Models for the free pseudosemilattices

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Abstract

In this paper we present two models for the free pseudosemilattices. We then use these models to study the relations ω^r, ω^l, ω, $\mathcal{R}$, and $\mathcal{L}$ on the free pseudosemilattices.

1 Introduction

Given a regular semigroup S, we denote by $E(S)$ the set of idempotents of S. The relation ω^l on $E(S)$ is defined by: for $e, f \in E(S)$,

$$e\,\omega^l f \quad \Leftrightarrow \quad e = ef.$$

We denote by ω^r the dual relation on $E(S)$, and by ω the relation $\omega^r \cap \omega^l$ on $E(S)$. Then ω^l and ω^r are quasi-orders, while ω is a partial order. For $\nu \in \{\omega^r, \omega^l, \omega\}$ let $\nu(f) = \{e \in E(S) \,|\, e\,\nu\, f\}$.

A locally inverse semigroup is a regular semigroup S such that all local submonoids eSe, $e \in E(S)$, are inverse semigroups. The locally inverse

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semigroups can be characterized as the regular semigroups such that for any $e, f \in E(S)$ there exists a unique element $e \wedge f \in E(S)$ satisfying

$$\omega^r(e) \cap \omega^l(f) = \omega(e \wedge f).$$

We can therefore define a new binary algebra $(E(S), \wedge)$ from any locally inverse semigroup S. These algebras are called pseudosemilattices and they were characterized abstractly by Nambooripad [5] who showed that an algebra $(E, \wedge)$ is a pseudosemilattice if and only if it satisfies the identities

$$\begin{aligned}
&x \wedge x \approx x,\\
&(x \wedge y) \wedge (x \wedge z) \approx (x \wedge y) \wedge z,\\
&((x \wedge y) \wedge (x \wedge z)) \wedge (x \wedge w) \approx (x \wedge y) \wedge ((x \wedge z) \wedge (x \wedge w)).
\end{aligned}$$

together with the right-left duals of the last two. We denote by **PS** the variety of all pseudosemilattices. This result was then generalized by Auinger [1] who showed that the mapping

$$\varphi : \mathcal{L}_e(\mathbf{LI}) \longrightarrow \mathcal{L}(\mathbf{PS}), \quad \mathbf{V} \longmapsto \{(E(S), \wedge) \mid S \in \mathbf{V}\}$$

is a surjective complete homomorphism from the lattice $\mathcal{L}_e(\mathbf{LI})$ of e-varieties of locally inverse semigroups to the lattice $\mathcal{L}(\mathbf{PS})$ of varieties of pseudosemilattices. The structure of the pseudosemilattices and its connections to the structure of locally inverse semigroups have been studied in [3, 8, 9, 10].

The relations ω^r, ω^l and ω can be defined in any pseudosemilattice $(E, \wedge)$ in terms of the operation $\wedge$ instead of the semigroup operation:

$$\begin{aligned}
e\,\omega^l f &\quad\Leftrightarrow\quad e \wedge f = e,\\
e\,\omega^r f &\quad\Leftrightarrow\quad f \wedge e = e,\\
e\,\omega f &\quad\Leftrightarrow\quad e \wedge f = e = f \wedge e.
\end{aligned}$$

Define the relation $\mathcal{R}$ and $\mathcal{L}$ on E as follows: $\mathcal{R} = \omega^r \cap (\omega^r)^{-1}$ and $\mathcal{L} = \omega^l \cap (\omega^l)^{-1}$. If S is a locally inverse semigroup such that $(E, \wedge)$ is isomorphic to $(E(S), \wedge)$, then $\mathcal{R}$ and $\mathcal{L}$ correspond to the restriction of the Green's relations with the same designation on S to $E(S)$.

Let $(F_2(X), \wedge)$ be the absolutely free binary algebra on X. For $u \in F_2(X)$, we denote by uR and uL the first and last letter of u, respectively. Let $\mathcal{B} = \{0, 1\}$ and $\preccurlyeq$ be the partially order on the free monoid $\mathcal{B}^*$ on $\mathcal{B}$ defined as follows: $a \preccurlyeq b$ if and only if b is a prefix of a. In [6] we associated in a natural fashion to each $u \in F_2(X)$ a finite partially ordered subset Γ of $(\mathcal{B}, \preccurlyeq)$ with its elements labeled by words from $F_2(X)$:

$$l : \Gamma \longrightarrow F_2(X), \quad a \longmapsto a_l.$$

We denoted by $\Gamma(u)$ the pair (Γ, l) and called it the labeled binary poset associated with u. We introduced also three ways to transform labeled binary posets into other labeled binary posets: $\mathcal{M}$, $\mathcal{E}$ and $\mathcal{S}$. Each one of this rules induces a binary relation on $F_2(X)$ which is denoted also by the same symbol.

Let σ be the relation on $F_2(X)$ defined by: for $u, v \in F_2(X)$, $u\sigma v$ if and only if $(u, v) \in \mathcal{S}^*$ or $c(u) = c(v) = \{x\}$ for some $x \in X$, where $\mathcal{S}^*$ denotes the reflexive and transitive closure of $\mathcal{S}$ and $c(u)$ denotes the content of u, that is, the set of all letters from X occurring in u. Then σ is a congruence relation on $F_2(X)$ and we denote by $[u]$ the σ-class of u. Consider also the reflexive and transitive closure δ of $\mathcal{M} \cup \mathcal{E}$. A word $u \in F_2(X)$ is called **reduced** if $u\delta v \in F_2(X)$ implies $u = v$. We denote by $RF_2(X)$ the set of all reduced words. By [6, Section 4 and Theorem 5.7] we have

Result 1.1 *Let $u, v \in F_2(X)$. Then*

(*i*) *there exists $u_1 \in RF_2(X)$ such that $u\delta u_1$,*

(*ii*) *if $[u] = [v]$ and $u\delta u_1$, $v\delta v_1$ for some $u_1, v_1 \in RF_2(X)$ then $[u_1] = [v_1]$,*

(*iii*) *the identity $u \approx v$ is satisfied by all pseudosemilattices if and only if $[u_1] = [v_1]$ for some [any] $u_1, v_1 \in RF_2(X)$ such that $u\delta u_1$, $v\delta v_1$.* ■

Thus, for each $u \in F_2(X)$, we can talk about the σ-class containing all reduced words v such that $u\delta v$. We denote this σ-class by $[u]_r$. We define $F_{PS}(X) = RF_2(X)/\sigma$ and introduce a binary operation $\wedge$ on $F_{PS}(X)$ as follows: for $u, v \in RF_2(X)$, $[u] \wedge [v] = [u \wedge v]_r$. This $\wedge$-operation is well-defined by Result 1.1(*ii*) and since σ is a congruence. By Result 1.1(*iii*) we have therefore that $(F_{PS}(X), \wedge)$ is a free pseudosemilattice on X. The free pseudosemilattices have been studied in [2, 4, 6].

2 Models for the free pseudosemilattices

For any set Y and any s in the free monoid Y^+ on Y, we denote by s_λ and s_τ the first and last letter of s, respectively. Let $\mathcal{B} = \{0, 1\}$ and $a \in \mathcal{B}^+$.

Then, there are $n \geq 1$, $e_i \in \mathcal{B}$ and $k_i \geq 1$ for $i \in \{1, \cdots, n\}$ such that $a = e_1^{k_1} \cdots e_n^{k_n}$ and $e_j \neq e_{j+1}$ for $j \in \{1, \cdots, n-1\}$. Let $a_0 = \iota$, where ι is the empty word, and $a_i = e_1^{k_1} \cdots e_i^{k_i}$ for $i \in \{1, \cdots, n\}$. We denote by $\nu(a)$ the vector $(a_0, a_1, \cdots, a_n)$. We define also $\nu(\iota) = (\iota)$.

Let $u \in F_2(X) \setminus X$, $\Gamma(u) = (\Gamma, l)$, and $a \in \Gamma \setminus \{\iota\}$. We denote by Γ_L the set of minimal elements of Γ and by $\overline{a}$ the element $ad^k \in \Gamma_L$ such that $a_\tau = d$. Let $\nu(a) = (a_0, \cdots, a_n)$ for some $n \geq 1$, and define $k_u(a) = (\overline{a_1})_l \cdots (\overline{a_n})_l$. Let

$$K_0(u) = \{k_u(a) \mid a \in \Gamma,\, a_\lambda = 0\} = \{k_u(a) \mid a \in \Gamma_L,\, a_\lambda = 0\},$$

$$K_1(u) = \{k_u(a) \mid a \in \Gamma,\, a_\lambda = 1\} = \{k_u(a) \mid a \in \Gamma_L,\, a_\lambda = 1\},$$

and consider $K(u) = (K_0(u), K_1(u))$. Define $K_0(x) = K_1(x) = \{x\}$ and $K(x) = (\{x\}, \{x\})$ for each $x \in X$.

By definition of $RF_2(X)$ and σ, we have $K_0(u) = K_0(v)$, $K_1(u) = K_1(v)$, and $K(u) = K(v)$ for any $u, v \in RF_2(X)$ such that $[u] = [v]$. We can define therefore $K_0([u]) = K_0(u)$, $K_1([u]) = K_1(u)$ and $K([u]) = K(u)$ for any $[u] \in F_{PS}(X)$.

Let P be the set of all $A \subseteq X^+$ satisfying the following conditions:

(C_1) A is a finite subset of xX^* for some $x \in X$.

(C_2) A is closed for prefixes from X^+.

(C_3) xyx is not a subword of s for any $x, y \in X$ and $s \in A$.

(C_4) If $sx^2 \in A$ for some $s \in X^*$ and $x \in X$, then there is $y \in X$ such that $sx^2y \in A$.

Let $\mathcal{K}$ be the set of all pairs $(A_0, A_1) \in P \times P$ such that

(C_5) For any $x, y \in X$ such that $x \in A_0$ and $y \in A_1$, we have that $xy \notin A_0$ and $yx \notin A_1$.

The next lemma follows easily from the definition of $\mathcal{M}$ and $\mathcal{E}$.

Lemma 2.1 *Let $u \in RF_2(X)$. Then $K([u]) \in \mathcal{K}$ and the mapping*

$$\varphi_k : F_{PS}(X) \longrightarrow \mathcal{K}, \quad [u] \longmapsto K([u])$$

is well defined. ∎

We show now that φ_k is in fact a bijection.

Lemma 2.2 *The mapping $\varphi_k : F_{PS}(X) \longrightarrow \mathcal{K}$ is a bijection.*

Proof: Let us show first that φ_k is surjective. Let $(A_0, A_1) \in \mathcal{K}$. We show there exists $u \in RF_2(X)$ such that $\varphi_k([u]) = (A_0, A_1)$ by induction on $n = |A_0| + |A_1|$. If $n = 2$, then $A_0 = \{x_0\}$ and $A_1 = \{x_1\}$. Take $u = x_0$ if $x_0 = x_1$, or take $u = x_0 \wedge x_1 \in RF_2(X)$ if $x_0 \neq x_1$.

Assume $n > 2$. Thus $|A_0| \geq 2$ or $|A_1| \geq 2$. We shall assume also that $|A_0| \geq 2$ since the case $|A_1| \geq 2$ can be shown dually. Then there exists $x_0x_1 \in A_0$ for some $x_0, x_1 \in X$. Let

$$A_{01} = \{s \,|\, s_\lambda = x_1,\, x_0 s \in A_0\}$$

and $A_{00} = A_0 \setminus x_0 A_{01}$, and observe that $(A_{00}, A_{01}) \in \mathcal{K}$. By induction we have $(A_{00}, A_{01}) = \varphi_k([u_0])$ for some $u_0 \in RF_2(X)$.

If $A_1 = \{y_1\}$, take $u_1 = y_1$. If $|A_1| \geq 2$, let $y_1 y_0 \in A_1$ for some $y_1, y_0 \in X$ and define $A_{10} = \{s \,|\, s_\lambda = y_0,\, y_1 s \in A_1\}$ and $A_{11} = A_1 \setminus y_1 A_{10}$. Again $(A_{10}, A_{11}) \in \mathcal{K}$ and by induction $(A_{10}, A_{11}) = \varphi_k([u_1])$ for some $u_1 \in RF_2(X)$.

Let $u = u_0 \wedge u_1$. Then $K_0(u) = A_{00} \cup x_0 A_{01} = A_0$ and $K_1(u) = A_1$, and so $u \in RF_2(X)$ since $u_0, u_1 \in RF_2(X)$ and A_0 and A_1 satisfy (C_4) and (C_5). Thus $\varphi_k([u]) = (A_0, A_1)$ and φ_k is a surjective mapping.

Let us show now that φ_k is injective. Let $u, v \in RF_2(X)$ such that $\varphi_k([u]) = (A_0, A_1) = \varphi_k([v])$. Obviously $uR = vR$ and $uL = vL$. We show that $[u] = [v]$ by induction on $n = |A_0| + |A_1|$. If $n = 2$, then $A_0 = \{x_0\}$ and $A_1 = \{x_1\}$. If $x_0 = x_1$ then $[u] = [x_0] = [v]$. If $x_0 \neq x_1$ then $u = x_0 \wedge x_1 = v$.

Assume $n > 2$. Then $u = u_0 \wedge u_1$ and $v = v_0 \wedge v_1$ for some $u_0, u_1, v_0, v_1 \in RF_2(X)$. Let $\Gamma(u) = (\Gamma, l)$ and $\Gamma(v) = (\Gamma_1, l_1)$. Let $y_0 = u_0 L$ and $w_0 \in RF_2(X)$ such that $[w_0] = [v_0] \wedge [y_0]$. Clearly, there is exactly one $i \geq 1$ such that $0^i 1 \in \Gamma_1$ and $k_v(0^i 1) = (uR)y_0$. Thus

$$K_1(w_0) = \{s \,|\, (uR)s \in A_0,\, s_\lambda = y_0\} = K_1(u_0)$$

and $K_0(w_0) = A_0 \setminus (uR)K_1(w_0) = K_0(u_0)$. By induction we conclude that $[u_0] = [w_0]$.

Let $x_1 = u_1 R$ and $w_1 \in RF_2(X)$ such that $[w_1] = [x_1] \wedge [v_1]$. Similarly we conclude that $[w_1] = [u_1]$, and so

$$[u] = [u_0] \wedge [u_1] = ([v_0] \wedge [y_0]) \wedge ([x_1] \wedge [v_1]) = [v_0] \wedge [v_1] = [v].$$

Consequently φ_k is an injective mapping. ∎

Let (A_0, A_1), $(B_0, B_1) \in \mathcal{K}$ and $u, v \in RF_2(X)$ such that $\varphi_k([u]) = (A_0, A_1)$ and $\varphi_k([v]) = (B_0, B_1)$. If we define a $\wedge$-operation on $\mathcal{K}$ by

$$(A_0, A_1) \wedge (B_0, B_1) = \varphi_k([u] \wedge [v]),$$

then φ_k becomes an isomorphism by Lemma 2.2, and thus $(\mathcal{K}, \wedge)$ is a model for the free pseudosemilattice on X. We can determine the operation $\wedge$ abstractly as follows. For $i \in \{0, 1\}$ let x_i, $y_i \in X$ such that $A_i \subseteq x_i X^*$ and $B_i \subseteq y_i X^*$, and for $j \in \{00, 01, 10, 11\}$ let A_j and B_j as the following sets:

- $A_{01} = \{s \,|\, x_0 s \in A_0,\ s_\lambda = y_1\}$ and $A_{00} = A_0 \setminus x_0 A_{01}$,
- if $A_1 = \{x_0\}$ then $A_{10} = A_{11} = \emptyset$, and if $A_1 \neq \{x_0\}$ then

$$A_{11} = \{s \in A_1 \,|\, s_\lambda = y_1\} \quad \text{and} \quad A_{10} = \{x_0 s \,|\, s \in A_1 \setminus A_{11}\},$$

- if $B_0 = \{y_1\}$ then $B_{01} = B_{00} = \emptyset$, and if $B_0 \neq \{y_1\}$ then

$$B_{00} = \{s \in B_0 \,|\, s_\lambda = x_0\} \quad \text{and} \quad B_{01} = \{y_1 s \,|\, s \in B_0 \setminus B_{00}\},$$

- $B_{10} = \{s \,|\, y_1 s \in B_1,\ s_\lambda = x_0\}$ and $B_{11} = B_1 \setminus y_1 B_{10}$.

Using the definition of $\mathcal{M}$ and $\mathcal{E}$ it is now straightforward to check that

$$\varphi_k([u] \wedge [v]) = (A_{00} \cup A_{10} \cup B_{00} \cup B_{10}, A_{01} \cup A_{11} \cup B_{01} \cup B_{11}).$$

Therefore we have

Theorem 2.3 *Using the notation above, for* (A_0, A_1), $(B_0, B_1) \in \mathcal{K}$ *define*

$$(A_0, A_1) \wedge (B_0, B_1) = (A_{00} \cup A_{10} \cup B_{00} \cup B_{10},\ A_{01} \cup A_{11} \cup B_{01} \cup B_{11}).$$

Then $\wedge$ *is a well-defined binary operation on* $\mathcal{K}$ *and* $(\mathcal{K}, \wedge)$ *is a model for the free pseudosemilattice on* X. ■

Let $\mathcal{H}$ be the set of all pairs (A, y) such that $A \subseteq X^+$ satisfies (C_1), (C_2), (C_3) and

(C_6) if $sx^2 \in A$ for some $s \in X^*$ and $x \in X$, then there is $x' \in X \setminus \{x\}$ such that (i) $s = \iota$ and $xx' \in A$, or (ii) $sx^2x' \in A$,

and $y \in X$ satisfies

(C_7) either $A=\{y\}$ or $xy \in A$ for some $x \in X$,

(C_8) if $x^2 \in A$ for some $x \in X$ and $x^2x' \notin A$ for any $x' \in X$, then $y=x$.

The next lemma is immediate:

Lemma 2.4 *Let* $(A_0, A_1) \in \mathcal{K}$ *such that* $(A_0, A_1) \neq (\{x\}, \{x\})$ *for any* $x \in X$*. Let* $x_i \in X$ *such that* $A_i \subseteq x_i X^*$ *for* $i \in \{0,1\}$*, and let* $A = A_0 \cup x_0 A_1$*. Then* $(A, x_1) \in \mathcal{H}$*.* ∎

Consider the mapping $\psi_h : \mathcal{K} \longrightarrow \mathcal{H}$ defined by $\psi_h((\{x\},\{x\})) = (\{x\}, x)$ for any $x \in X$ and by $\psi_h((A_0, A_1)) = (A_0 \cup x_0 A_1, x_1)$ for any $(A_0, A_1) \in \mathcal{K}$ such that $A_i \subseteq x_i X^*$ and $(A_0, A_1) \neq (\{x\},\{x\})$ for any $x \in X$. The mapping ψ_h is well defined due to the previous lemma. In fact ψ_h is a bijection whose inverse mapping is the mapping $\psi : \mathcal{H} \longrightarrow \mathcal{K}$ defined by $\psi((\{x\}, x)) = (\{x\},\{x\})$ for any $x \in X$ and by $\psi((A, x_1)) = (A \backslash x_0 A_1, A_1)$ for any $(A, x_1) \in \mathcal{H}$ such that $A \subseteq x_0 X^*$ and $|A| \geq 2$, where $A_1 = \{s \,|\, x_0 s \in A,\, s_\lambda = x_1\}$.

Let (A, x_1), $(B, y_1) \in \mathcal{H}$ such that $A \subseteq x_0 X^*$ and $B \subseteq y_0 X^*$ for some x_0, $y_0 \in X$, and define

- $A' = \begin{cases} A \setminus \{x_0^2\} & \text{if } x_0^2 \in A \text{ and } x_0^2 y \notin A \text{ for any } y \in X \\ A & \text{otherwise,} \end{cases}$

- $B_0' = \begin{cases} \emptyset & \text{if } B \subseteq \{y_1\} \cup y_1^2 X^* \\ \{s \in B \,|\, s \not\leq y_0 y_1\} & \text{if } B \nsubseteq \{y_1\} \cup y_1^2 X^* \text{ and } y_0 = x_0 \\ \{x_0 y_1 s \,|\, s \in B,\ s \not\leq y_0 y_1\} & \text{otherwise,} \end{cases}$

- $B_1' = \{x_0 y_1\} \cup \{s \,|\, y_0 y_1 s \in B,\ s_\lambda = x_0\} \cup \{x_0 y_1 s \,|\, y_0 y_1 s \in B,\ s_\lambda \neq x_0\}$,

- $C = \begin{cases} \{y_1\} & \text{if } x_0 = x_1 = y_0 = y_1 \text{ and } A = B = \{y_1\} \\ A' \cup B_0' \cup B_1' & \text{otherwise.} \end{cases}$

Theorem 2.5 *Let* $(A, x_1), (B, y_1) \in \mathcal{H}$ *such that* $A \subseteq x_0 X^*$ *and* $B \subseteq y_0 X^*$ *for some* $x_0, y_0 \in X$*, and define*

$$(A, x_1) \wedge (B, y_1) = (C, y_1).$$

Then $\wedge$ *is a well defined binary operation on* $\mathcal{H}$ *and* $(\mathcal{H}, \wedge)$ *is a model for the free pseudosemilattice on* X*.*

Proof: Let $\psi((A,x_1)) = (A_0,A_1)$ and $\psi((B,y_1)) = (B_0,B_1)$. If we show that

$$\psi_h((A_0,A_1) \wedge (B_0,B_1)) = (C,y_1), \tag{1}$$

then ψ_h is an isomorphism from $(\mathcal{K},\wedge)$ to $(\mathcal{H},\wedge)$ and so $(\mathcal{H},\wedge)$ is a model for the free pseudosemilattice on X.

Clearly (1) holds true for $A = B = \{y_1\}$. Assume $(A,B) \neq (\{x\},\{x\})$ for any $x \in X$. Then $A_i \subseteq x_iX^*$ and $B_i \subseteq y_iX^*$ for $i \in \{0,1\}$, and

$$\begin{aligned}\psi_h((A_0,A_1)\wedge(B_0,B_1)) &= \\ &= (A_{00} \cup A_{10} \cup B_{00} \cup B_{10} \cup x_0(A_{01} \cup A_{11} \cup B_{01} \cup B_{11}), y_1).\end{aligned}$$

We can check now that $A' = A_{00} \cup A_{10} \cup x_0(A_{01} \cup A_{11})$, $B_0' = B_{00} \cup x_0B_{01}$, and $B_1' = B_{10} \cup x_0B_{11}$. Consequently (1) holds true in this case also, and therefore ψ_h is an isomorphism. ∎

Observe that we can construct, in a dual manner, a third model by considering the pairs $(x_1A_0 \cup A_1, x_0)$ instead of the pairs $(A_0 \cup x_0A_1, x_1)$. This third model is the dual of $\mathcal{H}$.

Next we characterize the relations ω^r, ω^l, ω, $\mathcal{R}$, and $\mathcal{L}$ on $\mathcal{K}$.

Proposition 2.6 *Let (A_0,A_1), $(B_0,B_1) \in \mathcal{K}$ and $x_i, y_i \in X$ such that $A_i \subseteq x_iX^*$ and $B_i \subseteq y_iX^*$ for $i \in \{0,1\}$. Then*

- *(i)* $(B_0,B_1)\,\omega^r(A_0,A_1)$ *if and only if* $A_0 \cup x_0A_1 \cup \{x_0^2\} \subseteq B_0 \cup y_0B_1 \cup \{y_0^2\}$,
- *(ii)* $(B_0,B_1)\,\omega^l(A_0,A_1)$ *if and only if* $x_1A_0 \cup A_1 \cup \{x_1^2\} \subseteq y_1B_0 \cup B_1 \cup \{y_1^2\}$,
- *(iii)* $(B_0,B_1)\,\omega\,(A_0,A_1)$ *if and only if* $A_0 \subseteq B_0$ *and* $A_1 \subseteq B_1$,
- *(iv)* $(B_0,B_1)\,\mathcal{R}\,(A_0,A_1)$ *if and only if* $A_0 \cup x_0A_1 \cup \{x_0^2\} = B_0 \cup y_0B_1 \cup \{y_0^2\}$,
- *(v)* $(B_0,B_1)\,\mathcal{L}\,(A_0,A_1)$ *if and only if* $x_1A_0 \cup A_1 \cup \{x_1^2\} = y_1B_0 \cup B_1 \cup \{y_1^2\}$.

Proof: *(i)*. Let $A = A_{00} \cup A_{10} \cup B_{00} \cup B_{10}$ and $B = A_{01} \cup A_{11} \cup B_{01} \cup B_{11}$. Assume $(B_0,B_1)\,\omega^r(A_0,A_1)$. Then

$$(B_0,B_1) = (A_0,A_1) \wedge (B_0,B_1) = (A,B),$$

and $x_0 = y_0$. Furthermore

$$A_0 = A_{00} \cup x_0 A_{01} \subseteq B_0 \cup x_0 B_1,$$

$$x_0 A_1 \subseteq A_{10} \cup x_0 A_{11} \cup \{x_0^2\} \subseteq B_0 \cup x_0 B_1 \cup \{x_0^2\}.$$

Thus $A_0 \cup x_0 A_1 \cup \{x_0^2\} \subseteq B_0 \cup y_0 B_1 \cup \{y_0^2\}$.

Assume $A_0 \cup x_0 A_1 \cup \{x_0^2\} \subseteq B_0 \cup y_0 B_1 \cup \{y_0^2\}$. Then $x_0 = y_0$, $B_{10} = \emptyset$, $B_{00} \cup \{x_0\} = B_0$, and $x_0 \in A_{00} \cup A_{10} \subseteq B_0 \cup \{x_0^2\}$. Consequently $B_0 \subseteq A \subseteq B_0 \cup \{x_0^2\}$. If $x_0^2 \in A$, then $x_0^2 x \in A$ for some $x \in X$ by (C_4), and thus $x_0^2 x$ and x_0^2 belong to B_0. Hence $A = B_0$. We show that $B = B_1$ similarly, and therefore $(B_0, B_1)\, \omega^r (A_0, A_1)$.

(ii) is shown using dual arguments.

(iii). Assume that $(B_0, B_1)\, \omega\, (A_0, A_1)$. Then $A_0 \subseteq B_0 \cup y_0 B_1 \cup \{y_0^2\}$, $x_0 = y_0$ and $x_1 = y_1$ by *(i)* and *(ii)*. Thus $A_0 \cap y_0 B_1 = \emptyset$ due to (C_5) and (C_2). Consequently $A_0 \subseteq B_0 \cup \{y_0^2\}$. If $y_0^2 \in A_0$ then there is $x \in X$ such that $y_0^2 x \in A_0$ by (C_4), whence $y_0^2 x$ and y_0^2 belong to B_0. We have shown that $A_0 \subseteq B_0$. We show that $A_1 \subseteq B_1$ similarly. The reverse implication follows immediately from *(i)* and *(ii)*.

(iv) and *(v)* follow from *(i)* and *(ii)*, respectively. ■

Corollary 2.7 *Let (A, x_1), $(B, y_1) \in \mathcal{H}$ and $x_0, y_0 \in X$ such that $A \subseteq x_0 X^*$ and $B \subseteq y_0 X^*$. Then*

(i) $(B, y_1)\, \omega^r (A, x_1)$ *if and only if* $A \cup \{x_0^2\} \subseteq B \cup \{y_0^2\}$,

(ii) $(B, y_1)\, \omega\, (A, x_1)$ *if and only if* $A \subseteq B$ *and* $x_1 = y_1$,

(iii) $(B, y_1)\, \mathcal{R}\, (A, x_1)$ *if and only if* $A \cup \{x_0^2\} = B \cup \{y_0^2\}$.

Proof: Observe that $A \cup \{x_0^2\} = A_0 \cup x_0 A_1 \cup \{x_0^2\}$ if $(A_0, A_1) = \psi((A, x_1))$. Thus *(i)*, *(ii)* and *(iii)* follow from Proposition 2.6 *(i)*, *(iii)* and *(iv)*, respectively, via the pairwise inverse isomorphisms ψ and ψ_h. ■

We can also obtain conditions characterizing ω^l and $\mathcal{L}$ on $\mathcal{H}$. However, contrarily to ω^r, ω and $\mathcal{R}$, ω^l and $\mathcal{L}$ do not have a nice description in $\mathcal{H}$. The reader can find these conditions in [7, Proposition 4.2]. We remark also that ω^l, ω, and $\mathcal{L}$ have nice descriptions on the dual model of $\mathcal{H}$ mentioned before Proposition 2.6, but ω^r and $\mathcal{R}$ do not have such a nice description ([7, Proposition 4.4]).

3 The relations $\mathcal{S}_e^*$ and $\mathcal{E}_e^*$ on $F_2(X)$

We begin by defining the relation $\mathcal{S}_0$ on $F_2(X)$. $\mathcal{S}_0$ is a relation similar to the relation $\mathcal{S}$ introduced in [6]. Let $u \in F_2(X)$ and $\Gamma(u) = (\Gamma, l)$. Let $k \geq 0$ such that $0^{k+1} \in \Gamma$ and define

$$\Gamma' = \{a \in \Gamma \mid a \not\leq 1,\ a \not\leq 0^k 1\} \cup \{0^k 1b \mid 1b \in \Gamma\} \cup \{1c \mid 0^k 1c \in \Gamma\}$$

and $l' : \Gamma'_L \longrightarrow X$ as

$$a_{l'} = \begin{cases} a_l & \text{if } a \not\leq 1 \text{ and } a \not\leq 0^k 1 \\ (1b)_l & \text{if } a = 0^k 1b \in \Gamma'_L \\ (0^k 1c)_l & \text{if } a = 1c \in \Gamma'_L \end{cases}$$

Clearly (Γ', l') is a labeled binary poset, and let $v \in F_2(X)$ such that $\Gamma(v) = (\Gamma', l')$. We say that $\Gamma(v)$ is obtained from $\Gamma(u)$ by switching 1 with one of its neighbors. Let $\mathcal{S}_0$ be the set of all pairs $(u, v) \in F_2(X)$ such that $u = v = x \in X$ or $\Gamma(v)$ is obtained from $\Gamma(u)$ by switching 1 with one of its neighbors.

Similarly, we define the concept of switching 0 with one of its neighbors. The binary relation induced by this new concept is denoted by $\mathcal{S}_1$. Let

$$\mathcal{S}_e = \mathcal{S} \cup \mathcal{S}_0 \cup \mathcal{S}_1.$$

The relations $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{S}_e$ are clearly reflexive and symmetric.

For $u \in F_2(X)$ define

$$\gamma_0(u) = \begin{cases} u' & \text{if } u = u' \wedge (uR) \text{ for some } u' \in F_2(X) \\ u & \text{otherwise,} \end{cases}$$

$$\gamma_1(u) = \begin{cases} u' & \text{if } u = (uL) \wedge u' \text{ for some } u' \in F_2(X) \\ u & \text{otherwise,} \end{cases}$$

$$\gamma(u) = \begin{cases} u' & \text{if } u = u' \wedge (uR) \text{ or } u = (uL) \wedge u' \text{ for some } u' \in F_2(X) \\ u & \text{otherwise.} \end{cases}$$

Thus, if $u \in RF_2(X)$ then $\gamma_0(u)$, $\gamma_1(u)$, $\gamma(u) \in RF_2(X)$. Further, if $u\, \mathcal{S}_0\, v$ and $u = \gamma_0(u) \in RF_2(X)$, then $v \in RF_2(X)$. Similarly, if $u\, \mathcal{S}_1\, v$ and $u = \gamma_1(u) \in RF_2(X)$, then $v \in RF_2(X)$. Thus if $u\, \mathcal{S}_e\, v$ and $u = \gamma(u) \in RF_2(X)$, then $v \in RF_2(X)$. Let $\mathcal{S}_e^*$ denote the transitive closure of $\mathcal{S}_e$.

Let $u \in RF_2(X)$. We denote by $R_{[u]}$ and $L_{[u]}$ the $\mathcal{R}$-class and the $\mathcal{L}$-class, respectively, of $[u]$ in $F_{PS}(X)$.

Proposition 3.1 *If $u \in RF_2(X)$, then*

(*i*) $R_{[u]} = \{[v] \mid \gamma_0(u)\, \mathcal{S}_0\, v\} \cup \{[u] \wedge [uR]\}$;

(*ii*) $L_{[u]} = \{[v] \mid \gamma_1(u)\, \mathcal{S}_1\, v\} \cup \{[uL] \wedge [\gamma_1(u)]\}$.

Proof: (*i*). Clearly $[u]\, \mathcal{R}\, [\gamma_0(u)]$. Let $\Gamma(\gamma_0(u)) = (\Gamma, l)$, $uR = x = (\gamma_0(u))R$, and $\psi_h\varphi_k([\gamma_0(u)]) = (A, y)$. Then $\psi_h\varphi_k([v]) = (A, y')$ for any $v \in F_2(X)$ such that $\gamma_0(u)\, \mathcal{S}_0\, v$. Furthermore, $\psi_h\varphi_k([u_1]) = (A \cup \{x^2\}, x)$ for $u_1 \in RF_2(X)$ such that $[u_1] = [\gamma_0(u)] \wedge [uR] = [u] \wedge [uR]$. Therefore

$$R_{[u]} \supseteq \{[v] \mid \gamma_0(u)\, \mathcal{S}_0\, v\} \cup \{[u] \wedge [uR]\}$$

by Corollary 2.7.

Let $v \in RF_2(X)$ such that $[v] \in R_{[u]}$ and $[v] \neq [u] \wedge [uR]$. Let $\psi_h\varphi_k([v]) = (A', y')$. Then $A \cup \{x^2\} = A' \cup \{x^2\}$ due to Corollary 2.7, and $xy' \in A'$. Since $[v] \neq [u] \wedge [uR]$, we must have $A = A'$. Thus, there is $k \geq 0$ such that $0^k 1 \in \Gamma$ and $(0^k 1)_l^- = y'$. Let $u' \in RF_2(X)$ such that $\Gamma(u')$ is obtained from $\Gamma(\gamma_0(u))$ by switching 1 with $0^k 1$, and observe that $\psi_h\varphi_k([u']) = (A, y')$. Hence $[v] = [u']$ and

$$R_{[u]} = \{[v] \mid \gamma_0(u)\, \mathcal{S}_0\, v\} \cup \{[u] \wedge [uR]\}.$$

(*ii*) is shown using dual arguments. We need to consider the dual result of Corollary 2.7 for the dual model of $\mathcal{H}$. ■

Now, the following corollary can be proved easily:

Corollary 3.2 *Let $u \in RF_2(X)$, $uR = x$, $uL = y$, and $\Gamma(u) = (\Gamma, l)$. Let $k_0 = \max\{k \geq 0 \mid 0^k \in \Gamma\}$ and $k_1 = \max\{k \geq 0 \mid 1^k \in \Gamma\}$. Then*

(*i*) *$|R_{[u]}| = k_0$ if $(0^k 1)_l^- = x$ for some $0 \leq k < k_0$, or $|R_{[u]}| = k_0 + 1$ otherwise,*

(*ii*) *$|L_{[u]}| = k_1$ if $(1^k 0)_l^- = y$ for some $0 \leq k < k_1$, or $|L_{[u]}| = k_1 + 1$ otherwise.* ■

Let F be a pseudosemilattice. An E-**sequence** in F is a sequence $e_0, e_1, \cdots, e_n \in F$ such that $e_{i-1}\, (\mathcal{R} \cup \mathcal{L})\, e_i$ for $1 \leq i \leq n$. We say that an E-sequence $e_0, \cdots, e_n \in F$ is an E-**chain** if (*i*) $e_i \neq e_j$ for $i \neq j$, and (*ii*) $e_{i-1}\, \mathcal{R}\, e_i$ if and only if $e_i\, \mathcal{L}\, e_{i+1}$ for $1 \leq i \leq n-1$. Since $\mathcal{R}$ and $\mathcal{L}$ are equivalence relations in F, we can construct an E-chain from e to f from any E-sequence $e = e_0, e_1, \cdots, e_n = f$ by deleting some e_i. We say that $e, f \in F$ are **connected** if there is an E-chain (or an E-sequence) $e_0, e_1, \cdots, e_n \in F$ such that $e_0 = e$ and $e_n = f$.

Proposition 3.3 *Let $u, v \in RF_2(X)$. Then $[u]$ and $[v]$ are connected elements of $F_{PS}(X)$ if and only if $\gamma(u)\, \mathcal{S}_e^*\, \gamma(v)$.*

Proof: Assume that $[u]$ and $[v]$ are connected elements of $F_{PS}(X)$. Then $[\gamma(u)]$ and $[\gamma(v)]$ are connected too, and there exists an E-chain

$$[w_0],\ [w_1],\ \cdots,\ [w_n] \in F_{PS}(X)$$

such that $\gamma(u) = w_0$, $\gamma(v) = w_n$, and $w_i \in RF_2(X)$ for every $0 \leq i \leq n$.

Observe that if $\gamma(w) \neq w$ for some $w \in RF_2(X)$, then $|R_{[w]}| = 1$ or $|L_{[w]}| = 1$ due to Corollary 3.2. Thus, $\gamma(w_i) = w_i$ for any $0 \leq i \leq n$ since $[w_0], [w_1], \cdots, [w_n]$ is an E-chain. Hence, there is $w_i' \in [w_i]$ such that $w_{i-1}\, \mathcal{S}_e\, w_i'$ due to Proposition 3.1 for all $0 \leq j \leq n$. Therefore $w_{i-1}\, \mathcal{S}_e^*\, w_i$ for all $1 \leq i \leq n$, and so $\gamma(u)\, \mathcal{S}_e^*\, \gamma(v)$.

Assume $\gamma(u)\, \mathcal{S}_e^*\, \gamma(v)$. There are $w_0, \cdots, w_n \in RF_2(X)$ such that $w_0 = \gamma(u)$, $w_n = \gamma(v)$, and $w_{i-1}\, \mathcal{S}_e\, w_i$ for all $1 \leq i \leq n$. Thus $[w_{i-1}]\, \mathcal{R} \cup \mathcal{L}\, [w_i]$ by Proposition 3.1, and $[\gamma(u)]$ and $[\gamma(v)]$ are connected. Therefore $[u]$ and $[v]$ are connected too. ■

Next, we define the relation $\mathcal{E}_e$ on $F_2(X)$. The relation $\mathcal{E}_e$ is a generalization of the relation $\mathcal{E}$ introduced in [6]. Let $u \in F_2(X)$, $\Gamma(u) = (\Gamma, l)$, and $a = bd \in \Gamma$ such that $b \in \mathcal{B}^*$ and $d \in \mathcal{B}$. Let $e \in \mathcal{B} \setminus \{d\}$. Then we denote b and be by $\overleftarrow{a}$ and $\overleftrightarrow{a}$, respectively. Let now $a \in \Gamma$ such that 01 or 10 is a suffix of a. Define

$$\Gamma' = \{b \in \Gamma \mid b \not\leq \overleftarrow{a}\} \cup \{\overleftarrow{a}b' \mid \overleftrightarrow{a}b' \in \Gamma\}$$

and $l' : \Gamma'_L \longrightarrow X$ as follows:

$$c_{l'} = \begin{cases} c_l & \text{if } c \not\leq \overleftarrow{a} \\ (\overleftrightarrow{a}b')_l & \text{if } c = \overleftarrow{a}b' \end{cases}$$

Then (Γ', l') is a labeled binary poset, and let $v \in F_2(X)$ such that $\Gamma(v) = (\Gamma', l')$. We say that $\Gamma(v)$ is obtained from $\Gamma(u)$ by erasing the vertex a. We define $\mathcal{E}_e$ to be the set of all pairs $(u, v) \in F_2(X) \times F_2(X)$ such that $\Gamma(v)$ is obtained from $\Gamma(u)$ by erasing some vertex. We denote by $\mathcal{E}_e^*$ the reflexive and transitive closure of $\mathcal{E}_e$.

Proposition 3.4 *Let $u, v \in RF_2(X)$. Then $[v]\,\omega\,[u]$ in $F_{PS}(X)$ if and only if $v\,\mathcal{E}_e^*\,u'$ for some $u' \in [u]$.*

Proof: ($\Rightarrow$). Assume $[v]\,\omega\,[u]$. Then $uR = vR$, $uL = vL$, and $|c(v)| \geq |c(u)|$. If $|c(u)| = 1$, then $[u] = [x]$ and $x = vR = vL$ for some $x \in X$. This case is now immediate. We shall assume that $|c(u)| \geq 2$. Let $\varphi_k([u]) = (A_0, A_1)$ and $\varphi_k([v]) = (B_0, B_1)$. Thus $A_0 \subseteq B_0$ and $A_1 \subseteq B_1$ (Proposition 2.6(iii)).

Let $v_0 = v$ and $\Gamma(v_0) = (\Gamma_0, l_0)$. Assume $A_0 \neq B_0$ and let s_0 be a word of minimal length amongst all words from $B_0 \setminus A_0$. Clearly $|s_0| \geq 2$ and there exists $b_0 \in (\Gamma_0)_L$ such that $(b_0)_\lambda = 0$ and $k_{v_0}(b_0) = s_0$. Furthermore $b_0 = ad^k$ for some $k \geq 1$, $d \in \{0, 1\}$, and $a \neq \iota$ such that $a_\tau \neq d$. Let v_1 be such that $\Gamma(v_1)$ is obtained from $\Gamma(v_0)$ by erasing the vertex ad. Clearly $|c(v_1)| \geq |c(u)| \geq 1$. By the minimality of the length of s_0, observe that $v_1 \in RF_2(X)$ and $\varphi_k([v_1]) = (B'_0, B_1)$ for some $B'_0 \subseteq X^+$ such that $A_0 \subseteq B'_0 \subsetneqq B_0$.

The process explained above can be applied recursively to construct a sequence

$$v_0, v_1, \cdots, v_n$$

of words from $RF_2(X)$ such that $v_{i-1}\,\mathcal{E}_e\,v_i$ and $\varphi_k([v_n]) = (A_0, B_0)$. Let $u' = v_n$. Then $[u'] = [u]$ and $v\,\mathcal{E}_e^*\,u'$.

($\Leftarrow$). Assume $v\mathcal{E}_e^* u'$ for some $u' \in [u]$. Then $\Gamma(v)$ is obtained from $\Gamma(v \wedge u')$ by applying $\mathcal{M}$ and $\mathcal{E}$ successively, and so $[v] = [v] \wedge [u'] = [v] \wedge [u]$. Similarly $[u] \wedge [v] = [v]$ and consequently $[v]\,\omega\,[u]$. ∎

Proposition 3.5 *Let $u, v \in RF_2(X)$ such that $\varphi_k([u]) = (A_0, A_1)$ and $\varphi_k([v]) = (B_0, B_1)$. Let $uR = x_0$ and $uL = x_1$.*

(i) *$[v]\,\omega^r\,[u]$ if and only if*

- (a) *$x_0 = x_1$, or $x_1 = vL$, or $x_0x_1 \in B_0$,*
- (b) *there are $v' \in [v] \wedge [x_1]$ and $u' \in [u]$ such that $v'\,\mathcal{E}_e^*\,u'$.*

(ii) *$[v]\,\omega^l\,[u]$ if and only if*

- (c) *$x_0 = x_1$, or $x_0 = vR$, or $x_1x_0 \in B_1$,*
- (d) *there are $v' \in [x_0] \wedge [v]$ and $u' \in [u]$ such that $v'\,\mathcal{E}_e^*\,u'$.*

Proof: Observe that $[v]\,\omega^r\,[u]$ if and only if $[v]\,\mathcal{R}\,[v] \wedge [x_1]\,\omega[u]$. Thus ($i$) follows from Propositions 3.1 and 3.4. The proof of (ii) is similar. ∎

4 Morphisms.

In this section we introduce the notion of morphism between labeled binary posets. Let $u, v \in F_2(X)$. A **morphism** from $\Gamma(u) = (\Gamma, l)$ to $\Gamma(v) = (\Gamma', l')$ is a mapping $\varphi : \Gamma \longrightarrow \Gamma'$ such that

(M_1) for every $a \in \Gamma_L$, $(\varphi(a))_{l'} = a_l$,

(M_2) if $a \in \Gamma \setminus \Gamma_L$, then $\varphi(a) \notin \Gamma'_L$, $\varphi((a0)^-) = (\varphi(a)0)^-$ and $\varphi((a1)^-) = (\varphi(a)1)^-$.

The proof of the next result is long but straightforward, and we decided to omit it here. However a complete proof can be found in Lemmas 4.16, 4.17 and 4.18 of [7].

Lemma 4.1 *Let $\Gamma(u) = (\Gamma, l)$ and $\Gamma(v) = (\Gamma', l')$ for some $u, v \in F_2(X)$. Let $\varphi : \Gamma \longrightarrow \Gamma'$ be a morphism from $\Gamma(u)$ to $\Gamma(v)$.*

(i) φ is one-to-one if and only if $\varphi : \Gamma_L \longrightarrow \Gamma'_L$ is one-to-one.

(ii) φ is surjective if and only if $\varphi : \Gamma_L \longrightarrow \Gamma'_L$ is surjective.

(iii) If $u \in RF_2(X)$, then φ is one-to-one.

(iv) If $u \in RF_2(X)$ and φ is surjective, then $v \in RF_2(X)$ except if $\varphi(\iota) \neq \iota$ and $\gamma(u) \neq u$. ■

Let $u, v \in F_2(X)$ such that $u \,\mathcal{S}_e\, v$, $\Gamma(u) = (\Gamma, l)$ and $\Gamma(v) = (\Gamma', l')$. Let $a_0, a_1 \in \Gamma$ such that $\overleftarrow{a_0} \preccurlyeq \overleftarrow{a_1}$ and $\Gamma(v)$ is obtained from $\Gamma(u)$ by switching a_0 and a_1. Then either

- 01 or 10 is a suffix of both a_0 and a_1 (if $u \,\mathcal{S}\, v$), or
- $a_1 = 1$ and $a_0 = 0^k 1$ for some $k \geq 0$ (if $u \,\mathcal{S}_0\, v$), or
- $a_1 = 0$ and $a_0 = 1^k 0$ for some $k \geq 0$ (if $u \,\mathcal{S}_1\, v$).

Obviously $\Gamma' = \{a \in \Gamma \mid a \not\preccurlyeq a_0,\, a \not\preccurlyeq a_1\} \cup \{a_1 b \mid a_0 b \in \Gamma\} \cup \{a_0 b \mid a_1 b \in \Gamma\}$. Let $d = (a_0)_\tau = (a_1)_\tau$ and define the mapping $\varphi : \Gamma \longrightarrow \Gamma'$ as follows:

$$\varphi(c) = \begin{cases} \overleftarrow{a_1} c' & \text{if } c = \overleftarrow{a_0} c' \text{ for some } c' \in \{\iota\} \cup d\,\mathcal{B}^* \\ \overleftarrow{a_0} c' & \text{if } c = \overleftarrow{a_1} c' \text{ for some } c' \in \{\iota\} \cup d\,\mathcal{B}^* \\ c & \text{otherwise.} \end{cases}$$

The mapping φ is a natural isomorphism from $\Gamma(u)$ to $\Gamma(v)$. Furthermore, $\varphi(\iota) = \iota$, $\varphi(\overline{0}) = \overline{0}$ and $\varphi(\overline{1}) = \overline{1}$ if $u\,\mathcal{S}\,v$; $\varphi(\overline{0}) = \overline{0}$ if $u\,\mathcal{S}_0\,v$; and $\varphi(\overline{1}) = \overline{1}$ if $u\,\mathcal{S}_1\,v$.

Proposition 4.2 *Let $u, v \in F_2(X)$ such that $|c(u)| \geq 2$.*

(i) *There is an isomorphism φ from $\Gamma(u)$ to $\Gamma(v)$ such that $\varphi(\iota) = \iota$ if and only if $[u] = [v]$.*

(ii) *There is an isomorphism φ_0 from $\Gamma(u)$ to $\Gamma(v)$ such that $\varphi_0(\overline{0}) = \overline{0}$ if and only if $u\,\mathcal{S}_0\,v'$ for some $v' \in [v]$.*

(iii) *There is an isomorphism φ_1 from $\Gamma(u)$ to $\Gamma(v)$ such that $\varphi_1(\overline{1}) = \overline{1}$ if and only if $u\,\mathcal{S}_1\,v'$ for some $v' \in [v]$.*

Proof: Let $\Gamma(u) = (\Gamma, l)$ and $\Gamma(v) = (\Gamma', l')$.

(i). ($\Rightarrow$). Assume there is an isomorphism φ from $\Gamma(u)$ to $\Gamma(v)$ such that $\varphi(\iota) = \iota$. We show that $[u] = [v]$ by induction on $|\Gamma|$. The statement of this proposition assumes $|c(u)| \geq 2$. However, to help us implement the induction process, we do not consider this assumption on u for this implication. Thus, we show, by induction on $|\Gamma|$, that $[u] = [v]$ even if $|c(u)| = 1$. If $|\Gamma| = 1$, then $u = x = v$ for some $x \in X$.

Let $|\Gamma| = n \geq 2$. Then $|\Gamma'| = |\Gamma| \geq 2$. Let $u_0, u_1, v_0, v_1 \in F_2(X)$ such that $u = u_0 \wedge u_1$ and $v = v_0 \wedge v_1$. Let $\Gamma(u_i) = (\Gamma_i, l_i)$ and $\Gamma(v_i) = (\Gamma'_i, l'_i)$ for $i \in \{0, 1\}$. Since $\varphi(\iota) = \iota$, we must have $\varphi(\overline{0}) = \overline{0}$ and $\varphi(\overline{1}) = \overline{1}$.

Let $a \in \Gamma$ such that $a_\lambda = 0$. Let $\nu(a) = (a_0, a_1, \cdots, a_k)$ for some $k \geq 1$. If $k = 1$, then $a = 0^i$ and $\varphi(a) = 0^j$ for some $i, j \geq 1$. Assume $k > 1$ and $(\varphi(a_i))_\lambda = 0$ for all $1 \leq i < k$. Let $a = a_k = a_{k-1}d_k^{i_k}$ for some $i_k \geq 1$ and $d_k \in \{0, 1\}$. Then

$$\varphi(\overline{a_k}) = \varphi(a_{k-1})d_k^i \text{ and } \varphi(\overline{a_k}) = \varphi(a_k)d_k^j$$

for some $i \geq 1$ and $j \geq 0$, and $(\varphi(a_k))_\lambda = (\varphi(a_{k-1}))_\lambda = 0$ since $\varphi(a_k) \neq \iota$.

We have just shown (by induction on k) that $(\varphi(a))_\lambda = 0$ for every $a \in \Gamma$ such that $a_\lambda = 0$. Similarly, $(\varphi(a))_\lambda = 1$ for every $a \in \Gamma$ such that $a_\lambda = 1$. As a consequence we have

$$\{\varphi(0a) \mid 0a \in \Gamma\} = \{b \in \Gamma' \mid b \preccurlyeq 0\},$$

$$\{\varphi(1a) \mid 1a \in \Gamma\} = \{b \in \Gamma' \mid b \preccurlyeq 1\},$$

and thus $\varphi_0 : \Gamma_0 \longrightarrow \Gamma'_0$ and $\varphi_1 : \Gamma_1 \longrightarrow \Gamma'_1$ defined by $\varphi(0a) = 0\varphi_0(a)$ and $\varphi(1a) = 1\varphi_1(a)$, respectively, are well defined isomorphisms from $\Gamma(u_0)$ to $\Gamma(v_0)$ and from $\Gamma(u_1)$ to $\Gamma(v_1)$, respectively.

Let $b_0 = \varphi_0(\iota)$. Then $0b_0 = \varphi(0) = 0^k$ for some $k \geq 1$. Hence $b_0 = 0^{k_0}$ for some $k_0 \geq 0$. If $b_0 \neq \iota$, then let $v'_0 \in F_2(X)$ such that $\Gamma(v'_0)$ is obtained from $\Gamma(v_0)$ by switching $b_0 1$ and 1, and let ψ_0 be the natural isomorphism from $\Gamma(v_0)$ to $\Gamma(v'_0)$ as described prior to this proposition. If $b_0 = \iota$, let $v'_0 = v_0$ and ψ_0 be the identity automorphism of $\Gamma(v_0)$. Thus, in both cases, $v_0 \,\mathcal{S}_0\, v'_0$ and $\psi_0 \circ \varphi_0$ is an isomorphism from $\Gamma(u_0)$ to $\Gamma(v'_0)$ such that $\psi_0 \circ \varphi_0(\iota) = \iota$. By induction hypothesis ($|\Gamma_0| < |\Gamma|$) we have $[u_0] = [v'_0]$.

In the same way we can define a word v'_1 such that $v_1 \,\mathcal{S}_1\, v'_1$ and $[u_1] = [v'_1]$. Thus $[u] = [v'_0 \wedge v'_1] = [v_0 \wedge v_1] = [v]$.

($\Leftarrow$). If $[u] = [v]$, then $u \,\mathcal{S}^*\, v$ because $|c(u)| \geq 2$. This implication follows now from the observation made prior to this proposition.

(ii). ($\Rightarrow$). Assume there is an isomorphism φ_0 from $\Gamma(u)$ to $\Gamma(v)$ such that $\varphi_0(\overline{0}) = \overline{0}$. Then $\varphi_0^{-1}(\iota) = 0^k$ for some $k \geq 0$. Let v' such that $\Gamma(v')$ is obtained from $\Gamma(u)$ by switching 1 with $0^k 1$. Hence $u \,\mathcal{S}_0\, v'$. Let φ be the natural isomorphism from $\Gamma(u)$ to $\Gamma(v')$ as described prior to this proposition, and observe that $\varphi_0 \circ \varphi^{-1}$ is an isomorphism from $\Gamma(v')$ to $\Gamma(v)$ such that $\varphi_0 \circ \varphi^{-1}(\iota) = \iota$. Consequently $[v'] = [v]$ by *(i)*.

($\Leftarrow$). If $u \,\mathcal{S}_0\, v'$ for some $v' \in [v]$, then let φ be the natural isomorphism from $\Gamma(u)$ to $\Gamma(v')$, and let ψ_0 be the isomorphism from $\Gamma(v')$ to $\Gamma(v)$ given by *(i)*. Thus $\varphi(\overline{0}) = \overline{0}$, $\psi_0(\overline{0}) = \overline{0}$, and $\varphi_0 = \psi_0 \circ \varphi$ is an isomorphism from $\Gamma(u)$ to $\Gamma(v)$ such that $\varphi_0(\overline{0}) = \overline{0}$.

The proof of *(iii)* is analogous to the proof of *(ii)*. ∎

Proposition 4.3 *Let $u, v \in F_2(X)$ such that $|c(u)| \geq 2$. Then $u \,\mathcal{S}_e^*\, v$ if and only if $\Gamma(u)$ and $\Gamma(v)$ are isomorphic.*

Proof: The direct implication follows from Proposition 4.2. Assume that φ is an isomorphism from $\Gamma(u)$ to $\Gamma(v)$. Let $\Gamma(u) = (\Gamma, l)$ and $a \in \Gamma$ such that $\varphi(a) = \iota$. If $a = \iota$ then $u \,\mathcal{S}^*\, v$ by Proposition 4.2*(i)*.

We shall assume that $a \neq \iota$. Then $a = d_1^{k_1} \cdots d_n^{k_n}$ for some $d_i \in \{0, 1\}$ and $k_i \geq 1$ such that $d_i \neq d_{i-1}$. Let $d_{n+1} \in \{0, 1\}$ such that $d_n \neq d_{n+1}$. Observe that $ad_{n+1} \in \Gamma$. For $i \in \{0, \cdots, n\}$, define recursively $u_i \in F_2(X)$ as follows: $u_0 = u$ and $\Gamma(u_i)$ is obtained from $\Gamma(u_{i-1})$ by switching d_{i+1} with $d_i^{k_i} d_{i+1}$.

By construction we have $u_{i-1} \,\mathcal{S}_e\, u_i$, and so $u \,\mathcal{S}_e^*\, u_n$. Let ψ_i be the natural isomorphism from $\Gamma(u_{i-1})$ to $\Gamma(u_i)$. Then $\psi = \psi_n \circ \cdots \circ \psi_1$ is an isomorphism from $\Gamma(u)$ to $\Gamma(u_n)$ such that $\psi(a) = \iota$, and $\varphi' = \varphi \circ \psi^{-1}$ is an isomorphism from $\Gamma(u_n)$ to $\Gamma(v)$ such that $\varphi'(\iota) = \iota$. Thus $u_n \, \mathcal{S}^* \, v$ (Proposition 4.2(i)) and $u \, \mathcal{S}_e^* \, v$. ∎

The next result follows from Propositions 4.3 and 3.3.

Corollary 4.4 *Let $u, v \in RF_2(X)$. Then $[u]$ and $[v]$ are connected elements in $F_{PS}(X)$ if and only if $\Gamma(\gamma(u))$ and $\Gamma(\gamma(v))$ are isomorphic.* ∎

Proposition 4.5 *Let $u, v \in F_2(X)$ such that $|c(u)| \geq 2$. Then $v \, \mathcal{E}_e^* \, u'$ for some $u' \in [u]$ if and only if there is an embedding φ of $\Gamma(u)$ into $\Gamma(v)$ such that $\varphi(\iota) = \iota$.*

Proof: ($\Rightarrow$). Assume first that $v \, \mathcal{E}_e \, u'$ for some $u' \in [u]$. Then $\Gamma(u') = (\Gamma', l')$ is obtained from $\Gamma(v) = (\Gamma, l)$ by erasing some $a \in \Gamma$ such that 01 or 10 is a suffix of a. Let $d \in \mathcal{B}$ such that $d \neq a_\tau$ and define $\psi : \Gamma' \longrightarrow \Gamma$ by

$$\psi(b) = \begin{cases} b & \text{if } b \not\preccurlyeq \overleftarrow{a} \\ \overleftarrow{a}dc & \text{if } b = \overleftarrow{a}c \text{ for some } c \in \mathcal{B}^* \end{cases}$$

Observe that ψ is an embedding of $\Gamma(u')$ into $\Gamma(v)$ such that $\psi(\iota) = \iota$. By Proposition 4.2(i) there is an embedding φ of $\Gamma(u)$ into $\Gamma(v)$ such that $\varphi(\iota) = \iota$. The general case now follows immediately.

($\Leftarrow$). Let φ be an embedding of $\Gamma(u) = (\Gamma_1, l_1)$ into $\Gamma(v) = (\Gamma, l)$ such that $\varphi(\iota) = \iota$. Let

$$\Omega = \{a \in \Gamma \mid \varphi(b) \neq a \text{ for all } b \in \Gamma_1\}.$$

We show there is $u' \in [u]$ such that $v \, \mathcal{E}_e^* \, u'$ by induction on $|\Omega|$. If $\Omega = \emptyset$, then φ is an isomorphism and $v \in [u]$ (Proposition 4.2(i)). We shall assume $|\Omega| \geq 1$. Let $\Omega' = \{a \in \Omega \mid b \in \Omega \text{ for every } b \preccurlyeq a\}$. Clearly $\Omega' \neq \emptyset$. Let a be an element of minimal length amongst all elements of Ω'. Observe that 01 or 10 is a suffix of a since $\varphi(\iota) = \iota$. Let $v' \in F_2(X)$ such that $\Gamma(v') = (\Gamma', l')$ is obtained from $\Gamma(v)$ by erasing $a \in \Gamma$. Thus $v \, \mathcal{E}_e \, v'$. Furthermore, the mapping $\varphi' : \Gamma_1 \longrightarrow \Gamma'$ defined by

$$\varphi'(b) = \begin{cases} \varphi(b) & \text{if } \varphi(b) \not\preccurlyeq \overleftarrow{a} \\ \overleftarrow{a}c & \text{if } \varphi(b) = \overleftarrow{a}dc \text{ for some } d \in \mathcal{B} \text{ and } c \in \mathcal{B}^* \end{cases}$$

is an embedding of $\Gamma(u)$ into $\Gamma(v')$ such that $\varphi'(\iota) = \iota$. Since

$$|\Omega| > |\{c \in \Gamma' \mid \varphi'(b) \neq c \text{ for any } b \in \Gamma_1\}|\,,$$

there is $u' \in [u]$ such that $v' \,\mathcal{E}_e^*\, u'$ by induction hypothesis. Therefore $v \,\mathcal{E}_e^*\, u'$ for some $u' \in [u]$. ■

Corollary 4.6 *Let $u, v \in RF_2(X)$ such that $|c(u)| \geq 2$.*

(*i*) *$[v]\,\mathcal{R}\,[u]$ in $F_{PS}(X)$ if and only if there is an isomorphism φ from $\Gamma(\gamma_0(u))$ to $\Gamma(\gamma_0(v))$ such that $\varphi(\overline{0}) = \overline{0}$;*

(*ii*) *$[v]\,\mathcal{L}\,[u]$ in $F_{PS}(X)$ if and only if there is an isomorphism φ from $\Gamma(\gamma_1(u))$ to $\Gamma(\gamma_1(v))$ such that $\varphi(\overline{1}) = \overline{1}$;*

(*iii*) *$[v]\,\omega\,[u]$ in $F_{PS}(X)$ if and only if there is an embedding φ of $\Gamma(u)$ into $\Gamma(v)$ such that $\varphi(\iota) = \iota$;*

(*iv*) *$[v]\,\omega^r[u]$ in $F_{PS}(X)$ if and only if there is an embedding φ of $\Gamma(\gamma_0(u))$ into $\Gamma(\gamma_0(v))$ such that $\varphi(\overline{0}) = \overline{0}$;*

(*v*) *$[v]\,\omega^l[u]$ in $F_{PS}(X)$ if and only if there is an embedding φ of $\Gamma(\gamma_1(u))$ into $\Gamma(\gamma_1(v))$ such that $\varphi(\overline{1}) = \overline{1}$.*

Proof: From Proposition 3.1(*i*), $[v]\,\mathcal{R}\,[u]$ if and only if $\gamma_0(u)\,\mathcal{S}_0\,v'$ for some $v' \in [\gamma_0(v)]$. (*i*) follows now from Proposition 4.2(*ii*). The statement (*ii*) is the dual of (*i*). (*iii*) follows from Propositions 3.4 and 4.5.

(*iv*). ($\Rightarrow$). Assume $[v]\,\omega^r[u]$ in $F_{PS}(X)$. Then $[v]\,\mathcal{R}\,[w]\,\omega\,[u]$ for some $w \in RF_2(X)$ such that $[w] = [v] \wedge [uL]$. Clearly $|c(w)| \geq 2$. By (*i*), there is an isomorphism φ' from $\Gamma(\gamma_0(w))$ to $\Gamma(\gamma_0(v))$ such that $\varphi'(\overline{0}) = \overline{0}$; and by (*iii*), there is an embedding ψ' of $\Gamma(u)$ into $\Gamma(w)$ such that $\psi'(\iota) = \iota$. Since $\gamma_0(w) = w$ if $\gamma_0(u) = u$, ψ' induces an embedding ψ of $\Gamma(\gamma_0(u))$ into $\Gamma(\gamma_0(w))$ such that $\psi(\overline{0}) = \overline{0}$. Thus $\varphi = \varphi' \circ \psi$ is an embedding of $\Gamma(\gamma_0(u))$ into $\Gamma(\gamma_0(v))$ such that $\varphi(\overline{0}) = \overline{0}$.

($\Leftarrow$). Let φ be an embedding of $\Gamma(\gamma_0(u))$ into $\Gamma(\gamma_0(v))$ such that $\varphi(\overline{0}) = \overline{0}$. Let $a = \varphi(\iota)$. Then $a = 0^k$ for some $k \geq 0$. Let $w \in RF_2(X)$ such that $\Gamma(w)$ is obtained from $\Gamma(\gamma_0(v))$ by switching 1 with 0^k1 and let ψ be the natural isomorphism from $\Gamma(\gamma_0(v))$ to $\Gamma(w)$. Then $[w]\,\mathcal{R}\,[v]$ and $\varphi' = \psi \circ \varphi$ is an embedding of $\Gamma(\gamma_0(u))$ into $\Gamma(w)$ such that $\varphi'(\iota) = \iota$. Thus $[v]\,\mathcal{R}\,[w]\,\omega\,[\gamma_0(u)]\,\mathcal{R}\,[u]$ and $[v]\,\omega^r[u]$.

(*v*) is the dual of (*iv*). ■

5 Connectedness in $F_{PS}(X)$.

Let $w \in F_2(X)$ and $\Gamma(w) = (\Gamma, l)$. We say that $b \in \Gamma_L$ is **paired** in $\Gamma(w)$ if there is $a \in \Gamma \setminus \Gamma_L$ such that $(a0)_l^- = (a1)_l^-$ and $b \in \{(a0)^-, (a1)^-\}$. We denote the set of all paired elements of $\Gamma(w)$ by $\Gamma_p(w)$.

Let $u \in RF_2(X)$ such that $|c(u)| \geq 2$, and $\Gamma(\gamma(u)) = (\Gamma, l)$. Define $u_\iota = \gamma(u)$. Let $a \in \Gamma \setminus \Gamma_L$. Then $a = d_1^{k_1} \cdots d_n^{k_n}$ for some $n \geq 1$, $k_i \geq 1$, and $d_i \in \{0,1\}$ such that $d_{i-1} \neq d_i$, where $i \in \{1, \cdots, n\}$. Let $d_{n+1} \in \{0,1\}$ such that $d_n \neq d_{n+1}$. Recursively, define $u'_0 = \gamma(u)$ and $u'_i \in F_2(X)$ such that $\Gamma(u'_i)$ is obtained from $\Gamma(u'_{i-1})$ by switching d_{i+1} with $d_i^{k_i} d_{i+1}$, for $i \in \{1, \cdots, n\}$. Thus $\gamma(u) \, \mathcal{S}_e^* \, u'_n$, $u'_n \in RF_2(X)$, and $\gamma(u'_n) = u'_n$. Denote u'_n by u_a. Let φ'_i be the natural isomorphism from $\Gamma(u'_{i-1})$ to $\Gamma(u'_i)$. Then $\varphi_a = \varphi'_n \circ \cdots \circ \varphi'_1$ is an isomorphism from $\Gamma(\gamma(u))$ onto $\Gamma(u_a)$ such that $\varphi_a(a) = \iota$.

Consider now $b \in \Gamma_L$ and let $a \in \Gamma$ such that $b = ad$ for some $d \in \{0,1\}$. Let $x = b_l \in X$ and define $u_b \in F_2(X)$ as follows:

$$u_b = \begin{cases} u_a \wedge x & \text{if } d = 0 \\ x \wedge u_b & \text{if } d = 1. \end{cases}$$

In fact, if $b = cd^k$ for some $k \geq 1$ and

$$u' = \begin{cases} u_c \wedge x & \text{if } d = 0 \\ x \wedge u_c & \text{if } d = 1, \end{cases}$$

then $[u'] = [u_b]$. Observe that $\gamma(u_b) = u_a$ and that $u_b \in RF_2(X)$ if and only if $b \notin \Gamma_p(\gamma(u))$. Furthermore $u_b \in RF_2(X)$ if and only if $u' \in RF_2(X)$.

Let $D_{[u]} = \{[u_a] \in F_{PS}(X) \mid a \in \Gamma \setminus \Gamma_p(\gamma(u))\}$ for $u \in RF_2(X)$ such that $|c(u)| \geq 2$.

Proposition 5.1 *Let $u \in RF_2(X)$ such that $|c(u)| \geq 2$. Then*

$$D_{[u]} = \{[v] \in F_{PS}(X) \mid [u] \text{ and } [v] \text{ are connected}\}.$$

Proof: Let $\Gamma(\gamma(u)) = (\Gamma, l)$.

($\subseteq$). If $a \in \Gamma \setminus \Gamma_L$, then $\Gamma(\gamma(u))$ and $\Gamma(u_a)$ are isomorphic. Thus $[u]$ and $[u_a]$ are connected by Corollary 4.4. If $b \in \Gamma_L \setminus \Gamma_p(\gamma(u))$ and $b = cd$ for some $c \in \mathcal{B}^*$ and $d \in \{0,1\}$, then $[u]$ and $[u_b]$ are connected since $\gamma(u_b) = u_c$ and $[u]$ and $[u_c]$ are connected. Hence

$$D_{[u]} \subseteq \{[v] \in F_{PS}(X) \mid [u] \text{ and } [v] \text{ are connected}\}.$$

($\supseteq$). Let $v \in RF_2(X)$ such that $[u]$ and $[v]$ are connected and $\gamma(v) = v$. From Corollary 4.4, there is an isomorphism φ from $\Gamma(\gamma(u))$ to $\Gamma(v) = \Gamma(\gamma(v))$. Let $a \in \Gamma$ such that $\varphi(a) = \iota$. Then $\varphi_a \circ \varphi^{-1}$ is an isomorphism from $\Gamma(v)$ to $\Gamma(u_a)$ such that $\varphi_a \circ \varphi^{-1}(\iota) = \iota$. Consequently $[u_a] = [v]$ by Proposition 4.2(i), and $[v] \in D_{[u]}$.

If $\gamma(v) \neq v$, then $v = v' \wedge (vR)$ or $v = (vL) \wedge v'$ for some $v' \in RF_2(X)$. Thus $[u]$ and $[v']$ are connected and $\gamma(v') = v'$. Hence $[v'] = [u_c]$ for some $c \in \Gamma \setminus \Gamma_L$. If $v = v' \wedge (vR)$, then $[v] = [u_b]$ for $b = (c0)^-$. If $v = (vL) \wedge v'$, then $[v] = [u_b]$ for $b = (c1)^-$. Observe also that, in both cases, $b \notin \Gamma_p(\gamma(u))$. Consequently $[v] \in D_{[u]}$. ■

Define $D_{[x]} = \{[x]\}$ for any $x \in X$. Clearly $D_{[x]}$ is the set of all $[v] \in F_{PS}(X)$ such that $[x]$ and $[v]$ are connected. In the following lemma we call a word $u \in F_2(X)$ $\mathcal{M}$-reduced if we cannot apply $\mathcal{M}$ to $\Gamma(u)$.

Lemma 5.2 *If $u \in F_2(X)$ is $\mathcal{M}$-reduced, then there is no automorphism φ of $\Gamma(u)$ such that $\varphi(\iota) \neq \iota$.*

Proof: Assume the statement of this lemma is false and let A be the set of all $\mathcal{M}$-reduced words for which there exists an automorphism φ such that $\varphi(\iota) \neq \iota$. Let u be a word from A of minimal length amongst all words from A and let φ be an automorphism of $\Gamma(u) = (\Gamma, l)$ such that $\varphi(\iota) \neq \iota$. Since u is $\mathcal{M}$-reduced and $\varphi(\iota) \neq \iota$, either $\varphi(\iota) \preccurlyeq 0^l 1$ or $\varphi(\iota) \preccurlyeq 1^l 0$ for some $l \geq 1$. We shall assume $\varphi(\iota) \preccurlyeq 0^l 1$ for some $l \geq 1$. The case $\varphi(\iota) \preccurlyeq 1^l 0$ for some $l \geq 1$ is similar. Thus $(\varphi(0^l))^- = \varphi(\overline{0}) = (\varphi(\iota)0)^-$ and $\varphi(0^l) \preccurlyeq 0^l 1$. So $\varphi(0^l) \neq 0^l$.

Observe that $0^l 1^2 \in \Gamma$ since $\varphi(\iota) \notin \Gamma_L$. Let $u_1 \in F_2(X)$ such that $\Gamma(u_1)$ is obtained from $\Gamma(u)$ by switching 1 with $0^l 1$. Then u_1 is $\mathcal{M}$-reduced, $|u_1| = |u|$, and $0^2, 1^2 \in \Gamma_1$ for $\Gamma(u_1) = (\Gamma_1, l_1)$. Let ψ be the natural isomorphism from $\Gamma(u)$ to $\Gamma(u_1)$. Hence $\varphi_1 = \psi \circ \varphi \circ \psi^{-1}$ is an automorphism of $\Gamma(u_1)$ such that $\varphi_1(\iota) \neq \iota$ since $\varphi(0^l) \neq 0^l$. Thus $u_1 \in A$.

Let $a \in (\Gamma_1)_L$ with suffix 10 or 01. Let $a_0 = a$ and $a_i = \varphi_1^i(a)$ ($\varphi_1^i = \varphi_1 \circ \cdots \circ \varphi_1$ i-times). Then $a_i \in (\Gamma_1)_L$ for all i. Furthermore, 10 or 01 is a suffix of all a_i since φ_1 is an automorphism and $0^2, 1^2 \in \Gamma_1$. Thus there are $i \geq 0$ and $j \geq 1$ such that $a_i = a_{i+j}$, and so we can choose a sequence

$$b_0, \cdots, b_n \in (\Gamma_1)_L$$

for some $n \geq 0$ such that (i) $\varphi_1(b_{i-1}) = b_i$ for $1 \leq i \leq n$ and $\varphi_1(b_n) = b_0$, and (ii) 10 or 01 is a suffix of all b_i.

From (ii) we can apply $\mathcal{E}_e$ and erase all vertices b_i from $\Gamma(u_1)$. Let $u' \in F_2(X)$ such that $\Gamma(u')$ is obtained from $\Gamma(u_1)$ by erasing simultaneously all the vertices $b_0, \cdots, b_n$. Then u' is $\mathcal{M}$-reduced. Due to condition (i), observe that the automorphism φ_1 of $\Gamma(u_1)$ induces naturally an automorphism φ' of $\Gamma(u')$. Clearly $\varphi'(\iota) \neq \iota$ since $\varphi_1(\iota) \neq \iota$. Hence $u' \in A$ and $|u'| < |u_1| = |u|$ contradicting the choice of u. Therefore $A = \emptyset$ and the statement of this lemma is true. ■

Proposition 5.3 *If $u \in RF_2(X)$, then $|D_{[u]}| = |\Gamma(u)| - |\Gamma_p(u)|$.*

Proof: Clearly the result holds true for $|c(u)| = 1$. We shall assume that $|c(u)| \geq 2$. Let $\Gamma(\gamma(u)) = (\Gamma, l)$ and $a, b \in \Gamma \setminus \Gamma_p(\gamma(u))$ such that $[u_a] = [u_b]$. Due to Proposition 4.2(i), there is an isomorphism φ from $\Gamma(u_a)$ to $\Gamma(u_b)$ such that $\varphi(\iota) = \iota$. Thus either $a, b \in \Gamma_L$ or $a, b \notin \Gamma_L$ as otherwise $|u_a| \neq |u_b|$.

If $a, b \notin \Gamma_L$, then there are isomorphisms φ_a and φ_b from $\Gamma(\gamma(u))$ to $\Gamma(u_a)$ and to $\Gamma(u_b)$, respectively, such that $\varphi_a(a) = \iota$ and $\varphi_b(b) = \iota$. If $a \neq b$, then $\varphi' = \varphi \circ \varphi_a \circ \varphi_b^{-1}$ is an automorphism of $\Gamma(u_b)$ such that $\varphi'(\iota) \neq \iota$. Thus $a = b$ by Lemma 5.2.

If $a, b \in \Gamma_L$, then $a_\tau = b_\tau = d \in \{0, 1\}$. Let $a = a_1 d$. There are $b_1 \in \Gamma$ and isomorphism φ_1 from $\Gamma(u_{a_1})$ to $\Gamma(u_{b_1})$ such that $\varphi_1(\iota) = \iota$, for some $b_1 \in \Gamma$ such that $(b_1 d)^- = b$. From the previous paragraph, we conclude that $a_1 = b_1$ since $a_1, b_1 \notin \Gamma_L$. Thus $a = b_1 d = (b_1 d)^- = b$.

It is now clear that $|D_{[u]}| = |\Gamma(\gamma(u))| - |\Gamma_p(\gamma(u))|$. Finally, observe that if $\gamma(u) \neq u$, then $|\Gamma(u)| = |\Gamma(\gamma(u))| + 2$ and $|\Gamma_p(u)| = |\Gamma_p(\gamma(u))| + 2$. Therefore $|D_{[u]}| = |\Gamma(u)| - |\Gamma_p(u)|$. ■

The following corollary is now straightforward:

Corollary 5.4 *If $u \in RF_2(X)$ such that $u \neq x \wedge x$ for any $x \in X$ and $|u| = n$, then $n \leq |D_{[u]}| \leq 2n - 1$.* ■

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