A COHOMOLOGICAL APPROACH TO n-GERBES WITH CONNECTION

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Abstract. We work out the Chern–Weil theory for abelian n-gerbes and consider the problem of classifying n-gerbes with (flat) connections up to gauge equivalence.

1. Introduction

An abelian gerbe on a manifold M can be viewed as a geometrical realization of a cohomology class in $\check{H}^2(M, \underline{\mathbb{C}}^*)$, in the same way as a complex line bundle on M is a geometrical realization of a cohomology class in $\check{H}^1(M, \underline{\mathbb{C}}^*)$ (here $\underline{\mathbb{C}}^*$ denotes the sheaf of smooth $\mathbb{C}^*$-valued functions). The notion of a gerbe first arose in the context of non-abelian cohomology theory in Giraud's work [3]. Later, the subject gained renewed interest through the work of Brylinski [1] which focused on abelian gerbes and their differential geometry, introducing analogues of the notions of connections and curvature on line bundles. Among other things he worked out the analogue of Chern–Weil theory for line bundles, and also interpreted Dirac's magnetic monopole on the 3-sphere as a gerbe.

Since then, there has been an explosion of interest on this subject, not only from the point of view of geometry but also from that of theoretical physics, and we shall not attempt to give any comprehensive overview. We do, however, mention the work by Chatterjee and Hitchin [2, 4], which is particularly relevant to us. They introduced a very concrete point of view on abelian gerbes and their differential geometry, through the systematic use of Čech cohomology.

The theory of abelian gerbes generalizes to a theory of abelian n-gerbes, providing geometrical realizations of cohomology classes in $\check{H}^{n+1}(M, \underline{\mathbb{C}}^*)$ (in particular, what has above been called a gerbe is a 1-gerbe and a line bundle is a 0-gerbe). It was pointed out by Chatterjee that the Čech approach is particularly well suited for this purpose and for treating the corresponding differential geometry (i.e., the theory of connections and curvature), even though he did not carry out the details of this programme. In [5] Picken worked out in detail the Čech representation of an n-gerbe with connection and its curvature.

Here we adopt the Čech cohomological point of view on gerbes of Hitchin and Chatterjee. Our purpose is to work out the details of the Chern–Weil theory of n-gerbes and

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to describe as concretely as possible the set of gauge equivalence classes of gerbes with (flat) connections in cohomological terms.

The contents of this paper is as follows. In Section 2 we recall the basics of the theory of connnections on line bundles and in Section 3 its generalization to n-gerbes, following Chatterjee and Picken. Then, in Section 4, we construct a non-trivial n-gerbe on the $(n+2)$-sphere, generalizing an example given in Picken [5] (cf. also Brylinski [1]). In Section 5 we work out the details of the Chern–Weil theory for n-gerbes. Finally, in Section 6 we consider the classification problem for n-gerbes with connection: we prove that the set of isomorphism classes of flat n-gerbes on a manifold M can be identified with the cohomology group $\check{H}^{n+1}(M, \mathbb{C}^*)$ and we also consider the problem of classifying all connections on a given gerbe, up to gauge equivalence.

2. Line bundles with connection

We start by recalling some well-known facts about smooth complex line bundles. A smooth complex line bundle ξ over a smooth manifold M is given by a projection map $\pi : E \longrightarrow M$, where $\pi^{-1}(x) \simeq \mathbb{C}$ is the fibre over $x \in M$, E is a smooth manifold called the total space of ξ, such that for each $x \in M$ there is an open neighborhood U_x of x together with a local trivialization, i.e., a diffeomorfism $h_x : \pi^{-1}(U_x) \longrightarrow U_x \times \mathbb{C}$.

Consider an open cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of M for which there is a system of local trivializations

$$\{\varphi_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{C}\}_{i\in I}.$$

Given two open sets which intersect, we can define a transition function

$$g_{ij} : U_i \cap U_j \longrightarrow \mathbb{C}^*$$

by

$$g_{ij}(x) = (\varphi_i \circ \varphi_j^{-1})|_{\pi^{-1}(x)},\ \forall x \in U_i \cap U_j.$$

The functions g_{ij}, for any $i, j \in I$, are smooth, non-vanishing and satisfy:

$$\begin{cases} g_{ij}g_{ji} &= 1 \\ g_{ij}g_{jk}g_{ki} &= 1 \end{cases}$$

Using this characterization of line bundles we can also introduce connections in local terms. Consider a line bundle ξ over M and a system of transition functions

$$\{g_{ij} :\ i, j \in I\}$$

with respect to a cover $\mathfrak{U} = \{U_i\}_{i\in I}$. A connection over ξ is a collection $A = \{A_i\}_{i\in I}$ of 1-forms in $\Omega^1(U_i, \mathbb{C})$ such that on the overlap $U_i \cap U_j$,

$$A_i = A_j + g_{ij}^{-1} dg_{ij},\ \forall i, j \in I.$$

Note that, on $U_i \cap U_j$, we have $dA_i = dA_j$ so we can define a 2-form $B \in \Omega^2(M, \mathbb{C})$ by

$$B = dA$$

and this global 2-form is called the curvature of ξ associated to the connection A.

If we consider a different system of local trivializations

$$\{\varphi_i' : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{C}\}_{i\in I}$$

we get that they are related to the first ones through a set of funcions

$$\{f_i : U_i \longrightarrow \mathbb{C}^*; i \in I\}$$

by having $\varphi_i' = \varphi_i f_i$. Furthermore, we get new transition functions $\{g_{ij}'\}_{i,j\in I}$ and connection 1-forms $\{A_i'\}_{i\in I}$ given by

$$\left\{ \begin{array}{lcl} g_{ij}' & = & g_{ij} f_i f_j^{-1} \\ A_i' & = & A_i + f_i^{-1} df_i \end{array} \right.$$

which are said to be gauge equivalent to the original ones.

This information can be summarized in terms of Čech cochains: a line bundle with connection is given by an ordered pair (g, A) such that $g \in C^1(\mathfrak{U}, \underline{\mathbb{C}}^*)$ and $A \in C^0(\mathfrak{U}, \Omega^1)$ such that

$$\left\{ \begin{array}{lcl} (\delta g)_{ijk} & = & 1 \\ (\delta A)_{ij} & = & d \log g_{ij} \end{array} \right.$$

where $\underline{\mathbb{C}}^*$ represents the sheaf of smooth non-vanishing complex functions, Ω^1 represents the sheaf of differential 1-forms with values on $\mathbb{C}$ and $d\log$ is the logarithmic derivative. Two line bundles with connection (g, A) and (g', A') are gauge equivalent if they differ by $(\delta h, d \log h)$ where $h \in C^0(\mathfrak{U}, \underline{\mathbb{C}}^*)$. We do not need to mention the curvature as it is fixed by the choice of connection.

3. The local equations for n-gerbes with connection

Hitchin and Chatterjee's approach to gerbes (or 1-gerbes) consists in taking Čech theory quite literally, guided by the principle that gerbes are to line bundles what line bundles are to functions. Everything about gerbes is one step up, in number of data, number of open sets in an overlap or in form degree. Thus, 1-gerbes with connection have four data item: given a smooth manifold M with a cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of M, a gerbe with connection over M is given by a 4-tuple (g, A, B, C) where $g = \{g_{ijk}\}_{i,j,k\in I} \in C^2(\mathfrak{U}, \underline{\mathbb{C}}^*)$ is the cochain of "transition functions", $A = \{A_{ij}\}_{i,j\in I} \in C^1(\mathfrak{U}, \Omega^1)$ is the cochain of "connection 1-forms", and $B = \{B_i\}_{i\in I} \in C^0(\mathfrak{U}, \Omega^2)$ is the cochain of "connection 2-forms". These cochains must satisfy:

$$\left\{ \begin{array}{lcl} (\delta g)_{ijkl} & = & 1 \\ (\delta A)_{ijk} & = & -d \log g_{ijk} \\ (\delta B)_{ij} & = & -dA_{ij} \ . \end{array} \right.$$

Since $B_j - B_i = -dA_{ij}$ we get that $dB_j = dB_i$, so we can define the curvature as the global 3-form

$$C = dB \in \Omega^3(M, \mathbb{C}).$$

The notion of gauge equivalence is also a step higher: two gerbes with connection (g, A, B, C) and (g', A', B', C') they said to be gauge equivalent if there exists a pair (f, γ), such that $f = \{f_{ij}\}_{i,j\in I} \in C^1(\mathfrak{U}, \underline{\mathbb{C}}^0)$ and $\gamma = \{\gamma_i\}_{i\in I} \in C^0(\mathfrak{U}, \Omega^1)$, which satisfies the following conditions

$$\begin{cases} g'_{ijk} - g_{ijk} & = & (\delta f)_{ij} \\ A'_{ij} - A_{ij} & = & -d \log h_{ij} + (\delta\gamma)_{ij} \\ B'_i - B_i & = & dB_i \ . \end{cases}$$

Now we come to the systematic definition of n-gerbs (in Čech terms), following Picken [5]. Consider the “logarithmic” de Rham complex of sheaves

$$0 \longrightarrow \underline{\mathbb{C}}^* \longrightarrow \Omega^1 \longrightarrow \Omega^2 \longrightarrow \dots \ .$$

The associated Čech–de Rham double complex with respect to a cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of the manifold M is given by taking $C^n(\Omega^p)$ (with $n \geq 0$ and $p \geq 1$) to be the group of p-form valued Čech n-cochains on $(n+1)$-fold overlaps $U_{ijk\dots}$ (for $p = 0$, take $\mathbb{C}^*$-valued cochains). Now extend this by defining $C^{-1}(\Omega^p)$ to be the global complex p-forms on M and $C^{-1}(\underline{\mathbb{C}}^*)$ to be the global non-vanishing complex functions over M.

Writing cochains in the form $\alpha_{ijk\dots}$ where $ijk\dots$ ranges over the n-fold overlaps (or α without index if $n = 0$), we have the exterior derivative

$$d : C^n(\Omega^p) \longrightarrow C^n(\Omega^{p+1}), \quad \alpha_{ijk\dots} \longmapsto d\alpha_{ijk\dots} \qquad \forall p \geq 1$$

(for $p = 0$ take the logarithmic derivative) and the Čech coboundary operator

$$\delta : C^n(\Omega^p) \longrightarrow C^{n+1}(\Omega^p), \quad (\delta\alpha)_{i_0\dots i_{n+1}} = \sum_{r=0}^{n+1} (-1)^r \alpha_{i_1\dots\hat{i_r} i_{n+1}} \qquad \forall n \geq 0$$

(for $n = -1$ we take the inclusion maps). The extended Čech–de Rham double complex is the following diagram with anti-commuting squares

$$\begin{tikzcd}
\vdots & & & & & \\
C^{-1}(\Omega^4) \arrow[u] \arrow[r] & \dots & & & & \\
C^{-1}(\Omega^3) \arrow[u, "d"] \arrow[r, "i"] & C^0(\Omega^3) \arrow[u] \arrow[r] & \dots & & & \\
C^{-1}(\Omega^2) \arrow[u, "d"] \arrow[r, "i"] & C^0(\Omega^2) \arrow[u, "-d"] \arrow[r, "\delta"] & C^1(\Omega^2) \arrow[u] \arrow[r] & \dots & & \\
C^{-1}(\Omega^1) \arrow[u, "d"] \arrow[r, "i"] & C^0(\Omega^1) \arrow[u, "-d"] \arrow[r, "\delta"] & C^1(\Omega^1) \arrow[u, "d"] \arrow[r, "\delta"] & C^2(\Omega^1) \arrow[u] \arrow[r] & \dots & \\
C^{-1}(\underline{\mathbb{C}}^*) \arrow[u, "d\log"] \arrow[r, "i"] & C^0(\underline{\mathbb{C}}^*) \arrow[u, "-d\log"] \arrow[r, "\delta"] & C^1(\underline{\mathbb{C}}^*) \arrow[u, "d\log"] \arrow[r, "\delta"] & C^2(\underline{\mathbb{C}}^*) \arrow[u, "-d\log"] \arrow[r, "\delta"] & C^3(\underline{\mathbb{C}}^*) \arrow[u] \arrow[r] & \dots
\end{tikzcd}$$

The associated total complex $(\Lambda^\bullet, D)$ is given by

$$\Lambda^k = C^k(\underline{\mathbb{C}}^*) \oplus C^{k-1}(\Omega^1) \oplus \dots \oplus C^{-1}(\Omega^{k+1})$$

where the operator $D : \Lambda^k \longrightarrow \Lambda^{k+1}$ is defined by

$$D = \delta + (-1)^n d \ .$$

In this setup, a line bundle with connection may be defined to be an element $\mathcal{B} \in \Lambda^1$ given by

$$\mathcal{B} = g_{ij} + A_i - B$$

satisfying $D\mathcal{B} = 0$, and a gerbe with connection may be defined as an element

$$\mathcal{G} = g_{ijk} + A_{ij} + B_i - C$$

in Λ^2 such that $D\mathcal{G} = 0$.

Having written the equations in this handy way, they can be extended to elements in $\Lambda^\bullet$ of any degree. Thus, an n-gerbe with connection is defined to be an element

$$\mathcal{G}_n \in \ker D : \Lambda^{n+1} \longrightarrow \Lambda^{n+2}$$

so, in these terms, a gerbe is a 1-gerbe, and a line bundle is a 0-gerbe. For $n \geq 1$, an n-gerbe with connection can be written

$$\mathcal{G}_n = g + A^1 + A^2 + \dots A^n + A^{n+1} - \Theta$$

where the "bare" n-gerbe (without connection) is the $C^{n+1}(\underline{\mathbb{C}}^*)$ part g, the multilayered connection consists of the $C^n(\Omega^1)$ to $C^0(\Omega^{n+1})$ parts $A^1, \dots, A^{n+1}$ of $\mathcal{G}_n$, and the curvature is minus the globally defined $(n+2)$-form part Θ of $\mathcal{G}_n$.

Note that a -1-gerbe is an element

$$\mathcal{F} = f_i - A \in \Lambda^0, \qquad D\mathcal{F} = 0$$

that is, the -1-gerbe is a global function, there are no connections, and the curvature is a global 1-form given by $A_i = d \log f_i$ on each open set U_i of the covering of M.

The notion of gauge equivalence between line bundles and 1-gerbes with connection can also be expressed in this setup. For two line bundles $\mathcal{B}$ and $\mathcal{B}'$, equivalence can be expressed as

$$\mathcal{B}' \sim \mathcal{B} \Longleftrightarrow \mathcal{B}' - \mathcal{B} = Dh, \qquad h \in C^0(\underline{\mathbb{C}}^*),$$

and for two gerbes $\mathcal{G}$ and $\mathcal{G}'$ equivalence means that

$$\mathcal{G}' \sim \mathcal{G} \Longleftrightarrow \mathcal{G}' - \mathcal{G} = D(f_{ij} + \gamma_i), \qquad f_{ij} + \gamma_i \in C^1(\underline{\mathbb{C}}^*) \oplus C^1(\Omega^0).$$

Thus, as pointed out by Picken [5], it is natural to define gauge equivalence for n-gerbes $\mathcal{H}$ and $\mathcal{H}' \in \Lambda^{n+1}$ as follows

$$\mathcal{H}' \sim \mathcal{H} \Longleftrightarrow \mathcal{H}' - \mathcal{H} = D\mathcal{F}, \qquad \mathcal{F} \in \Lambda^n_0,$$

where Λ^n_0 represents Λ^n *without the* $C^{-1}(\Omega^\bullet)$ *part*, that is to say, we disregard the first column.

This definition says that the set of gauge equivalence classes of n-gerbes with connection is

$$\ker(D \colon \Lambda^{n+1} \to \Lambda^{n+2}) / \mathrm{Im}(D \colon \Lambda^n_0 \to \Lambda^{n+1}) \ .$$

and thus this set projects onto the cohomology of the double complex $(\Lambda^\bullet, D)$. Indeed, as noted by Picken, this cohomology classifies gerbes with connection in a coarser way than the standard gauge equivalence, considering them as equivalent if their curvature agrees in de Rham cohomology.

One might also consider the effect of leaving out altogether the first column of the double complex, in other words, considering the hypercohomology of the complex of sheaves

$$\mathcal{K}^\bullet\colon 0 \longrightarrow \underline{\mathbb{C}}^* \xrightarrow{d\log} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \dots$$

As one easily sees, this amounts to demanding that the curvature Θ vanishes. Hence the hypercohomology $\mathbb{H}^{n+1}(\mathcal{K}^\bullet)$ classifies *flat* n-gerbes with connection (this was already pointed out in [2]).

4. An example

In [5], Picken has made a generalization of the standard monopole bundle on the 2-sphere to a gerbe on the 3-sphere (and, as noted in the Introduction, a similar construction was already made by Brylinski [1]). In this section, we see how Picken's construction easily generalizes to a constructing an monopole like n-gerbe on the $(n+2)$-sphere.

We start by recalling the monopole bundle on S^2 as described by Picken. Consider a cover $\mathfrak{U} = \{U_1, U_2\}$ of S^2 such that $U_1 \cap U_2$ is a band around the equator, diffeomorphic to $S^1\times]0,1[$. Take the transition function g_{12} to be

$$\begin{array}{rccc} g_{12}: & S^1\times]0,1[& \longrightarrow & \mathbb{C}^* \\ & (\theta,t) & \longmapsto & \theta \end{array}.$$

Define the connection 1-form to be $A_1 = 0$ on U_1 and on U_2 let A_2 be equal to $d\log g_{12}$ on the overlap U_{12} and extend it in some way to U_2. Then the equation for a line bundle with connection

$$A_2 - A_1 = d\log g_{12}$$

is satisfied U_{12}. The curvature B is given by $B = dA$ and its support is contained in U_2. Using Stokes' Theorem and the winding number of a loop around a point, one calculates that

$$\int_{S^2} \frac{B}{2\pi i} = 1.$$

We shall now describe an analogous construction of an n-gerbe over the $(n+2)$-sphere S^{n+2} (or "n-gerbopole" in the language of Picken). We use induction on n. The case $n = 0$ has already been done. Suppose that for some k we have defined the $(k-1)$-gerbopole over S^{k+1}, i.e., we have a cover $\mathfrak{U} = \{U_1, U_2, \dots, U_{k+1}\}$ of S^{k+1} such that there is a k-multilayered connection $(A^1_{i_1\dots A_{i_k}}, A^2_{i_1\dots i_{k-1}}, \dots, A^k_{i_1})$ and a $(k+1)$-curvature form $B = dA^k$ such that

$$\int_{S^{k+1}} (-1)^{k-1}\frac{B}{2\pi i} = 1.$$

We now define the k-gerbopole over S^{k+2}. Consider the cover $\mathfrak{U}' = \{U'_1, U'_2, \dots, U'_{k+2}\}$ as follows: the open set U'_{k+2} covers one half of S^{k+2} and the equator, $U'_1, \dots, U'_{k+1}$ together cover the other half of S^{k+2} and the equator in a way that

$$\begin{array}{l} U'_{i,k+2} \simeq U_i\times]0,1[, \quad i = 1, \dots, k+1 \\ U'_{1\dots k+2} \simeq S^1\times]0,1[^{k+1} \\ U'_{k+2} \cap (U'_1 \cup \dots \cup U'_{k+1}) \simeq S^{k+1}\times]0,1[\end{array}$$

and we require that $U'_{k+2} \cap (U'_1 \cup \cdots \cup U'_{k+1})$ be a spherical shell (not too wide). We take the transition function

$$\begin{array}{rccc} g'_{12\dots k+2}: & S^1\times]0,1[^{k+1} & \longrightarrow & \mathbb{C}^* \\ & (\theta, t_1, \dots, t_{k+1}) & \longmapsto & \theta \end{array}$$

and the connection 1-forms, defined on $(k+1)$-fold overlaps, to be:

$$\begin{array}{l} A^{1'}_{1\dots k+1} = 0 \\ A^{1'}_{i_1\dots i_k, k+2} = -A^1_{i_1\dots i_k} \end{array}$$

for all indexes $i_1 \dots i_k$ ranging in $\{1, 2, \dots, k+1\}$. The other "auxiliary" connection r-forms, where $2 \leq r \leq k$, on the $(k+2-r)$-fold overlaps, are defined likewise:

$$\begin{array}{l} A^{r'}_{i_1\dots i_{k+2-r}} = 0, \quad \text{if } k+2 \notin \{i, \dots, i_{k+2-r}\} \\ A^{r'}_{i_1\dots i_{k+1-r}, k+2} = -A^r_{i_1\dots i_{k+1-r}} \end{array}$$

taking, as before, the r-forms constant along transversal directions. As for the "true" connection $(k+1)$-forms defined on the open patches U'_i, $i = 1, \dots + 2$, consider

$$\begin{array}{l} A^{k+1}_i = 0, \quad \text{if } i \neq k+2 \\ A^{k+1}_i = -\overline{B}, \quad \text{if } i = k+2 \end{array}$$

where $\overline{B}$ means that it is equal to B on $U'_{1,k+2} \cup \cdots \cup U'_{k+1,k+2}$, constant along transversal directions and extended to U'_{n+2} in some way. The curvature of the k-gerbopole is, of course, $B' = dA^{k+1}$. It is not difficult to check that this data is a k-gerbe in the terms of the definition and that

$$\int_{S^{k+2}} (-1)^k \frac{B}{2\pi i} = 1.$$

5. The Chern class of an n-gerbe and its relation to curvature

We will be considering n-gerbes without differential structure for a while. The abelian groups of equivalence classes of "bare" n-gerbes for a fixed cover $\mathfrak{U}$ of a given manifold M, are the Čech cohomology groups $\check{H}^{n+1}(\mathfrak{U}, \underline{\mathbb{C}}^*)$. If we do not want to mention the cover, we take the direct limit of all covers of M, and get the group of equivalence classes of n-gerbes to be $\check{H}^{n+1}(M, \underline{\mathbb{C}}^*)$.

Let $\mathbb{Z}$ denote the constant sheaf of integers and $\underline{\mathbb{C}}$ denote the sheaf of complex smooth functions. We have the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \underline{\mathbb{C}} \longrightarrow \underline{\mathbb{C}}^* \longrightarrow 0$$

and the co-boundary isomorphisms

$$\partial^p : \check{H}^p(M, \underline{\mathbb{C}}^*) \longrightarrow \check{H}^{p+1}(M, \mathbb{Z}),$$

for every $p \geq 1$. Given an n-gerbe $\mathcal{G}_n$ over the manifold M, the Chern class of $\mathcal{G}_n$, denoted by $c(\mathcal{G}_n)$, is defined to be $\partial^{p+1}([\mathcal{G}_n])$, where $[\mathcal{G}_n]$ is the equivalence class of $\mathcal{G}_n$. Abusing notation, we shall write

$$c(\mathcal{G}_n) \in H^{n+2}_{dR}(M, \mathbb{C})$$

for the image of $c(\mathcal{G}_n)$ under the natural homomorphism

$$\check{H}^{n+2}(M, \mathbb{Z}) \longrightarrow \check{H}^{n+2}(M, \mathbb{C}) \ ,$$

identifying $\check{H}^{n+2}(M,\mathbb{C})$ with $H^{n+2}_{dR}(M,\mathbb{C})$ via de Rham's Theorem.

It is a classical result that for a line bundle with connection, we have that its Chern class is identified with its curvature form in de Rham cohomology. Chatterjee [2] proved a similar result for 1-gerbes. The strategy of both proofs is, to follow the sheaf theoretic proof of de Rham's Theorem. We now use the same method to make the generalization for n-gerbes. We assume that $\mathfrak{U}=\{U_i\}_{i\in I}$ is a good cover of the manifold M (so that we can use Leray's Theorem).

Proposition 5.1. *Let $\mathcal{G}_n$ be a n-gerbe with connection over a manifold M and Θ its curvature form. Then Θ is a closed $(n+2)$-form, its de Rham cohomology class is independent of choice of the multilayered connection, and*

$$c(\mathcal{G}_n)=\left[\frac{(-1)^n}{2\pi i}\Theta\right]\in H^{n+2}_{dR}(M,\mathbb{C}).$$

Proof. Suppose that $\mathcal{G}_n$ is given by the following data:

$$g+A^1+\cdots+A^{n+1}-\Theta\ .$$

Since Θ is the global $(n+2)$-form $d(A^{n+1})$, where $A^{n+1}=\{A_i^{n+1}\}_{i\in I}$ is the "last" connection, clearly Θ is a closed form.

The sequence

$$0\longrightarrow\mathbb{C}\longrightarrow\Omega^0\xrightarrow{d}\Omega^1\xrightarrow{d}\dots$$

is an acyclic resolution of the constant sheaf $\mathbb{C}$ which splits into the short exact sequences

$$\begin{array}{c}
0\longrightarrow\mathbb{C}\longrightarrow\Omega^0\xrightarrow{d}\mathcal{Z}^1\longrightarrow 0\\
0\longrightarrow\mathcal{Z}^1\longrightarrow\Omega^1\xrightarrow{d}\mathcal{Z}^2\longrightarrow 0\\
\vdots\\
0\longrightarrow\mathcal{Z}^{n+1}\longrightarrow\Omega^{n+1}\xrightarrow{d}\mathcal{Z}^{n+2}\longrightarrow 0
\end{array}$$

where $\mathcal{Z}^\bullet$ is the sheaf of closed forms, producing the following co-boundary isomorphisms

$$\begin{array}{c}
\partial^1:\check{H}^{n+1}(M,\mathcal{Z}^1)\longrightarrow\check{H}^{n+2}(M,\mathbb{C})\\
\partial^2:\check{H}^{n}(M,\mathcal{Z}^2)\longrightarrow\check{H}^{n+1}(M,\mathcal{Z}^1)\\
\vdots\\
\partial^{n+2}:\frac{\check{H}^0(M,\mathcal{Z}^{n+2})}{d\check{H}^0(M,\Omega^{n+1})}\longrightarrow\check{H}^1(M,\mathcal{Z}^{n+1}).
\end{array}$$

Note that the composite $\partial^1\circ\partial^2\circ\cdots\circ\partial^{n+1}\circ\partial^{n+2}$ is the de Rham's isomorphism. We get that

$$\begin{aligned}
\partial^1\circ\partial^2\circ\cdots\circ\partial^{n+2}(\Theta) &= \partial^1\circ\partial^2\circ\cdots\circ\partial^{n+2}(dA^{n+1})\\
&= \partial^1\circ\partial^2\circ\cdots\circ\partial^{n+1}(\delta A^{n+1})\\
&= \partial^1\circ\partial^2\circ\cdots\circ\partial^{n+1}(-dA^{n})\\
&= \partial^1\circ\partial^2\circ\cdots\circ\partial^{n}(-\delta A^{n})\\
&\ \ \vdots\\
&= \partial^1((-1)^n d\log g)\\
&= (-1)^n\delta\log g,
\end{aligned}$$

where, in the last line, we use the principal branch of the logarithm (this makes sense because we are working with a good cover). Now observe that $\{\delta\log g\}_{i_1\dots i_{n+2}}=2\pi i\{\alpha\}_{i_1\dots i_{n+1}}$, where $\alpha_{i_1\dots i_{n+2}}$ is an integer, so

$$\partial^1\circ\partial^2\circ\cdots\circ\partial^{n+2}(\Theta)=(-1)^n 2\pi i\ \alpha.$$

It remains to show that $\alpha = \dfrac{1}{2\pi i}\delta \log g$ is a representative of the Chern class $c(\mathcal{G}_n)$. But it is obvious that the pre-image of α under the co-boundary isomorphism

$$\partial : \check{H}^{n+1}(M, \underline{\mathbb{C}}^*) \longrightarrow \check{H}^{n+2}(M, \mathbb{Z})$$

represents the equivalnce class of $\mathcal{G}_n$. $\square$

Poincaré duality tells us that $H^{n+2}_{dR}(S^{n+2}, \mathbb{C})$ is canonically identified with $\mathbb{C}$ under the correspondence $[\varphi] \longmapsto \displaystyle\int_{S^p} \varphi$. So the curvature class of the n-gerbopole constructed in Section 4 is identified with $1 \in \mathbb{C}$ and by means of the above Proposition the equivalence class of the "bare" n-gerbopole is $1 \in \mathbb{Z}$, that is to say, a generator of the set of equivalence classes of bare gerbes $\check{H}^{n+1}(S^{n+2}, \underline{\mathbb{C}}^*)$.

6. Classification of n-gerbes with connection

In this Section, we consider the classification problem for n-gerbes with connection. We begin by considering *flat* gerbes.

As explained at the end of Section 3, the group of gauge equivalence classes of n-gerbes with flat connections can be identified with the hypercohomology $\mathbb{H}^{n+1}(\mathcal{K}^\bullet)$ of the complex of sheaves

$$\mathcal{K}^\bullet : 0 \longrightarrow \underline{\mathbb{C}}^* \xrightarrow{d\log} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \ldots .$$

We shall show that these hypercohomology groups can be calculated as ordinary Čech cohomology groups with values in the constant sheaf $\mathbb{C}^*$.

Proposition 6.1. *With the above notation there is an isomorphism*

$$\mathbb{H}^p(M, \mathcal{K}^\bullet) \cong \check{H}^p(M, \mathbb{C}^*) \ .$$

Proof. Let $\mathcal{L}^\bullet$ be the complex of sheaves:

$$\mathcal{L}^\bullet : 0 \longrightarrow \mathbb{C}^* \longrightarrow 0 \longrightarrow 0 \ldots$$

and consider the inclusion map $\iota : \mathcal{L}^\bullet \longrightarrow \mathcal{K}^\bullet$.

Let $\mathcal{H}^\bullet$ denote the cohomology of a complex of sheaves. Clearly,

$$\mathcal{H}^q(\mathcal{L}^\bullet) = \begin{cases} \mathbb{C}^*, & \text{if } q = 0 \\ 0, & \text{if } q > 0 \end{cases}$$

It is also clear that $\mathcal{H}^0(\mathcal{K}^\bullet) = \mathbb{C}^*$ and, by the Poincaré lemma, that if, $p > 1$, $\mathcal{H}^p(\mathcal{K}^\bullet) = 0$. As for $p = 1$, the pre-sheaf associated with $\mathcal{H}^1(\mathcal{K}^\bullet)$ is given by

$$U \longmapsto \frac{\ker d^1|_U}{\mathrm{Im}\ \mathrm{dlog}|_U},$$

and taking a co-final system of simply-connected open neighborhoods of $x \in M$, we get that if $\xi \in \ker d^1$, then $\xi = d(f) = d\log(\exp f)$, for some $f \in \underline{\mathbb{C}}$. Thus, the stalk $(\mathcal{H}^1(\mathcal{K}^\bullet))_x = 0$ and $\mathcal{H}^1(\mathcal{K}^\bullet) = 0$. Hence, $\mathcal{H}^\bullet(\mathcal{L}^\bullet) = \mathcal{H}^\bullet(\mathcal{K}^\bullet)$ and so $\iota : \mathcal{L}^\bullet \longrightarrow \mathcal{K}^\bullet$ is a quasi-isomorphism, then the induced map on hypercohomology $\iota_\bullet : \mathbb{H}^\bullet(M, \mathcal{L}^\bullet) \longrightarrow \mathbb{H}^\bullet(M, \mathcal{K}^\bullet)$ is an isomorphism. It is easy to check that $\mathbb{H}^p(M, \mathcal{L}^\bullet) = \check{H}^p(M, \mathbb{C}^*)$, so $\mathbb{H}^p(M, \mathcal{K}^\bullet) = \check{H}^p(M, \mathbb{C}^*)$. $\square$

Finally we turn our attention to the problem of describing the set of isomorphism classes of *all* gerbes with connection (not just flat ones).

We start by the simplest case, describing the set of equivalence classes of -1-gerbes. We fix the cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of the manifold M. A -1-gerbe is given by the data

$$\mathcal{G}_{-1} = f - \Theta \in \check{C}^0(\mathfrak{U}, \underline{\mathbb{C}}^*) \oplus \Omega^1(M).$$

Using the gerbe equations, we get that a -1-gerbe is fully determined by a global function $f : M \longrightarrow \mathbb{C}^*$ and that if two -1-gerbes are gauge equivalent then their global functions differ by a non-zero constant. Hence, the set of equivalence classes of -1-gerbes is given by $\dfrac{\underline{\mathbb{C}}^*(M)}{\mathbb{C}^*(M)}$.

Now, we consider 0-gerbes, i.e., line bundles. Suppose that we fix the line bundle itself, in other words, we fix the transition functions. We may ask how many connections such a line bundle admits up to gauge equivalence. Two line bundles $\mathcal{G}_0 = (g, A, \Theta)$ and $\mathcal{G}_0' = (g', A', \Theta')$ are equivalent if they differ by $(\delta f, d\log f, 0)$, where $f \in C^0(\mathfrak{U}, \underline{\mathbb{C}}^*)$. As $g = g'$ then $\delta f = 0$, so f is a global function. It is a well-known fact that two connections on the same line bundle differ by a 1-form. Thus, two connections are equivalent is they differ by $d\log f$, where $f \in \underline{\mathbb{C}}^*$. Therefore, we get that the set we are looking for is $\dfrac{\Omega^1(M)}{d\log(\underline{\mathbb{C}}^*(M))}$. If a line bundle admits flat connections then the set of flat connections up to gauge equivalence is $\dfrac{\mathcal{Z}^1(M)}{d\log(\underline{\mathbb{C}}^*(M))}$.

Now let

$$\mathcal{G}_n = g + A^1 + A^2 + \cdots + A^n + A^{n+1} - \Theta$$

be a n-gerbe, where g gives the transition functions, $A^1, \ldots, A^{n+1}$ is the multilayered connection and Θ is the curvature. Consider that the "bare" gerbe is fixed and that the only gauge transformation allowed on this level is identity. Suppose also that all connections are fixed except for the last one. Then, the result analogous to the one just given for line bundles is the following:

Proposition 6.2. *The set of gauge equivalence classes of n-gerbes with connection, such that the lower order data $g + A^1 + A^2 + \cdots + A^n$ is fixed, can be identified with* $\dfrac{\Omega^{n+1}(M)}{d(\Omega^n(M))}$.

Proof. Let $\mathcal{G}_n'$ be another n-gerbe given by the following data:

$$\mathcal{G}_n' = g' + A'^1 + A'^2 + \ldots A'^n + A'^{n+1} - \Theta'$$

If $\mathcal{G}_n$ and $\mathcal{G}_n'$ are equivalent and obey the imposed restrictions then they differ by

$$0 + \delta\alpha^1 + [(-1)^{n-1}d\alpha^1 + \delta\alpha^2] + \cdots + [(-1)^{n-1}d\alpha^{n-1} + \delta\alpha^n] + d\alpha^n + 0$$

and

$$\begin{aligned}
&\delta\alpha^1 = 0\\
&(-1)^{n-1}d\alpha^1 + \delta\alpha^2 = 0\\
&\qquad\vdots\\
&(-1)^{n-1}d\alpha^{n-1} + \delta\alpha^n = 0
\end{aligned}$$

Since the sheaves $\Omega^\bullet(M)$ are all fine and therefore acyclic we have that $\delta\alpha^1 = 0$ implies that $\alpha^1 = \delta\xi^1$. Substituting in the next equation, $(-1)^{n-1}d\delta\xi^1 + \delta\alpha^2 = \delta((-1)^{n-1}d\xi + \delta\alpha^2) = 0$, so $(-1)^{n-1}d\xi + \delta\alpha^2 = \delta\xi^2$. With similar reasoning, we finally conclude that $\delta((-1)^{n-1}d\xi^{n-1} + \alpha^n) = 0$. Take f to be $(-1)^{n-1}d\xi^{n-1} + \alpha^n$. By the previous calculations $\delta f = 0$, which means that f is a global n-form. Also

$$A'^{n+1} - A^{n+1} = d\alpha^n = d((-1)^{n-1}d\xi^{n-1} + \alpha^n),$$

so two connections in these terms differ by an exact n-form on M. Note that, by the gerbe equations, as $\delta A'^{n+1} = \delta A^{n+1} = dA^n$, any two connections compatible with the previous ones differ by a global n-form. Hence, we get the desired result. □

To conclude, we remark that if the bare n-gerbe admits flat connections then the corresponding set of gauge equivalence classes of flat $(n+1)$-connections is $H^{n+1}_{dR}(M)$.

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