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Multiscale Analysis of Convective Magnetic Systems in a Horizontal Layer

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1.1 Introduction

Boussinesq hydromagnetic convection obeys the Navier-Stokes equation with Lorentz and Archimedes forces for the flow, the magnetic induction equation for the magnetic field, and the heat transfer equation for temperature (ν, η, k, α are, respectively, the molecular viscosity, magnetic diffusivity, thermal diffusivity and thermal compressibility; $\tilde{\mathbf{F}}$, $\tilde{\mathbf{R}}$, $\tilde{S}$ are external source terms):

$$\begin{aligned}
&\partial_t \mathbf{V} = \mathbf{V} \times (\partial \times \mathbf{V}) - \partial p + \nu \partial^2 \mathbf{V} - \mathbf{H} \times (\partial \times \mathbf{H}) - \alpha (T - T_0)\mathbf{G} + \tilde{\mathbf{F}},\\
&\partial_t \mathbf{H} = \partial \times (\mathbf{V} \times \mathbf{H}) + \eta \partial^2 \mathbf{H} + \tilde{\mathbf{R}}, \qquad \partial \cdot \mathbf{H} = 0, \qquad \partial \cdot \mathbf{V} = 0, \qquad (1.1)\\
&\partial_t T = -(\mathbf{V} \cdot \partial) T + k \partial^2 T + \tilde{S}.
\end{aligned}$$

They may be used to simulate the evolution of astrophysical convective hydromagnetic (CHM) systems. Accurate simulations for geo- and astrophysical parameter values are close to impossible, due to the limited power of available computers. A semi-analytic approach, based on multiscale analysis, can be applied to this system, in particular to evaluate eddy diffusivity [1, 2, 4, 3]. Negative values of eddy diffusivity indicate that the system is unstable to perturbations involving large scales. This instability is a possible mechanism for kinematic magnetic field generation (non-convective dynamos exploiting this mechanism were studied in [5, 6]).

1.2 Multiscale analysis

The modes of perturbations of steady CHM states obey the equation $\mathbf{PAW} = \lambda \mathbf{W}$, where λ is the growth rate of perturbations, $\mathbf{W} = \left[\ \mathbf{W}^{\mathbf{V}}\ \mathbf{W}^{\mathbf{H}}\ \mathbf{W}^{T}\ \right]^{\mathrm{t}}$ is a block vector representing their spatial profile, $\mathbf{A}$ is the linearised CHM operator and $\mathbf{P}$ is the projection onto the subspace of solenoidal vector fields [1].

In the two-scale expansion, slow variables $\mathbf{X} = \varepsilon\mathbf{x}$ are introduced to describe large-scale dynamics. Expanding perturbations and growth rates in power series in the scale ratio ε, substituting them in the eigenvalue equation and equating the terms in ε^n at each order n, a hierarchy of equations of the form $\mathbf{PA}^{(0)}\mathbf{Pf} = \mathbf{Pg}$ is obtained. Such an equation has a solution, as long as its right hand side is orthogonal to the kernel of the adjoint operator, $\mathbf{PA}^{(0)^*}\mathbf{P}$. For $n = 0$ and $n = 1$, this solvability condition is trivially satisfied, if the perturbed CHM state possesses certain symmetries. For $n = 2$, it yields an eigenvalue equation for the eddy diffusivity operator in slow variables.

1.3 Numerical results and discussion

To model magnetic instabilities in turbulent convective flows, periodic steady CHM states can be randomly generated in the Fourier space [1, 6]. These states satisfy (1.1) for the appropriate source terms. Symmetry and solenoidality conditions must be imposed on the Fourier coefficients, which are normalised afterwards to have a prescribed decaying energy spectrum and the r.m.s. average one. Auxiliary problems are solved numerically in the Fourier space by pseudo-spectral methods (sine or cosine transforms are applied in the vertical direction, in accordance with boundary conditions for respective components of vector fields). Algebraic ($E(k) \sim k^{-\xi}$) or exponential ($E(k) \sim \exp(-\xi k)$) spectra were considered in [6], where flows with exponentially decaying spectra were found to be statistically better dynamos.

Simulations have been carried out for $\nu = \eta = k = 0.5$, and $\alpha = 1$ for the periodicity box of size $2\pi \times 2\pi \times \pi$, with the resolution of $32 \times 32 \times 16$ Fourier harmonics. An ensemble of 1000 instances of CHM steady states, involving Fourier harmonics with wave numbers not exceeding 7, has been generated for both algebraic and exponentially decaying spectra, assuming $\xi = 4$ in both cases. For algebraic spectra, it turns out that 110 out of 1000 (11%) generated flows exhibit negative combined eddy diffusivity. The number rises to 131 (13%) for exponential spectra (see Fig. 1.1 (a)-(b)). Steady states leading to negative eddy diffusivity are unstable to large-scale perturbations. The growth rate of the perturbation is quadratic in the scale ratio ε. Therefore, this instability can be observed only if the considered CHM steady state is stable to short-scale perturbations, which would have larger growth rates otherwise.

For one of the generated CHM states, one of the molecular diffusivities has been varied, keeping all the other parameters equal to the previously used values (see Fig. 1.1 (c)-(d)). The combined eddy diffusivity depends explicitly on the molecular diffusivities ν and η, and on a correction involving the solutions of the auxiliary problems. Molecular diffusivities ν, η and k are also present in the linearised operator and affect the solutions of these auxiliary problems. If no correction is present, the maximum growth rate is negative and equal to $-\min(\nu, \eta)$. For large molecular diffusivities, the growth rate remains negative, since the correction is of the order of $1/\min(\nu, \eta, k)$ and cannot outweigh the additive contribution of ν and η. Beyond the interval of this asymptotic behaviour, the correction may grow in amplitude, leading eventually to positive growth rates λ_2. Whether this happens depends in particular on the spectral properties of the linearisation of (1.1). Thus the influence of molecular diffusivities on the growth rate of the dominant mode of large-scale perturbations is difficult to predict.

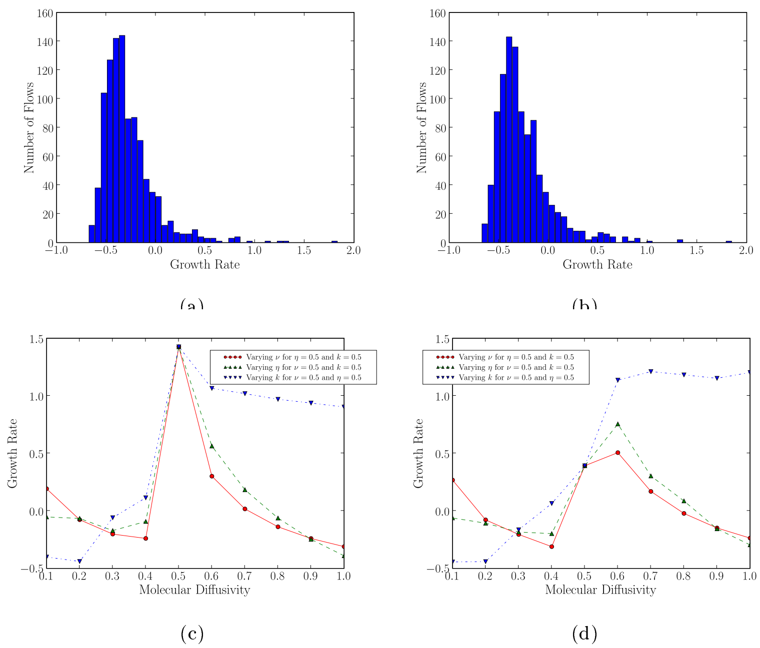

Figure 1.1. Statistics of growth rates (opposite of eddy diffusivity), for algebraic (a) and exponential (b) spectra; growth rates as function of molecular diffusivities, for algebraic (c) and exponential (d) spectra.

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