set A, and let $s(u, v)$ denote the distance on A_R^* given by the length of the shortest undirected path connecting u and v in Γ. We can view A_R^ω as the *space of ends* of Γ. By [7, Theorems 5.7 and 5.12], the metric d on A_R^ω induces the *Gromov topology* on the space of ends of the hyperbolic metric space (A_R^*, s).

We recall that $x \in X$ is an *adherence value* of $(u_n)_n$ if:

$$\forall \varepsilon > 0\, \forall n \in \mathbb{N}\, \exists m \geq n : d(u_m, x) < \varepsilon.$$

This is equivalent to say that there exists some infinite subsequence of $(u_n)_n$ converging to x. We denote the set of all adherence values of $(u_n)_n$ by $\mathrm{Ad(u_n)_n}$.

Given a mapping $\varphi : X \to X$, we say that $x \in X$ is *φ-periodic* if $x = x\varphi^m$ for some $m \in \mathbb{N}$. If $m = 1$, we say that x is a *fixed point* for φ. We denote by $\mathrm{Per}(\varphi)$ (respectively $\mathrm{Fix}(\varphi)$) the set of all periodic (respectively fixed) points of φ.

The following result is essential when considering rational languages in the context of special confluent rewriting systems (see also [6, Theorems 4.1.2 and 4.2.4]):

Theorem 2.1 *[1, 5] Let R be a finite special confluent rewriting system on A and let $L \subseteq A^*$ be rational. Then:*

(i) $\overline{L}$ is rational;

(ii) $D_L = \{u \in A^* \mid \overline{u} \in \overline{L}\}$ is deterministic context-free.

Moreover, both $\overline{L}$ and D_L are effectively constructible from L.

We present now a series of results from [7] that will prove useful in the forthcoming sections.

We fix $R = \{(r_1, 1), (r_2, 1), \ldots, (r_n, 1)\}$ and write $t_R = \max\{|\mathrm{r_1}|, |\mathrm{r_2}|, \ldots, |\mathrm{r_n}|\}$.

Lemma 2.2 *[7, Lemma 4.1] Let $u, v, w \in A_R^*$ be such that $|v| \geq |w|(t_R - 1)$ and $uv \in A_R^*$. Then $\overline{uvw} = u\overline{vw}$.*

Lemma 2.3 *[7, dual of Lemma 4.2] For all $u, v \in A_R^*$,*

(i) $u = u'u''$ and $v = v'v''$ with $\overline{uv} = u'v''$ and $|u''v'| \leq min\{|u|, |v|\} \cdot t_R$.

(ii) $|\overline{uv}| \geq \max\{|\mathrm{v}| - (\mathrm{t_R} - 1)|\mathrm{u}|, |\mathrm{u}| - (\mathrm{t_R} - 1)|\mathrm{v}|\}$.

Lemma 2.4 *[7, Lemma 5.8] Let $L \subseteq A_R^*$ be rational and let $\varphi : A_R^* \to A_R^*$ be an endomorphism. Then $L\varphi$ is rational and effectively constructible from L.*

We generalize the concept of ω-semigroup [16, Chapter I.4] as follows. A *partial ω-monoid* is a structure of the form $(M_1, M_2, \cdot, \circ, \pi)$, where $\cdot : M_1 \times M_1 \to M_1$ and $\circ : M_1 \times M_2 \to M_2$ are binary operations and $\pi : M_1^\omega = M_1 \times M_1 \times \ldots \to M_1 \cup M_2$ is a surjective partial map, such that:

(w1) $(M_1, \cdot)$ is a monoid;

(w2) if $(u_1, u_2, \ldots)\pi$ is defined and $i_1 < i_2 < \ldots$ is a sequence in $\mathbb{N}$, then $(u_1 \ldots u_{i_1}, u_{i_1+1} \ldots u_{i_2}, u_{i_2+1} \ldots u_{i_3}, \ldots)\pi$ is defined and equal to $(u_1, u_2, \ldots)\pi$;

(w3) if $(u_1, u_2, \ldots)\pi$ is defined and $v \in M_1$, then $(v, u_1, u_2, \ldots)\pi$ is defined and equal to $v \circ ((u_1, u_2, \ldots)\pi)$;

(w4) $(1, 1, \ldots)\pi$ is defined and equals 1.

We noted in [7] that these axioms imply the mixed associative law given by

$$u \circ (v \circ \alpha) = (u \cdot v) \circ \alpha$$

for all $u, v \in M_1$ and $\alpha \in M_2$.

If $M_1 \cup M_2$ is endowed with a distance d such that:

- the operations $\cdot$ and $\circ$ are continuous (considering the product topology on $M_1 \times (M_1 \cup M_2)$, for instance via the *max* metric on the components);
- $(u_1, u_2, \ldots)\pi$ is defined if and only if $\lim_{n\to\infty} u_1 u_2 \ldots u_n$ exists, in which case they coincide;

then we have a *metric* partial ω-monoid.

It follows easily from (w3) and (w2) that the identity of M_1 is a left identity for the mixed product $\circ$. If π is a full map, we have the standard concept of ω-monoid (ω-semigroup if we don't require $(M_1, \cdot)$ to have an identity).

If $u \in M_1$ and $(u, u, u, \ldots)\pi$ is defined, we denote it by u^ω.

An endomorphism of $(M_1, M_2, \cdot, \circ, \pi)$ is a mapping $\varphi : M_1 \cup M_2 \to M_1 \cup M_2$ such that:

(h1) $M_1\varphi \subseteq M_1$;

(h2) for all $u, v \in M_1$, $(u \cdot v)\varphi = (u\varphi) \cdot (v\varphi)$;

(h3) for all $u \in M_1$ and $\alpha \in M_2$,

$$(u \circ \alpha)\varphi = \begin{cases} (u\varphi) \cdot (\alpha\varphi) & \text{if } \alpha\varphi \in M_1 \\ (u\varphi) \circ (\alpha\varphi) & \text{if } \alpha\varphi \in M_2 \end{cases}$$

(h4) if $(u_1, u_2, \ldots)\pi$ is defined, then $(u_1\varphi, u_2\varphi, \ldots)\pi$ is defined and equal to $(u_1, u_2, \ldots)\pi\varphi$.

An endomorphism is said to be *proper* if $M_2\varphi \subseteq M_2$.

We define a binary operation

$$\begin{aligned} \circ : A_R^* \times A_R^\omega &\to A_R^\omega \\ (u, \alpha) &\mapsto \overline{u\alpha} \end{aligned}$$

by $\overline{u\alpha} = \lim_{n\to\infty} \overline{u\alpha^{[n]}}$.

The partial operation $\pi : (A_R^*)^\omega \to A_R^\infty$ is defined as follows: for every sequence $(u_n)_n \in (A_R^*)^\omega$, $(u_1, u_2, \ldots)\pi$ is defined if and only if $(\overline{u_1 \ldots u_n})_n$ converges. In such a case, we have

$$(u_1, u_2, \ldots)\pi = \lim_{n\to\infty} \overline{u_1 \ldots u_n}.$$

Theorem 2.5 *[7, Theorem 4.4] With the ultrametric d, $(A_R^*, A_R^\omega, \cdot, \circ, \pi)$ is a metric partial ω-monoid.*

The next result shows necessary and sufficient conditions for the existence of a continuous extension to A_R^∞ of an endomorphism φ of A_R^*. We refer to the constant homomorphism as the *trivial* homomorphism.

Theorem 2.6 *[7, Theorems 8.4 and 8.7] Let φ be a nontrivial endomorphism of A_R^*. Then the following conditions are equivalent and decidable:*

(i) φ can be extended to a continuous mapping $\Phi : A_R^\infty \to A_R^\infty$;

(ii) φ can be extended to a proper uniformly continuous endomorphism of the metric partial ω-monoid A_R^∞;

(iii) φ is uniformly continuous;

(iv) $w\varphi^{-1}$ is finite for every $w \in A_R^*$.

Moreover, if these conditions hold the continuous mapping Φ is unique and given by $\alpha\Phi = \lim_{n\to\infty} \alpha^{[n]}\varphi$.

Surjectivity of Φ is determined by the surjectivity of φ:

Proposition 2.7 *Let φ be a uniformly continuous endomorphism of A_R^* and let $\Phi : A_R^\infty \to A_R^\infty$ be its continuous extension. Then Φ is onto if and only if φ is onto.*

Proof. By Theorem 2.6, Φ is proper and so if Φ is onto, φ must be onto as well.

Conversely, assume that φ is onto and let $\alpha \in A_R^\omega$. Let

$$X = \bigcup_{n\geq 1} \alpha^{[n]}\varphi^{-1}.$$

Since φ is onto and by Theorem 2.6, $\alpha^{[n]}\varphi^{-1}$ is a finite nonempty subset of A_R^* for every $n \geq 1$. Thus X is a countable infinite subset of the compact space (A_R^∞, d) and so must have some adherence value in A_R^∞, that is,

$$\exists \beta \in A_R^\infty \; \forall \varepsilon > 0 \; \exists w \in X : \; 0 < d(w, \beta) < \varepsilon.$$

We show that $\beta\Phi = \alpha$. Suppose that $\alpha^{(m)} \neq (\beta\Phi)^{(m)}$ for some $m \geq 1$. Since Φ is uniformly continuous,

$$\exists M \geq 1 \; \forall \alpha_1, \alpha_2 \in A_R^\infty \; (r(\alpha_1, \alpha_2) > M \Rightarrow r(\alpha_1\Phi, \alpha_2\Phi) > m).$$

Let $w \in X$ be such that $0 < d(w, \beta) < 2^{-M}$. Since there are infinitely many such w and each $\alpha^{[n]}\varphi^{-1}$ is finite, we may assume that $w\varphi = \alpha^{[n]}$ with $n \geq m$. Hence $r(w, \beta) > M$ and so $r(\alpha^{[n]}, \beta\Phi) = r(w\varphi, \beta\Phi) > m$. Since $n \geq m$, it follows that $\alpha^{(m)} = (\beta\Phi)^{(m)}$, a contradiction. Thus $\beta\Phi = \alpha$ and Φ is onto. $\square$

3 Endomorphism dynamics

We fix a nontrivial uniformly continuous endomorphism φ of A_R^*. Let Φ be its continuous extension to A_R^∞. We intend to classify Φ-periodic points from a dynamical viewpoint. Clearly, given $\alpha \in A_R^\infty$, we consider $\{\alpha\Phi^n \mid n \in \mathbb{N}\}$ to be the *orbit* of α. Then α is Φ-periodic if and only $\alpha = \alpha\Phi^p$ for some $p \geq 1$. This is of course equivalent to α being a fixed point for the power endomorphism Φ^p, and most of the terminology we are about to introduce is usually defined for fixed points. The smallest such p is said to be the *period* of α.

Given $\alpha \in \mathrm{Per}(\Phi)$, we define the *attraction basin* to be

$$\mathrm{Att}(\alpha) = \{\beta \in A_R^\infty \mid \alpha \in \mathrm{Ad}(\beta\Phi^{\mathrm{n}})_{\mathrm{n}}\}.$$

If Φ is onto, it makes sense to define the *repulsion basin* of α to be

$$\mathrm{Rep}(\alpha) = \{\alpha\} \cup \{\beta \in A_R^\infty \mid \forall \varepsilon > 0 \, \forall n \in \mathbb{N} \, \exists m \geq n \, \exists \gamma \in \beta\Phi^{-m} : d(\alpha, \gamma) < \varepsilon.\}$$

Note that Φ is onto if and only if φ is onto by Proposition 2.7. In terms of a dynamical system, we can say that the future of α – the orbit $(\alpha\Phi, \alpha\Phi^2, \ldots)$ – is uniquely determined but the past of α may be not so (unless Φ is one-to-one). In that case, its past is a ramified tree corresponding to the various elements of $\alpha\Phi^{-1}, \alpha\Phi^{-2}, \ldots$ The idea is to collect in the repulsion basin of α all those words that *could* have been arbitrarily close to α in the past but got away from it (and also α for technical reasons).

We say that α is *singular* if α belongs to the topological closure of $\mathrm{Per}(\varphi)$. Otherwise, we say that α is *regular*. We denote the set of all regular (respectively singular) Φ-periodic points of A_R^∞ by $\mathrm{Per_r}(\Phi)$ (respectively $\mathrm{Per_s}(\Phi)$). Clearly, $\mathrm{Per_r}(\Phi) \subseteq \mathrm{A_R^\omega}$.

We say that $\alpha \in \mathrm{Per_r}(\Phi)$ is

- an *attractor* if some neighbourhood of α is contained in $\mathrm{Att}(\alpha)$.

If φ is onto, we say also that $\alpha \in \mathrm{Per_r}(\Phi)$ is

- a *repeller* if some neighbourhood of α is contained in $\mathrm{Rep}(\alpha)$;
- *hyperbolic* if α is neither an attractor nor a repeller, but some neighbourhood of α is contained in $\mathrm{Att}(\alpha) \cup \mathrm{Rep}(\alpha)$;
- *degenerate* if no neighbourhood of α is contained in $\mathrm{Att}(\alpha) \cup \mathrm{Rep}(\alpha)$.

If Φ is an automorphism, then $\mathrm{Per}(\Phi^{-1}) = \mathrm{Per}(\Phi)$ and it is straightforward that the attraction basin of $\alpha \in \mathrm{Per}(\Phi)$ relatively to Φ is the repulsion basin of α relatively to Φ^{-1}. Hence $\alpha \in \mathrm{Per}(\Phi)$ is an attractor for Φ if and only if it is a repeller for Φ^{-1} and vice-versa.

The dynamical study of automorphisms of the free group has been carried on by different authors (e.g. [3, 4, 9, 10, 11, 12]). In particular, it is known that:

Theorem 3.1 *[9, 11, 12] Let A_R^* be a free group of rank k and let φ be an automorphism of A_R^*. Then:*

(i) There are at least two infinite Φ-periodic points of period $\leq 2k$. If A_R^ω has a single orbit of Φ-periodic points, then this orbit has period 2.

(ii) The period of $\alpha \in A_R^\infty$ is bounded by some constant M_k depending only on k, and veryfing $M_k \sim \sqrt{k \log(k)}$ when $k \to +\infty$.

(iii) Every regular Φ-periodic point is either an attractor or a repeller.

We intend to deal with a more general situation, going beyond the free group and beyond automorphisms. It is therefore natural that the condition (iii) of the theorem does not hold any longer, as we show in the next example. In the presence of formal inverses in an alphabet, we say that an endomorphism is *matched* if it preserves (formal) inverses.

Example 3.2 *Let $A = \{a, b, c, b^{-1}\}$ and $R = \{(bb^{-1}, 1), (b^{-1}b, 1)\}$. Let $\varphi : A_R^* \to A_R^*$ be the matched endomorphism defined by*

$$a\varphi = ab, \quad b\varphi = b, \quad c\varphi = b^{-2}c.$$

Then A_R^ω contains (regular) hyperbolic Φ-periodic points.

Proof. It is clear that ab^ω and $a(b^{-1})^\omega$ are Φ-periodic. Suppose that $u \in A_R^+$ is Φ-periodic. If $u \notin b^* \cup (b^{-1})^*$, we may write $u = b^k xv$ for some $k \in \mathbf{Z}$ and $x \in \{a, c\}$. Clearly, $x = c$ implies $u\varphi^n = b^{k-2n}c \ldots$ for every $n \in \mathbb{N}$, hence $x = a$. If $u = b^k ab^m$, then $u\varphi^n = b^k ab^{m+n}$ for every $n \in \mathbb{N}$ and u is not periodic. If $u = b^k ab^m av$, then $u\varphi^n = b^k ab^{m+n} a \ldots$ and u is not periodic either. Finally, if $u = b^k ab^m cv$, then $u\varphi^n = b^k ab^{m-n} c \ldots$. Thus $\mathrm{Per}(\varphi) = \mathrm{b}^* \cup (\mathrm{b}^{-1})^*$ and so ab^ω and $a(b^{-1})^\omega$ are regular.

Since $\lim_{n\to\infty}(ab^k c)\varphi^n = a(b^{-1})^\omega$, we have $ab^k c \notin \mathrm{Att}(ab^\omega)$ for every $k \in \mathbf{Z}$. Since every neighbourhood of ab^ω must contain some word of the form $ab^k c$, it follows that ab^ω is not an attractor.

Clearly, φ is onto. Suppose that $ab^k \in \mathrm{Rep}(ab^\omega)$ with $k > 0$. Then in particular

$$\exists m \geq k \; \exists w \in (ab^k)\Phi^{-m} : \; r(ab^\omega, w) > k,$$

yielding a contradiction since $(ab^k)\Phi^{-m} = ab^{k-m}$. Thus $ab^k \notin \mathrm{Rep}(ab^\omega)$ for every $k > 0$. Since every neighbourhood of ab^ω must contain some word of the form ab^k, it follows that ab^ω is not a repeller.

Similarly, we can check that

$$ab^* \cup ab^*aA_R^* \subseteq \mathrm{Att}(ab^\omega),$$

$$ab^*cA_R^* \subseteq \mathrm{Rep}(ab^\omega),$$

thus

$$d(\beta, ab^\omega) < 2^{-2} \Rightarrow \beta = ab \ldots \Rightarrow \beta \in \mathrm{Att}(ab^\omega) \cup \mathrm{Rep}(ab^\omega)$$

and ab^ω is hyperbolic. □

In the following example, we present a case where all the regular periodic points are degenerate.

Example 3.3 *Let $A = \{a, b, c, b^{-1}\}$ and $R = \{(bb^{-1}, 1), (b^{-1}b, 1)\}$. Let $\varphi : A_R^* \to A_R^*$ be the matched endomorphism defined by*

$$a\varphi = ab, \quad b\varphi = b, \quad c\varphi = b^{-1}cb.$$

Then $Per_r(\Phi)$ is infinite and all its elements are degenerate.

Proof. First we show that

$$(u \leq v \ \wedge \ u\varphi = ub) \ \Rightarrow \ v \notin \mathrm{Per}(\varphi). \tag{1}$$

Indeed, assume that $v \in \mathrm{Per}(\varphi)$ possesses a prefix u such that $u\varphi = ub$. We may assume that u is maximal for this property. Since v cannot be of the form $v = ub^k$ since $\varphi^n(ub^k) = ub^{k+n}$ nor $v = ub^kaw$ since $\varphi^n(ub^kaw) \in ub^{k+n}aA_R^*$, it follows that $v = ub^kcw$ for some $k \in \mathbf{Z}$ and $w \in A_R^*$. However, this contradicts the maximality of u since $ub^kc \leq v$ and $(ub^kc)\varphi = ub^kcb$. Thus (1) holds.

Adapting the argument in Example 3.2, it is now easy to prove that $\mathrm{Per}(\varphi) = \mathrm{b}^* \cup (\mathrm{b}^{-1})^*$. Hence $\mathrm{Per_s}(\Phi) = \{\mathrm{b}^\omega, (\mathrm{b}^{-1})^\omega\}$. Write $B = b^* \cup (b^{-1})^*$. We show next that

$$\mathrm{Per_r}(\Phi) = \mathrm{Ba}(\mathrm{Bc})^*\{\mathrm{b}^\omega, (\mathrm{b}^{-1})^\omega\} \ \cup \ \mathrm{Ba}(\mathrm{Bc})^\omega. \tag{2}$$

Let $\alpha \in \mathrm{Per_r}(\Phi)$ and write $\alpha = b_0x_1b_1x_2b_3 \ldots$ where the x_i represent all the occurrences of either a or c. Similar arguments to those we used in the finite case show that $x_ix_{i+1} \in \{a^2, ca\}$ contradicts $\alpha \in \mathrm{Per_r}(\Phi)$. Thus $x_ix_{i+1} \in \{ac, c^2\}$ for every i and so the direct inclusion of (2) holds. The opposite inclusion is easily verified and so (2) holds.

In particular, $\mathrm{Per_r}(\Phi)$ is infinite. Take $\alpha \in \mathrm{Per_r}(\Phi)$. It follows from (2) that $\mathrm{Per_r}(\Phi)$ contains no isolated points, so there exists a sequence $(\alpha_i)_i$ of distinct elements of $\mathrm{Per_r}(\Phi)$ such that $\alpha = \lim_{i\to\infty} \alpha_i$. We may assume that no element of this sequence is in the orbit of α. Since $\alpha_i \in \mathrm{Per_r}(\Phi)$, it follows that $\alpha_i \notin \mathrm{Att}(\alpha)$ for every i. Since φ is an automorphism, $\mathrm{Rep}(\alpha)$ relatively to Φ equals $\mathrm{Att}(\alpha)$ relatively to Φ^{-1}. Since $\alpha_i \in \mathrm{Per_r}(\Phi^{-1})$, it follows by duality that $\alpha_i \notin \mathrm{Rep}(\alpha)$ for every i. Therefore every neighbourhood of α contains some $\alpha_i \notin \mathrm{Att}(\alpha) \cup \mathrm{Rep}(\alpha)$ and so α is degenerate. □

4 Preparatory results

We fix an endomorphism φ of A_R^* throughout the section. We define

$$\mathrm{Fin}(\varphi) = \{\mathrm{u} \in \mathrm{A_R^*} \mid \{\mathrm{u}\varphi^{\mathrm{n}} \mid \mathrm{n} \geq 1\} \text{ is finite }\} = (\mathrm{Per}(\varphi))(\varphi^{-1})^*,$$

$$\mathrm{Inf}(\varphi) = \mathrm{A_R^*} \backslash \mathrm{Fin}(\varphi), \quad \mathrm{A_0} = \mathrm{A} \cap \mathrm{Fin}(\varphi), \quad \mathrm{A_1} = \mathrm{A} \setminus \mathrm{A_0}.$$

Clearly, $\mathrm{Fin}(\varphi)$ is a submonoid of A_R^* and

$$(\mathrm{Fin}(\varphi))\varphi \cup (\mathrm{Fin}(\varphi))\varphi^{-1} \subseteq \mathrm{Fin}(\varphi).$$

In particular, $\overline{A_0^*} \subseteq \mathrm{Fin}(\varphi)$ and $\overline{A_0^*}\varphi \subseteq \mathrm{Fin}(\varphi)$.

Lemma 4.1 $\overline{(\mathrm{Fin}(\varphi))(\mathrm{Inf}(\varphi))} \subseteq \mathrm{Inf}(\varphi)$.

Proof. Let $u \in \mathrm{Fin}(\varphi)$ and $v \in \mathrm{Inf}(\varphi)$. Since $(\overline{uv})\varphi^n = \overline{(u\varphi^n)(v\varphi^n)}$, it follows from Lemma 2.3(ii) that $(\overline{uv})\varphi^n$ has unbounded length and so $\overline{uv} \in \mathrm{Inf}(\varphi)$. $\square$

By Theorem 2.6, φ admits a continuous extension Φ to A_R^∞ if and only if it is uniformly continuous. Henceforth, we shall assume that, whenever φ is uniformly continuous, Φ denotes its (unique) continuous extension to A_R^∞.

Lemma 4.2 *Let φ be uniformly continuous and let $u \in A_R^*$. Then the following conditions are equivalent:*

(i) $\mathrm{Ad}(\mathrm{u}\varphi^{\mathrm{n}})_{\mathrm{n}}$ *is finite;*

(ii) $(u\varphi^{nk})_n$ *converges for some* $k > 0$

Moreover, if $\lim_{n\to\infty} u\varphi^{nk} = \alpha$, *then* $\alpha\Phi^k = \alpha$ *and*

$$\mathrm{Ad}(\mathrm{u}\varphi^{\mathrm{n}})_{\mathrm{n}} = \{\alpha, \alpha\Phi, \ldots, \alpha\Phi^{\mathrm{k}-1}\} \subseteq \mathrm{Per}(\Phi).$$

Proof. Assume that $\mathrm{Ad}(\mathrm{u}\varphi^{\mathrm{n}})_{\mathrm{n}}$ is finite. We observe that

$$\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N} \; \forall n \geq n_0 \; \exists \alpha_n \in \mathrm{Ad}(\mathrm{u}\varphi^{\mathrm{n}})_{\mathrm{n}} : \; \mathrm{d}(\mathrm{u}\varphi^{\mathrm{n}}, \alpha_{\mathrm{n}}) < \varepsilon.$$

Indeed, let $\varepsilon > 0$. Since A_R^∞ is compact, we have $A_R^\infty = \cup_{i=1}^m B_{\varepsilon/2}(\beta_i)$ for some $\beta_1, \ldots, \beta_m \in A_R^\infty$. If $u\varphi^n \in B_{\varepsilon/2}(\beta_i)$ for infinitely many values of n, then $B_\varepsilon(\beta_i) \cap \mathrm{Ad}(\mathrm{u}\varphi^{\mathrm{n}})_{\mathrm{n}} \neq \emptyset$. Hence we take n_0 such that, whenever $u\varphi^n \in B_{\varepsilon/2}(\beta_i)$ for only finitely many values of n, then $n_0 > n$ for all such n.

If ε is chosen so that

$$\varepsilon \leq \varepsilon_1 = \frac{1}{2}\min\{d(\alpha, \beta) \mid \alpha, \beta \in \mathrm{Ad}(\mathrm{u}\varphi^{\mathrm{n}})_{\mathrm{n}}, \alpha \neq \beta\},$$

then α_n is uniquely defined. Now, as Φ is uniformly continuous, there exists $\varepsilon_2 > 0$ such that

$$\forall \alpha_1, \alpha_2 \in A_R^\infty \; (d(\alpha_1, \alpha_2) < \varepsilon_2 \Rightarrow d(\alpha_1\Phi, \alpha_2\Phi) < \varepsilon_1).$$

Taking $\varepsilon_3 = \min\{\varepsilon_1, \varepsilon_2\}$, $d(u\varphi^n, \alpha_n) < \varepsilon_3$ yields $d(u\varphi^{n+1}, \alpha_n\Phi) < \varepsilon_1$. Since $d(u\varphi^{n+1}, \alpha_{n+1}) < \varepsilon_1$ and $\alpha_n \in \mathrm{Ad}(u\varphi^n)_n$ yields $\alpha_n\Phi \in \mathrm{Ad}(u\varphi^n)_n$, we obtain $\alpha_n\Phi = \alpha_{n+1}$ by uniqueness. Since $\mathrm{Ad}(u\varphi^n)_n$ is finite, there exists some $k \geq n_0$ such that $\alpha_k = \alpha_{2k} = \alpha_k\Phi^k$. Thus

$$\forall \varepsilon \in]0, \varepsilon_3]\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0 : \quad d(u\varphi^{nk}, \alpha_k) < \varepsilon$$

and so $\lim_{n\to\infty} u\varphi^{nk} = \alpha_k$. Thus (ii) holds.

Conversely, assume that $\lim_{n\to\infty} u\varphi^{nk} = \alpha$. Since Φ is continuous, it commutes with limits. It follows that, for $i = 0, \ldots, k-1$,

$$\lim_{n\to\infty} u\varphi^{i+nk} = \lim_{n\to\infty} u\varphi^{nk}\Phi^i = (\lim_{n\to\infty} u\varphi^{nk})\Phi^i = \alpha\Phi^i.$$

Thus $\alpha\Phi^i \in \mathrm{Ad}(u\varphi^n)_n \cap A_R^\omega$ for $i = 0, \ldots, k-1$.

Suppose now that $\beta \in \mathrm{Ad}(u\varphi^n)_n$. Then $\beta = \lim_{n\to\infty} u\varphi^{j_n}$ for some infinite subsequence $(u\varphi^{j_n})_n$ of $(u\varphi^n)_n$. Clearly, there exists some $i \in \{0, \ldots, k-1\}$ such that $\{j_n \mid n \in \mathbb{N}\} \cap \{i + nk \mid n \in \mathbb{N}\}$ is infinite. Thus

$$\beta = \lim_{n\to\infty} u\varphi^{j_n} = \lim_{n\to\infty} u\varphi^{i+nk} = \alpha\Phi^i$$

and so

$$\mathrm{Ad}(u\varphi^n)_n = \{\alpha\Phi^i \mid i = 0, \ldots, k-1\}.$$

Finally, we remark that

$$(\alpha\Phi^i)\Phi^k = (\lim_{n\to\infty} u\varphi^{i+nk})\Phi^k = \lim_{n\to\infty} u\varphi^{i+(n+1)k} = \alpha\Phi^i$$

and so $\alpha\Phi^i \in \mathrm{Per}(\Phi)$. $\square$

Lemma 4.3 *Let $\Phi : A_R^\infty \to A_R^\infty$ be an endomorphism. If $u \in A_R^*$ and $\alpha \in A_R^\omega$ are Φ-periodic, so is $\overline{u\alpha}$.*

Proof. If $u\Phi^p = u$ and $\alpha\Phi^q = \alpha$ for some $p, q \in \mathbb{N}$, then

$$\overline{u\alpha}\Phi^{pq} = \overline{(u\Phi^{pq})(\alpha\Phi^{pq})} = \overline{u\alpha}$$

as required. $\square$

Lemma 4.4 *Let $(u_n)_n$ be a sequence in A_R^* with $|u_n|$ bounded and let $(v_n)_n, (w_n)_n$ be sequences in A_R^∞ such that*

$$\forall k \in \mathbb{N}\ \exists l \in \mathbb{N}\ \forall n \geq l\ \ r(v_n, w_n) > k.$$

Then

$$\forall k \in \mathbb{N}\ \exists l \in \mathbb{N}\ \forall n \geq l\ \ r(\overline{u_n v_n}, \overline{u_n w_n}) > k.$$

Moreover, $\mathrm{Ad}(\overline{u_n v_n})_n = \mathrm{Ad}(\overline{u_n w_n})_n$.

Proof. Assume that $M = \max\{|\mathrm{u_n}| : \mathrm{n} \in \mathbb{N}\}$. Let $k \in \mathbb{N}$. Then there exists $l \in \mathbb{N}$ such that $r(v_n, w_n) > k + M(t_R - 1)$ for every $n \geq l$. It follows from Lemma 2.3 that $r(\overline{u_n v_n}, \overline{u_n w_n}) > k$ as required. Hence

$$\forall \varepsilon > 0\ \exists l \in \mathbb{N}\ \forall n \geq l\ \ d(\overline{u_n v_n}, \overline{u_n w_n}) < \varepsilon.$$

If $\alpha \in \mathrm{Ad}(\overline{\mathrm{u_n v_n}})_\mathrm{n}$, then $\alpha = \lim_{n\to\infty} \overline{u_{i_n} v_{i_n}}$ for some increasing sequence $(i_n)_n$ in $\mathbb{N}$. Thus

$$\forall \varepsilon > 0\ \exists q \in \mathbb{N}\ \forall n \geq q\ \ d(\overline{u_{i_n} v_{i_n}}, \alpha) < \varepsilon.$$

Since d is an ultra-metric, it follows that

$$\forall \varepsilon > 0\ \exists h \in \mathbb{N}\ \forall n \geq h\ \ d(\overline{u_{i_n} w_{i_n}}, \alpha) < \varepsilon$$

and so $\alpha \in \mathrm{Ad}(\overline{\mathrm{u_n w_n}})_\mathrm{n}$. Thus $\mathrm{Ad}(\overline{\mathrm{u_n v_n}})_\mathrm{n} \subseteq \mathrm{Ad}(\overline{\mathrm{u_n w_n}})_\mathrm{n}$ and the lemma follows by symmetry. $\square$

Next we fix

$$p = \min\{n \geq 1 \mid \forall a \in A_0\ a\varphi^{2p} = a\varphi^p\}. \tag{3}$$

Since A_0 is finite and $\{a\varphi^n \mid n \geq 1\}$ is finite for every $a \in A_0$, p is well defined. If $u = a_1 \dots a_n \in \overline{A_0^*}$ with $a_i \in A_0$, it follows that

$$u\varphi^{2p} = \overline{(a_1\varphi^{2p}) \dots (a_n\varphi^{2p})} = \overline{(a_1\varphi^p) \dots (a_n\varphi^p)} = u\varphi^p. \tag{4}$$

As a consequence, we obtain:

Lemma 4.5 *$\overline{A_0^*}\varphi^p \subseteq \mathrm{Per}(\varphi)$.*

We note that if $A_1 = \emptyset$, we can always identify all Φ-periodic points:

Proposition 4.6 *Let φ be nontrivial and uniformly continuous. If $A = A_0$, then:*

(i) $\mathrm{Per}(\Phi) = \mathrm{A_R^*}\varphi^\mathrm{p} \cup \mathrm{A_R^\omega}\Phi^\mathrm{p}$*;*

(ii) if A_R^* is infinite, there exist Φ-periodic words in both A_R^+ and A_R^ω.

Proof. (i) By Lemma 4.5, we have $A_R^*\varphi^p \subseteq \mathrm{Per}(\varphi)$. Conversely, let $u \in \mathrm{Per}(\varphi)$. Then $u = u\varphi^n$ for some $n \geq 1$ and so $u = u\varphi^{np}$. On the other hand, since $a\varphi^{2p} = a\varphi^p$ for every $a \in A$, we get $u\varphi^{2p} = u\varphi^p$. Thus $u = u\varphi^{np} = u\varphi^p$ and so $u \in A_R^*\varphi^p$. Therefore $\mathrm{Per}(\varphi) = \mathrm{A_R^*}\varphi^\mathrm{p}$.

Take $\alpha \in A_R^\omega$ and let $\beta = \alpha\Phi^p$. Since Φ is proper, then $\beta \in A_R^\omega$. Moreover,

$$\begin{aligned}\beta\Phi^p &= \alpha\Phi^{2p} = (\lim_{n\to\infty} \alpha^{[n]})\Phi^{2p} = \lim_{n\to\infty} \alpha^{[n]}\varphi^{2p} \\ &= \lim_{n\to\infty} \alpha^{[n]}\varphi^p = (\lim_{n\to\infty} \alpha^{[n]})\Phi^p = \alpha\Phi^p = \beta\end{aligned}$$

and so $A_R^\omega\Phi^p \subseteq \mathrm{Per(Phi)}$. The inclusion $\mathrm{Per(Phi)} \cap \mathrm{A_R^\omega} \subseteq \mathrm{A_R^\omega\Phi^p}$ is proved similarly to the finite case.

(ii) Assume that A_R^* is infinite. By Theorem 2.6, $w\varphi^{-1}$ is finite for every $w \in A_R^*$. Iteration of this argument shows that $1(\varphi^p)^{-1}$ must be finite and thus a proper subset of A_R^*. Therefore $u\varphi^p \neq 1$ for some $u \in A_R^*$ and so A_R^+ contains φ-periodic words, infinitely many in fact.

On te other hand, A_R^* infinite implies $A_R^\omega \neq \emptyset$. Since Φ is proper, it follows that $\mathrm{Per(Phi)} \cap \mathrm{A_R^\omega} = \mathrm{A_R^\omega\Phi^p}$ is nonempty as well. $\square$

Given $u \in A_R^* \setminus A_0^*$, let $u\theta$ denote the (unique) prefix of u in $A_0^* A_1$. We define

$$A_2 = \{a \in A_1 : \exists m \in \mathbb{N}\ \forall n \in \mathbb{N}\ \ |a\varphi^n\theta| \leq m\}.$$

Lemma 4.7 *If $\overline{A_0^*}$ is finite, then $A_2 = A_1$.*

Proof. For every $a \in A_1$, we have $a\varphi^n\theta \in \overline{A_0^*}A_1$ and therefore $|a\varphi^n\theta|$ is bounded if $\overline{A_0^*}$ is finite. Hence $A_2 = A_1$. □

5 Prefix-convergent endomorphisms

We fix an endomorphism φ of A_R^* throughout the section and adopt all the notation introduced in Section 4.

We say that the endomorphism φ is *prefix-convergent* if:

$$\forall a \in A_1\ \forall k \in \mathbb{N}\ \exists m \in \mathbb{N}\ \forall v \in A_R^*\ \forall n \geq m\ (a \leq v \Rightarrow r(a\varphi^n, v\varphi^n) > k). \tag{5}$$

The concept expresses the fact that, for every $a \in A_1$, the sequences $(d(a\varphi^n, v\varphi^n))_n$ converge uniformly to 0 for all v having a as prefix.

Lemma 5.1 *If φ is prefix-convergent, then:*

(i) $\mathrm{Fin}(\varphi) = \overline{A_0^*}$;

(ii) $\overline{A_0^*}\varphi \subseteq \overline{A_0^*}$;

(iii) $\forall u \in \overline{A_0^*}\ \forall v \in A_R^* \setminus \overline{A_0^*}\quad \overline{uv} \notin \overline{A_0^*}$.

Proof. (i) By a previous remark, we only have to show that $\mathrm{Fin}(\varphi) \subseteq \overline{A_0^*}$. Let $u \in \mathrm{Fin}(\varphi)$ and suppose that $u \notin \overline{A_0^*}$. Then we may write $u = vaw$ with $v \in \overline{A_0^*}$, $a \in A_1$ and $w \in A_R^*$. Applying (5) to a and aw, we get

$$\forall k \in \mathbb{N}\ \exists m \in \mathbb{N}\ \forall n \geq m\ \ r(a\varphi^n, (aw)\varphi^n) > k.$$

Thus, as $a \in \mathrm{Inf}(\varphi)$, we also have $aw \in \mathrm{Inf}(\varphi)$. Since $v \in \mathrm{Fin}(\varphi)$, we obtain $u = vaw \in \mathrm{Inf}(\varphi)$ by Lemma 4.1, a contradiction. Therefore $u \in \overline{A_0^*}$ as required.

(ii) follows from (i) and $\overline{A_0^*}\varphi \subseteq \mathrm{Fin}(\varphi)$.

(iii) follows from (i) and Lemma 4.1. □

Lemma 5.2 *Let φ be prefix-convergent uniformly continuous and let $\alpha = ua\beta \in A_R^\infty$ with $u \in A_0^*$ and $a \in A_1$. Then*

$$\forall k \in \mathbb{N}\ \exists l \in \mathbb{N}\ \forall n \geq l\ \ r((ua)\varphi^n, \alpha\Phi^n) > k.$$

Moreover, $\mathrm{Ad}((ua)\varphi^n)_n = \mathrm{Ad}(\alpha\Phi^n)_n$.

Proof. Let $k \in \mathbb{N}$. Since $a \in A_1$ and φ is prefix-convergent, there exists some $l \in \mathbb{N}$ such that:

$$\forall n \geq l \, \forall s \in \mathbb{N} \;\; r(a\varphi^n, (a\beta^{[s]})\varphi^n) > k+1.$$

Let $n \geq l$. Since Φ^n is continuous, $\lim_{s\to\infty} a\beta^{[s]} = a\beta$ yields $\lim_{s\to\infty}(a\beta^{[s]})\varphi^n = (a\beta)\Phi^n$. Thus $d(a\varphi^n, (a\beta^{[s]})\varphi^n) < 2^{-k-1}$ for every $s \in \mathbb{N}$ yields $d(a\varphi^n, (a\beta)\Phi^n) \leq 2^{-k-1} < 2^{-k}$ and so $r(a\varphi^n, (a\beta)\Phi^n) > k$. Since $|u\varphi^n|$ is bounded, Lemma 4.4 yields

$$\forall k \in \mathbb{N} \, \exists l \in \mathbb{N} \, \forall n \geq l \;\; r((ua)\varphi^n, \alpha\Phi^n) > k$$

and $\mathrm{Ad}((\mathrm{ua})\varphi^{\mathrm{n}})_{\mathrm{n}} = \mathrm{Ad}(\alpha\Phi^{\mathrm{n}})_{\mathrm{n}}$. □

We discuss next the periodic points of φ, recalling the definition of p in (3).

Lemma 5.3 *If φ is prefix-convergent, then* $\mathrm{Per}(\varphi) = \overline{\mathrm{A}_0^*}\varphi^{\mathrm{p}}$.

Proof. Assume that $u \in \mathrm{Per}(\varphi)$. Then $u = u\varphi^q$ for some $q \geq 1$. Let $v = u\varphi^{p(q-1)}$. Then $u = u\varphi^{pq} = v\varphi^p$. Suppose that $v = v'av''$ with $v' \in \overline{A_0^*}$ and $a \in A_1$. Since φ is prefix-convergent,

$$\forall k > 0 \, \exists m \in \mathbb{N} \, \forall n \geq m \;\; r(a\varphi^n, (av'')\varphi^n) > k.$$

Since $v, v' \in \mathrm{Fin}(\varphi)$, we have $av'' \in \mathrm{Fin}(\varphi)$ as well by Lemma 4.1. Thus

$$\exists m \in \mathbb{N} \, \forall n \geq m \;\; a\varphi^n = (av'')\varphi^n$$

and so $a \in \mathrm{Fin}(\varphi)$, contradicting $a \in A_1$. Therefore $v \in \overline{A_0^*}$ and so $Per(\varphi) \subseteq \overline{A_0^*}\varphi^p$.

The opposite inclusion follows from Lemma 4.5. □

We can now determine the singular Φ-periodic points.

Theorem 5.4 *If φ is prefix-convergent uniformly continuous, then*

$$\mathrm{Per_s}(\Phi) = \mathrm{Per}(\Phi) \cap \mathrm{A}_0^\infty = (\mathrm{A}_\mathrm{R}^\infty \cap \mathrm{A}_0^\infty)\Phi^\mathrm{p}.$$

Moreover, every $\alpha \in \mathrm{Per_s}(\Phi)$ has period $\leq p$.

Proof. Assume that $\alpha \in \mathrm{Per_s}(\Phi)$. Then

$$\forall k > 0 \, \exists m \geq k \;\; \alpha^{[m]} \in \mathrm{Per}(\varphi).$$

It follows that $\alpha^{[m]} \in \overline{A_0^*}\varphi^p \subseteq \overline{A_0^*}$ by Lemmas 5.1(ii) and 5.3. Hence $\alpha \in \mathrm{Per}(\Phi) \cap \mathrm{A}_0^\infty$.

Assume next that $\alpha \in \mathrm{Per}(\Phi) \cap \mathrm{A}_0^\infty$. Then $\alpha = \alpha\Phi^q$ for some $q \geq 1$, and so $\alpha = \alpha\Phi^{pq}$. Write $\beta = \alpha\Phi^{p(q-1)}$. Since

$$\beta = \alpha\Phi^{p(q-1)} = (\lim_{n\to\infty} \alpha^{[n]})\Phi^{p(q-1)} = \lim_{n\to\infty} \alpha^{[n]}\varphi^{p(q-1)}$$

by continuity of Φ and $\alpha^{[n]}\varphi^{p(q-1)} \in \overline{A_0^*}$ by Lemma 5.1(ii), we obtain $\beta \in A_0^\infty$ and so $\alpha \in (A_R^\infty \cap A_0^\infty)\Phi^p$.

Finally, assume that $\alpha = \beta\Phi^p$ for some $\beta \in A_R^\infty \cap A_0^\infty$. We have

$$\begin{aligned} \alpha\Phi^p = \beta\Phi^{2p} &= (\lim_{n\to\infty} \beta^{[n]})\Phi^{2p} = \lim_{n\to\infty} \beta^{[n]}\varphi^{2p} \\ &= \lim_{n\to\infty} \beta^{[n]}\varphi^p = (\lim_{n\to\infty} \beta^{[n]})\Phi^p = \beta\Phi^p = \alpha \end{aligned}$$

by continuity of Φ and (4), hence $\alpha \in \mathrm{Per}(\varphi)$. By Lemma 5.3, $\alpha = \lim_{n\to\infty} \beta^{[n]}\varphi^p$ is singular.

Since $\alpha\Phi^p = \alpha$, the lemma is proved. □

Next we determine the regular Φ-periodic points.

Theorem 5.5 *If φ is prefix-convergent uniformly continuous, then*

$$\mathrm{Per_r}(\Phi) = \bigcup_{\mathrm{a} \in \mathrm{A}_2} \overline{(\mathrm{A}_0^* \varphi^{\mathrm{p}})\mathrm{Ad}(\mathrm{a}\varphi^{\mathrm{n}})_{\mathrm{n}}} \tag{6}$$

Moreover, every $\alpha \in \mathrm{Per_r}(\Phi)$ is an attractor and there exists some $M \in \mathbb{N}$ such that any $\alpha \in \mathrm{Per}(\Phi)$ has period $\leq M$.

Proof. Let $a \in A_2$. For every $n \geq 1$, write $a\varphi^n = u_n a_n v_n$ with $u_n \in A_0^*$ and $a_n \in A_1$. Since $a \in A_2$, we have $u_r a_r = u_{r+q} a_{r+q}$ for some $q, r \geq 1$. We show that $(a\varphi^{r+nq})_n$ converges. Indeed, since φ is prefix-convergent, we get

$$\forall k > 0 \; \exists m \in \mathbb{N} \; \forall n \geq m \quad r(a_r\varphi^{nq}, (a_r v_r)\varphi^{nq}), \, r(a_r\varphi^{nq}, (a_r v_{r+q})\varphi^{nq}) > k.$$

Thus

$$\forall k > 0 \; \exists m \in \mathbb{N} \; \forall n \geq m \quad r((a_r v_r)\varphi^{nq}), (a_r v_{r+q})\varphi^{nq}) > k.$$

Since $u_r \in \mathrm{Fin}(\varphi)$ and k is arbitrary, we get

$$\forall k > 0 \; \exists m \in \mathbb{N} \; \forall n \geq m \quad r(a\varphi^{r+nq}, a\varphi^{r+q+nq}) = r((u_r a_r v_r)\varphi^{nq}), (u_r a_r v_{r+q})\varphi^{nq}) > k$$

and so

$$\forall k > 0 \; \exists m \in \mathbb{N} \; \forall n, n' > m \;\; r(a\varphi^{r+nq}, a\varphi^{r+n'q}) > k.$$

Thus $(a\varphi^{r+nq})_n$ is a Cauchy sequence and therefore converges since A_R^∞ is compact. Applying Lemma 4.2 with $u = a\varphi^r$, we get $\mathrm{Ad}(\mathrm{a}\varphi^{\mathrm{r+n}})_{\mathrm{n}} = \mathrm{Ad}(\mathrm{a}\varphi^{\mathrm{n}})_{\mathrm{n}} \subseteq \mathrm{Per}(\Phi)$. Then Lemmas 4.3 and 5.3 yield

$$\cup_{a \in A_2} \overline{(A_0^* \varphi^p)\mathrm{Ad}(\mathrm{a}\varphi^{\mathrm{n}})_{\mathrm{n}}} \subseteq \mathrm{Per}(\Phi).$$

Since $a\varphi^n = u_n a_n v_n$ and $|u_n|$ is bounded, we get $\mathrm{Ad}(\mathrm{a}\varphi^{\mathrm{n}})_{\mathrm{n}} \cap \mathrm{A}_0^\infty = \emptyset$ and so

$$(\cup_{a \in A_2} \overline{(A_0^* \varphi^p)\mathrm{Ad}(\mathrm{a}\varphi^{\mathrm{n}})_{\mathrm{n}}}) \cap A_0^\infty = \emptyset$$

by Lemma 5.1(iii). In view of Theorem 5.4, we obtain

$$\cup_{a \in A_2} \overline{(A_0^* \varphi^p)\mathrm{Ad}(\mathrm{a}\varphi^{\mathrm{n}})_{\mathrm{n}}} \subseteq \mathrm{Per_r}(\Phi).$$

Conversely, let $\alpha \in \mathrm{Per_r}(\Phi)$ satisfy $\alpha = \alpha\Phi^q$. By Theorem 5.4, we may write $\alpha = ua\beta$ with $u \in A_0^*$ and $a \in A_1$. Then Lemma 5.2 yields $\mathrm{Ad}((\mathrm{ua})\varphi^{\mathrm{n}})_{\mathrm{n}} = \mathrm{Ad}(\alpha\Phi^{\mathrm{n}})_{\mathrm{n}}$, which is finite since $\alpha \in \mathrm{Per}(\Phi)$. Write $\alpha\Phi^n = u_n a_n \beta_n$ with $u_n \in A_0^*$ and $a_n \in A_1$. Let $k = \max\{|\mathrm{u_n}|;\ \mathrm{n} \in \mathbb{N}\}$. Since in a compact space any infinite sequence has an adherence value and any convergent subsequence of $((ua)\varphi^n)_n$ must converge to some $\alpha\Phi^i$ ($i \in \{0, \ldots, q-1\}$), it follows that

$$\exists l \in \mathbb{N} \; \forall n \geq l \; \exists i \in \{0, \ldots, q-1\} \quad r((ua)\varphi^n, u_i a_i \beta_i) > k.$$

Then $|(ua)\varphi^n\theta| = |u_i a_i| \leq k+1$. As $|u\varphi^n|$ is bounded, it follows from Lemma 2.3 that $|a\varphi^n\theta|$ is bounded, hence $a \in A_2$. By the first part of the proof, it follows that some subsequence $(a\varphi^{nr})_n$ converges and so does its subsequence $(a\varphi^{npqr})_n$. Thus

$$\alpha = \lim_{n\to\infty} \overline{(u\varphi^p)(a\varphi^{npqr})} = \overline{(u\varphi^p) \lim_{n\to\infty} a\varphi^{nr}} \in \overline{(\overline{A_0^*}\varphi^p)\mathrm{Ad}(\mathrm{a}\varphi^{\mathrm{n}})_{\mathrm{n}}}$$

and so (6) holds.

Let $\alpha \in \mathrm{Per_r}(\Phi)$. We show that α is an attractor. As we have already proved, we may write $\alpha = ua\beta$ with $u \in A_0^*$ and $a \in A_2$. It is enough to show that $\alpha' \in \mathrm{Att}(\alpha)$ whenever $r(\alpha, \alpha') > |u| + 1$.

Indeed, if $r(\alpha, \alpha') > |u|+1$, then we may write $\alpha' = ua\beta'$ for some $\beta' \in A_R^\infty$. By Lemma 5.2, we have $\mathrm{Ad}(\alpha\Phi^n)_n = \mathrm{Ad}((ua)\varphi^n)_n = \mathrm{Ad}(\alpha'\Phi^n)_n$ and so $\alpha \in \mathrm{Ad}(\alpha'\Phi^n)_n$ as required.

Since $a\varphi^{2p} = a\varphi^p$ for every $a \in A_0$, to show that the period of $\alpha \in \mathrm{Per}(\Phi)$ is bounded, it is enough to prove it for $\alpha \in \mathrm{Ad}(a\varphi^n)_n$. By Lemma 4.2 and by the early part of the proof, the period of α is bounded by $|\{a\varphi^n\theta;\ n \geq 1\}|$. $\square$

Corollary 5.6 *Let φ be prefix-convergent uniformly continuous. If $\overline{A_0^*}$ is finite and $A_1 \neq \emptyset$, then $\mathrm{Per}(\Phi) \cap \mathrm{A_R^\omega}$ is a finite nonempty set of (regular) attractors.*

Proof. Since $\overline{A_0^*}$ is finite, it follows from Theorem 5.4 that $\mathrm{Per}(\Phi) \cap \mathrm{A_R^\omega} = \mathrm{Per_r}(\Phi)$. By Lemma 4.7, we have $A_1 = A_2$ and so

$$\mathrm{Per_r}(\Phi) = \bigcup_{\mathrm{a} \in \mathrm{A_1}} \overline{(\overline{\mathrm{A_0^*}}\varphi^{\mathrm{p}})\mathrm{Ad}(\mathrm{a}\varphi^{\mathrm{n}})_{\mathrm{n}}}$$

by Theorem 5.5. As we saw in the proof of Theorem 5.5, we may apply Lemma 4.2 to conclude that $\mathrm{Ad}(\mathrm{a}\varphi^{\mathrm{n}})_{\mathrm{n}}$ is finite and nonempty. Thus $\mathrm{Per}(\Phi) \cap \mathrm{A_R^\omega} = \mathrm{Per_r}(\Phi)$ is finite and nonempty since $A_1 \neq \emptyset$. All its elements are attractors by Theorem 5.5. $\square$

The existence of fixed points follows from the following condition:

Theorem 5.7 *Let φ be prefix-convergent uniformly continuous. Let $u \in A_R^+ \setminus \overline{A_0^*}$ and $w \in A_R^+$ be such that $u\varphi = uw$. Then $\lim_{n\to\infty} u\varphi^n \in Fix(\Phi)$.*

Proof. Write $u = u'au''$ with $u' \in \overline{A_0^*}$ and $a \in A_1$. Since φ is prefix-convergent and $uw = u'au''w$ is irreducible, we have

$$\forall k \in \mathbb{N}\ \exists m \in \mathbb{N}\ \forall n \geq m\ (r(a\varphi^n, (au'')\varphi^n) > k \ \ \wedge\ \ r(a\varphi^n, (au''w)\varphi^n) > k).$$

Since $|u'\varphi^n|$ is bounded and k is arbitrary, we get

$$\forall k \in \mathbb{N}\ \exists m \in \mathbb{N}\ \forall n \geq m\ (r((u'a)\varphi^n, (u'au'')\varphi^n) > k \ \ \wedge\ \ r((u'a)\varphi^n, (u'au''w)\varphi^n) > k),$$

that is,

$$\forall k \in \mathbb{N}\ \exists m \in \mathbb{N}\ \forall n \geq m\ (r((u'a)\varphi^n, u\varphi^n) > k \ \ \wedge\ \ r((u'a)\varphi^n, u\varphi^{n+1}) > k).$$

Hence

$$\forall k \in \mathbb{N}\ \exists m \in \mathbb{N}\ \forall n \geq m\ \ r(u\varphi^n, u\varphi^{n+1}) > k$$

and so

$$\forall k \in \mathbb{N}\ \exists m \in \mathbb{N}\ \forall n, n' \geq m\ \ r(u\varphi^n, u\varphi^{n'}) > k.$$

Thus $(u\varphi^n)_n$ is a Cauchy sequence and therefore converges to some $\alpha \in A_R^\omega$ by compactness of A_R^∞. By Lemma 4.2, $\alpha \in \mathrm{Fix}(\Phi)$. $\square$

We proceed now to discuss prefix-convergency through examples and particular cases.

We say that an endomorphism φ of A_R^* *preserves prefixes* if, for every prefix v of $u \in A_R^*$, $v\varphi$ is still a prefix of $u\varphi$.

Lemma 5.8 *If φ preserves prefixes, it is prefix-convergent.*

Proof. Let $a \in A_1$ and $k \in \mathbb{N}$. Then there exists some $m \in \mathbb{N}$ such that $|a\varphi^n| \geq k$ for every $n \geq m$. Since φ preserves prefixes, so does φ^n. Thus, whenever $n \geq m$,

$$a \leq v \Rightarrow a\varphi^n \leq v\varphi^n \Rightarrow r(a\varphi^n, v\varphi^n) = |a\varphi^n| + 1 > k.$$

Therefore φ is prefix-convergent. $\square$

We show next that preserving prefixes is a decidable property.

Lemma 5.9 *For every $v \in A_R^*$, the language $L = \{u \in A_R^* \mid u \leq \overline{uv}\}$ is rational and effectively constructible.*

Proof. Let $k = (t_R - 1)|v|$, $L' = \{u \in L : |u| < k\}$ and $L'' = \{u \in L : |u| = k\}$. We show that

$$L = L' \cup ((A_R^* L'') \cap A_R^*). \tag{7}$$

Let $w \in A_R^*$ and $u \in L''$ with wu irreducible. Then $u \leq \overline{uv}$ and so $wu \leq w\overline{uv}$. By the dual of Lemma 2.2, we get $wu \leq w\overline{uv} = \overline{wuv}$ and so $wu \in L$. Thus $L' \cup ((A_R^* L'') \cap A_R^*) \subseteq L$.

Conversely, let $u \in L$. We may assume that $|u| \geq k$ and write $u = wz$ with $|z| = k$. Since $wz = u \leq \overline{uv} = \overline{wzv}$ and $\overline{wzv} = w\overline{zv}$ by the dual of Lemma 2.2, it follows that $z \leq \overline{zv}$ and so $z \in L''$. Therefore $u \in (A_R^* L'') \cap A_R^*$ and (7) holds. It follows that L is rational.

Since L' and L'' can be effectively computed, L is effectively constructible. $\square$

Proposition 5.10 *It is decidable whether or not an endomorphism φ of A_R^* preserves prefixes.*

Proof. We remark that φ preserves prefixes if and only if

$$\forall u \in A_R^* \, \forall a \in A \, (ua \in A_R^* \Rightarrow u\varphi \leq (ua)\varphi). \tag{8}$$

Indeed, if (8) holds and $ua_1 \ldots a_n \in A_R^*$ ($a_i \in A$), successive application of (8) yields

$$u\varphi \leq (ua_1)\varphi \leq (ua_1a_2)\varphi \leq \ldots \leq (ua_1 \ldots a_n)\varphi.$$

For every $a \in A$, let

$$L_a = \{u \in A_R^* \mid u \leq \overline{u(a\varphi)}\},$$
$$K_a = \{u \in A_R^* \mid ua \in A_R^*\}.$$

Then φ preserves prefixes if and only if

$$\forall u \in A_R^* \, \forall a \in A \, (u \in K_a \Rightarrow u\varphi \in L_a),$$

or equivalently,

$$\forall a \in A \; K_a\varphi \subseteq L_a. \tag{9}$$

Now $K_a = (A_R^* \cap (A^*a))a^{-1}$ is rational and effectively constructible by the standard closure properties of rational languages and so is $K_a\varphi$ by Lemma 2.4. Since L_a is rational and effectively constructible by Lemma 5.9, it follows that (9) is decidable as required. $\square$

We shall present now a number of examples. Most of them involve a free group on 2 generators, so the following lemma will come handy:

Lemma 5.11 *Let A_R^* be the free group on the alphabet $A = \{a, b, a^{-1}, b^{-1}\}$ and let φ be an endomorphism of A_R^*. Then φ is uniformly continuous if and only if $(ab)\varphi \neq (ba)\varphi$.*

Proof. By Theorem 2.6, φ is uniformly continuous if and only if $w\varphi^{-1}$ is finite for every $w \in A_R^*$. Since A_R^* is a group, this is equivalent to $1\varphi^{-1}$ being finite. Since the unique finite subgroup of a free group is the trivial subgroup, this is equivalent to φ being injective.

Clearly, if φ is injective then $(ab)\varphi \neq (ba)\varphi$. Conversely, assume that $(ab)\varphi \neq (ba)\varphi$. By the Nielsen-Schreier Theorem [15, Section I.2], $A_R^*\varphi$ is a free group that must therefore have rank 2 since it is nonabelian and so is the quotient group $A_R^*/1\varphi^{-1}$. Since a finitely generated free group cannot be isomorphic to a proper quotient [15, Proposition I.3.5], it follows that φ is injective. $\square$

In the examples to follow, when we say that A_R^* be the free group on B, we assume that $A = B \cup B^{-1}$.

Example 5.12 *Let A_R^* be the free group on $\{a, b\}$ and let φ be the endomorphism of A_R defined by $a\varphi = ab$, $b\varphi = ba$. Then φ preserves prefixes, is uniformly continuous and* $|\mathrm{Per}(\Phi)| = 5$*. There are no finite nontrivial Φ-periodic points.*

Proof. Let $x, y \in A$. If xy is irreducible, so is $(x\varphi)(y\varphi)$. Hence φ preserves prefixes. By Lemma 5.11, φ is uniformly continuous.

Clearly, the sequences $(a\varphi^n)_n$ and $(b\varphi^n)_n$ converge to some instance of the Thue-Morse infinite word [13, Section 2.2]. Since $a^{-1}\varphi^2 = a^{-1}b^{-2}a^{-1}$ and $b^{-1}\varphi^2 = b^{-1}a^{-2}b^{-1}$, it is a simple exercise to check that

$$\mathrm{Ad}(\mathrm{a}^{-1}\varphi^{\mathrm{n}})_{\mathrm{n}} = \mathrm{Ad}(\mathrm{b}^{-1}\varphi^{\mathrm{n}})_{\mathrm{n}} = \{\mathrm{a}^{-1}\mathrm{b}^{-1}\mathrm{b}^{-1}\mathrm{a}^{-1}\ldots, \mathrm{b}^{-1}\mathrm{a}^{-1}\mathrm{a}^{-1}\mathrm{b}^{-1}\ldots\}$$

consist of two further instances of the Thue-Morse word, hence $A_0 = \emptyset$ and $\mathrm{Per}(\varphi) = \{1\}$ by Lemma 5.3. By Theorem 5.5, the 4 instances of the Thue-Morse word

$$(abba\ldots,\ baab\ldots,\ a^{-1}b^{-1}b^{-1}a^{-1}\ldots,\ b^{-1}a^{-1}a^{-1}b^{-1}\ldots)$$

are the unique infinite Φ-periodic points. $\square$

The next example shows an instance of Corollary 5.6.

Example 5.13 *Let A_R^* be the free group on $\{a, b\}$ and let φ be the endomorphism of A_R^* defined by $a\varphi = a^{-2}ba^2$, $b\varphi = a^{-1}ba$. Then φ is prefix-convergent uniformly continuous and $A_2 = A$. Moreover,* $\mathrm{Per}(\varphi) = \{1\}$ *and* $|\mathrm{Per_r}(\Phi)| = 1$*.*

Proof. By Lemma 5.11, φ is uniformly continuous.

It is easy to see that if $u = x_1 \ldots x_n \in A_R^+$ with $x_1, \ldots, x_n \in A$, then $u\varphi = a^{r_0}b^{s_1}a^{r_1} \ldots b^{s_n}a^{r_n}$ with

$$s_i \in \{-1, 1\}\ (i = 1, \ldots, n), \qquad r_0 \in \{-2, -1\},$$
$$r_i \in \{-1, 0, 1\}\ (i = 1, \ldots, n-1), \qquad r_n \in \{1, 2\}.$$

Let $u' = a^{r_0}b^{s_1}a^{r_1}\dots b^{s_n}$. Then $v\varphi \in u'A^*$ whenever $u \leq v$. Moreover, $|u| < |u'|$. Thus $|u\varphi^n| \geq |u| + n$ for every n and so $A_0 = \emptyset$. By Lemma 4.7, we obtain $A = A_2$. We show that

$$\forall u, v \in A_R^+ \ (u \neq v \ \Rightarrow \ r(u,v) < r(u\varphi, v\varphi)). \tag{10}$$

Indeed, if $u = wu_0$, $v = wv_0$ with $w \neq 1$, then w' is a common prefix of $w\varphi$, $u\varphi$ and $v\varphi$. Since $|w| < |w'|$, (10) holds if $r(u,v) > 1$. Since $r(u\varphi, v\varphi) > 1$ in any case, (10) holds. Thus

$$\forall u, v \in A_R^+ \ \forall n \in \mathbb{N} \ r(u\varphi^n, v\varphi^n) > n \tag{11}$$

and so φ is prefix-convergent.

Since $A_0 = \emptyset$, Lemma 5.3 yields $\mathrm{Per}(\varphi) = \{1\}$. By Theorems 5.4 and 5.5, we get $\mathrm{Per_s}(\Phi) = \emptyset$ and $\mathrm{Per_r}(\Phi) = \cup_{\mathrm{a}\in\mathrm{A}} \mathrm{Ad}(\mathrm{a}\varphi^{\mathrm{n}})_{\mathrm{n}}$. As all sequences converge to the same point by (11), we get $|\mathrm{Per_r}(\Phi)| = 1$. $\square$

Note that in the preceding example φ does not preserve prefixes.

The next example shows an instance of Theorem 5.5 with $\overline{A_0^*}$ infinite.

Example 5.14 *Let A_R^* be the free group on $\{a,b\}$ and let φ be the endomorphism of A_R defined by $a\varphi = aba^{-1}$, $b\varphi = b$. Then φ is prefix-convergent uniformly continuous and $A_1 = A_2 = \{a, a^{-1}\}$.*

Proof. By Lemma 5.11, φ is uniformly continuous.

It is easy to show by induction that, for every $n \geq 1$,

$$a\varphi^n = ab^{\varepsilon_1}a^{-1}b^{\varepsilon_2}ab^{\varepsilon_3}a^{-1}\dots ab^{\varepsilon_{2^n-1}}a^{-1}$$

with $\varepsilon_i = \pm 1$ for $i = 1, \dots, 2^n - 1$. Thus $A_1 = \{a, a^{-1}\}$ since $b\varphi^n = b$ for every $n \in \mathbb{N}$. Since $a\varphi^n\theta = a^{-1}\varphi^n\theta = a$ for every $n \geq 1$, we obtain $A_2 = A_1$.

For every $q \geq 1$, we have

$$a^q\varphi^n = (a\varphi^{n-1})b^q(a^{-1}\varphi^{n-1}) \tag{12}$$

Let $x \in A_2$ and let $k \in \mathbb{N}$. Take $m = k$ and assume that xw is irreducible. If $w \notin (a \cup a^{-1})A_R^*$, then $(xw)\varphi^n = (x\varphi^n)(w\varphi^n)$ and so

$$r(x\varphi^n, (xw)\varphi^n) = |x\varphi^n| + 1 = 2^{n+1} > k$$

whenever $n \geq m$.

Otherwise, $w = x^q w'$ for some $q \geq 1$, $w' \notin (a \cup a^{-1})A_R^*$ and so in view of (12)

$$r(x\varphi^n, (xw)\varphi^n) = r(x\varphi^n, x^{q+1}\varphi^n) = |a\varphi^{n-1}| + 2 = 2^n + 1 > k$$

whenever $n \geq m$. Thus φ is prefix-convergent. $\square$

The next example shows that, as far as fixed points are concerned, we cannot expect reduction to finite fixed points and $\mathrm{Ad}(\mathrm{a}\varphi^{\mathrm{n}})_{\mathrm{n}}$ in the spirit of Theorem 5.5.

Example 5.15 *Let $A = \{a,b\}$ and $R = \{(a^3, 1)\}$. Let $\varphi : A_R^* \to A_R^*$ be the endomorphism defined by $a\varphi = a^2$ and $b\varphi = aba^2b$. Then φ is prefix-convergent and uniformly continuous and $Fix(\Phi) = \{1, (a^2b)^\omega\}$.*

Proof. Clearly, $A_0 = \{a\}$ and $(a^2b)\varphi = (a^2b)^2$. A simple induction shows that

$$\forall n \in \mathbb{N} \quad (b\varphi^{2n} = b(a^2b)^{2^{2n}-1} \;\wedge\; b\varphi^{2n+1} = ab(a^2b)^{2^{2n+1}-1}),$$

hence

$$\mathrm{Ad(b\varphi^n)_n = \{b(a^2b)^\omega, ab(a^2b)^\omega\}}.$$

Since the number of ocurrences of b increases in each iteration of φ, it follows easily that φ is prefix-convergent and uniformly continuous. It is immediate that $\mathrm{Fix}(\varphi) = \{1\}$. Suppose that $\alpha = (a^2b)^k a^l b\beta \in \mathrm{Fix}(\Phi)$ with $l \in \{0,1\}$. Then $\alpha = \alpha\Phi = (a^2b)^{2k}aba^2b(\beta\Phi)$ if $l = 0$ and $\alpha = \alpha\Phi = (a^2b)^{2k}ba^2b(\beta\Phi)$ if $l = 1$, a contradiction in any case. Thus $\mathrm{Fix}(\Phi) = \{1, (a^2b)^\omega\}$. $\square$

A non prefix-convergent endomorphism does not have to produce regular periodic points:

Example 5.16 *Let A_R^* be the free group on $\{a,b\}$ and let φ be the endomorphism of A_R defined by $a\varphi = ab$, $b\varphi = ab^{-1}a^{-1}$. Then φ is not prefix-convergent but is uniformly continuous and $A_0 = \emptyset$. There exist just finitely many infinite Φ-periodic points, but no regular ones.*

Proof. By Lemma 5.11, φ is uniformly continuous.

Let $z = aba$. Since $z\varphi = z$, we have $z^\omega, (z^{-1})^\omega \in \mathrm{Per}(\Phi)$. A simple induction shows that

$$a\varphi^{2n+1} = z^n abz^{-n}, \quad b\varphi^{2n+1} = z^n ab^{-1}a^{-1}z^{-n}, \quad b\varphi^{2n} = z^n bz^{-n} \quad (n \geq 0),$$

$$a\varphi^{2n} = z^n b^{-1}a^{-1}z^{-(n-1)} \quad (n \geq 1),$$

hence $\lim_{n\to\infty} x\varphi^n = z^\omega$ for every $x \in A$. It follows that $A_0 = \emptyset$ and so $A = A_2$ by Lemma 4.7. Since $r(a\varphi^n, z\varphi^n) \leq 4$ for every $n \in \mathbb{N}$, it follows that φ is not prefix-convergent.

We show now that z^ω and $(z^{-1})^\omega$ are the unique infinite Φ-periodic points. Let $\alpha \in \mathrm{Per}(\Phi) \cap \mathrm{A_R^\omega}$. Then $\alpha = \alpha\Phi^p$ for some $p > 0$. Since $x\varphi^{2p} = \overline{z^p x z^{-p}}$ for every $x \in A$, we get

$$\begin{aligned}\alpha = \alpha\Phi^{2p} &= (\lim_{n\to\infty} \alpha^{[n]})\Phi^{2p} = \lim_{n\to\infty}(\alpha^{[n]}\varphi^{2p}) \\ &= \lim_{n\to\infty} \overline{z^p\alpha^{[n]}z^{-p}} = \lim_{n\to\infty} \overline{z^p\alpha^{[n]}} = \overline{z^p\alpha}.\end{aligned}$$

We may write $z^p = uv$, $\alpha = v^{-1}\beta$ with $\overline{z^p\alpha} = u\beta$. Thus $v^{-1}\beta = \alpha = \overline{z^p\alpha} = u\beta$. Since $uv = z^p$, we must have either $u = 1$ or $v = 1$.

If $u = 1$, then $\beta = z^{-p}\beta$ yields $\beta = (z^{-p})^\omega$ and so $\alpha = z^{-p}\beta = (z^{-1})^\omega$. Otherwise $\beta = \alpha$ and so $\alpha = z^p\alpha$ yields $\alpha = z^\omega$.

Since z^ω and $(z^{-1})^\omega$ are both singular, we conclude that there exist no regular Φ-periodic points. $\square$

We end this sequence of examples by considering the famous Fibonacci endomorphism:

Example 5.17 *Let A_R^* be the free group on $\{a,b\}$ and let φ be the endomorphism of A_R defined by $a\varphi = ab$, $b\varphi = a$. Then φ is not prefix-convergent but it is uniformly continuous and $A_0 = \emptyset$.*

Proof. By Lemma 5.11, φ is uniformly continuous.

Clearly, $\lim_{n\to\infty} a\varphi^n = \lim_{n\to\infty} b\varphi^n = \alpha$, where $\alpha = abaab\ldots$ denotes the *Fibonacci (infinite) word* [14, Section 2.1]. It follows that $A_0 = \emptyset$.

We have

$$(a^{-1}ba)\varphi^2 = (b^{-1}ab)\varphi = ba,$$

hence $(a^{-1}ba)\varphi^n \in (a \cup b)A_R^*$ for $n \geq 2$. Since $a^{-1}\varphi^n \in (a^{-1} \cup b^{-1})A_R^*$ for every $n \in \mathbb{N}$, it follows that $r(a^{-1}\varphi^n, (a^{-1}ba)\varphi^n) = 1$ for every $n \geq 2$. Thus φ is not prefix-convergent. $\square$

However, if we consider the Fibonacci endomorphism for $A = \{a, b, a^{-1}, b^{-1}\}$ and $R = \{(aa^{-1}, 1), (bb^{-1}, 1)\}$, it preserves prefixes since xy irreducible implies $(x\varphi)(y\varphi)$ irreducible for all $x, y \in A$. We still have uniform continuity and $A_0 = \emptyset$, therefore Corollary 5.6 applies. It is a simple exercise to check that

$$Per(\Phi) = \{\lim_{n\to\infty} a\varphi^n, \lim_{n\to\infty} a^{-1}\varphi^{2n}, \lim_{n\to\infty} b^{-1}\varphi^{2n}\}.$$

6 Length-increasing endomorphisms

Let φ be an endomorphism of A_R^*. We say that φ is

- *length-increasing* if
$$\forall u \in A_R^+ \ \ |u\varphi| \ > \ |u|;$$
- *eventually length-increasing* if
$$\exists m \in \mathbb{N}\ \forall u \in A_R^+ \ \ (|u| \geq m \Rightarrow |u\varphi| \ > \ |u|); \tag{13}$$
- *expanding* if
$$\forall k \in \mathbb{N}\ \exists m \in \mathbb{N}\ \forall u \in A_R^+ \ (|u| \geq m \Rightarrow |u\varphi| \geq |u| + k). \tag{14}$$

Obviously, if φ is either length-increasing or expanding, then it is eventually length-increasing. Examples 6.13 and 6.14 show that length-increasing and expanding are independent properties.

Lemma 6.1 *If φ is eventually length-increasing, then it is uniformly continuous.*

Proof. Assume that φ is eventually length-increasing. Then $w\varphi^{-1}$ is finite for every $w \in A_R^*$ and so φ is uniformly continuous by Theorem 2.6. $\square$

We fix φ and

$$h = \max\{|\mathrm{a}\varphi|;\ \mathrm{a} \in \mathrm{A}\}.$$

Lemma 6.2 *Let φ be uniformly continuous. Then*

$$\exists M \in \mathbb{N}\ \forall u \in A_R^* \ \ (|u| \geq M \Rightarrow |u\varphi| \geq 2h). \tag{15}$$

Proof. Suppose not. Then

$$\forall n \in \mathbb{N}\ \exists u_n \in A_R^* \ \ (|u_n| \geq n \wedge |u_n\varphi| < 2h).$$

Since there are only finitely many words v of length $< 2h$, it follows that $v\varphi^{-1}$ is infinite for some v, contradicting Theorem 2.6. $\square$

We fix $M \geq \max\{\mathrm{t_R}, 2\mathrm{h}\}$ satisfying (15).

Lemma 6.3 *Let $uvw \in A_R^*$ with $|v| \geq M$. Then there exists a factorization $v\varphi = v_1v_2v_3$ such that $v_2 \neq 1$ and*

$$(uvw)\varphi = \overline{(u\varphi)v_1}\, v_2\, \overline{v_3(w\varphi)}.$$

Proof. Let $v \in A_R^*$ be such that $|v| \geq M$. Suppose the lemma fails for some choice of u and w. We may assume that $|uw|$ is minimal.

Assume that $u, w \neq 1$. Let u_0 denote the first letter of u and write $u = u_0u_1$. Let w_0 denote the last letter of w and write $w = w_1w_0$. By minimality of $|uw|$, we have a factorization $v = v_1v_2v_3$ with $v_2 \neq 1$ and $(u_1vw_1)\varphi = \overline{(u_1\varphi)v_1}\, v_2\, \overline{v_3(w_1\varphi)}$. Write $x_1 = \overline{(u_1\varphi)v_1}$, $y_1 = \overline{v_3(w_1\varphi)}$. We discuss the reduction process in

$$(uvw)\varphi = \overline{(u_0\varphi)x_1v_2y_1(w_0\varphi)}$$

and show that $(uvw)\varphi$ is the product of a proper prefix of $u_0\varphi$ by a proper suffix of $w_0\varphi$. By minimality of $|uw|$, the factor v_2 cannot be fully cancelled in the reduction $\overline{(u_0\varphi)x_1v_2y_1}$. We consider two cases:

Case I: some part of the factor v_2 is cancelled.

Then we may write $\overline{(u_0\varphi)x_1v_2y_1} = u_2v_3y_1$ where $u_2 < u_0\varphi$ and v_3 is a proper suffix of v_2. Note that $v_3 \neq 1$, otherwise w_0 would be superfluous. Now in the reduction of $(uvw)\varphi = \overline{u_2v_3y_1(w_0\varphi)}$ we get a word of the form u_3w_2 with $u_3 \leq u_2$ and w_2 a proper suffix of $w_0\varphi$.

Case II: the factor v_2 remains intact.

Then we may write $\overline{(u_0\varphi)x_1v_2y_1} = u_2x_2v_2y_1$ where $u_2 \leq u_0\varphi$ and x_2 is a suffix of x_1. Now in the reduction of $(uvw)\varphi = \overline{u_2x_2v_2y_1(w_0\varphi)}$ part of u_2 must be cancelled, otherwise u_0 would be superfluous. Hence we get a word of the form u_3w_2 with $u_3 < u_2$ and w_2 a proper suffix of $w_0\varphi$.

In any case, we get $|(uvw)\varphi| < |u_0\varphi| + |w_0\varphi| \leq 2h$ and so $|v| \leq |uvw| < M$ by (15), a contradiction.

The cases $u = 1$ or $w = 1$ are actually a simplification of the case discussed and can therefore be omitted. □

Lemma 6.4 *Let φ be expanding. Then*

$$\exists m \in \mathbb{N}\ \forall u, v \in A_R^+\ (r(u,v) \in\,]m, +\infty[\ \Rightarrow r(u\varphi, v\varphi) > r(u,v)).$$

Proof. Since φ is expanding,

$$\exists m \geq M\ \forall u \in A_R^+\ (|u| \geq m \Rightarrow |u\varphi| \geq |u| + hM(t_R - 1)). \tag{16}$$

Let $u, v \in A_R^+$ be distinct with $r(u,v) > m$. Let w be the longest common prefix of u and v. Write $u = wu'$. We show that there exists a factorization $w\varphi = w_1w_2$ such that

$$u\varphi = w_1(\overline{w_2(u'\varphi)}) \qquad \text{and} \qquad |w_1| > |w|. \tag{17}$$

Since $|w| \geq m \geq M$, it follows from Lemma 6.3 that there exists a factorization $w\varphi = w_1w_2$ such that $w_1 \neq 1$ and $u\varphi = w_1(\overline{w_2(u'\varphi)})$. We assume that w_1 has maximal length.

Suppose that $|w_1| \leq |w|$. By (16), we have

$$|w| + |w_2| \geq |w_1| + |w_2| = |w\varphi| \geq |w| + hM(t_R - 1)$$

and so $|w_2| \geq hM(t_R - 1)$. By maximality of $|w_1|$, w_2 must be fully cancelled in the reduction $\overline{w_2(u'\varphi)}$. Let u'' be the shortest prefix of u' such that w_2 is fully cancelled in the reduction $\overline{w_2(u''\varphi)}$. Since the image of the last letter of u'' must necessarily help to cancel the first letter of w_2, we conclude that $|\overline{w_2(u''\varphi)}| < h$. Now let w_0 be the shortest suffix of w such that w_2 is a suffix of $w_0\varphi$. Since the first letter of w_2 must necessarily originate from the image of the first letter of w_0, we conclude that $|w_0\varphi| < |w_2| + h$. Since w_0u'' is a factor of $wu' = u$, it is reduced. Writing $w_0\varphi = xw_2$, we have $(w_0u'')\varphi = \overline{xw_2(u''\varphi)}$ and so

$$|(w_0u'')\varphi| = |\overline{xw_2(u''\varphi)}| \leq |x| + |\overline{w_2(u''\varphi)}| < 2h.$$

By (15), it follows that $|w_0u''| < M$ and so $|u''| < M$. Hence $|u''\varphi| < hM$. By Lemma 2.3(ii), it follows that $u''\varphi$ cannot cancel a word with length $\geq hM(t_R - 1)$ and so $|w_2| < hM(t_R - 1)$, a contradiction. Thus $|w_1| > |w|$ and (17) holds.

Applying the same argument to $v\varphi$, we conclude that $u\varphi$ and $v\varphi$ have a common prefix of length $|w| + 1$ and so $r(u\varphi, v\varphi) > r(u, v)$ as required. □

Lemma 6.5 *Let φ be expanding. Then $\mathrm{Ad}(\mathrm{u}\varphi^{\mathrm{n}})_{\mathrm{n}}$ is finite and nonempty for every $u \in A_R^*$.*

Proof. By Lemma 6.4,

$$\exists m \in \mathbb{N}\ \forall u, v \in A_R^+\ \forall n \in \mathbb{N}\ (r(u, v) > m \Rightarrow r(u\varphi^n, v\varphi^n) > n). \tag{18}$$

Let $u \in A_R^*$. Since $(|u\varphi^n|)_n$ is increasing, there exist some $r, q \geq 1$ such that $u\varphi^r$ and $u\varphi^{r+q}$ have a common prefix w of length m. Write $u\varphi^r = wv$ and $u\varphi^{r+q} = wv'$. By (18), we have

$$r(w\varphi^n, u\varphi^{r+n}) = r(w\varphi^n, (wv)\varphi^n) > n,$$

$$r(w\varphi^n, u\varphi^{r+q+n}) = r(w\varphi^n, (wv')\varphi^n) > n,$$

hence $r(u\varphi^{r+n}, u\varphi^{r+q+n}) > n$ for every $n \in \mathbb{N}$. It follows that $r(u\varphi^{r+nq}, u\varphi^{r+(n+1)q}) \geq n$ for every $n \geq 1$ and so

$$\forall k \in \mathbb{N}\ \forall n, n' > k\ \ r(u\varphi^{r+nq}, u\varphi^{r+n'q}) > k.$$

Thus $(u\varphi^{r+nq})_n$ is a Cauchy sequence and therefore convergent since A_R^∞ is compact. Thus $\mathrm{Ad}(\mathrm{u}\varphi^{\mathrm{n}})_{\mathrm{n}}$ is nonempty. By Lemma 6.1, we may apply Lemma 4.2 and conclude that $\mathrm{Ad}(\mathrm{u}\varphi^{\mathrm{n}})_{\mathrm{n}}$ is finite. □

We are now ready for the characterization of the periodic points. We denote as usual by Φ the continuous extension of φ to A_R^∞.

Theorem 6.6 *Let φ be expanding. Then $\mathrm{Per}_{\mathrm{s}}(\Phi) = \mathrm{Per}(\varphi)$ and there exists some $m \in \mathbb{N}$ such that*

$$\mathrm{Per}_{\mathrm{r}}(\Phi) = \bigcup_{|\mathrm{u}|=\mathrm{m}} \mathrm{Ad}(\mathrm{u}\varphi^{\mathrm{n}})_{\mathrm{n}} \tag{19}$$

is a finite nonempty set of attractors.

Proof. Since φ is expanding, $\mathrm{Per}(\varphi)$ is finite and so $\mathrm{Per_s}(\Phi) = \mathrm{Per}(\varphi)$.

We take m from (18). By Lemmas 6.1, 4.2 and 6.5, $\cup_{|u|=m}\mathrm{Ad(u\varphi^n)_n}$ is finite nonempty and contained in $\mathrm{Per}(\Phi)$. Since φ is expanding and m in (18) originates from (16), then $\mathrm{Ad(u\varphi^n)_n} \subseteq \mathrm{A^\omega_R}$ and so $\mathrm{Ad(u\varphi^n)_n} \subseteq \mathrm{Per_r}(\Phi)$.

Conversely, let $\alpha \in \mathrm{Per_r}(\Phi)$. We may write $\alpha = u\beta$ for some $u \in A_R^*$ of length m and some $\beta \in A_R^\omega$. For all $n, k \in \mathbb{N}$, we have $r(u\varphi^n, \alpha^{[m+k]}\varphi^n) > n$ by (18) and since Φ^n is continuous we get

$$\alpha\Phi^n = (\lim_{k\to\infty} \alpha^{[m+k]})\Phi^n = \lim_{k\to\infty} \alpha^{[m+k]}\varphi^n,$$

hence $r(u\varphi^n, \alpha\Phi^n) > n$ for every n. Since $\alpha \in \mathrm{Per}(\Phi)$, we have $\alpha = \alpha\Phi^p$ for some $p \geq 1$. Hence $r(u\varphi^{np}, \alpha) > np$ for every n and so $\alpha = \lim_{n\to\infty} u\varphi^{np} \in \mathrm{Ad(u\varphi^n)_n}$. Thus (19) holds.

To show that α is an attractor, we check the inclusion $B_{2^{-m}}(\alpha) \subseteq \mathrm{Att}(\alpha)$. Let $\gamma \in B_{2^{-m}}(\alpha)$. Then $r(\alpha, \gamma) > m$ and so $\gamma = u\beta'$ for some $\beta' \in A_R^\infty$. Since $r(u\varphi^n, \gamma^{[m+k]}\varphi^n) > n$ for all n and k by (18), we get $r(u\varphi^n, \gamma\Phi^n) > n$ similarly to the case of α. In view of $r(u\varphi^{np}, \alpha) > np$, it follows that $r(\gamma\Phi^{np}, \alpha) > n$ for every n and so $\lim_{n\to\infty} \gamma\Phi^{np} = \alpha$. Thus $\alpha \in \mathrm{Ad(\gamma\Phi^n)_n}$ and so $\gamma \in \mathrm{Att}(\alpha)$. Therefore $B_{2^{-m}}(\alpha) \subseteq \mathrm{Att}(\alpha)$ and α is an attractor. □

We address now the decidability question.

Given $u \in A^*$, let $u\xi$ denote the suffix of length M of u if $|u| > M$. Otherwise, let $u\xi = u$.

We define a finite $(A, \mathbf{Z})$-transducer $\mathcal{T}_\varphi = (Q, q_0, T, E)$ as follows:

$Q = \{u \in A_R^* : |u| \leq M\}$ is the set of states;

$q_0 = 1$ is the initial state;

$T = Q \setminus \{1\}$ is the set of terminal states;

$E = \{(u, a, n, v) \in Q \times A \times \mathbf{Z} \times Q : ua \in A_R^*,\ n = |(ua)\varphi| - |u\varphi| - 1,\ v = (ua)\xi\}$ is the set of edges.

The label of a path p in $\mathcal{T}_\varphi$ is denoted by $p\lambda$ and its projections on A^* and $\mathbf{Z}$ by $p\lambda_1$ and $p\lambda_2$, respectively.

For details on automata and transducers, the reader is referred to [2].

Lemma 6.7 *Let $uv, vw \in A_R^*$ with $|v| \geq M$. Then $uvw \in A_R^*$ and*

$$|(uvw)\varphi| - |(uv)\varphi| = |(vw)\varphi| - |v\varphi|.$$

Proof. Since $|v| \geq t_R$, we have $uvw \in A_R^*$. By Lemma 6.3, there exists a factorization $v\varphi = v_1v_2v_3$ such that $v_2 \neq 1$ and

$$(uvw)\varphi = \overline{(u\varphi)v_1}\, v_2\, \overline{v_3(w\varphi)}.$$

Hence

$$\begin{aligned}|(uvw)\varphi| - |(uv)\varphi| &= |\overline{(u\varphi)v_1}| + |v_2| + |\overline{v_3(w\varphi)}| - |\overline{(u\varphi)v_1}| - |v_2v_3| \\ &= |v_1v_2| + |\overline{v_3(w\varphi)}| - |v_1v_2v_3| = |(vw)\varphi| - |v\varphi|\end{aligned}$$

and the lemma holds. □

Let

$$m_\varphi = \min\{p\lambda_2;\ p \text{ is a cycle-free successful path in } \mathcal{T}_\varphi\}.$$

Theorem 6.8 *The following conditions are equivalent:*

(i) φ *is eventually length-increasing;*

(ii) $c\lambda_2 \geq 0$ *for every cycle* c *in* $\mathcal{T}_\varphi$ *and* $p\lambda_2 > 0$ *for every successful path* p *with*

$$(\max\{0, 2 - \mathrm{m}_\varphi\})|\mathrm{Q}| \leq |\mathrm{p}| < (1 + \max\{0, 2 - \mathrm{m}_\varphi\})|\mathrm{Q}|.$$

Proof. Suppose that

$$p : q_0 \xrightarrow{(a_1,n_1)} q_1 \xrightarrow{(a_2,n_2)} \dots \xrightarrow{(a_k,n_k)} q_k$$

is a successful path in $\mathcal{T}_\varphi$. For every $i \in \{1, \dots, k\}$, we have $n_i = |(q_{i-1}a_i)\varphi| - |q_{i-1}\varphi| - 1$ and $q_i = (q_{i-1}a_i)\xi$. We show that $a_1 \dots a_i \in A_R^*$ and

$$n_i = |(a_1 \dots a_i)\varphi| - |(a_1 \dots a_{i-1})\varphi| - 1. \tag{20}$$

Assume first that $i \leq M$. Then $q_j = q_{j-1}a_j$ for every $j \leq i$ and the claims follow immediately. Assume now that $i > M$ and the claims hold for $i-1$. Let $u = a_1 \dots a_{i-1-M}$. By the induction hypothesis, we have $a_1 \dots a_{i-1} \in A_R^*$. It is easy to check that $q_j = (q_{j-1}a_j)\xi$ for every $j \leq i$ yields $a_1 \dots a_{i-1} = uq_{i-1}$. Now $i > M$ implies $|q_{i-1}| = M$. Since $uq_{i-1}, q_{i-1}a_i \in A_R^*$, we may apply Lemma 6.7 and obtain $a_1 \dots a_i = uq_{i-1}a_i \in A_R^*$ and also

$$\begin{aligned} n_i &= |(q_{i-1}a_i)\varphi| - |q_{i-1}\varphi| - 1 = |(u_{i-1}q_{i-1}a_i)\varphi| - |(u_{i-1}q_{i-1})\varphi| - 1 \\ &= |(a_1 \dots a_i)\varphi| - |(a_1 \dots a_{i-1})\varphi| - 1. \end{aligned}$$

The induction is therefore complete and so (20) holds.

It follows that

$$p\lambda_2 = \sum_{i=1}^{k} n_i = \sum_{i=1}^{k} (|(a_1 \dots a_i)\varphi| - |(a_1 \dots a_{i-1})\varphi| - 1) = |(a_1 \dots a_k)\varphi| - k$$

and so

$$p\lambda_2 = |p\lambda_1\varphi| - |p\lambda_1|. \tag{21}$$

Notice that

$$\{p\lambda_1;\ p \text{ is a successful path in } \mathcal{T}_\varphi\} = A_R^+. \tag{22}$$

Indeed, we have just proved the direct inclusion and the opposite one follows from the following inductive argument: if $ua \in A_R^*$ and we assume that there is a successful path p with $p\lambda_1 = u$, then we can always extend p by means of some edge of the form (u, a, n, v). Therefore (22) holds.

Assume now that (i) holds and let c be a cycle in $\mathcal{T}_\varphi$. Suppose that $c\lambda_2 < 0$. Since $\mathcal{T}_\varphi$ is trim, we have a successful path of the form pc and so pc^n is also a successful path for every $n \in \mathbb{N}$. Let $u = p\lambda_1$ and $v = c\lambda_1$. Then uv^* is an infinite subset of A_R^+ by (22), and (21) implies

$$|(uv^n)\varphi| - |uv^n| = (pc^n)\lambda_2 = p\lambda_2 + n(c\lambda_2) \leq p\lambda_2 - n$$

for every n, yielding $|(uv^n)\varphi| < |uv^n|$ for infinitely many n, contradicting (i). Thus $c\lambda_2 \geq 0$.

Next we show that

$$p\lambda_2 \geq \min\{1, m_\varphi + \frac{|p|}{|Q|} - 1\} \tag{23}$$

for every successful path p by induction on $|p|$. Let p be a successful path and assume the claim holds for all shorter paths. If p is cycle-free, then $|p| < |Q|$ and $p\lambda_2 \geq m_\varphi + \frac{|p|}{|Q|} - 1$ follows from the definition of m_φ. Hence we may assume that $p = p_1 c p_2$ for some cycle c. Since $c\lambda_2 \geq 0$, we may assume that

$$(p_1 p_2)\lambda_2 < 1, \tag{24}$$

otherwise we are done. Suppose first that $c\lambda_2 > 0$. Then $|c| \leq |Q|$ yields

$$\frac{|p|}{|Q|} = \frac{|p_1 p_2|}{|Q|} + \frac{|c|}{|Q|} \leq \frac{|p_1 p_2|}{|Q|} + 1$$

and by (24) the induction hypothesis yields

$$p\lambda_2 = (p_1 p_2)\lambda_2 + c\lambda_2 \geq (p_1 p_2)\lambda_2 + 1 \geq m_\varphi + \frac{|p_1 p_2|}{|Q|} \geq m_\varphi + \frac{|p|}{|Q|} - 1$$

and (23) holds. Assume now that $c\lambda_2 = 0$. Then $(p_1 p_2)\lambda_2 > 0$, otherwise $p_1 c^* p_2$ is an infinite set of successful paths with $(p_1 c^* p_2)\lambda_2 \subseteq]-\infty, 0]$, contradicting (i). This contradicts (24), hence (23) holds in any case.

Thus, if we take a successful path p with $(\max\{0, 2 - \mathrm{m}_\varphi\})|\mathrm{Q}| \leq |\mathrm{p}| < (1 + \max\{0, 2 - \mathrm{m}_\varphi\})|\mathrm{Q}|$, we get

$$m_\varphi + \frac{|p|}{|Q|} - 1 \geq m_\varphi + 2 - m_\varphi - 1 = 1$$

if $m_\varphi \leq 1$ and

$$m_\varphi + \frac{|p|}{|Q|} - 1 \geq 2 + 0 - 1 = 1$$

if $m_\varphi \geq 2$, hence in any case (23) yields $p\lambda_2 > 0$ and so (ii) holds.

Conversely, assume that (ii) holds. Let $n = (\max\{0, 2 - \mathrm{m}_\varphi\})|\mathrm{Q}|$. Then $p\lambda_2 > 0$ for every successful path p with $n \leq |p| < n + |Q|$. We show that $p\lambda_2 > 0$ for every successful path p with $n + |Q| \leq |p|$ by induction on $|p|$. Assume that $n + |Q| \leq |p|$ and the claim holds for shorter paths of that form. We may factor $p = p_1 c p_2$ for some cycle c. Since $n \leq |p_1 p_2| < |p|$ it follows from (ii) and the induction hypothesis that $(p_1 p_2)\lambda_2 > 0$. Since $c\lambda_2 \geq 0$ by (ii), we obtain $p\lambda_2 = (p_1 p_2)\lambda_2 + c\lambda_2 > 0$.

Therefore $p\lambda_2 > 0$ for every successful path p with $n \leq |p|$ and so (i) holds by (22) and (21). $\square$

It follows from the proof of Theorem 6.8 that

Corollary 6.9 *If φ is eventually length-increasing, then*

$$\forall u \in A_R^+ \;\; (|u| \geq (\max\{0, 2 - \mathrm{m}_\varphi\})|\mathrm{Q}| \Rightarrow |\mathrm{u}| < |\mathrm{u}\varphi|).$$

Theorem 6.10 *The following conditions are equivalent:*

(i) φ *is expanding;*

(ii) $c\lambda_2 > 0$ *for every cycle* c *in* $\mathcal{T}_\varphi$.

Proof. Assume that (i) holds and let c be a cycle in $\mathcal{T}_\varphi$. Suppose that $c\lambda_2 \leq 0$. Since $\mathcal{T}_\varphi$ is trim, we have a successful path of the form pc and so pc^n is also a successful path for every $n \in \mathbb{N}$. Let $u = p\lambda_1$ and $v = c\lambda_1$. Then uv^* is an infinite subset of A_R^+ by (22), and (21) implies

$$|(uv^n)\varphi| - |uv^n| = (pc^n)\lambda_2 = p\lambda_2 + n(c\lambda_2) \leq p\lambda_2$$

for every n, contradicting (i). Thus (ii) holds.

Conversely, assume that (ii) holds. We show that

$$p\lambda_2 \geq m_\varphi + \frac{|p|}{|Q|} - 1 \tag{25}$$

for every successful path p by induction on $|p|$. Let p be a successful path and assume the claim holds for all shorter paths. If p is cycle-free, then $|p| < |Q|$ and (25) follows from the definition of m_φ. Hence we may assume that $p = p_1cp_2$ for some cycle c. Since $c\lambda_2 > 0$ and $|c| \leq |Q|$, we get

$$\frac{|p|}{|Q|} = \frac{|p_1p_2|}{|Q|} + \frac{|c|}{|Q|} \leq \frac{|p_1p_2|}{|Q|} + 1$$

and the induction hypothesis yields

$$\begin{aligned} p\lambda_2 &= (p_1p_2)\lambda_2 + c\lambda_2 \geq (p_1p_2)\lambda_2 + 1 \\ &\geq m_\varphi + \tfrac{|p_1p_2|}{|Q|} \geq m_\varphi + \tfrac{|p|}{|Q|} - 1 \end{aligned}$$

and so (25) holds.

Let $k > 0$. We show that

$$\forall u \in A_R^* \, (|u| \geq (k - m_\varphi + 1)|Q| \Rightarrow |u\varphi| \geq |u| + k). \tag{26}$$

Let $u \in A_R^*$ be such that $|u| \geq (k - m_\varphi + 1)|Q|$. By (22), there exists a successful path p such that $p\lambda_1 = u$. Since $|p| \geq (k - m_\varphi + 1)|Q|$, (21) and (25) yield

$$|u\varphi| - |u| = p\lambda_2 \geq m_\varphi + \frac{|p|}{|Q|} - 1 \geq m_\varphi + k - m_\varphi + 1 - 1 = k$$

and so (26) holds. Therefore φ is expanding. $\square$

Corollary 6.11 *It is decidable whether or not an arbitrary endomorphism* φ *is*

(i) length-increasing;

(ii) eventually length-increasing;

(iii) expanding.

Proof. By Theorem 2.6, we may decide whether or not φ is uniformly continuous. This is a necessary condition for φ to be eventually length-increasing by Lemma 6.1. Thus we may assume that φ is uniformly continuous. As it is proved in [7, Theorem 8.7], we may effectively compute $w\varphi^{-1}$ for any given word w. It follows that M can be effectively computed. Now decidability of (ii) and (iii) follows from Theorems 6.8 and 6.10 since we can construct the transducer $\mathcal{T}$ and then check if conditions 6.8(ii), 6.10(ii) hold. In view of Corollary 6.9, decidability of (i) follows from (ii) since we only need to test finitely many short words. $\square$

We can use previous results to bound the periods:

Corollary 6.12 *If φ is expanding and $\alpha \in \mathrm{Per}(\Phi)$, then the period of α is bounded by $|A|^{\max\{\mathrm{M},(\mathrm{hM}(\mathrm{t_R}-1)-\mathrm{m}_\varphi)|\mathrm{Q}|\}}$.*

Proof. Assume first that $\alpha \in A_R^\omega$. By Theorem 6.6, we have $\alpha \in \mathrm{Ad}(\mathrm{u}\varphi^{\mathrm{n}})_{\mathrm{n}}$ for some $u \in A_R^*$. By the proof of Lemma 6.5 and Lemma 4.2, $\alpha\Phi^p = \alpha$ if $r(u\varphi^r, u\varphi^{r+p}) > m$ where $r \in \mathbb{N}$ and m is given by (18). By the proofs of Lemmas 6.4 and 6.5, m originates from (16). By (26), we can take $m = \max\{\mathrm{M}, (\mathrm{hM}(\mathrm{t_R}-1)-\mathrm{m}_\varphi)|\mathrm{Q}|\}$. Since $(u\varphi^n)^{[m]}$ with length m can take at most $|A|^m$ values, it follows that $\alpha\Phi^p = \alpha$ for some

$$p \leq |A|^m \leq |A|^{\max\{\mathrm{M},(\mathrm{hM}(\mathrm{t_R}-1)-\mathrm{m}_\varphi)|\mathrm{Q}|\}}.$$

Assume now that $\alpha \in A_R^*$. Then $|u\varphi^n| < (2-m_\varphi)|Q|$ for every $n \in \mathbb{N}$ by Corollary 6.9 and so $\alpha\Phi^p = \alpha$ for some

$$p \leq |A|^{(2-m_\varphi)|Q|} \leq |A|^{(hM(t_R-1)-m_\varphi)|Q|} \leq |A|^m$$

as required. $\square$

We end the section by presenting some examples.

Example 6.13 *Let $A = \{a, b, b^{-1}\}$ and $R = \{(bb^{-1}, 1)\}$. Let $\varphi : A_R^* \to A_R^*$ be the matched endomorphism defined by $a\varphi = b^{-1}ab$ and $b\varphi = b^3$. Then φ is length-increasing but not expanding.*

Proof. Given $u = x_1 \dots x_n \in A_R^+$ with $x_1, \dots, x_n \in A$, we have $|(x_1\varphi)\dots(x_n\varphi)| = 3n$. It is easy to check that the maximum number of letters that can be cancelled in the reduction of $(x_1\varphi)\dots(x_n\varphi)$ is $2(n-1)$, hence $|u\varphi| \geq 3n - 2(n-1) = n + 2 = |u| + 2$ and so φ is length-increasing.

Since $a^n\varphi = b^{-1}a^nb$ for every $n \geq 1$, φ is not expanding. $\square$

For the next counterexample, we reuse the endomorphism from Example 5.15.

Example 6.14 *Let $A = \{a, b\}$ and $R = \{(a^3, 1)\}$. Let $\varphi : A_R^* \to A_R^*$ be the endomorphism defined by $a\varphi = a^2$ and $b\varphi = aba^2b$. Then φ is expanding but not length-increasing.*

Proof. As $|a^2\varphi| < |a^2|$, φ is not length-increasing. Every $u \in A_R^*$ can be written as $u = u_1 \dots u_n v$ with $u_i \in \{b, ab, a^2b\}$ and $v \in \{1, a, a^2\}$. Then $u\varphi = u'_1 \dots u'_n v$ with $u'_i \in \{aba^2b, ba^2b, a^2ba^2b\}$ and $v' \in \{1, a, a^2\}$, so $|u\varphi| \geq 2|u| - 3$ and φ is expanding. $\square$

Given $u \in A^*$ and $a \in A$, we denote by $|u|_a$ the number of occurrences of the letter a in u.

Example 6.15 *Let A_R^* be the free group on $\{a, b, c\}$ and let φ be the endomorphism of A_R defined by $a\varphi = ac$, $b\varphi = c^{-1}a^{-1}b^3$ and $c\varphi = ca$. Then φ is expanding and length-increasing but not prefix-convergent.*

Proof. The unique reducible words of the form $(x\varphi)(y\varphi)$ with $x, y \in A$ are $(a\varphi)(b\varphi)$ and $(b^{-1}\varphi)(a^{-1}\varphi)$. Let $u \in A_R^*$. We replace any occurrence of ab (respectively $b^{-1}a^{-1}$) by d (respectively d^{-1}) to get a reduced word u' in the free group F on $B = \{a, b, c, d\}$. Write $u' = x_1 \dots x_n$ with $x_1, \dots, x_n \in B \cup B^{-1}$. We extend φ to an endomorphism $\widehat{\varphi}$ of F by taking $d\widehat{\varphi} = (ab)\varphi = b^3$. It is easy to check that

$$u\varphi = u'\widehat{\varphi} = (x_1 \dots x_n)\varphi = (x_1\varphi) \dots (x_n\varphi)$$

and

$$|u\varphi| = |x_1\varphi| + \dots + |x_n\varphi| \geq 2n + |u'|_d + |u'|_{d^{-1}} = n + |u| \geq \frac{3}{2}|u|.$$

Thus φ is expanding and length-increasing.

Since φ is length-increasing, we have $A_0 = \emptyset$ and so $A_2 = A$ by Lemma 4.7. We have $a \leq ab$ and $a\varphi^n \in aA^*$ for every $n \in \mathbb{N}$. However, $(ab)\varphi = b^3$ and a simple induction shows that $(ab)\varphi^n \in \{a^{-1}, c^{-1}, b\}A^*$ for every $n \geq 1$. Thus $r(a\varphi^n, (ab)\varphi^n) = 1$ for every $n \geq 1$ and so φ is not prefix-convergent. $\square$

Concerning fixed points, we see in the next example they do not necessarily exist, even when $a\varphi = aw$ for some $a \in A$.

Example 6.16 *Let A_R^* be the free group on $\{a, b\}$ and let φ be the endomorphism of A_R defined by $a\varphi = ab$ and $b\varphi = b^{-1}a^{-2}b^3a$. Then φ is expanding and length-increasing but has no nontrivial fixed points.*

Proof. The unique reducible words of the form $(x\varphi)(y\varphi)$ with $x, y \in A$ are $(a\varphi)(b\varphi)$ and $(b^{-1}\varphi)(a^{-1}\varphi)$.

Let $u \in A_R^*$. We replace any occurrence of ab (respectively $b^{-1}a^{-1}$) by c (respectively c^{-1}) to get a reduced word u' in the free group F on $B = \{a, b, c\}$. Write $u' = x_1 \dots x_n$ with $x_1, \dots, x_n \in B \cup B^{-1}$. We extend φ to an endomorphism $\widehat{\varphi}$ of F by taking $c\widehat{\varphi} = (ab)\varphi = a^{-1}b^3a$. It is easy to see that $(x_i\varphi)(x_{i+1}\varphi)$ is reducible if and only if $x_ix_{i+1} \in \{bc, c^{-1}b^{-1}, c^2, c^{-2}\}$ and in that case reduction goes no further than $aa^{-1} = 1$. Since $u\varphi = (x_1\widehat{\varphi}) \dots (x_n\widehat{\varphi})$, we get

$$|u\varphi| \geq 3n - |u'|_a - |u'|_{a^{-1}} = 3(|u'|_b + |u'|_{b^{-1}} + |u'|_c + |u'|_{c^{-1}}) + 2(|u'|_a + |u'|_{a^{-1}})$$

$$\geq \tfrac{3}{2}(|u'|_b + |u'|_{b^{-1}} + |u'|_c + |u'|_{c^{-1}}) + \tfrac{3}{2}(|u'|_a + |u'|_c + |u'|_{a^{-1}} + |u'|_{c^{-1}})$$

$$= \tfrac{3}{2}(|u|_b + |u|_{b^{-1}} + |u|_a + |u|_{a^{-1}}) = \tfrac{3}{2}|u|.$$

Thus φ is expanding and length-increasing.

Suppose that $\alpha \in x_1x_2A_R^\infty$ is a fixed point with $x_1, x_2 \in A$. Let $\alpha' = x_1'x_2'\beta$ be the word on $B \cup B^{-1}$ obtained as before. If $x_1' = a$, then $x_1 = a$ and $x_2 \neq b$ and so $\alpha\Phi \in abA_R^\infty$, a contradiction. If $x_1' = c$, then $x_1x_2 = ab$ and so $\alpha\Phi \in a^{-1}A_R^\infty$, a contradiction. We omit the remaining cases, that confirm that $\mathrm{Fix}(\Phi) = \{1\}$. $\square$

7 The free monoid case

We develop now the particular case of the free monoid, making more explicit existence results such as Konig's Lemma [16]. It is well known that infinite fixed points for free monoid endomorphisms are of the form $u \lim_{n\to\infty} a\varphi^n$, where $u \in \mathrm{Fix}(\varphi)$, $a \in A$, $a\varphi \in aA^+$ and $|a\varphi^n|$ is not bounded [14]. Hence we concentrate our efforts once again in the periodic case. In view of Theorem 2.6, we remark that an endomorphism φ of A^* is uniformly continuous if and only if $w\varphi^{-1}$ is finite for every $w \in A^*$, and this is clearly equivalent to have $a\varphi \neq 1$ for every $a \in A$. Moreover, Theorem 2.6 asserts that this is equivalent to the existence of a (proper) endomorphism extension of φ to A^∞, henceforth denoted by Φ. We keep the notation $A = A_0 \cup A_1$ introduced in Section 5.

By Lemma 5.8, we obtain:

Lemma 7.1 *Let $\varphi : A^* \to A^*$ be a uniformly continuous endomorphism. Then φ is prefix-convergent.*

We further introduce

$$A_3 = \{a \in A : \forall n \in \mathbb{N}\ |a\varphi^n| = 1\} \subseteq A_0.$$

We define also a directed graph $G(\varphi)$ by

$$\begin{aligned} V(G(\varphi)) &= A_1 \cup A_3; \\ E(G(\varphi)) &= \{(a,b) \in A_1 \times A_1 \mid a\varphi \in A_0^* b A^*\} \\ &\quad \cup \{(a,b) \in A_3 \times A_3 \mid a\varphi = b\}. \end{aligned}$$

As usual, a cycle is a closed path

$$a_0 \to a_1 \to \ldots \to a_n = a_0$$

satisfying

$$\forall i,j \in \{0,\ldots,n-1\}\ (a_i = a_j \Rightarrow i = j).$$

Since $G(\varphi)$ is finite, it has only finitely many cycles. Moreover, there exists at least one cycle since, in each $(a,b) \in E(G(\varphi))$, b is uniquely determined by a and so $|E(G(\varphi))| = |V(G(\varphi))|$. We define

$$l_3 = \mathrm{lcm}\{|c|;\ \text{c is an } A_3\text{-cycle in } G(\varphi)\}$$

and

$$L(\varphi) = \max\{\mathrm{lcm}(l_3, |c|);\ \text{c is an } A_1\text{-cycle in } G(\varphi)\}.$$

Clearly, $L(\varphi) \leq |A_1||A_3|!$.

Lemma 7.2 *Let φ be a uniformly continuous endomorphism of A^* and let $a \in A_0$. Then every letter of $a\varphi^{|A_0|-1}$ lies in an A_3-cycle.*

Proof. We use induction on $|A_0|$. If $|A_0| = 1$, then $A_0 = \{a\} = A_3$ since $a\varphi = a$, thus the claim holds. Assume now that $|A_0| > 1$ and the claim holds for smaller values of $|A_0|$.

If $a\varphi^q = uav$ for some $q \in \mathbb{N}$, then $a \in A_0$ yields $u = v = 1$ and so $a \in A_3$ since φ is uniformly continuous. Therefore a lies in an A_3-cycle.

Otherwise, let A_0' be the set of letters occurring in $\{a\varphi^n \mid n \geq 1\}$. We may apply the induction hypothesis to $\varphi|_{A_0'}$ to conclude that every letter of $b\varphi^{|A_0'|-1}$ lies in an A_3-cycle for every b occurring in $a\varphi$. Since $|A_0'| < |A_0|$, this completes the proof. □

Theorem 7.3 *Let φ be a uniformly continuous endomorphism of A^* and let $a_0 \in A_1$. Then*

$$0 < |\mathrm{Ad(a_0\varphi^n)_n}| \leq \mathrm{L}(\varphi).$$

Proof. By Lemma 7.2, for every $a \in A_0$, every letter of $a\varphi^{|A|}$ lies in an A_3-cycle. Thus

$$\forall a \in A_0 \; a\varphi^{|A|+l_3} = a\varphi^{|A|}. \tag{27}$$

In fact, since every letter of $a\varphi^{|A|}$ lies in an A_3-cycle and l_3 is a multiple of the length of any A_3-cycle, (27) holds.

For every $a \in A_1$, we have $a\varphi \in A_0^* A_1 A^*$. Hence, for every $n \in \mathbb{N}$, we may write $a_0\varphi^n = u_n a_n u_n'$ for some $u_n \in A_0^*$, $a_n \in A_1$ and $u_n' \in A^*$. Clearly, there exists some A_1-cycle c in $G(\varphi)$ such that

$$\forall j \geq |A_1| \; a_{j+|c|} = a_j. \tag{28}$$

In fact, $j \geq |A_1|$ implies that a_j lies in some A_1-cycle. Hence (28) holds.

Let $l = \mathrm{lcm(l_3, |c|)}$. We show that:

$$\forall i \in \{0, \ldots, l-1\} \; (a_0\varphi^{|A|+i+ln})_n \text{ converges}. \tag{29}$$

Let $i \in \{0, \ldots, l-1\}$ and $k = |A| + i$. We have $a_0\varphi^k = u_k a_k u_k'$ and (28) yields

$$\begin{aligned} u_{k+l} a_k u_{k+l}' &= u_{k+l} a_{k+l} u_{k+l}' = a_0\varphi^{k+l} = (u_k a_k u_k')\varphi^l \\ &= (u_k\varphi^l)(a_k\varphi^l)(u_k'\varphi^l). \end{aligned}$$

Since $u_{k+l}, u_k\varphi^l \in A_0^*$, it follows that $u_{k+l} = (u_k\varphi^l)v$ and $a_k\varphi^l = v a_k w$ for some $v \in A_0^*$ and $w \in A^*$.

Suppose first that $v \neq 1$. For every $n \in \mathbb{N}$, we have

$$a_0\varphi^{k+ln} = (u_k a_k u_k')\varphi^{ln} = (u_k\varphi^{ln})(v\varphi^{l(n-1)})(v\varphi^{l(n-2)}) \ldots (v\varphi^l) v a_k \ldots$$

Since $u_k, v \in A_0^*$, it follows from (27) that, for $n \geq |A|$, $u_k\varphi^{ln} = u_k\varphi^{l|A|}$ and $v\varphi^{ln} = v\varphi^{l|A|}$. Moreover, if $n > |A| + n_0$,

$$a_0\varphi^{k+ln} = (u_k\varphi^{l|A|})(v\varphi^{l|A|})^{n_0} \ldots$$

Since $v\varphi^{l|A|} \neq 1$, it is immediate that

$$\lim_{n\to\infty} a_0\varphi^{|A|+i+ln} = (u_k\varphi^{l|A|})(v\varphi^{l|A|})^{\omega}.$$

Suppose now that $v = 1$. Since $a_k \in A_1$, we have $w \neq 1$. For every $n \in \mathbb{N}$, we get

$$a_0\varphi^{k+ln} = (u_k a_k u_k')\varphi^{ln} = (u_k\varphi^{ln}) a_k w (w\varphi^l)(w\varphi^{2l}) \ldots (w\varphi^{l(n-1)})(u_k'\varphi^{ln})$$

Since $u_k \in A_0^*$, it follows from (27) that, for $n \geq |A|$, $u_k\varphi^{ln} = u_k\varphi^{l|A|}$. Since $w\varphi^{ln} \neq 1$ for every $n \in \mathbb{N}$, it is immediate that

$$\lim_{n\to\infty} a_0\varphi^{|A|+i+ln} = (u_k\varphi^{l|A|}) a_k w (w\varphi^l)(w\varphi^{2l})(w\varphi^{3l}) \ldots$$

Therefore (29) holds.

It is straightforward that the limits of the subsequences in (29) are the only adherence values of $(a_0\varphi^n)_n$ since any such adherence value must be an adherence value for one of the l subsequences of the partition, and a convergent sequence has its limit as its only adherence value. Therefore

$$0 < |\mathrm{Ad(a_0\varphi^n)_n}| \leq \mathrm{l} \leq \mathrm{L}(\varphi)$$

as required. □

The next example shows that the bound $L(\varphi)$ is in some sense tight.

Example 7.4 *Let $A = \{a_1, \ldots, a_8\}$ and let φ be the (uniformly continuous) endomorphism of A^* defined by:*

$$a_1\varphi = a_2a_5, \quad a_4\varphi = a_2, \quad a_8\varphi = a_5^2, \quad a_i\varphi = a_{i+1} \ (i \in \{2,3,5,6,7\}).$$

Then $a_1 \in A_1$ and $|\mathrm{Ad}(a_1\varphi^n)_n| = \mathrm{L}(\varphi)$.

Proof. It is easy to see that

$$A_1 = \{a_1, a_5, a_6, a_7, a_8\}, \quad A_3 = \{a_2, a_3, a_4\},$$

hence $G(\varphi)$ is the graph described by

$$\begin{tikzcd}
a_1 \arrow[r] & a_5 \arrow[r] & a_6 \arrow[d] & & a_2 \arrow[r] & a_3 \arrow[d] \\
 & a_8 \arrow[u] & a_7 \arrow[l] & & & a_4 \arrow[ul]
\end{tikzcd}$$

hence $L(\varphi) = \mathrm{lcm}(3,4) = 12$. We have

$$a_i\varphi^{12} = \begin{cases} a_i & \text{for } i = 2,3,4 \\ a_i^8 & \text{for } i = 5,6,7,8. \end{cases}$$

A straightforward induction shows that

$$\begin{array}{lll}
a_1\varphi^{12n} = a_4a_8^{8^{n-1}4} & a_1\varphi^{12n+4} = a_2a_8^{8^n} & a_1\varphi^{12n+8} = a_3a_8^{8^n2} \\
a_1\varphi^{12n+1} = a_2a_5^{8^n} & a_1\varphi^{12n+5} = a_3a_5^{8^n2} & a_1\varphi^{12n+9} = a_4a_5^{8^n4} \\
a_1\varphi^{12n+2} = a_3a_6^{8^n} & a_1\varphi^{12n+6} = a_4a_6^{8^n2} & a_1\varphi^{12n+10} = a_2a_6^{8^n4} \\
a_1\varphi^{12n+3} = a_4a_7^{8^n} & a_1\varphi^{12n+7} = a_2a_7^{8^n2} & a_1\varphi^{12n+11} = a_3a_7^{8^n4}.
\end{array}$$

In particular, $a_1 \in A_1$ and

$$\mathrm{Ad}(a_1\varphi^n)_n = \{a_ia_j^\omega \mid i \in \{2,3,4\}, j \in \{5,6,7,8\}\}$$

has 12 elements. □

We can now identify all the Φ-periodic points:

Theorem 7.5 *Let φ be uniformly continuous. Let $B = \{a \in A \mid a\varphi^{l_3} = a\}$. Then*

$$\mathrm{Per}(\Phi) = B^\infty \cup (\bigcup_{a \in A_2} B^*\mathrm{Ad}(a\varphi^n)_n) \tag{30}$$

and each $\alpha \in \mathrm{Per}(\Phi)$ has period $\leq L(\varphi)$. Moreover, if α is regular then it is an attractor.

Proof. Since φ is uniformly continuous, we have $B \subseteq A_3$. Given $u \in \mathrm{Per}(\varphi)$, we must have $u \in A_0^*$ and so $u\varphi^{|A|+l_3} = u\varphi^{|A|}$ by (27). If $u\varphi^n = u$, we get

$$u = u\varphi^{n|A|} = u\varphi^{n|A|+l_3} = u\varphi^{l_3}.$$

Thus $\mathrm{Per}(\varphi) \subseteq \mathrm{B}^*$ since φ is uniformly continuous. The opposite inclusion being obvious, we obtain $\mathrm{Per}(\varphi) = \mathrm{B}^*$. By Lemma 7.1, we may apply Lemma 5.3 and Theorem 5.4 to get $B^* = A_0^*\varphi^p$ and $\mathrm{Per_s}(\Phi) = \mathrm{A_0^\infty \Phi^p}$. Hence $\mathrm{Per_s}(\Phi) = \mathrm{B}^\infty$.

Now Theorem 5.5 yields

$$\mathrm{Per_r}(\Phi) = \bigcup_{\mathrm{a} \in \mathrm{A}_2} \mathrm{B^*Ad(a\varphi^n)_n}$$

and implies that all these regular periodic points are attractors.

Finally, we bound the period. It is immediate that $\alpha\Phi^{l_3} = \alpha$ for every $\alpha \in B^\infty$, hence we only need to show that every $\alpha \in \mathrm{Ad(a\varphi^n)_n}$ ($a \in A_1$) satisfies $\alpha\Phi^l = \alpha$ for some $l \leq L(\varphi)$ with $l_3|l$. By the remark at the end of the proof of Theorem 7.3, $\mathrm{Ad(a\varphi^n)_n}$ consists of the limits of the subsequences in (29), thus

$$\alpha = \lim_{n\to\infty} a\varphi^{k+nl}$$

for some $k \in \mathbb{N}$ and $l \leq L(\varphi)$ with $l_3|l$. Since Φ is a continuous endomorphism of A^∞ extending φ, it follows that

$$\begin{aligned} \alpha\Phi^l &= (\lim\nolimits_{n\to\infty} a\varphi^{k+nl})\Phi^l = \lim\nolimits_{n\to\infty}(a\varphi^{k+nl}\varphi^l) \\ &= \lim\nolimits_{n\to\infty} a\varphi^{k+(n+1)l} = \lim\nolimits_{n\to\infty} a\varphi^{k+nl} = \alpha. \end{aligned}$$

Therefore each $\alpha \in \mathrm{Per}(\Phi)$ has period $\leq L(\varphi)$. □

Write

$$A_4 = \{a \in A_1 :\ a\varphi^{k_a} \in aA^+ \text{ for some } k_a \geq 1\}.$$

Alternatively, A_4 is the set of vertices of $G(\varphi)$ lying in some A_1-cycle.

Corollary 7.6 *Let φ be uniformly continuous and let $B = \{a \in A \mid a\varphi^{L(\varphi)} = a\}$. Then there exists a finite subset F of A^ω with $|F| = |A_4|$ such that*

$$\mathrm{Per}(\Phi) = \mathrm{B}^\infty \cup \mathrm{B^*F} \tag{31}$$

and all elements of B^*F are attractors.

Proof. For every $a \in A_4$, let $\alpha_a = \lim_{n\to\infty} a\varphi^{nk_a}$. Let

$$F = \{\alpha_a; a \in A_4\}.$$

We show that (31) holds.

Assume that $\alpha \in \mathrm{Per}(\Phi)$ and $\alpha\Phi^m = \alpha$ ($m \geq 1$). By Theorem 7.5, we have $\mathrm{Per}(\Phi) = \mathrm{B}^\infty \cup \mathrm{B^*F_1}$ with $F_1 = \cup_{a\in A_1}\mathrm{Ad(a\varphi^n)_n}$. Hence we may assume that $\alpha \in B^*F_1$. Write $\alpha = ua\beta$ with $u \in A_0^*$ and $a \in A_1$. Since $\alpha\Phi^m = \alpha$, $A_0\varphi \subseteq A_0^*$ and $a\varphi^m \neq a$, it follows that $u\varphi^m = u$ and $a\varphi^m = av$ for some $v \in A^+$. By Theorem 5.7, $\lim_{n\to\infty} a\varphi^{nm} \in \mathrm{Fix}(\Phi^m)$ and it is now straighforward to check that

$$\alpha = \alpha\Phi^m = uav(v\varphi^m)(v\varphi^{2m})\ldots = u\lim_{n\to\infty} a\varphi^{nm}.$$

Moreover, $a \in A_4$ and $\alpha = u\alpha_a$ since two convergent sequences sharing an infinite subsequence must share the same limit. Thus $\mathrm{Per}(\Phi) \subseteq \mathrm{B}^\infty \cup \mathrm{B^*F}$.

Trivially, $F \subseteq F_1$ and so (31) follows from Theorem 7.5. Thus (31) holds.

Since all the α_a start with different letters, we get $|F| = |A_4|$. Finally, all elements of B^*F are attractors by Theorem 7.5. □

Acknowledgement

The second author acknowledges support from Project ASA (PTDC/MAT/65481/2006) and C.M.U.P., financed by F.C.T. (Portugal) through the programmes POCTI and POSI, with national and European Community structural funds.

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