On the Maximal Invariant set*

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Abstract

We discuss the relation between the maximal invariant set of an endomorphism $f : M \to M$ and the intersection of forward iterates of M.

Given an abstract endomorhism on a set $\mathcal{M}$, $f : \mathcal{M} \to \mathcal{M}$, let the **Maximal Invariant set**, $\mathfrak{M}$, be the smallest set containing all sets A such that $f(A) = A$.

First of all we note that $\mathfrak{M}$ exists by Zorn's Lemma provided the set $\{X \subset \mathcal{M} : f(X) = X\}$ is nonempty. Another important issue is that if one considers the invariance condition on A to be either $f(A) \subset A$ or both $f(A) \subset A$ and $f^{-1}(A) \subset A$ then $\mathcal{M}$ is always the maximal invariant set. However, the question is not trivial if we impose the condition: $f(A) = A$.

In this small note we will establish that in at least two different settings - which most maps f considered in applications satisfy - we have that,

$$\cap_{n=0}^{\infty} f^n(\mathcal{M}) = \mathfrak{M}.$$

Nevertheless, in Figure 1 we point out that $\mathfrak{M}$ may be strictly contained in $\cap_{n=0}^{\infty} f^n(\mathcal{M})$.

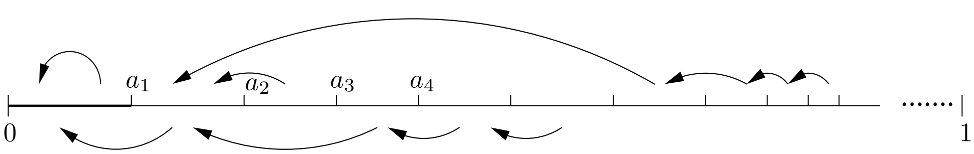

Figure 1: Arrows indicate the dynamics of the map. Note that the interval $[a_1, a_2[$ has preimages of all order but it is not contained in the maximal invariant set, which is $[0, a_1[$.

We say that $\overline{x} = \{x_{-k}\}_{k \in \Lambda}$, where, $\Lambda = \{1, ..., n\}$ for $n \in \mathbb{N}$, or $\Lambda = \mathbb{N}$, is a *backward chain* under the map f of x if $x_0 = x$ and $x_{-k} \in f^{-1}(x_{-k-1})$. The *length* of a chain, $|\overline{x}|$, is the cardinality of Λ. If $\Lambda = \mathbb{N}$ then $|\overline{x}| = \infty$ and then, we say that x admits an *infinite backward chain.*

The concept of backward chain can be used to characterize $\mathfrak{M}$ and will allow us to show that, for a general class of maps which includes injective endomorphisms, $\cap_{n=0}^{\infty} f^n(\mathcal{M})$ is the maximal invariant set.

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Lemma 1 $\mathfrak{M} = \{x \in \mathcal{M} : x$ *admits an infinite backward chain*$\}$.

Proof. Firstly, if x belongs to some invariant set I then there exists a $x_{-1} \in I$ such that $f(x_{-1}) = x$ since $f_{|I}$ is surjective and by the same argument there also exists a preimage x_{-2}, of x_{-1}, and so on.

Let $\mathbb{I} = \{x \in \mathcal{M} : x$ admits an infinite backward chain$\}$. It suffices to show that $\mathbb{I}$ is an invariant set. Since any infinite backward chain $\overline{x}$ can be extended forwards by adding $f(x_0)$ we conclude that $f(\mathbb{I}) \subset \mathbb{I}$. Finally, for every $x \in \mathbb{I}$ take $y = x_{-1}$. It follows that, y also admits an infinite backward chain $\bar{y} = \{y_{-k}\}_{k\geq 0} \equiv \{x_{-n}\}_{n\geq 2}$ and therefore $y \in \mathbb{I}$ which shows that $f_{|\mathbb{I}}$ is surjective since, by definition, $f(y) = x$. ■

In Example 1 one might assert that, although $\mathcal{N} = \cap_{n=0}^{\infty} f^n(\mathcal{M})$ is not the maximal invariant set, the other set $\cap_{n=0}^{\infty} f^n(\mathcal{N})$ is. Therefore, let us consider the following collection of sets: let $\mathcal{M}_0 := \mathcal{M}$ and $\mathcal{M}_n := \cap_{n=0}^{\infty} f^n(\mathcal{M}_{n-1})$ for every $n \in \mathbb{N}$. Note that $f(\mathcal{M}_n) \subset \mathcal{M}_n$ and thus, $\mathcal{M}_{n+1} \subset \mathcal{M}_n$. More importantly, notice that $\mathfrak{M} \subset \mathcal{M}_n$ for all $n \in \mathbb{N}$, hence $\mathfrak{M} \subset \mathcal{M}_\infty := \cap_{n=0}^{\infty} \mathcal{M}_n$.

We say that f is *finite-to-one* if for every $x \in \mathcal{M}$ the cardinality of the set $f^{-1}(x)$ is finite.

Proposition 2 *Let $\mathcal{M}$ be a manifold and $f : \mathcal{M} \to \mathcal{M}$ a map such that, for some $n \in \mathbb{N}$, $f_{|\mathcal{M}_n}$ is finite-to-one. Then $\mathcal{M}_{n+1} = \mathfrak{M}$. In particular, if f is finite-to-one then $\mathfrak{M} = \cap_{n=0}^{\infty} f^n(\mathcal{M})$.*

Proof. This observation follows trivially from the fact that every point $x \in \mathcal{M}_{n+1}$ has backward chains of every length in $\mathcal{M}_n$ since $x \in f^k(\mathcal{M}_n)$ for all $k \in \mathbb{N}$. Therefore, there exists an infinite number of backward chains. Let $x_0 = x$. Since $f_{|\mathcal{M}_n}$ is finite-to-one, we know that there exist an infinite number of backward chains $\overline{x}^i = \{x_k^i\}_k$ that must agree for $k = -1$. Take $x_{-1} = x_{-1}^{i_1} = ... = x_{-1}^{i_m} = ...$ and so on. At each stage s we have only a finite number of preimages and therefore the above argument always applies and x_{-s} can always be found. Since we always have left an infinite number of backward chains with unbounded set of lengths, we may conclude that this procedure does not stop. Consequently, $\overline{x} = \{x_{-s}\}_{s\in\mathbb{N}}$ is an infinite backward chain for x. ■

The converse is not true, though, for we can modify the first example slightly, in such a way that $\cap_{n=0}^{\infty} f^n(\mathcal{M})$ is the maximal invariant set and yet f remains infinite-to-one.

Another setting where $\mathfrak{M} = \cap_{n=0}^{\infty} f^n(\mathcal{M})$ is explained in the following result.

Proposition 3 *Let $\mathcal{M}$ be a compact manifold and $f : \mathcal{M} \to \mathcal{M}$ a continuous map. Then, $\mathfrak{M} = \cap_{n=0}^{\infty} f^n(\mathcal{M}) \neq \emptyset$.*

Proof. By the Fixed Point theorem we know that $\mathfrak{M} \neq \emptyset$. It suffices to prove that $\mathcal{M}_1 \subset f(\mathcal{M}_1)$. Let $y \in \mathcal{M}_1 = \cap_{n=0}^{\infty} \mathcal{F}_n$ where $\mathcal{F}_n = f^n(\mathcal{M})$. Consequently, there exists a sequence of points $z_n \in \mathcal{F}_n$ such that $f^n(z_n) = y$. Let $w_n = f^{n-1}(z_n)$. It follows that, for all n, $f(w_n) = y$. By compactness of $\mathcal{M}$ we must have a converging subsequence $\{w_{n_k}\}_k$ to a point $x \in \mathcal{M}$ and also, by continuity of f it must be true that $f(x) = y$. Moreover, since f is continuous and $\mathcal{M}$ is compact every iterate of $\mathcal{M}$ is compact as well. The fact that $\lim_{k\to\infty} w_{n_k} = x$, $w_{n_k} \in \mathcal{F}_{n_k}$ and every $\mathcal{F}_{n_k}$ is compact implies that $x \in \cap_{k=0}^{\infty} \mathcal{F}_{n_k}$. Finally, $x \in \cap_{n=0}^{\infty} \mathcal{F}_n$ because $\{\mathcal{F}_n\}_n$ is a decreasing sequence of sets. ■

One question that arises now is whether one can construct a map such that for every $n \in \mathbb{N}$, $\mathcal{M}_n \neq \mathfrak{M}$. This is equivalent to saying that the sequence of sets $\{\mathcal{M}_n\}_n$ is infinite. From

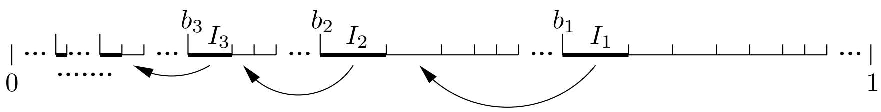

Figure 2: An example of a map whose maximal invariant set differs from $\mathcal{M}_n$ for every n.

Proposition 2, if one aims to build such map, it must be so that $f_{|\mathcal{M}_n}$ is infinite-to-one for all $n \in \mathbb{N}$.

The main idea in our next construction is to glue up countably many copies of intervals on which f acts in a similar fashion to that in our first example. Namely, take a sequence $\{b_n\}_{n\in\mathbb{N}}$ of strictly decreasing numbers in the unit interval $[0,1]$ such that $b_0 = 1$, $\lim_{n\to\infty} b_n = 0$. We then divide each interval of the form $[b_n, b_{n-1}[$ in subintervals of the form $[b_n^i, b_n^{i+1}[$ where $\{b_n^i\}_i$ is an increasing sequence of numbers in the interval $[b_n, b_{n-1}[$ such that $b_n^0 = b_n$ and $\lim_{i\to\infty} b_n^i = b_{n-1}$. Followingly, we define the action of $f_{|[b_n,b_{n-1}[}$ similarly to that shown in Figure 1 for the interval $[a_1, 1[$ (via some homeomorphism from $[a_1, 1[$ to $[b_n, b_{n-1}[$) for all subintervals but $I_n := [b_n^0, b_n^1[$ which is the homeomorphic copy of $[a_1, a_2[$ in Figure 1 within $[b_n, b_{n-1}[$. We then define, for every $n \in \mathbb{N}$, $f(I_n) = [b_{n+1}, b_n[$ and finally, let $f(0) = 0$ and $f(1) = 0$. In Figure 2 we show a graphic example of the map. It follows from this construction that $f_{|\mathcal{M}_n}$ is infinite-to-one for every $n \in \mathbb{N}$ and $\mathcal{M}_n = [0, b_n[\cup I_n$. More importantly, we have that $\mathcal{M}_\infty = \{0\}$ hence $\mathfrak{M} = \{0\}$.

Open question. In our previous example we have concluded that $\mathcal{M}_\infty = \mathfrak{M}$. However, there is no evidence that this is the case for a given map f in general. It could happen that $f(\mathcal{M}_\infty)$ is strictly contained in $\mathcal{M}_\infty$ and the process of approximating $\mathfrak{M}$ starts all over again: $\mathcal{M}_{\infty,0} = \mathcal{M}_\infty$, $\mathcal{M}_{\infty,1} = \cap_{n=0}^{\infty} f^n(\mathcal{M}_{\infty,0}), \ldots, \mathcal{M}_{\infty,k} = \cap_{n=0}^{\infty} f^n(\mathcal{M}_{\infty,k-1})$ and $\mathcal{M}_{\infty,\infty} = \cap_{n=0}^{\infty} \mathcal{M}_{\infty,n}$. And so on. Is there an example of an abstract map f on a set $\mathcal{M}$ such that $\mathfrak{M}$ can never be attained under this process?

Final remark. In the setting of invertible endomorphisms we can consider the maximal invariant set as being the maximal set satisfying both $f(A) = A$ and $f^{-1}(A) = A$. Since the map is invertible both f and f^{-1} are one-to-one. Therefore, the maximal invariant set under this definition corresponds to the intersection of the maximal invariant sets of f and f^{-1} which turns out to be, from Proposition 2, $\mathfrak{M} = \cap_{n=-\infty}^{\infty} f^n(\mathcal{M})$.

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