HOMOTOPY GROUPS OF MODULI SPACES OF REPRESENTATIONS

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Abstract. We calculate certain homotopy groups of the moduli spaces for representations of a compact oriented surface in the Lie groups $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{U}(p,q)$. Our approach relies on the interpretation of these representations in terms of Higgs bundles and uses Bott–Morse theory on the corresponding moduli spaces.

1. Introduction

Given a closed oriented surface, X, and a Lie group G, moduli spaces of surface group representations in G have rich geometric and topological structure which reflects properties of both X and G. In this paper we consider the cases where G is $\mathrm{GL}(n,\mathbb{C})$ or $\mathrm{U}(p,q)$.

Our main tools rely on an interpretation of the moduli spaces in terms of holomorphic bundles. Such an interpretation starts from the basic correspondence between representations of the fundamental group and flat principal bundles. Holomorphic bundles enter the picture if we fix a complex structure on the surface X — thereby turning it into a Riemann surface. By results of Hitchin [16], Donaldson [11], Simpson [21] and Corlette [8], if G is complex semisimple then the flat principal G-bundles corresponding to semisimple representations of $\pi_1 X$ in G are equivalent to polystable G-Higgs bundles over the Riemann surface. More generally, such Higgs bundles exist if G is complex reductive, in which case the polystable G-Higgs bundles correspond to semisimple representations not of $\pi_1 X$ but of a central extension of the fundamental group.

Referring to $\pi_1 X$ and its central extensions as surface groups, we can thus identify the moduli spaces of surface group representations with moduli spaces of polystable Higgs bundles. This identification puts a natural Kähler structure on the moduli spaces and also reveals a compatible $\mathbb{C}^*$-action. The restriction of this action to S^1 leads to a symplectic moment map whose squared norm serves as a proper Morse function. In a striking example of the interplay between geometry and topology, these geometric features on the moduli space of Higgs bundles provide powerful tools for studying the topology of the underlying moduli spaces of surface group representations.

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Holomorphic bundle techniques can also be adapted to the case in which G is a real reductive Lie group, in particular when G is a real form of a complex reductive group. If G is the compact real form of a complex reductive group $G_{\mathbb{C}}$, then the theorem of Narasimhan and Seshadri [19] and its generalization by Ramanathan [20] identify representations into G with polystable principal $G_{\mathbb{C}}$-bundles. For non-compact real forms the basic ideas were first introduced by Nigel Hitchin. In [17] he outlined how to define the appropriate Higgs bundles and applied his methods to the case $G = \mathrm{SL}(n, \mathbb{R})$ and also to other split real forms. Other special cases have been considered in a similar way[1]. In [4] we began an in-depth study of the groups $\mathrm{U}(p, q)$ and their adjoint forms $\mathrm{PU}(p, q)$ (for any p and q) from this point of view. This paper is a continuation of that work.

The most primitive topological feature of the moduli spaces is their number of connected components, i.e. π_0. The above methods have been effective in addressing this question, mainly by exploiting the properness of the above mentioned Morse function. This transfers questions about π_0 for the moduli spaces into questions about the connected components of the minimal submanifolds for the Morse function.

In good cases, there is additional useful Morse theoretic information which has thus far gone unexploited. Our goal is to correct this oversight. In particular, using information about the Morse indices of non-minimal critical points, we can relate higher homotopy groups for the full moduli spaces to those of their minimal submanifolds. For the latter we rely on the calculations by Daskalopoulos and Uhlenbeck [9] for higher homotopy groups of the moduli space of stable vector bundles. Our main results for the moduli spaces of $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{U}(p, q)$ Higgs bundles, and hence for the corresponding moduli spaces of representations, are given in Theorems 4.4 and 4.19 respectively.

2. Surface group representations and Higgs bundles

For a more thorough account of the material in this section see [4].

2.1. **Surface group representations.** Let X be a smooth closed oriented surface of genus $g \geqslant 2$. The fundamental group, $\pi_1 X$, of X is a finitely generated group generated by $2g$ generators, say $A_1, B_1, \ldots, A_g, B_g$, subject to the single relation $\prod_{i=1}^{g} [A_i, B_i] = 1$. It has a universal central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Gamma \longrightarrow \pi_1 X \longrightarrow 1 \tag{2.1}$$

generated by the same generators as $\pi_1 X$, together with a central element J subject to the relation $\prod_{i=1}^{g} [A_i, B_i] = J$.

By a representation of Γ in $\mathrm{GL}(n, \mathbb{C})$ we mean a homomorphism $\rho \colon \Gamma \to \mathrm{GL}(n, \mathbb{C})$. We say that a representation of Γ in $\mathrm{GL}(n, \mathbb{C})$ is *semisimple* if the $\mathbb{C}^n$-representation of Γ induced by the fundamental representation of $\mathrm{GL}(n, \mathbb{C})$ is semisimple[2]. The group

[1] Notably $\mathrm{Sp}(4, \mathbb{R})$ and $\mathrm{SU}(2, 2)$ [14, 13], $\mathrm{U}(2, 1)$ [15], $\mathrm{U}(p, q)$ and $\mathrm{PU}(p, q)$ [4], $\mathrm{GL}(n, \mathbb{R})$ [6]. Higgs bundle methods have also been applied, albeit in a more algebraic way in the cases $\mathrm{U}(p, 1)$ [24], $\mathrm{PU}(2, 1)$ [25], and $\mathrm{PU}(p, p)$ [18].

[2] In general a representation of Γ in a reductive Lie group G is said to be semisimple if the induced (adjoint) representation on the Lie algebra of G is semisimple. For $G = \mathrm{GL}(n, \mathbb{C})$ this is equivalent to the definition given here.

$\mathrm{GL}(n,\mathbb{C})$ acts on the set of representations via conjugation. Restricting to the semisimple representations, denoted by $\mathrm{Hom}^+(\Gamma,\mathrm{GL}(n,\mathbb{C}))$, we get the *moduli space* of representations of Γ in $\mathrm{GL}(n,\mathbb{C})$,

$$\mathcal{R}(\Gamma,\mathrm{GL}(n,\mathbb{C})) = \mathrm{Hom}^+(\Gamma,\mathrm{GL}(n,\mathbb{C}))/\mathrm{GL}(n,\mathbb{C}). \tag{2.2}$$

The set $\mathrm{Hom}^+(\Gamma,\mathrm{GL}(n,\mathbb{C}))$ can be embedded in $\mathrm{GL}(n,\mathbb{C})^{2g+1}$ via the map

$$\begin{aligned}\mathrm{Hom}^+(\Gamma,\mathrm{GL}(n,\mathbb{C})) &\to \mathrm{GL}(n,\mathbb{C})^{2g+1}\\ \rho &\mapsto (\rho(A_1),\dots\rho(B_g),\rho(J)).\end{aligned}$$

We can then give $\mathrm{Hom}^+(\Gamma,\mathrm{GL}(n,\mathbb{C}))$ the subspace topology and $\mathcal{R}(\Gamma,\mathrm{GL}(n,\mathbb{C}))$ the quotient topology. This topology is Hausdorff because we have restricted attention to semisimple representations.

There is a topological invariant of a representation $\rho\in\mathcal{R}(\Gamma,\mathrm{GL}(n,\mathbb{C}))$ given by $\rho(J)$, which coincides with the first Chern class of the vector bundle with central curvature associated to ρ. Fixing this invariant, we define

$$\mathcal{R}(n,d) := \{\rho\in\mathcal{R}(\Gamma,\mathrm{GL}(n,\mathbb{C})) \mid \rho(J)\in\mathbb{Z}_d\subset Z(\mathrm{GL}(n,\mathbb{C})\}.$$

In particular the representations with vanishing degree correspond to representations of the fundamental group of X, that is,

$$\mathcal{R}(n,0) = \mathcal{R}(\pi_1 X,\mathrm{GL}(n,\mathbb{C})) := \mathrm{Hom}^+(\pi_1 X,\mathrm{GL}(n,\mathbb{C}))/\mathrm{GL}(n,\mathbb{C})\ . \tag{2.3}$$

Similarly to the case of $\mathrm{GL}(n,\mathbb{C})$ we consider the moduli space

$$\mathcal{R}(\Gamma,\mathrm{U}(p,q)) = \mathrm{Hom}^+(\Gamma,\mathrm{U}(p,q))/\mathrm{U}(p,q)\ . \tag{2.4}$$

The moduli space $\mathcal{R}(\Gamma,\mathrm{U}(p,q))$ can be identified with the moduli space of $\mathrm{U}(p,q)$-bundles on X with projectively flat connections. Taking a reduction to the maximal compact $\mathrm{U}(p)\times\mathrm{U}(q)$, we thus associate to each class $\rho\in\mathcal{R}(\Gamma,\mathrm{U}(p,q))$ a vector bundle of the form $V\oplus W$, where V and W are rank p and q respectively, and thus a pair of integers $(a,b)=(\deg(V),\deg(W))$. There is thus a map

$$c\colon \mathcal{R}(\Gamma,\mathrm{U}(p,q)) \to \mathbb{Z}\oplus\mathbb{Z}$$

given by $c(\rho)=(a,b)$. The corresponding map on $\mathrm{Hom}^+(\Gamma,\mathrm{U}(p,q))$ is clearly continuous and thus locally constant. Since $\mathrm{U}(p,q)$ is connected, the map c is likewise continuous and thus constant on connected components. The subspace of $\mathcal{R}(\Gamma,(\mathrm{U}(p,q))$ corresponding to representations with invariants (a,b) is denoted by

$$\mathcal{R}(p,q,a,b) = c^{-1}(a,b) = \{\rho\in\mathcal{R}(\Gamma,\mathrm{U}(p,q)) \mid c(\rho)=(a,b)\in\mathbb{Z}\oplus\mathbb{Z}\}\ . \tag{2.5}$$

The representations for which $a+b=0$ correspond to representations of the fundamental group of X, that is,

$$\mathcal{R}(p,q,a,-a) = c^{-1}(a,-a) = \{\rho\in\mathcal{R}(\pi_1 X,\mathrm{U}(p,q)) \mid c(\rho)=(a,-a)\in\mathbb{Z}\oplus\mathbb{Z}\}\ . \tag{2.6}$$

2.2. $\mathrm{GL}(n,\mathbb{C})$**-Higgs bundles.** A $\mathrm{GL}(n,\mathbb{C})$-Higgs bundle on a compact Riemann surface X is a pair (E,Φ), where E is a rank n holomorphic vector bundle over X and $\Phi \in H^0(\mathrm{End}(E)\otimes K)$ is a holomorphic endomorphism of E twisted by the canonical bundle K of X. The $\mathrm{GL}(n,\mathbb{C})$-Higgs bundle (E,Φ) is *stable* if the slope stability condition

$$\mu(E') < \mu(E) \tag{2.7}$$

holds for all proper Φ-invariant subbundles E' of E. Here the *slope* is defined by $\mu(E) = \deg(E)/\operatorname{rk}(E)$ and Φ-*invariance* means that $\Phi(E') \subset E'\otimes K$. *Semistability* is defined by replacing the above strict inequality with a weak inequality. A Higgs bundle is called *polystable* if it is the direct sum of stable Higgs bundles with the same slope.

Given a hermitian metric on E, let A denote the unique unitary connection compatible with the holomorphic structure, and let F_A be its curvature. *Hitchin's equations* on (E,Φ) are

$$\begin{aligned} F_A + [\Phi,\Phi^*] &= -\sqrt{-1}\mu\mathrm{Id}_E\omega,\\ \bar\partial_A\Phi &= 0, \end{aligned} \tag{2.8}$$

where ω is the Kähler form on X, Id_E is the identity on E, $\mu = \mu(E)$ and $\bar\partial_A$ is the anti-holomorphic part of the covariant derivative d_A. A solution to Hitchin's equations is *irreducible* if there is no proper subbundle of E preserved by A and Φ.

If we define a Higgs connection (as in [22]) by

$$D = d_A + \theta \tag{2.9}$$

where $\theta = \Phi + \Phi^*$, then Hitchin's equations are equivalent to the conditions

$$\begin{aligned} F_D &= -\sqrt{-1}\mu\mathrm{Id}_E\omega,\\ d_A^*\theta &= 0. \end{aligned} \tag{2.10}$$

In particular, the first equation says that D is a projectively flat connection[3]. If $\deg(E) = 0$ then D is actually flat. It follows that in this case the pair (E,D) defines a representation of $\pi_1 X$ in $\mathrm{GL}(n,\mathbb{C})$. If $\deg(E)\neq 0$, then the pair (E,D) defines a representation of $\pi_1 X$ in $\mathrm{PGL}(n,\mathbb{C})$, or equivalently, a representation of Γ in $\mathrm{GL}(n,\mathbb{C})$. By the theorem of Corlette ([8]), every semisimple representation of Γ (and therefore every semisimple representation of $\pi_1 X$) arises in this way.

If we fix the rank and degree (say n and d respectively) of the bundle E, i.e. on bundles of fixed topological type, the isomorphism classes of polystable Higgs bundles are parameterized by a quasi-projective variety of dimension $2+2n^2(g-1)$. We denote this moduli space of rank n degree d polystable Higgs bundles by $\mathcal{M}(n,d)$.

If we fix a hermitian metric on a smooth rank n degree d complex vector bundle on X, then there is a gauge theoretic moduli space of pairs (A,Φ), consisting of a unitary connection A and an endomorphism valued $(1,0)$-form Φ, which are solutions to Hitchin's equations (2.8), modulo $\mathrm{U}(n)$-gauge equivalence.

The gauge theory moduli space and $\mathcal{M}(n,d)$ are related by virtue of the Hitchin-Kobayashi correspondence: a $\mathrm{GL}(n,\mathbb{C})$-Higgs bundle (E,Φ) is polystable if and only if it admits a hermitian metric such that Hitchin's equations (2.8) are satisfied, and (E,Φ) is stable if and only if the corresponding solution is irreducible. There is, moreover,

[3]The other equation is an harmonicity constraint.

a map from the gauge theoretic moduli space to this moduli space given by taking a solution (A, Φ) to Hitchin's equations to the Higgs bundle (E, Φ), where the holomorphic structure on E is given by $\bar{\partial}_A$. This map is a homeomorphism, and a diffeomorphism on the smooth locus.

In view of the relation between Hitchin's equations and projectively flat connections, this correspondence gives rise to a homeomorphism between $\mathcal{M}(n,d)$ and the component $\mathcal{R}(n,d)$ of the moduli space of semisimple representations of Γ in $\mathrm{GL}(n,\mathbb{C})$. If the degree of the Higgs bundle is zero, then the moduli space $\mathcal{M}(n,0)$ is homeomorphic to the moduli space of representations of $\pi_1 X$ in $\mathrm{GL}(n,\mathbb{C})$.

Theorem 2.1. *If (n,d) is such that $\mathrm{GCD}(n,d)=1$ then the moduli space $\mathcal{M}(n,d)$ is a non-empty connected smooth hyperkähler manifold.*

2.3. $\mathrm{U}(p,q)$**-Higgs bundles.** There is a special class of $\mathrm{GL}(n,\mathbb{C})$-Higgs bundles, related to representations in $\mathrm{U}(p,q)$ given by the requirements that

$$E = V \oplus W$$

$$\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \tag{2.11}$$

where V and W are holomorphic vector bundles of rank p and q respectively and the non-zero components in the Higgs field are $\beta \in H^0(\mathrm{Hom}(W,V)\otimes K)$, and $\gamma \in H^0(\mathrm{Hom}(V,W)\otimes K)$. We say (E,Φ) is a *stable* $\mathrm{U}(p,q)$-Higgs bundle if the slope stability condition $\mu(E') < \mu(E)$, is satisfied for all Φ-invariant subbundles $E' = V' \oplus W'$, i.e. for all subbundles $V' \subset V$ and $W' \subset W$ such that

$$\beta : W' \longrightarrow V' \otimes K \tag{2.12}$$

$$\gamma : V' \longrightarrow W' \otimes K\ . \tag{2.13}$$

Semistability and polystability are defined analogously to the way they are defined for $\mathrm{GL}(n,\mathbb{C})$-Higgs bundles.

Hitchin's equations make sense for $\mathrm{U}(p,q)$-Higgs bundles, with a $\mathrm{U}(p,q)$ solution being a metric with respect to which $E = V \oplus W$ is an orthogonal decomposition. With Φ as in (2.11) and $\theta = \Phi + \Phi^*$, the corresponding $\mathrm{U}(p,q)$-Higgs connection $D = d_A + \theta$ is not only projectively flat but has $\mathrm{U}(p,q)$ holonomy. This provides the link between $\mathrm{U}(p,q)$-Higgs bundles and surface group representations in $\mathrm{U}(p,q)$, leading to:

Theorem 2.2. *Let $\mathcal{M}(p,q,a,b)$ be the moduli space of polystable $\mathrm{U}(p,q)$-Higgs bundles with $\deg(V) = a$ and $\deg W = b$. Then with $\mathcal{R}(p,q,a,b)$ as in (2.5) there is a homeomorphism $\mathcal{M}(p,q,a,b) \cong \mathcal{R}(p,q,a,b)$.*

The *Toledo invariant* of the representation corresponding to $(E = V\oplus W, \Phi)$ is defined by

$$\tau = \tau(p,q,a,b) = 2\frac{qa-pb}{p+q} \tag{2.14}$$

where $a = \deg(V)$ and $b = \deg(W)$. This invariant satisfies the following Milnor-Wood-type inequality (proved by Domic and Toledo [10])

$$|\tau(p,q,a,b)| \leqslant \min\{p,q\}(2g-2)\ . \tag{2.15}$$

Theorem 2.3. [4] *Let (p,q,a,b) such that* $\mathrm{GCD}(p+q,a+b)=1$*. Then $\mathcal{M}(p,q,a,b)$ (and hence $\mathcal{R}(p,q,a,b)$) is a connected smooth Kähler manifold which is non-empty if and only if $|\tau(p,q,a,b)| \leqslant \min\{p,q\}(2g-2)$.*

3. Morse theory on the moduli space

3.1. **The Morse function.** Let $\mathcal{M}$ be either $\mathcal{M}(n,d)$ or $\mathcal{M}(p,q,a,b)$. We will assume that $\mathrm{GCD}(n,d)=1$ and $\mathrm{GCD}(p+q,a+b)=1$. Under this coprimality condition, there are no strictly semistable Higgs bundles and the moduli space $\mathcal{M}$ is smooth. The non-zero complex numbers $\mathbb{C}^*$ act on $\mathcal{M}$ via the map $\lambda\cdot(E,\Phi)=(E,\lambda\Phi)$. However, to have an action on the gauge theory moduli space (i.e. on the set of solutions to Hitchin's equations (2.8), cf. Section 2), one must restrict to the action of $S^1\subset\mathbb{C}^*$. This is a Hamiltonian action and the associated moment map is

$$[(A,\Phi)] \mapsto -\tfrac{1}{2}\|\Phi\|^2 = -i\int_X \mathrm{tr}(\Phi\Phi^*)$$

where the adjoint Φ^* is taken with respect to the hermitian metric on E. We shall, however, prefer to consider the positive function

$$f([A,\Phi]) = \tfrac{1}{2}\|\Phi\|^2. \tag{3.1}$$

Next we recall a general result of Frankel [12], which was first used in the context of moduli spaces of Higgs bundles by Hitchin [16].

Theorem 3.1. *Let $\tilde{f}\colon M\to\mathbb{R}$ be a proper moment map for a Hamiltonian circle action on a Kähler manifold M. Then $\tilde{f}$ is a perfect Bott–Morse function.*

3.2. **Morse theory and homotopy groups.** In this Section we recall some basic facts of Bott–Morse theory. Let $\mathcal{M}_l\subset\mathcal{M}$ be the critical submanifolds of f and $\nu(\mathcal{M}_l)$ be the normal bundle of $\mathcal{M}_l$ in $\mathcal{M}$. The Hessian of f is non-degenerate on $\nu(\mathcal{M}_l)$ and we have the decomposition in positive and negative eigenspace bundles

$$\nu(\mathcal{M}_l) = \nu^+(\mathcal{M}_l)\oplus\nu^-(\mathcal{M}_l).$$

The index of $\mathcal{M}_l$ is defined as

$$\mathrm{index}(\mathcal{M}_l) := \mathrm{rk}\,\nu^-(\mathcal{M}_l).$$

Let $\mathcal{M}_l^+$ be the stable set of $\mathcal{M}_l$, i.e., the subset of $\mathcal{M}$ defined by the points of $\mathcal{M}$ which flow to $\mathcal{M}_l$. It follows from Bott–Morse theory that $\mathcal{M}_l^+$ is a submanifold of $\mathcal{M}$ of codimension

$$\mathrm{codim}_{\mathbb{R}}(\mathcal{M}_l^+) = \mathrm{index}(\mathcal{M}_l), \tag{3.2}$$

and that there is a stratification

$$\mathcal{M} = \bigcup_l \mathcal{M}_l^+. \tag{3.3}$$

Proposition 3.2. *Let $\mathcal{N}=\mathcal{M}_0\subset\mathcal{M}$ be the submanifold of local minima of f. If* $\mathrm{index}(\mathcal{M}_l)\geqslant m\geqslant 2$ *for every $l\neq 0$ then*

$$\pi_i(\mathcal{M})\cong\pi_i(\mathcal{N}) \quad \textit{for} \quad i\leqslant m-2.$$

Proof. The stratification (3.3) shows that

$$\mathcal{M}_0^+ = \mathcal{M} \smallsetminus \bigcup_{l\neq 0} \mathcal{M}_l^+$$

and the Morse flow defines a retraction from $\mathcal{M}_0^+$ to $\mathcal{N} = \mathcal{M}_0$. Thus the result is an immediate consequence of standard homotopy theory, using (3.2). □

3.3. **Deformation theory of Higgs bundles.** In the following we recall some standard facts about the deformation theory of Higgs bundles (this has been treated in many places, a convenient reference is Biswas–Ramanan [2]). In order to describe the results in a uniform way for a G-Higgs bundle (E,Φ) when $G = \mathrm{GL}(n,\mathbb{C})$ or $\mathrm{U}(p,q)$, we introduce bundles U_G^+, U_G^- and U_G defined by

$$\begin{aligned} U^+_{\mathrm{GL}(n,\mathbb{C})} &= U^-_{\mathrm{GL}(n,\mathbb{C})} = U_{\mathrm{GL}(n,\mathbb{C})} = \mathrm{End}(E),\\ U^+_{\mathrm{U}(p,q)} &= \mathrm{End}(V) \oplus \mathrm{End}(W),\\ U^-_{\mathrm{U}(p,q)} &= \mathrm{Hom}(W,V) \oplus \mathrm{Hom}(V,W),\\ U_{\mathrm{U}(p,q)} &= U^+_{\mathrm{U}(p,q)} \oplus U^-_{\mathrm{U}(p,q)} = \mathrm{End}(V\oplus W), \end{aligned}$$

where the bundles V and W are as in Section 2.3. Note that, with this notation, $\Phi \in H^0(U_G^- \otimes K)$.

Remark 3.3. Both for $G = \mathrm{GL}(n,\mathbb{C})$ and for $G = \mathrm{U}(p,q)$, there is an inner product on U_G which is invariant under the adjoint action of U_G, i.e.,

$$\langle \mathrm{ad}(\psi)x, y\rangle + \langle x, \mathrm{ad}(\psi)y\rangle = 0 \tag{3.4}$$

for local sections x, y and ψ of U_G. This inner product restricts to an inner product on U_G^- and U_G^+, giving rise to an isomorphism

$$U_G^{\pm} \xrightarrow{\cong} (U_G^{\pm})^*. \tag{3.5}$$

Note that under this duality

$$\mathrm{ad}(\Phi)^t = -\,\mathrm{ad}(\Phi) \otimes 1_{K^{-1}}.$$

Proposition 3.4. *Let (E,Φ) be a G-Higgs bundle for $G = \mathrm{GL}(n,\mathbb{C})$ or $G = \mathrm{U}(p,q)$ and define the following complex of sheaves*

$$C_G^\bullet(E,\Phi)\colon U_G^+ \xrightarrow{\mathrm{ad}(\Phi)} U_G^- \otimes K.$$

Then the following holds:

1. *The space of endomorphisms of (E,Φ) is naturally isomorphic to $\mathbb{H}^0(C_G^\bullet)$.*
2. *The infinitesimal deformation space of (E,Φ) is naturally isomorphic to $\mathbb{H}^1(C_G^\bullet)$.*

The following proposition is simply a statement of the fact that a stable Higgs bundle is simple.

Proposition 3.5. *Let (E,Φ) be a stable G-Higgs bundle for $G = \mathrm{GL}(n,\mathbb{C})$ or $G = \mathrm{U}(p,q)$. Then*

$$\mathbb{H}^0(C_G^\bullet(E,\Phi)) \cong \mathbb{C},$$

generated by the scalar multiples of the identity morphism.

3.4. **Critical points and Morse indices.** In the following (E, Φ) continues to denote a G-Higgs bundle for $G = \mathrm{GL}(n, \mathbb{C})$ or $G = \mathrm{U}(p, q)$ and for ease of notation we omit the subscript G on the bundles $U^{\pm}_G$ and the complex $C^{\bullet}_G$. The critical points of the function f are exactly the fixed points of the S^1-action on $\mathcal{M}$. This allows one to describe the corresponding Higgs bundles as "complex variations of Hodge structure", as follows (cf. Hitchin [16, 17] and also Simpson [22]).

Proposition 3.6. *If (E, Φ) corresponds to a critical point of f, then there is a semisimple element $\psi \in H^0(U^+)$ and a corresponding decomposition in eigenspace bundles*

$$U^{\pm}_G = \bigoplus_k U^{\pm}_k \tag{3.6}$$

for the adjoint action of ψ, such that $\mathrm{ad}(\psi)$ has eigenvalue ik on $U^{\pm}_k$. Furthermore, $[\psi, \Phi] = i\Phi$, i.e.,

$$\Phi \in H^0(U^-_1 \otimes K).$$

In particular, this means that the deformation complex of (E, Φ) decomposes as

$$C^{\bullet}(E, \Phi) = \bigoplus_k C^{\bullet}_k(E, \Phi), \tag{3.7}$$

where we have defined for each k the complex

$$C^{\bullet}_k(E, \Phi) \colon U^+_k \xrightarrow{\mathrm{ad}(\Phi)} U^-_{k+1} \otimes K.$$

Thus the tangent space to $\mathcal{M}$ at (E, Φ) has a decomposition

$$\mathbb{H}^1(C^{\bullet}(E, \Phi)) = \bigoplus_k \mathbb{H}^1(C^{\bullet}_k(E, \Phi)). \tag{3.8}$$

Remark 3.7. Using the definition of the U_k and (3.4), we have that

$$U^{\pm}_k \cong U^{\pm,*}_{-k}$$

under the duality (3.5). Moreover, writing

$$\mathrm{ad}(\Phi)^{\pm}_k = \mathrm{ad}(\Phi)_{|U^{\pm}_k} \colon U^{\pm}_k \to U^{\mp}_{k+1} \otimes K,$$

we have

$$\mathrm{ad}(\Phi)^{\pm}_{k,t} = (\mathrm{ad}(\Phi)^{\mp}_{-k-1}) \otimes 1_{K^{-1}}.$$

The calculations of Hitchin [17, §8] show that eigenvalues of the Hessian of f at a critical point can be calculated as follows.

Proposition 3.8. *Let (E, Φ) be a stable G-Higgs bundle which corresponds to a critical point of f, for $G = \mathrm{GL}(n, \mathbb{C})$ or $G = \mathrm{U}(p, q)$. In the decomposition (3.8) the eigenvalue $-k$ subspace for the Hessian of f is isomorphic to $\mathbb{H}^1(C^{\bullet}_k(E, \Phi))$. In particular, the negative eigenspace at (E, Φ) for the Hessian is given by*

$$\nu^-_{(E,\Phi)}(\mathcal{M}_l) \cong \bigoplus_{k>0} \mathbb{H}^1(C^{\bullet}_k).$$

Lemma 3.9. *Let (E, Φ) be a stable G-Higgs bundle which corresponds to a critical point of f. Then*

$$\mathbb{H}^0(C^{\bullet}_k(E, \Phi)) = 0 \qquad \text{and} \qquad \mathbb{H}^2(C^{\bullet}_k(E, \Phi)) = 0$$

for $k > 0$.

Proof. From (3.7) we have a decomposition

$$\mathbb{H}^0(C^\bullet(E,\Phi)) = \bigoplus_k \mathbb{H}^0(C^\bullet_k(E,\Phi)).$$

and we know from Proposition 3.5 that the only trivial endomorphisms of (E,Φ) are the scalars, which have weight zero in this decomposition. This gives the vanishing of $\mathbb{H}^0$.

For the vanishing of $\mathbb{H}^2$, consider first the case $G = \mathrm{GL}(n,\mathbb{C})$. Then $U^+_k = U^-_k$ and, using Remark 3.7, we see that the dual complex of $C^\bullet_k(E,\Phi)$ is isomorphic to the complex

$$C^\bullet_{-k-1}(E,\Phi)\otimes K^{-1}\colon U^+_{-k-1}\otimes K^{-1} \xrightarrow{-\operatorname{ad}(\Phi)} U^-_{-k}.$$

The change in sign of $\operatorname{ad}(\Phi)$ does not influence the cohomology and hence Serre duality for hypercohomology gives

$$\mathbb{H}^2(C^\bullet_k(E,\Phi)) \cong \mathbb{H}^0(C^\bullet_{-k-1}(E,\Phi))^*.$$

It follows that $\mathbb{H}^2(C^\bullet_k(E,\Phi))$ vanishes for $k\neq -1$. The case $G=\mathrm{U}(p,q)$ follows essentially from this, by using the fact that stability as a $\mathrm{U}(p,q)$-Higgs bundle implies stability as a $\mathrm{GL}(n,\mathbb{C})$-Higgs bundle (see [4, Proposition 3.19] for a detailed argument). □

Proposition 3.10. *Let (E,Φ) be a stable G-Higgs bundle which corresponds to a critical point of f. Then the Morse index of the corresponding critical submanifold $\mathcal{M}_l$ is*

$$\operatorname{index}(\mathcal{M}_l) = \sum_{k>0} \dim \mathbb{H}^1(C^\bullet_k(E,\Phi)),$$

where

$$\dim \mathbb{H}^1(C^\bullet_k(E,\Phi)) = -\chi(C^\bullet_k(E,\Phi)).$$

Proof. This is immediate from Proposition 3.8 and the vanishing of Lemma 3.9. □

The following lemma is essentially Proposition 4.14 of [4]. We provide a complete proof, taking this opportunity to correct some inaccuracies in the argument given in [4].

Lemma 3.11. *Let (E,Φ) be a stable G-Higgs bundle which corresponds to a critical point of f, for $G=\mathrm{GL}(n,\mathbb{C})$ or $G=\mathrm{U}(p,q)$. Then*

$$\chi(C^\bullet_k(E,\Phi)) \leqslant (g-1)\big(2\operatorname{rk}(\operatorname{ad}(\Phi)^+_k) - \operatorname{rk}(U^+_k) - \operatorname{rk}(U^-_{k+1})\big).$$

Thus, in particular, we have:

1. *If $\chi(C^\bullet_k(E,\Phi))\neq 0$ then $\chi(C^\bullet_k(E,\Phi)) \leqslant -(g-1)$.*
2. *The vanishing $\chi(C^\bullet_k(E,\Phi))=0$ holds if and only if $\operatorname{ad}(\Phi)^+_k\colon U^+_k\to U^-_{k+1}\otimes K$ is an isomorphism.*

Proof. In the following we shall use the abbreviated notations $C^\bullet_k = C^\bullet_k(E,\Phi)$ and

$$\Phi^\pm_k = \operatorname{ad}(\Phi)^\pm_k\colon U^+_k \to U^-_{k-1}\otimes K.$$

By the Riemann–Roch formula we have

$$\chi(C^\bullet_k) = (1-g)\big(\operatorname{rk}(U^+_k)+\operatorname{rk}(U^-_{k+1})\big) + \deg(U^+_k) - \deg(U^-_{k+1}), \tag{3.9}$$

thus we can prove the inequality stated in the Lemma by estimating the difference $\deg(U_k^+) - \deg(U_k^-)$. In order to do this, we note first that there are short exact sequences of sheaves

$$0 \to \ker(\Phi_k^+) \to U_k^+ \to \operatorname{im}(\Phi_k^+) \to 0$$

and

$$0 \to \operatorname{im}(\Phi_k^+) \to U_{k+1}^- \otimes K \to \operatorname{coker}(\Phi_k^+) \to 0.$$

It follows that

$$\deg(U_k^+) - \deg(U_{k+1}^-) = \deg(\ker(\Phi_k^+)) + (2g-2)\operatorname{rk}(U_{k+1}^-) - \deg(\operatorname{coker}(\Phi_k^+)). \tag{3.10}$$

We shall prove the following inequalities below.

$$\deg(\ker(\Phi_k^+)) \leqslant 0, \tag{3.11}$$

$$-\deg(\operatorname{coker}(\Phi_k^+)) \leqslant (2g-2)\big(-\operatorname{rk}(U_{k+1}^-) + \operatorname{rk}(\Phi_k^+)\big). \tag{3.12}$$

Combining these with (3.10) we obtain

$$\deg(U_k^+) - \deg(U_{k+1}^-) \leqslant (2g-2)\operatorname{rk}(\Phi_k^+),$$

which, together with (3.9), proves the inequality stated in the Lemma.

It remains to prove (3.11) and (3.12). For this we use the fact that the adjoint Higgs bundle $(U_G, \operatorname{ad}(\Phi))$ is semistable (one way of seeing this is to note that it supports a solution to Hitchin's equations). Clearly, the subbundle $\ker(\Phi_k^+) \subseteq U_G$ is $\operatorname{ad}(\Phi)$-invariant and hence

$$\deg(\ker(\Phi_k^+)) \leqslant \deg(U_G) = 0,$$

thus proving (3.11).

In order to prove (3.12) a bit more work needs to be done. Consider the dual of Φ_k^+,

$$\Phi_k^{+,t} \colon U_{k+1}^{-,*} \otimes K^{-1} \to U_k^{+,*},$$

and note that the image of Φ_k^+ goes to zero under the restriction map

$$U_{k+1}^- \otimes K \to \ker(\Phi_k^{+,t})^*.$$

Hence there is an induced map

$$\operatorname{coker}(\Phi_k^+) \to \ker(\Phi_k^{+,t})^*$$

which is generically an isomorphism — in fact, its kernel is the torsion subsheaf of $\operatorname{coker}(\Phi_k^+)$. It follows that

$$\deg(\operatorname{coker}(\Phi_k^+)) \geqslant \deg(\ker(\Phi_k^{+,t})^*).$$

Since $\ker(\Phi_k^{+,t})$ is locally free (in fact a subbundle) this shows that

$$-\deg(\operatorname{coker}(\Phi_k^+)) \leqslant \deg(\ker(\Phi_k^{+,t})), \tag{3.13}$$

the difference being the degree of the torsion subsheaf of $\operatorname{coker}(\Phi_k^+)$. Now Remark 3.7 tells us that we have a commutative diagram

$$\begin{array}{ccc}
U_{k+1}^{-,*} \otimes K^{-1} & \xrightarrow{\Phi_k^{+,t}} & U_k^{+,*} \\
\cong \big\downarrow & & \cong \big\downarrow \\
U_{-k-1}^{-} \otimes K^{-1} & \xrightarrow{-\Phi_{-k-1}^{-} \otimes 1_{K^{-1}}} & U_k^{+},
\end{array}$$

and thus

$$\ker(\Phi_k^{+,t}) \cong \ker(\Phi_{-k-1}^-) \otimes K^{-1}$$

from which we conclude that

$$\deg(\ker(\Phi_k^{+,t})) = \deg(\ker(\Phi_{-k-1}^-)) - (2g-2)\,\mathrm{rk}(\ker(\Phi_{-k-1}^-)).$$

Again we apply semistability of $(U_G, \mathrm{ad}(\Phi))$ to the $\mathrm{ad}(\Phi)$-invariant subbundle $\ker(\Phi_{-k-1}^-)$ to obtain

$$\deg(\ker(\Phi_k^{+,t})) \leqslant -(2g-2)\,\mathrm{rk}(\ker(\Phi_{-k-1}^-)). \tag{3.14}$$

But clearly, $\mathrm{rk}(\Phi_k^+) = \mathrm{rk}(\Phi_k^{+,t}) = \mathrm{rk}(\Phi_{-k-1}^-)$ and $\mathrm{rk}(U_{k+1}^-) = \mathrm{rk}(U_{-k-1}^{-,*}) = \mathrm{rk}(U_{-k-1}^-)$ so

$$\mathrm{rk}(\ker(\Phi_{-k-1}^-)) = \mathrm{rk}(U_{k+1}^-) - \mathrm{rk}(\Phi_k^+).$$

Combining this fact with (3.13) and (3.14) concludes the proof of (3.12).

Finally, (1) of the statement of the Lemma is an immediate consequence of the inequality proved, while to prove (2) we note that if $\chi(C_k^\bullet) = 0$ then $\mathrm{rk}(\Phi_k^+) = \mathrm{rk}(U_k^+) = \mathrm{rk}(U_{k+1}^- \otimes K)$ and equality holds in (3.13), thus showing that Φ_k^+ is an isomorphism. □

Proposition 3.12. *For both $\mathcal{M} = \mathcal{M}(n,d)$ and $\mathcal{M} = \mathcal{M}(p,q,a,b)$*

$$\mathrm{index}(\mathcal{M}_l) \geqslant 2g-2$$

for every non-minimal critical submanifold $\mathcal{M}_l \subset \mathcal{M}$.

Proof. It is immediate from Proposition 3.10 that

$$\dim \mathbb{H}^1(C_k^\bullet(E,\Phi)) = -\chi(C_k^\bullet(E,\Phi)) > 0$$

for at least one $k > 0$ when $\mathcal{M}_l$ is non-minimal. Now the result follows from (2) of Lemma 3.11. □

Remark 3.13. The preceding argument applies uniformly to $G = \mathrm{U}(p,q)$ and $G = \mathrm{GL}(n,\mathbb{C})$; in fact, it applies to the moduli of G-Higgs bundles for any real reductive G (cf. [4]). But it should be noted that a simpler argument is possible for $G = \mathrm{GL}(n,\mathbb{C})$: using semistability of $(U_G, \mathrm{ad}(\Phi))$ one sees quite easily that $\dim \mathbb{H}^1(C_{k_0}^\bullet) \geqslant g-1$ for the highest weight k_0.

3.5. **Local Minima.** The minima of the Morse function on $\mathcal{M}(n,d)$ is given by the following [16].

Proposition 3.14. *Let $\mathcal{N}(n,d) \subset \mathcal{M}(n,d)$ be the set of local minima. Then*

$$\mathcal{N}(n,d) = \{(E,\Phi) \in \mathcal{M}(n,d) \ \mid \ \Phi = 0\}.$$

Hence $\mathcal{N}(n,d)$ coincides with $M(n,d)$, the moduli space of semistable vector bundles of rank n and degree d, which equals the subvariety $M^s(n,d) \subset M(n,d)$ corresponding to stable bundles if $\mathrm{GCD}(n,d) = 1$.

The minima of the Morse function on $\mathcal{M}(p,q,a,b)$ have been characterized in [4]. One has the following.

Proposition 3.15. *Let $\mathcal{N}(p,q,a,b) \subset \mathcal{M}(p,q,a,b)$ be the set of local minima. Then*

$$\mathcal{N}(p,q,a,b) = \{(E,\Phi) \in \mathcal{M}(p,q,a,b) \ \mid \ \beta = 0 \ \text{ or } \ \gamma = 0\}.$$

More precisely, let $(E,\Phi) \in \mathcal{N}(p,q,a,b)$. Then

(1) $\beta = 0$ *if and only if* $a/p > b/q$ (*i.e.* $\tau < 0$).
(2) $\gamma = 0$ *if and only if* $a/p < b/q$ (*i.e.* $\tau > 0$).

Remark 3.16. Since we are assuming $\mathrm{GCD}(p+q, a+b) = 1$ then $\tau \neq 0$.

4. Homotopy groups

4.1. **Homotopy groups of $\mathcal{M}(n,d)$.** Combining Propositions 3.2, 3.12 and 3.14 we have the following.

Theorem 4.1. *Let* $\mathrm{GCD}(n,d) = 1$. *Then*

$$\pi_i(\mathcal{M}(n,d)) \cong \pi_i(M(n,d)), \quad for \quad i \leqslant 2g-4.$$

Now, the homotopy groups of $M(n,d)$ have been computed by Daskalopoulos and Uhlenbeck [9] (here n and d are not assumed to be coprime). Their result is the following.

Theorem 4.2. *Let $M^s(n,d)$ be the moduli space of stable vector bundles of rank n and degree d. Let $(n,g) \neq (2,2)$. Then*

(1) $\pi_1(M^s(n,d)) \cong H_1(X, \mathbb{Z})$;
(2) $\pi_2(M^s(n,d)) \cong \mathbb{Z} \oplus \mathbb{Z}_{\mathrm{GCD}}(n,d)$;
(3) $\pi_i(M^s(n,d)) \cong \pi_{i-1}(\mathcal{G})$, *for* $2 < i \leqslant 2(g-1)(n-1)-2$, *where $\mathcal{G}$ is the unitary gauge group.*

Remark 4.3. The proof of (1) when n and d are coprime is given by Atiyah–Bott [1].

As a corollary of Theorems 4.1 and 4.2 we have the following.

Theorem 4.4. *Let* $\mathrm{GCD}(n,d) = 1$ *and let* $g \geqslant 3$. *Then*

(1) $\pi_1(\mathcal{M}(n,d)) \cong H_1(X, \mathbb{Z})$;
(2) $\pi_2(\mathcal{M}(n,d)) \cong \mathbb{Z}$;
(3) $\pi_i(\mathcal{M}(n,d)) \cong \pi_{i-1}(\mathcal{G})$, *for* $2 < i \leqslant 2g-4$.

Remark 4.5. As a consequence of Theorem 4.1 and the connectedness of $M(n,d)$ [19] one obtains that $\mathcal{M}(n,d)$ is also connected [16, 23].

A proof of (1) when $n = 2$ is given by Hitchin [16].

4.2. **Moduli space of triples.** The next step is to identify the spaces $\mathcal{N}(p,q,a,b)$ as moduli spaces in their own right. They turn out to be examples of the moduli spaces of triples studied in [3] [4] and [5]. We briefly recall the relevant definitions and results. See [4] for details.

A *holomorphic triple* on X, $T = (E_1, E_2, \phi)$, consists of two holomorphic vector bundles E_1 and E_2 on X and a holomorphic map $\phi\colon E_2 \to E_1$. Denoting the ranks E_1 and E_2 by n_1 and n_2, and their degrees by d_1 and d_2, we refer to (n_1, n_2, d_1, d_2) as the *type* of the triple.

A homomorphism from $T' = (E_1', E_2', \phi')$ to $T = (E_1, E_2, \phi)$ is a commutative diagram

$$\begin{tikzcd} E_2' \arrow[r, "\phi'"] \arrow[d] & E_1' \arrow[d] \\ E_2 \arrow[r, "\phi"] & E_1. \end{tikzcd}$$

$T' = (E_1', E_2', \phi')$ is a subtriple of $T = (E_1, E_2, \phi)$ if the homomorphisms of sheaves $E_1' \to E_1$ and $E_2' \to E_2$ are injective.

For any $\alpha \in \mathbb{R}$ the *α-degree* and *α-slope* of T are defined to be

$$\begin{aligned} \deg_\alpha(T) &= \deg(E_1) + \deg(E_2) + \alpha \operatorname{rk}(E_2), \\ \mu_\alpha(T) &= \frac{\deg_\alpha(T)}{\operatorname{rk}(E_1) + \operatorname{rk}(E_2)} \\ &= \mu(E_1 \oplus E_2) + \alpha \frac{\operatorname{rk}(E_2)}{\operatorname{rk}(E_1) + \operatorname{rk}(E_2)}. \end{aligned}$$

The triple $T = (E_1, E_2, \phi)$ is *α-stable* if

$$\mu_\alpha(T') < \mu_\alpha(T) \tag{4.1}$$

for any proper sub-triple $T' = (E_1', E_2', \phi')$. Define *α-semistability* by replacing (4.1) with a weak inequality. A triple is called *α-polystable* if it is the direct sum of α-stable triples of the same α-slope. It is *strictly α-semistable (polystable)* if it is α-semistable (polystable) but not α-stable.

We denote the moduli space of isomorphism classes of *α-polystable* triples of type (n_1, n_2, d_1, d_2) by

$$\mathcal{N}_\alpha = \mathcal{N}_\alpha(n_1, n_2, d_1, d_2). \tag{4.2}$$

Using Jordan–Hölder filtrations of α-semistable triples one can define S-equivalence, and view $\mathcal{N}_\alpha$ as the moduli space of S-equivalence classes of α-semistable triples. The isomorphism classes of *α-stable* triples form a subspace which we denoted by $\mathcal{N}_\alpha^s$.

Proposition 4.6 ([3])**.** *The moduli space $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$ is a complex projective variety. A necessary condition for the moduli space $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$ to be non-empty is*

$$\begin{cases} 0 \leqslant \alpha_m \leqslant \alpha \leqslant \alpha_M & \text{if } n_1 \neq n_2 \\ 0 \leqslant \alpha_m \leqslant \alpha & \text{if } n_1 = n_2 \end{cases} \tag{4.3}$$

where

$$\alpha_m = \mu_1 - \mu_2, \tag{4.4}$$

$$\alpha_M = \Big(1 + \frac{n_1 + n_2}{|n_1 - n_2|}\Big)(\mu_1 - \mu_2) \tag{4.5}$$

and $\mu_1 = \frac{d_1}{n_1}$, $\mu_2 = \frac{d_2}{n_2}$.

Whenever necessary we shall indicate the dependence of α_m and α_M on (n_1, n_2, d_1, d_2) by writing $\alpha_m = \alpha_m(n_1, n_2, d_1, d_2)$, and similarly for α_M.

Within the allowed range for α there is a discrete set of *critical values*. These are the values of α for which it is numerically possible to have a subtriple $T' = (E_1', E_2', \phi')$ such that $\mu(E_1' \oplus E_2') \neq \mu(E_1 \oplus E_2)$ but $\mu_\alpha(T') = \mu_\alpha(T')$. All other values of α are called *generic*. The critical values of α are precisely the values for α at which the stability properties of a triple can change, i.e. there can be triples which are strictly α-semistable, but either α'-stable or α'-unstable for $\alpha' \neq \alpha$.

The following result relates the stability conditions for holomorphic triples and that for $\mathrm{U}(p, q)$-Higgs bundles.

Proposition 4.7. *A* $\mathrm{U}(p,q)$*-Higgs bundle* (E,Φ) *with* $\beta=0$ *or* $\gamma=0$ *is (semi)stable if and only if the corresponding holomorphic triple is* α*-(semi)stable for* $\alpha=2g-2$*.*

Combining Propositions 3.15 and 4.7, we have the following characterization of the subspace of local minima $\mathcal{N}(p,q,a,b)$.

Theorem 4.8. *Let* $\mathcal{N}(p,q,a,b)$ *be the subspace of local minima of* f *on* $\mathcal{M}(p,q,a,b)$ *and let* τ *be the Toledo invariant.*

If $a/p<b/q$*, or equivalently if* $\tau<0$*, then* $\mathcal{N}(p,q,a,b)$ *can be identified with the moduli space of* α*-polystable triples of type* $(p,q,a+p(2g-2),b)$*, with* $\alpha=2g-2$*.*

If $a/p>b/q$*, or equivalently if* $\tau>0$*, then* $\mathcal{N}(p,q,a,b)$ *can be identified with the moduli space of* α*-polystable triples of type* $(q,p,b+q(2g-2),a)$*, with* $\alpha=2g-2$*.*

That is,

$$\mathcal{N}(p,q,a,b)\cong\begin{cases}\mathcal{N}_{2g-2}(p,q,a+p(2g-2),b) & \text{if } a/p<b/q \text{ (equivalently } \tau<0)\\ \mathcal{N}_{2g-2}(q,p,b+q(2g-2),a) & \text{if } a/p>b/q \text{ (equivalently } \tau>0)\end{cases}$$

In view of Theorem 4.8 it is important to understand where $2g-2$ lies in relation to the range (given by Proposition 4.6) for the stability parameter α. One has the following.

Proposition 4.9. *Fix* (p,q,a,b)*. Then*

$$0\leqslant|\tau|\leqslant\min\{p,q\}(2g-2)\Leftrightarrow 0<\alpha_m(p,q,a,b)\leqslant 2g-2\leqslant\alpha_M(p,q,a,b) \text{ if } p\neq q \tag{4.6}$$

Proposition 4.9 shows that in order to study $\mathcal{N}(p,q,a,b)$ for different values of the Toledo invariant, we need to understand the moduli spaces of triples for values of α that may lie anywhere (including at the extremes α_m and α_M) in the α-range given in Proposition 4.6.

We can assume $n_1>n_2$, since by triples duality one has the following.

Proposition 4.10. $\mathcal{N}_\alpha(n_1,n_2,d_1,d_2)\cong\mathcal{N}_\alpha(n_2,n_1,-d_2,-d_1)$.

Recall that the allowed range for the stability parameter is $\alpha_m\leqslant\alpha\leqslant\alpha_M$, where $\alpha_m=\mu_1-\mu_2$ and $\alpha_M=\frac{2n_1}{n_1-n_2}\alpha_m$, and we assume that $\mu_1-\mu_2>0$. We describe the moduli space $\mathcal{N}_\alpha$ for $2g-2\leqslant\alpha<\alpha_M$.

Let α_L be the largest critical value in (α_m,α_M), and let $\mathcal{N}_L$ (respectively $\mathcal{N}_L^s$) denote the moduli space of α-polystable (respectively α-stable) triples for $\alpha_L<\alpha<\alpha_M$. We refer to $\mathcal{N}_L$ as the 'large α' moduli space.

Proposition 4.11. *Let* $T=(E_1,E_2,\phi)$ *be an* α*-semistable triple for some* α *in the range* $\alpha_L<\alpha<\alpha_M$*. Then* T *is of the form*

$$0\longrightarrow E_2\xrightarrow{\phi}E_1\longrightarrow F\longrightarrow 0, \tag{4.7}$$

with F *locally free, and* E_2 *and* F *are semistable.*

In the converse direction we have:

Proposition 4.12. *Let* $T=(E_1,E_2,\phi)$ *be a triple of the form*

$$0\longrightarrow E_2\xrightarrow{\phi}E_1\longrightarrow F\longrightarrow 0,$$

with F *locally free. If* E_2 *is stable and* F *is stable then* T *is* α*-stable for* $\alpha=\alpha_M-\epsilon$ *in the range* $\alpha_L<\alpha<\alpha_M$*.*

Theorem 4.13. *Assume that* $n_1 > n_2$ *and* $d_1/n_1 > d_2/n_2$. *Then the moduli space* $\mathcal{N}_L^s = \mathcal{N}_L^s(n_1, n_2, d_1, d_2)$ *is smooth of dimension*

$$(g-1)(n_1^2 + n_2^2 - n_1 n_2) - n_1 d_2 + n_2 d_1 + 1,$$

and includes a $\mathbb{P}^N$*-fibration* $\mathcal{P}$ *over* $M^s(n_1 - n_2, d_1 - d_2) \times M^s(n_2, d_2)$, *where* $M^s(n, d)$ *is the moduli space of stable bundles of rank* n *and degree* d, *and* $N = n_2 d_1 - n_1 d_2 + n_1(n_1 - n_2)(g-1) - 1$. *Moreover, the complex codimension of* $\mathcal{N}_L^s \smallsetminus \mathcal{P}$ *is greater or equal than* $g - 1$. *In particular,* $\mathcal{N}_L^s(n_1, n_2, d_1, d_2)$ *is non-empty and irreducible.*

If $\mathrm{GCD}(n_1 - n_2, d_1 - d_2) = 1$ *and* $\mathrm{GCD}(n_2, d_2) = 1$, *the birational equivalence is an isomorphism.*

Proof. The birational equivalence between $\mathcal{P}$ and $\mathcal{N}_L^s$ is proved in [5]. To obtain the precise estimate of the codimension of $\mathcal{N}_L^s \smallsetminus \mathcal{P}$ in $\mathcal{N}_L^s$ we see that, by Proposition 4.11, it suffices to estimate the dimension of stable triples like (4.7) with E_2 and F semistable.

Now, for any family of semistable bundles the complex codimension of the set of strictly semistable bundles is at least $g - 1$. A computation of the precise estimate can be found in [7]. The proof is finished by observing that for a stable triple of the form (4.7) $H^0(X, E_2 \otimes F^*) = 0$ (see [5]). □

The following is proved in [5].

Theorem 4.14. *Let* α *be any value in the range* $\alpha_m < 2g - 2 \leqslant \alpha < \alpha_M$. *Then* $\mathcal{N}_\alpha^s$ *is birationally equivalent to* $\mathcal{N}_L^s$. *Moreover, they are isomorphic outside of a set of complex codimension greater or equal than* $g-1$. *In particular,* $\mathcal{N}_\alpha^s$ *is non-empty and irreducible.*

4.3. **Homotopy groups of moduli spaces of triples.** The strategy to compute the homotopy groups of $\mathcal{N}(p, q, a, b)$ is to compute first those of the moduli space of α-stable triples $\mathcal{N}_\alpha^s$ for large α.

Let $n_1 > n_2$ and let $\mathcal{P} \subset \mathcal{N}_L$ be the $\mathbb{P}^N$-fibration over $M^s(n_1 - n_2, d_1 - d_2) \times M^s(n_2, d_2)$ given in Theorem 4.14. As a consequence of Theorems 4.13 and 4.14 we have the following.

Proposition 4.15. *Let* $2g - 2 \leqslant \alpha < \alpha_M$. *Then*

$$\pi_i(\mathcal{N}_\alpha^s(n_1, n_2, d_1, d_2)) \cong \pi_i(\mathcal{N}_L^s(n_1, n_2, d_1, d_2)) \cong \pi_i(\mathcal{P}) \quad \text{for} \quad i \geqslant 2g - 4.$$

Associated to the $\mathbb{P}^n$-fibration $\mathcal{P}$ over $M^s(n_1 - n_2, d_1 - d_2) \times M^s(n_2, d_2)$ there is a homotopy sequence

$$\begin{aligned} (4.8) \quad \cdots \longrightarrow \pi_i(\mathbb{P}^N) \longrightarrow \pi_i(\mathcal{P}) \longrightarrow \pi_i(M^s(n_2, d_2)) \times \pi_i(M^s(n_1 - n_2, d_1 - d_2)) \\ \longrightarrow \pi_{i-1}(\mathbb{P}^N) \longrightarrow \cdots \end{aligned}$$

Proposition 4.16. *Let* $n_1 > n_2$. *Assume that* $(n_2, g) \neq (2, 2)$ *and* $(n_1 - n_2, g) \neq (2, 2)$ *(for our applications we will actually assume* $g \neq 3$*). Then*

(1) $\pi_1(\mathcal{P}) \cong \pi_1(M^s(n_2, d_2)) \times \pi_1(M^s(n_1 - n_2, d_1 - d_2)) \cong H_1(X, \mathbb{Z}) \oplus H_1(X, \mathbb{Z})$;
(2) $\pi_2(\mathcal{P}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_{\mathrm{GCD}(n_2, d_2)} \oplus \mathbb{Z}_{\mathrm{GCD}(n_1 - n_2, d_1 - d_2)}$.

Proof. From the homotopy sequence (4.8), since $\pi_0(\mathbb{P}^N) = \pi_1(\mathbb{P}^N) = 0$, we deduce that $\pi_1(\mathcal{P}) \cong \pi_1(M^s(n_2, d_2)) \times \pi_1(M^s(n_1 - n_2, d_1 - d_2))$. Statement (1) follows from Theorem 4.2.

Since $\pi_1(\mathbb{P}^N) = 0$, (4.8) gives

$$\text{(4.9)} \quad ... \longrightarrow \pi_2(\mathbb{P}^N) \longrightarrow \pi_2(\mathcal{P}) \longrightarrow \pi_2(M^s(n_2, d_2)) \times \pi_2(M^s(n_1 - n_2, d_1 - d_2)) \longrightarrow 0.$$

On the other hand, by Hurewicz' theorem $\pi_2(\mathbb{P}^N)) \cong H_2(\mathbb{P}^N, \mathbb{Z}) \cong \mathbb{Z}$. Now, the map $f : \mathbb{Z} \cong \pi_2(\mathbb{P}^N) \longrightarrow \pi_2(\mathcal{P})$ in (4.9) is injective since one has the commutative diagram

$$\begin{array}{ccc} \mathbb{Z} \cong \pi_2(\mathbb{P}^N) & \xrightarrow{f} & \pi_2(\mathcal{P}) \\ \| \downarrow & & \downarrow \\ H_2(\mathbb{P}^N, \mathbb{Z}) & \longrightarrow & H_2(\mathcal{P}, \mathbb{Z}). \end{array}$$

and $H_2(\mathbb{P}^N, \mathbb{Z}) \longrightarrow H_2(\mathcal{P}, \mathbb{Z})$ must be injective because the restriction of an ample line bundle over $\mathcal{P} \subset \mathcal{N}_L$ to $\mathbb{P}^N$ must give an ample line bundle. Note that the natural map $H_2(\mathbb{P}^N) \longrightarrow H_2(\mathcal{N}_L)$ is injective and factors through $H_2(\mathbb{P}^N) \longrightarrow H_2(\mathcal{P}) \longrightarrow H_2(\mathcal{N}_L)$. Now, we obtain (2) from Theorem 4.2. $\square$

As a corollary of Proposition 4.15 and Proposition 4.16 we have the following.

Theorem 4.17. *Let $n_1 > n_2$, let $g \geqslant 3$ and let $2g - 2 \leqslant \alpha < \alpha_M$. Then*

(1) $\pi_1(\mathcal{N}_\alpha^s) \cong H_1(X, \mathbb{Z}) \oplus H_1(X, \mathbb{Z})$;
(2) $\pi_2(\mathcal{N}_\alpha^s) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_{\mathrm{GCD}(n_2,d_2)} \oplus \mathbb{Z}_{\mathrm{GCD}(n_1-n_2,d_1-d_2)}$.

4.4. **Homotopy groups of $\mathcal{M}(p, q, a, b)$.** Combining Propositions 3.2, 3.12 and 3.15 we have the following.

Theorem 4.18. *Let* $\mathrm{GCD}(p + q, a + b) = 1$. *Then*

$$\pi_i(\mathcal{M}(p, q, a, b)) \cong \pi_i(\mathcal{N}(p, q, a, b)), \quad for \quad i \leqslant 2g - 4.$$

As a corollary of Theorems 4.18, 4.8 and 4.17 and Proposition 4.10, we conclude the following.

Theorem 4.19. *Let $p \neq q$ and* $\mathrm{GCD}(p + q, a + b) = 1$ *and let $g \geqslant 3$. Then*

(1) $\pi_1(\mathcal{M}(p, q, a, b)) \cong H_1(X, \mathbb{Z}) \oplus H_1(X, \mathbb{Z})$;
(2) $\pi_2(\mathcal{M}(p, q, a, b)) \cong \mathbb{Z}^{\oplus 3} \oplus \mathbb{Z}_m \oplus \mathbb{Z}_l$, *where*
   - $m = \mathrm{GCD}(p, -a - p(2g - 2))$ *and* $l = \mathrm{GCD}(q - p, a - b + p(2g - 2))$ *for* $\tau < 0$ *and* $p < q$;
   - $m = \mathrm{GCD}(q, b)$ *and* $l = \mathrm{GCD}(p - q, a - b + p(2g - 2))$ *for* $\tau < 0$ *and* $p > q$;
   - $m = \mathrm{GCD}(p, a)$ *and* $l = \mathrm{GCD}(q - p, b - a + q(2g - 2))$ *for* $\tau > 0$ *and* $p < q$;
   - $m = \mathrm{GCD}(q, -b - q(2g - 2))$ *and* $l = \mathrm{GCD}(p - q, b - a + q(2g - 2))$ *for* $\tau > 0$ *and* $p > q$.

Remark 4.20. As a consequence of Theorem 4.18 and the connectedness of $\mathcal{N}(p, q, a, b)$ we have that $\mathcal{M}(p, q, a, b)$ is also connected, as proved in [4].

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