A SURVEY OF RECENT RESULTS ON STATISTICAL FEATURES OF NON-UNIFORMLY EXPANDING MAPS

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Abstract. The aim of this work is to present a survey of recent results on the statistical properties of non-uniformly expanding maps on finite-dimensional Riemannian manifolds. We will mostly focus on the existence of SRB measures, continuity of the SRB measure and its entropy, decay of correlations and stochastic stability.

Introduction

In broad terms, one may say that Dynamical Systems theory has two major goals: *i*) to describe the typical behavior of trajectories, specially as time goes to infinity; *ii*) to understand how this behavior changes when the system is modified, and to what extent it is stable under small perturbations. Even in cases of very simple evolution laws the orbits may have a rather complicated behavior. Moreover, systems may display sensitivity on the initial conditions, i.e. a small variation on the initial point may give rise to a completely different behavior of its orbit. Among many others, these are obstacles that we have to deal with when we try to predict the long-term behavior of a system and its stability.

Related to the first goal that we have mentioned above are the *Sinai-Ruelle-Bowen (SRB)* or *physical measures* which characterize asymptotically, in time average, a large set of orbits in the phase space. These measures may be understood as equilibrium states for a probabilistic description of the dynamical system. Significant information on the dynamical properties of a system is given by the *correlation decay* of an SRB measure, which in particular tells the velocity at which the equilibrium can be reached. Connected to this is the *statistical stability* of a system, which means continuous variation of the SRB measures under small modifications of the law that governs the system; this naturally points in the direction of the second goal, as does the *stochastic stability* of a system. Briefly, this may be understood as the characterization of the stability of the statistical properties of the system when small random errors are incorporated in each iteration.

The *entropy* of a dynamical system can be regarded as a measure of unpredictability of the system. *Topological entropy* measures the complexity of a dynamical system in terms of the exponential growth rate of the number of orbits distinguishable over long time intervals. *Metric entropy* quantifies the average level of uncertainty every time we iterate, in terms of exponential growth rate of the number of statistically significant paths an orbit can follow. Here we will be particularly interested in the metric entropy with respect to an SRB measure of the system.

Dynamical systems in Riemannian manifolds displaying uniformly expanding behavior or combining uniformly expanding behavior in several directions with uniformly contracting behavior in other directions (hyperbolic systems) have been exhaustively studied in the last decades, and many results on their statistical properties have been obtained, starting with Sinai, Krzyzewski, Szlenk, Ruelle and Bowen; see [B1, B2, BR, KS, R1, S1, S2] and also

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[Ki1, Ki2, Y1]. The study of the statistical features of certain dynamical systems exhibiting expansion only in asymptotic terms has been done in the pioneer work of Jakobson [Ja], where the existence of SRB measures for many quadratic transformations of the interval was established; see also [BeC1, BeY1]. Decay of correlations and stochastic stability for this kind of systems have been obtained by Baladi, Benedicks, Carleson, Viana and Young [BaY, BaV, BeC1, BeY1]. Also related to this is the remarkable work of Benedicks and Carleson [BeC2] for Hénon two dimensional maps exhibiting strange attractors; see also [BeV1, BeV2, BeY2, BeY3, MV, Vi1].

Motivated by the results obtained in [Al1, Vi2] for Viana maps, general conclusions on the existence of SRB measures for multidimensional non-uniformly expanding dynamical systems are drawn in [ABV]; see also [Car, BoV] for related results. Subsequent works gave rise to a good knowledge on the properties of those SRB measures; see for instance [AA1, ALP, AV, Cas, Go, Ol, Va].

Concerning the stability of the statistical behavior of dynamical systems in a wide sense, we are also interested in the variation of entropy. The question of the continuity of the entropy (topological or metric) is an old issue, going back to the work of Newhouse [Ne], for instance. By means of the continuity of the (unique) SRB measure for uniformly expanding transformations and the entropy formula for these transformations one easily obtains the continuity of the SRB entropy for that kind of transformations. For Axiom A diffeomorphisms the continuity of SRB measures and even more regularity is established in [R3] and [Ma3]. The analyticity of the metric entropy for Anosov diffeomorphisms is proved in [Po]. General conclusions on the continuity of the SRB metric entropy for non-uniformly expanding maps were established in [AOT].

An expanded version of these notes with a detailed presentation of almost all subjects including proofs of the results can be found in [Al3]. An exception concerns the results on the continuity of entropy, whose proofs can be found in [AOT].

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1. Non-uniform expansion

Let M be a compact finite-dimensional Riemannian manifold and let m be a normalized Riemannian volume form on M that we call *Lebesgue measure*. Let $f\colon M\to M$ be a continuous transformation, which is a local diffeomorphism in the whole manifold but in a critical set $\mathcal{C}\subset M$ with zero Lebesgue measure.

Definition 1.1. We say that $\mathcal{C}$ is a *non-degenerate critical set* if conditions (c_1)-(c_3) below are satisfied. The first one essentially means that $\|Df\|$ behaves as power of the distance close to the critical set: there are $B>1$ and $\beta>0$ such that for all $x\in M\setminus\mathcal{C}$ and $v\in T_xM$

$$(c_1)\qquad \frac{1}{B}\operatorname{dist}(x,\mathcal{C})^{\beta}\le\frac{\|Df(x)v\|}{\|v\|}\le B\operatorname{dist}(x,\mathcal{C})^{-\beta}.$$

Furthermore, we want the functions $\log|\det Df|$ and $\log\|Df^{-1}\|$ to be *locally Lipschitz* with Lipschitz constants depending on the distance to the critical set: for all $x,y\in M\setminus\mathcal{C}$ with $\operatorname{dist}(x,y)<\operatorname{dist}(x,\mathcal{C})/2$ we have

$$(c_2)\qquad \big|\log\|Df(x)^{-1}\|-\log\|Df(y)^{-1}\|\,\big|\le\frac{B}{\operatorname{dist}(x,\mathcal{C})^{\beta}}\operatorname{dist}(x,y);$$

$$(c_3)\qquad \big|\log|\det Df(x)|-\log|\det Df(y)|\,\big|\le\frac{B}{\operatorname{dist}(x,\mathcal{C})^{\beta}}\operatorname{dist}(x,y).$$

For the next definition we need to introduce the following notion. Given $\delta>0$ and $x\in M\setminus\mathcal{C}$, we define the *$\delta$-truncated distance* from x to $\mathcal{C}$ as

$$\operatorname{dist}_\delta(x,\mathcal{C})=\begin{cases}1, & \text{if } \operatorname{dist}(x,\mathcal{C})\ge\delta;\\ \operatorname{dist}(x,\mathcal{C}), & \text{otherwise.}\end{cases}$$

Definition 1.2. Let $f\colon M\to M$ be a C^2 local diffeomorphism in $M\setminus\mathcal{C}$, where $\mathcal{C}$ is a non-degenerate critical set with zero Lebesgue measure. We say that f is *non-uniformly expanding* if the following conditions hold:

(n_1) there is $\lambda>0$ such that for Lebesgue almost all $x\in M$

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log\|Df(f^i(x))^{-1}\|<-\lambda;$$

(n_2) for all $\epsilon>0$ there is $\delta>0$ such that for Lebesgue almost every $x\in M$

$$\limsup_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}-\log\operatorname{dist}_\delta(f^j(x),\mathcal{C})<\epsilon.$$

We shall refer to condition (n_2) as *slow recurrence* to the critical set. We naturally ignore that condition whenever $\mathcal{C}=\emptyset$.

Condition (n_1) assures that the *expansion time* function

$$\mathcal{E}(x)=\min\left\{N\ge1:\frac{1}{n}\sum_{i=0}^{n-1}\log\|Df^{-1}_{f^i(x)}\|<-\lambda,\quad\text{for all } n\ge N\right\}$$

is well defined and finite Lebesgue almost everywhere in M.

Slow recurrence condition is not needed in all its strength for our purposes. Actually, the only place where it is needed is in the proof of Proposition 2.5, and as observed in [Al4, Remark 3.8] it is enough that it holds for some sufficiently small $\epsilon>0$ and $\delta>0$

conveniently chosen. We shall fix once and for all $\epsilon > 0$ and $\delta > 0$ in those conditions. Thus the *recurrence time*

$$\mathcal{R}(x) = \min\left\{ N \geq 1 : \frac{1}{n}\sum_{i=0}^{n-1} -\log \operatorname{dist}_\delta(f^j(x), \mathcal{C}) < \epsilon, \quad \text{for all } n \geq N \right\},$$

is also well defined and finite Lebesgue almost everywhere in M, by (n_1).

We define the *tail set (at time n)*

$$\Gamma_n = \big\{ x \in M \colon \mathcal{E}(x) > n \ \text{ or } \ \mathcal{R}(x) > n \big\}. \tag{1}$$

This is the set of points that at time n have not reached yet the exponential growth or slow recurrence guaranteed by conditions (n_1) and (n_2).

It is proved in [Ma1] that a non-uniformly expanding local diffeomorphism in dimension one is necessarily *uniformly expanding*: there is $\sigma > 1$ such that for some choice of a Riemannian metric $\|\ \|$ in M

$$\|Df(x)v\| \geq \sigma\|v\|, \quad \text{for all } x \in M \text{ and all } v \in T_xM.$$

Such a result is not valid in higher dimensions – the non-uniformly expanding local diffeomorphisms that we present in §1.2 are not uniformly expanding in M, since saddle points are allowed.

The next result shows that apparently weaker forms of non-uniform expansion imply uniform expansion. Recall that a set is said to have *total probability* if it has probability one for every invariant probability measure.

Theorem 1.3. [AAS] *Let $f : M \to M$ be a C^1 local diffeomorphism. If for all x in a subset of M with total probability*

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log \|Df(f^i(x))^{-1}\| < 0, \tag{2}$$

then f is uniformly expanding.

Observe that (2) above is even weaker than condition (n_1) in the definition of a non-uniformly expanding map. A similar result for maps with critical sets cannot hold, for obvious reasons.

We present next three classes of non-uniformly transformations. The first one is the well-known family of one-dimensional quadratic maps; the second one is a family of local diffeomorphisms in higher dimensions from [ABV]; and finally, the family of Viana maps which has been introduced in [Vi2].

1.1. **Quadratic maps.** Consider $f_a \colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = a - x^2$, with $a \in \mathbb{R}$. The most interesting part on study of the dynamics of f_a occurs for $a \in (-1/4, 2]$, with f_a restricted to the interval $[f^2(0), f(0)]$. Benedicks and Carleson proved in [BeC1, BeC2] that there is a positive Lebesgue measure set $\Omega \subset (-1/4, 2]$ for which f_a with $a \in \Omega$ satisfies for all $n \geq 1$ both the *Collet-Eckmann condition*,

$$|(f_a^n)'(f_a(0))| \geq e^{cn}, \quad c > 0,$$

and the *basic assumption*,

$$|f_a^n(0)| \geq e^{-\alpha n}, \quad \alpha > 0.$$

Maps satisfying both Collet-Eckmann condition and the basic assumption are necessarily non-uniformly expanding, as the next result shows.

Theorem 1.4. [Fr] *If $a \in \Omega$, then f_a is non-uniformly expanding. Moreover, there are $C, \tau > 0$ such that $m(\Gamma_n) \le Ce^{-\tau\sqrt{n}}$ for all $n \ge 1$.*

The results in [Fr] give in particular that the set of points x for which $\mathcal{E}(x) > n$ decays exponentially fast with n.

1.2. **Local diffeomorphisms.** Let $f_0 \colon M \to M$ be a uniformly expanding transformation of the finite dimensional Riemannian manifold M, and take $V \subset M$ some compact domain for which $f_0|_V$ is injective. We take f close to f_0 such that for $0 < \sigma_0 < 1 < \sigma_1$ and $\delta > 0$ small:

(1) $|\det Df(x)| > \sigma_1$ for all $x \in M$;
(2) $\|Df(x)^{-1}\| < \sigma_0$ for all $x \in M \setminus V$;
(3) $\|Df(x)^{-1}\| < 1 + \delta$ for all $x \in V$.

In the first condition we require that f expands volume in M, while in the second one we impose f to be uniformly expanding $M \setminus V$. The last condition means that f does not contract to much in V. We have defined in this way an open set in the C^1 topology of transformations from M into itself. Note that the conditions above allow us to create saddle points for f, thus having transformations in this set which are not uniformly expanding.

Theorem 1.5. [ABV] *If $\delta > 0$ is sufficiently small, then f is non-uniformly expanding. Moreover, there are $C, c > 0$ such that $m(\Gamma_n) \le Ce^{-cn}$ for all $n \ge 1$.*

The main ingredient for the proof of this theorem consists in the study of the asymptotic frequency of visits of typical orbits to the set V. The fact that f is volume expanding implies that such frequency is a small fraction of time.

1.3. **Viana maps.** Let $a_0 \in (1, 2)$ be a parameter for which $x = 0$ is pre-periodic for the quadratic transformation $Q(x) = a_0 - x^2$. Take $S^1 = \mathbb{R}/\mathbb{Z}$ and $b : S^1 \to \mathbb{R}$ a Morse function, for instance $b(s) = \sin(2\pi s)$. Consider, for small $\alpha > 0$, the cylinder transformation $\hat{f} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ given by

$$\hat{f}(s, x) = \big(\hat{g}(s), \hat{q}(s, x)\big),$$

where $\hat{g}$ is an expanding map of the circle $\hat{g}(s) = ds \pmod{\mathbb{Z}}$, for some $d \ge 2$ and

$$\hat{q}(s, x) = a(s) - x^2, \quad \text{with } a(s) = a_0 + \alpha b(s).$$

In the former work of Viana it was imposed that $d \ge 16$ for technical reasons. However, the later [BST] showed that $d \ge 2$ is enough for the results we have in mind.

If f is sufficiently close to $\hat{f}$ in the C^2 topology, then f has a non-degenerate critical set close to $\{x = 0\}$. Moreover, for small $\alpha > 0$ there is an interval $I \subset (-2, 2)$ such that $\hat{f}(S^1 \times I)$ is contained in the interior of $S^1 \times I$. Thus, any f suficiently close to $\hat{f}$ in the C^0 topology has $S^1 \times I$ as a forward invariant region. Furthermore, the results in [AV] give that $\Lambda = f^2(S^1 \times I)$ is an attractor for f on which f is *topologically mixing*: given any open set $A \subset \Lambda$ there is $n \ge 1$ such that $f^n(A) = \Lambda$. In particular, we have the transitivity of f when restricted to Λ.

Theorem 1.6. [Vi2] *If f is close to $\hat{f}$ in the C^3 topology, then f is non-uniformly expanding. Moreover, there are $C, \gamma > 0$ such that $m(\Gamma_n) \le Ce^{-\gamma\sqrt{n}}$ for all $n \ge 1$.*

Similar examples can be obtained in higher dimensions, considering transformations $\hat{f} \colon T^m \times \mathbb{R} \longrightarrow T^m \times \mathbb{R}$, where T^m is the m-dimensional torus and $\hat{f}$ is given by $\hat{f}(\Theta, x) = (\hat{g}(\Theta), \hat{h}(\Theta, x))$, where $\hat{g}$ is an expanding map of T^m and $\hat{h}(\Theta, x) = a_0 + \alpha b(\Theta) - x^2$. The non-uniform expansion of these maps is also proved in [Vi2].

2. SRB measures

Definition 2.1. A probability measure μ invariant by $f\colon M \to M$ is said to be an *SRB measure* for f if for a positive Lebesgue measure set of points $x \in M$

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi\, d\mu, \quad \text{for every continuous } \varphi : M \to \mathbb{R}. \tag{3}$$

We define $B(\mu)$, the *basin* of μ, as the set of those points $x \in M$ for which (3) holds.

It follows easily from Birkhoff's Ergodic Theorem that if μ is an ergodic probability measure which is absolutely continuous with respect to the Lebesgue measure, then μ is an SRB measure. Actually, if μ is ergodic, then by Birkhoff's Ergodic Theorem $B(\mu)$ has full μ measure. The absolute continuity of μ with respect to Lebesgue implies that $B(\mu)$ cannot have zero Lebesgue measure.

Theorem 2.2. [ABV] *Let $f\colon M \to M$ be a C^2 non-uniformly expanding map. Then there are absolutely continuous (with respect to Lebesgue) ergodic probability measures $\mu_1, \dots, \mu_p$ whose basins cover a full Lebesgue measure subset of M.*

One of the main ingredients in the proof of this theorem presented in [ABV] is the existence of *hyperbolic times*. This notion has been introduced in [Al1] for proving the existence of SRB measures for Viana maps, and put into an abstract setting in [ABV]. The applications of hyperbolic times go far beyond the proof of the existence of SRB measures for non-uniformly expanding maps, as it will become clear along these notes.

Definition 2.3. Given $\sigma < 1$ and $\delta > 0$, we say that n is a (σ, δ)-*hyperbolic time* for $x \in M$, if for all $1 \le k \le n$,

$$\prod_{j=n-k}^{n-1} \|Df(f^j(x))^{-1}\| \le \sigma^k \quad \text{and} \quad \operatorname{dist}_\delta(f^{n-k}(x), \mathcal{C}) \ge \sigma^{bk}, \tag{4}$$

where $b > 0$ is a small constant. If $\mathcal{C} = \emptyset$, then the definition of (σ, δ)-hyperbolic time reduces to the first condition in (4) and we call it simply σ-hyperbolic time.

We present below results which show that: *i*) if n is a hyperbolic time for x, then f^n is a diffeomorphism with bounded distortion (not depending on the point nor on the iterate) in a neighborhood of x which is sent into a ball of uniform radius; *ii*) almost every point has infinitely many hyperbolic times.

Proposition 2.4. [ABV] *Given $0 < \sigma < 1$ and $\delta > 0$, there are $\delta_1, D_1 > 0$ such that if n is a (σ, δ)-hyperbolic time for $x \in M$, then there is a neighborhood $V_n(x)$ of x for which:*
1. *f^n sends $V_n(x)$ diffeomorphically onto the ball $B(f^n(x), \delta_1)$;*
2. $\operatorname{dist}(f^{n-k}(y), f^{n-k}(z)) \le \sigma^{k/2} \operatorname{dist}(f^n(y), f^n(z))$ *for all $1 \le k < n$ and $y, z \in V_n(x)$;*
3. *f^n has distortion bounded by D_1 in $V_n(x)$: for all $y, z \in V$ we have*
$$\frac{1}{D_1} \le \frac{|\det Df^n(y)|}{|\det Df^n(z)|} \le D_1\,.$$

We shall refer to a set $V_n(x)$ given in the previous proposition as a *hyperbolic pre-ball*, and to its image $f^n(V_n(x))$ as a *hyperbolic ball*. The latter are indeed balls of radius $\delta_1 > 0$.

Observe that if f is a uniformly expanding transformation, then every n is a hyperbolic time for every $x \in M$. Our next issue concerns the existence of hyperbolic times for non-uniformly expanding maps. In such context hyperbolic times exist for almost all points with positive frequency as the next result shows.

Proposition 2.5. [ABV] *There are $\theta, \delta > 0$ such that for Lebesgue almost every $x \in M$ and large $n \in \mathbb{N}$ there are (σ, δ)-hyperbolic times $1 \le n_1 < \cdots < n_l \le n$ for x with $l \ge \theta n$.*

In order to illustrate an application of hyperbolic times, let us sketch a proof of Theorem 2.2. We follow the approach in [ABV]. Let $(\mu_n)_n$ be the sequence of measures in M

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j m. \tag{5}$$

Letting H_j denote the set of those points $x \in M$ for which j is a (σ, δ)-hyperbolic time, we define

$$\nu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j (m \mid H_j).$$

It follows from Proposition 2.5 that there is $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$

$$\nu_n(M) \ge \frac{1}{n}\sum_{i=0}^{n-1} m(H_i) \ge \frac{\theta}{2} m(M). \tag{6}$$

Moreover, using the bounded distortion property at hyperbolic times, we have

$$f_*^j(m|H_j) \ll m, \quad \text{for all } j \ge 1,$$

with density uniformly bounded from above, and hence the same holding for ν_n. Now we take $n_k \to \infty$ such that

$$\mu_{n_k} \to \mu \quad \text{and} \quad \nu_{n_k} \to \nu$$

in the weak* topology. Then μ is an invariant probability measure that may be written as $\mu = \nu + \eta$, for some measure η, with $\nu \ll m$ and $\nu(M) > 0$ by (6). Considering the Lebesgue decomposition

$$\eta = \eta_{ac} + \eta_s, \quad \text{with } \eta_s \perp m \quad \text{and} \quad \eta_{ac} \ll m,$$

then $\mu_{ac} = \nu + \eta_{ac}$ gives the absolutely continuous component for the decomposition of μ. By uniqueness of the decomposition and the fact that f preserves null Lebesgue measure subsets, we conclude that μ_{ac} is a finite measure invariant by f. Additionally,

$$\mu_{ac}(M) \ge \nu(M) > 0.$$

Normalizing μ_{ac} we obtain an absolutely continuous invariant probability measure.

The next result plays a crucial role in the decomposition of the measure μ_{ac} into a sum of finitely many ergodic components.

Lemma 2.6. [ABV] *Let $G \subset M$ with $m(G) > 0$ be forward invariant by f. Then there is some disk Δ (with radius depending only on δ_1) such that $m(\Delta \setminus G) = 0$.*

The idea of the proof of this lemma is simple. By Proposition 2.5 almost every point in M has infinitely many hyperbolic times. Thus, taking arbitrarily large hyperbolic times we obtain arbitrarily small hyperbolic pre-balls as in Proposition 2.4, with arbitrarily large (close to 1) density of points in G. For the conclusion of the lemma one just have to use bounded distortion and take an accumulation ball of the hyperbolic balls.

Let us now finish the sketch of the proof of Theorem 2.2. Let μ_0 be an absolutely continuous invariant probability measure. If μ_0 is not ergodic, then there are invariant sets H_1 and H_2 with $M = H_1 \dot\cup H_2$ and $\mu_0(H_1)\mu_0(H_2) > 0$. In particular, H_1 and H_2 have positive Lebesgue measure. Let

$$\mu_1 = \mu_0(H_1)^{-1}(\mu_0|H_1), \quad \text{and} \quad \mu_2 = \mu_0(H_2)^{-1}(\mu_0|H_2).$$

Clearly, μ_1 and μ_2 are also absolutely continuous invariant measures. If they are not ergodic, then we decompose it in the same way as we did it for μ_0. Lemma 2.6 assures that this decomposition must stop after a finite number of steps, and so

$$\mu_0 = \sum_{i=1}^{s} \mu_0(H_i)\mu_i,$$

where $H_1, \dots, H_s$ is a partition of M into subsets of positive Lebesgue measure, and each

$$\mu_i = \frac{1}{\mu_0(H_i)}(\mu_0|H_i)$$

is an ergodic probability measure. Uniqueness of the SRB measure can be assured if one assumes transitivity.

Corollary 2.7. *Assume that $f\colon M \to M$ is a C^2 non-uniformly expanding map. If f is transitive, then M is covered (Lebesgue mod 0) by the basin of a unique SRB measure, which is ergodic and absolutely continuous.*

3. Hyperbolic times: frequency versus integrability

We say that f has *positive frequency* of (σ, δ)-hyperbolic times if the conclusion of Lema 2.5 holds almost everywhere in M, i.e. there is $\theta > 0$ such that for Lebesgue almost every $x \in M$ and large $n \in \mathbb{N}$ there are $\ell \geq \theta n$ and iterates $1 \leq n_1 < n_2 \cdots < n_\ell \leq n$ which are (σ, δ)-hyperbolic times for x. In the proof of Theorem 2.2 sketched above one uses non-uniform expansion just to assure the existence of hyperbolic times with positive frequency. Moreover, for the existence of at least one absolutely continuous ergodic probability measure, it is enough to have non-uniform expansion holding in a subset with positive Lebesgue measure. Thus we have the following result:

Theorem 3.1. *Let $f\colon M \to M$ be a C^2 local diffeomorphism in $M \setminus \mathcal{C}$, where $\mathcal{C} \subset M$ is a non-degenerate critical set. If f has positive frequency of (σ, δ)-hyperbolic times in a set with positive Lebesgue measure, then f has an absolutely continuous invariant probability measure.*

The existence of hyperbolic times for Lebesgue almost all points allows us to introduce a function $h\colon M \to \mathbb{N}$ (defined Lebesgue almost everywhere), associating the first hyperbolic time at the points where it exists. The same conclusion of Theorem 3.1 may be drawn assuming the integrability of the first hyperbolic time function.

Theorem 3.2. [AA2] *Let $f\colon M \to M$ be a C^2 local diffeomorphism in $M \setminus \mathcal{C}$, where $\mathcal{C} \subset M$ is a non-degenerate critical set. If $h : M \to \mathbb{Z}^+$ is integrable with respect to Lebesgue measure, then f has some absolutely continuous invariant probability measure.*

Actually, the proof of this last theorem gives something even stronger than Theorem 3.1: assuming the integrability of h with respect to the Lebesgue measure, it is possible to prove that every weak* accumulation point of (5) is absolutely continuous with respect to Lebesgue measure.

At this point it is natural to try to find any relation between the existence positive frequency of hyperbolic times Lebesgue almost everywhere and the integrability of the first hyperbolic time function with respect to the Lebesgue measure. We present in Example 3.5 a map with positive frequency of hyperbolic times Lebesgue almost everywhere such that no first hyperbolic time function is integrable with respect to the Lebesgue measure. Such example has a nonempty critical set. It is not known any example like that with no critical set (a local diffeomorphism). In the opposite direction we have the following result for local diffeomorphisms:

Theorem 3.3. [AA2] *Let $f\colon M \to M$ be a C^2 local diffeomorphism. If the first σ-hyperbolic time function $h\colon M \to \mathbb{N}$ is integrable with respect to Lebesgue measure for some $0 < \sigma < 1$, then there is $\sigma < \bar\sigma < 1$ such that Lebesgue almost every point has positive frequency of $\bar\sigma$-hyperbolic times.*

Let us consider now the case of $f : M \to M$ with a nonempty critical set. In this case we need to assume some stronger integrability of the first hyperbolic time function in order to guarantee the same conclusion of Theorem 3.3. We do not know if this stronger hypothesis is really necessary or it is just a weakness of the proof.

Theorem 3.4. [AA2] *Let $f\colon M \to M$ be a C^2 local diffeomorphism in $M\setminus\mathcal{C}$, where $\mathcal{C} \subset M$ is a non-degenerate critical set. If the first (σ,δ)-hyperbolic time function $h\colon M \to \mathbb{N}$ belongs to $L^p(m)$ with $p > 4$, then there is some $\sigma < \bar\sigma < 1$ such that Lebesgue almost every point has positive frequency of $(\bar\sigma,\delta)$-hyperbolic times.*

We finish this section with an example of a transformation presented in [AA2] with positive frequency of hyperbolic times almost everywhere for which no first hyperbolic time function is Lebesgue integrable.

Example 3.5. Consider a function $f\colon I \to I$ from the interval $I = [-1,1]$ into itself such that $f(x) = 2\sqrt{x} - 1$, for $x \ge 0$, and $f(x) = 1 - 2\sqrt{|x|}$, otherwise. Since $f(-1) = -1$ and $f(1) = 1$ this induces a transformation $f : S^1 \to S^1$, differentiable everywhere except at the point 0 having $\mathcal{C} = \{0, \pm 1\}$ as a non-degenerate critical set. Moreover, $\log \operatorname{dist}(x,\mathcal{C})$ and $\log|(f'(x))^{-1}|$ are integrable with respect to Lebesgue measure, and $\int_{S^1} \log|(f'(x))^{-1}|\,dx < 0$. Using the fact that f preserves Lebesgue measure, Birkhoff's Ergodic Theorem implies the existence of a subset of points in S^1 with positive Lebesgue measure where f is non-uniformly expanding. Using mixing properties of the dynamical system, it is proved in [AA2] that f is actually non-uniformly expanding, and so it has positive frequency of hyperbolic times.

For the non-integrability of the first hyperbolic time map, we start by observing that if n is a (σ,δ)-hyperbolic time for $x \in S^1$, then we have $|f'(f^{n-1}(x))| \ge \sigma^{-1} > 1$, and so the first (σ,δ)-hyperbolic time for x is at least the number of iterates the orbit of that point needs to hit a neighborhood of 0. Thus, considering a point $x_1 \in (0,1)$ and the sequence $(x_n)_{n\ge1}$ of its pre-images in $(0,1)$, we have

$$\int_{S^1} h\,dm \gtrsim \sum_{n\ge1} n(x_{n+1} - x_n).$$

Noting that $x_{n+1} - x_n \approx 1/n^2$, we obtain the non-integrability of h with respect to the Lebesgue measure.

4. Induced transformations

A method which proves usefulness in many contexts for deriving some ergodic properties of certain dynamical systems consists in replacing the original dynamics by an induced (return) map in some region of the phase space. This is the method used in [Al1] for constructing SRB measures for Viana maps, which contains in particular a result on the existence of absolutely continuous invariant probability measures for piecewise expanding maps. The proof of this result uses bounded variation functions in higher dimensions, and generalizes a result from [GB] to the setting of maps with infinitely many smoothness domains.

The method used in [ABV] for the construction of absolutely continuous invariant measures is more flexible than that in [Al1], having however the weakness that it does not give much information on the properties of the absolutely continuous invariant measures.

The results in [ALP] show that induced transformations with good properties exist in general for non-uniformly expanding maps. The initial motivation in [ALP] for the cosntruction of such structures was to obtain estimates on the decay of correlations for non-uniformly expanding maps, as we shall see in §7. However, the existence of those induced transformations prove usefulness also in the study of the continuity of SRB measures and their entropy, as we shall see in §5 and §6.

Definition 4.1. Consider $F : \Delta \to \Delta$ defined in a region $\Delta \subset M$. We say that F is an *induced transformation* for $f\colon M \to M$ if there is a countable partition $\mathcal{P}$ of a full Lebesgue measure subset of Δ, and there is a *return time* function $\tau\colon \mathcal{P} \to \mathbb{Z}^+$ such that $F|_\omega = f^{\tau(\omega)}|_\omega$ for each $\omega \in \mathcal{P}$. Consider also the following conditions:

(i_1) Markov*: $F|_\omega$ is a C^2 diffeomorphism onto Δ, for each $\omega \in \mathcal{P}$.*

(i_2) Expansion*: there is $0 < \kappa < 1$ such that $\|DF(x)^{-1}\| < \kappa$ for all $\omega \in \mathcal{P}$ and $x \in \omega$.*

(i_3) Bounded distortion*: there is $K > 0$ such that for all $\omega \in \mathcal{P}$ and $x, y \in \omega$*

$$\log \left| \frac{\det DF(x)}{\det DF(y)} \right| \le K \operatorname{dist}(F(x), F(y)).$$

It is well known that a transformation in these conditions has a unique ergodic absolutely continuous invariant probability measure. Furthermore, such probability measure is equivalent to Lebesgue measure on Δ, with density bounded from above and from below by constants; see [Y3, Y4], for instance.

The integrability of the return time function $\tau\colon \Delta \to \mathbb{Z}^+$ with respect to the Lebesgue measure is sufficient for deriving the existence of an absolutely continuous invariant probability measure for the original dynamics f. Indeed, if μ_F is the absolutely continuous invariant probability measure for F, then

$$\mu_f^* = \sum_{j=0}^{\infty} f_*^j \left(\mu_F \mid \{\tau_f > j\}\right) \tag{7}$$

is an invariant measure absolutely continuous with respect to the Lebesgue measure, which is finite if τ is Lebesgue integrable. We shall denote by μ_f the probability measure which consists of the normalization of μ_f^*.

Using a technique inspired in that developed for Axiom A attractors in [Y3], it was possible to prove in [ALP] the existence of induced transformations satisfying (i_1)-(i_3)

for transitive non-uniformly expanding maps, thus generalizing the result in [Al1] on the existence of such structures for Viana maps.

Theorem 4.2. [ALP] *Let $f : M \to M$ be a transitive non-uniformly expanding C^2 transformation. Then f induces some transformation in a disk $\Delta \subset M$ satisfying (i_1), (i_2) and (i_3). Moreover, if $m(\Gamma_n) \lesssim n^{-\gamma}$ for some $\gamma > 0$, then $m\{\tau > n\} \lesssim n^{-\gamma}$.*

Gouezel has recently announced in [Go] similar results for subexponential and exponential decay of the tail set.

The naive strategy for proving the last theorem can be described as follows. We start by choosing a point $p \in M$ with dense pre-images and some sufficiently small ball Δ_0 around this point. This will be the domain of definition of our induced map. We then attempt to implement the naive strategy of iterating Δ_0 until we find some good return iterate n_0 such that $f^{n_0}(\Delta_0)$ completely covers Δ_0 and some bounded distortion property is satisfied. There exists then some topological ball $U \subset \Delta_0$ such that $f^{n_0}(U) = \Delta_0$. This ball is then by definition an element of the final partition of Δ_0 for the induced Markov map and has an associated return time n_0. We then continue iterating the complement $\Delta_0 \setminus U$ until more good returns occur.

5. Statistical stability

We aim at studying the continuous variation of SRB measures with respect to the dynamics in certain families of non-uniformly expanding maps.

Definition 5.1. Let $\mathcal{F}$ be a family of C^k transformations, for some $k \geq 2$, from a manifold M into itself, and consider $\mathcal{F}$ endowed with the C^k topology. We assume that each $f \in \mathcal{F}$ admits a unique f-invariant probability measure μ_f. We say that $f_0 \in \mathcal{F}$ is *statistically stable* (in $\mathcal{F}$) if

$$\mathcal{F} \ni f \longmapsto \frac{d\mu_f}{dm}$$

is continuous at f_0 with respect to the $L^1(m)$-norm in the space of densities.

Using induced transformations we present below sufficient conditions for the statistical stability of transformations in certain families. Assume that we may associate to each $f \in \mathcal{F}$ an induced transformation $F_f\colon \Delta \to \Delta$ satisfying (i_1), (i_2) and (i_3). We thus implicitly have for each $f \in \mathcal{F}$ a partition $\mathcal{P}_f$ in the smoothness domains of F_f, and the corresponding return time function $\tau_f\colon \mathcal{P}_f \to \mathbb{Z}^+$. If $\tau_f \in L^1(\Delta)$, then μ_f^* defined as in (7) is a finite absolutely continuous f-invariant measure, where μ_F is the unique absolutely continuous F-invariant probability measure. Consider the following *uniformity conditions*:

(u_1) τ_f varies continuously with $f \in \mathcal{F}$ in the C^k-topology;

(u_2) κ and K as in (i_2) and (i_3) may be taken the same for every $f \in \mathcal{F}$.

Theorem 5.2. [AV] *If $\mathcal{F}$ is a family of transformations satisfying (u_1) and (u_2), then each $f \in \mathcal{F}$ is statistically stable.*

Consider now some family $\mathcal{F}$ of C^k non-uniformly expanding transformations (with the same constants ϵ, δ and λ) for some $k \geq 2$. We associate to each $f \in \mathcal{F}$ its tail set Γ_n^f defined as in (1).

Theorem 5.3. [Al2] *Let $\mathcal{F}$ be a family of transitive non-uniformly expanding C^k transformations for some $k \geq 2$. If there are $C > 0$ and $\gamma > 1$ such that $m(\Gamma_n^f) \leq Cn^{-\gamma}$ for all $f \in \mathcal{F}$ and $n \geq 1$, then we may construct induced transformations for the elements of $\mathcal{F}$ satisfying (u_1) and (u_2).*

The transitivity assumption is imposed for we be able to use Theorem 4.2, thus guaranteeing the existence of induced transformations. As an immediate consequence of Theorem 5.2 and Theorem 5.3 we have:

Corollary 5.4. *If there are $C > 0$ and $\gamma > 1$ such that $m(\Gamma_n^f) \leq Cn^{-\gamma}$ for all $f \in \mathcal{F}$ and $n \geq 1$, then $f \in \mathcal{F}$ is statistically stable.*

The quadratic transformations with parameters in the Benedicks-Carleson set, the local diffeomorphisms from §1.2 and Viana maps all satisfy the assumptions of Corollary 5.4; see [AV, Al4, Fr]. Thus we have that the maps in those three classes are statistically stable.

6. Continuity of entropy

In this section we address ourselves to the study of the continuity of the entropy with respect to the SRB measure, for certain families of non-uniformly expanding maps, following the approach of [AOT].

For families of uniformly expanding maps $f\colon M \to M$, the continuity of the entropy $h_{\mu_f}(f)$ with respect to the (unique) SRB measure μ_f is an immediate consequence of the continuity of the SRB measure, together with the *entropy formula*

$$h_{\mu_f}(f) = \int \log|\det Df(x)|\, d\mu_f, \tag{8}$$

since $\log|\det Df(x)|$ is continuous.

Let us remark that for maps with critical points the continuity of the SRB measure does not necessarily imply the continuity of its entropy. In fact, in the quadratic family $f_a(x) = 4ax(1-x)$ we may find parameters a for which f_a has an absolutely continuous invariant measure and a sequence of parameters a_n converging to a with the maps f_{a_n} having a unique SRB measure which is a singular measure supported on an attracting periodic orbit (sink). Moreover, the measures supported on these sinks have zero entropy and converge to the SRB measure of f_a, which has strictly positive entropy.

Let $\mathcal{F}$ be a class of non-uniformly expanding C^k maps, for some $k \geq 2$, for which each $f \in \mathcal{F}$ admits a unique SRB measure μ_f. A formula similar to the one displayed in (8) holds for C^2 endomorphisms f of a compact manifold M with respect to an absolutely continuous invariant probability measure μ_f. In fact, by [Li, Remark 1.2] the Jacobian function $\log|\det Df(x)|$ is always integrable with respect to μ_f. Thus, by [QZ, Theorem 1.1], if

$$\lambda_1(x) \leq \cdots \leq \lambda_s(x) \leq 0 \leq \lambda_{s+1}(x) \leq \cdots \leq \lambda_d(x)$$

are the Lyapunov exponents at x, then

$$h_{\mu_f}(f) = \int_M \sum_{i=s+1}^{d} \lambda_i(x)\, d\mu_f(x). \tag{9}$$

In our situation f has all its Lyapunov exponents positive with respect to μ_f. Hence, by Oseledets Theorem and the integrability of the Jacobian of f with respect to μ_f, we have that the entropy formula (8) holds for each $f \in \mathcal{F}$.

Assume now that we have a family $\mathcal{F}$ as in §5. In particular, each $f \in \mathcal{F}$ induces some transformation $F_f : \Delta \to \Delta$ that we assume satisfying (i_1), (i_2) and (i_3) as in §4. For each $f \in \mathcal{F}$ we consider the partition $\mathcal{P}_f$ of Δ and the return time function $\tau_f\colon \mathcal{P}_f \to \mathbb{Z}^+$ as before. Let μ_F be the unique absolutely continuous F_f-invariant probability measure. Assuming that $\tau_f \in L^1(\Delta)$, we define μ_f^* as in (7) and μ_f the respective normalization. Our first result establishes a relation between the entropies of μ_F and μ_f; this is a classical well-known result whose proof may be found in [AOT].

Theorem 6.1. *If F is an induced transformation for f, then*

$$h_{\mu_f}(f) = \frac{1}{\mu_f^*(M)} h_{\mu_F}(F).$$

As we have seen in Theorem 5.2, the SRB measure varies continuously with $f \in \mathcal{F}$ under certain assumptions on the family $\mathcal{F}$. Under the same assumptions we obtain the continuity of the SRB entropy.

Theorem 6.2. [AOT] *If $\mathcal{F}$ is a family satisfying (u_1) and (u_2), then the entropy $h_{\mu_f}(f)$ varies continuously with $f \in \mathcal{F}$.*

The proof of this result uses Theorem 6.1 and the fact that $\mu_f^*(M)$ and the entropy $h_{\mu_F}(F)$ vary continuously with $f \in \mathcal{F}$. Similarly to Corollary 5.4, we obtain continuity of the SRB entropy for families of non-uniformly expanding transformations, whenever the decay of the Lebesgue measure of the tail set occurs with some uniformity in $f \in \mathcal{F}$.

Corollary 6.3. *Let $\mathcal{F}$ be a family of transitive non-uniformly expanding C^k maps, for some $k \geq 2$. If there are $C > 0, \gamma > 1$ such that $m(\Gamma_n^f) \leq Cn^{-\gamma}$ for all $f \in \mathcal{F}$ and $n \geq 1$, then the entropy $h_{\mu_f}(f)$ varies continuously with $f \in \mathcal{F}$.*

Once more, we have imposed the transitivity assumption in order to be able to use Theorem 4.2 and assure the existence of induced transformations.

The quadratic transformations with parameter in the Benedicks-Carleson set, the local diffeomorphisms from §1.2 and Viana maps all satisfy the assumptions of Corollary 6.3; see [AV, Al4, Fr]. Thus we have that the SRB entropy depends continuously on the transformation in those three classes.

7. Decay of correlations

A transformation $f\colon M \to M$ is said to be *mixing* with respect to an invariant measure μ, if for all measurable sets $A, B \subset M$ we have $\mu(f^{-n}(A) \cap B) \to \mu(A)\mu(B)$ when $n \to \infty$. Defining for $\varphi, \psi\colon M \to \mathbb{R}$ the *correlation function*

$$C_n(\varphi, \psi) = \left| \int (\varphi \circ f^n)\psi d\mu - \int \varphi d\mu \int \psi d\mu \right|,$$

we have that f is mixing if and only if

$$C_n(\chi_A, \chi_B) \to 0 \quad \text{when } n \to \infty,$$

for all measurable sets $A, B \subset M$. We are interested in determining the velocity at which $C_n(\varphi, \psi) \to 0$ when $n \to \infty$ (*decay of correlations*) and to assure the validity of the *Central Limit Theorem (CLT)*: Given $\varphi\colon M \to \mathbb{R}$ Hölder continuous which is not a co-boundary ($\varphi \neq \psi \circ f - \psi$ for any $\psi \in L^2(\mu)$), there is $\sigma > 0$ such that for each interval $J \subset \mathbb{R}$,

$$\mu\left\{x \in X : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left(\varphi(f^j(x)) - \int \varphi d\mu\right) \in J\right\} \to \frac{1}{\sigma\sqrt{2\pi}} \int_J e^{-t^2/2\sigma^2} dt.$$

Theorem 7.1. [ALP] *Let $f : M \to M$ be a transitive non-uniformly expanding C^2 map. If there is some $\gamma > 1$ such that $m(\Gamma_n) \leq \mathcal{O}(n^{-\gamma})$, then some power of f is mixing with respect to the (unique) SRB measure, and the correlation function for $\varphi \in L^\infty(m)$ and ψ Hölder continuous satisfies $C_n(\varphi, \psi) \leq \mathcal{O}(n^{-\gamma+1})$. Moreover, if $\gamma > 2$ then CLT holds.*

Using Gouezel results one obtains similar conclusions for subexponential and exponential decay of the tail set; see [Go].

Corollary 2.7 assures that in the conditions of the last theorem f has a unique probability measure μ which is ergodic and absolutely continuous with respect to Lebesgue measure. The proof of Theorem 7.1 consists in combining Theorem 4.2 with results from [Y4] for Markovian towers, as we illustrate next. By Theorem 4.2 we know that there is some disk $\Delta_0 \subset M$, a countable partition $\mathcal{P}$ into subdisks of Δ_0, up to a zero Lebesgue measure subset, and a return time function $\tau : \Delta_0 \to \mathbb{N}$ constant in the elements of $\mathcal{P}$ such that (i_1), (i_2) and (i_3) hold for the induced transformation $F : \Delta_0 \to \Delta_0$ given by $F(x) = f^{\tau(x)}(x)$. We introduce a *tower*

$$\Delta = \big\{(x, n) \in \Delta_0 \times \mathbb{N}:\ 0 \le n < \tau(x)\big\},$$

and the respective *tower map* $T\colon \Delta \to \Delta$, defined as

$$T(x,n) = \begin{cases} (x, n+1) & \text{if } n+1 < \tau(x), \\ (F(x), 0) & \text{if } n+1 = \tau(x). \end{cases}$$

We have by construction

$$T^{\tau(x)}(x, 0) = (F(x), 0) = (f^{\tau(x)}(x), 0).$$

Let m_0 be the Lebesgue measure in Δ_0 and $\bar{m}$ the measure obtained from it by pushing forward m_0 to the upper levels of the tower. Note that each level of the tower is naturally identified with a subset of Δ_0. Let

$$\mathcal{H}_\beta = \left\{\varphi : \Delta \to \mathbb{R} \mid\ \exists C > 0 \text{ such that } |\varphi(x) - \varphi(y)| \le C\beta^{s(x,y)}\ \forall x, y \in \Delta\right\},$$

where $s(x, y)$ is the *separation time* defined as the biggest $n \ge 0$ such that $F^i(x)$ and $F^i(y)$ lie in the same element of $\mathcal{P}$ for $1 \le i \le n$. We define $R_1, R_2, \dots$ as the sequence of return times to the ground level of the tower.

Theorem 7.2. [Y4] *Assume that $gcd\{R_i\} = 1$. If $\varphi \in L^\infty(\bar{m})$ and $\psi \in \mathcal{H}_\beta$, then*

1. *T has an invariant probability measure ν equivalent to $\bar{m}$;*
2. *if $m\{R > n\} = \mathcal{O}(n^{-\gamma})$ for some $\gamma > 1$, then $C_n(\varphi, \psi) = \mathcal{O}(n^{-\gamma+1})$;*
3. *if $m\{R > n\} = \mathcal{O}(n^{-\gamma})$ for some $\gamma > 2$, then CLT holds.*

Since we have no a priori knowledge of $\gcd\{R_i\}$, it might happen that we have to take some power of f in order to assure that $\gcd\{R_i\} = 1$. Consider the projection from the tower into the manifold

$$\pi\colon \Delta \to \bigcup_{n \ge 0} f^n(\Delta_0)$$

defined as $\pi(x, n) = f^n(x)$. This projection π satisfies $f \circ \pi = \pi \circ T$. Taking $\mu^* = \pi_*\nu$, we easily see that μ^* is an absolutely continuous f-invariant probability measure. Hence, μ^* coincides with the unique absolutely continuous f invariant probability measure μ.

Take $\varphi \in L^\infty(m)$ and $\varphi\colon M \to \mathbb{R}$ Hölder continuous with exponent $\eta > 0$. Defining $\bar{\psi} = \psi \circ \pi$ and $\bar{\varphi} = \varphi \circ \pi$, then $\bar{\psi} \in L^\infty(\bar{m})$ and $\bar{\varphi} \in \mathcal{H}_\beta$, where $\beta = \max \|(Df^R)^{-1}\|^\eta$. Moreover,

$$\int (\varphi \circ f^n)\psi d\mu - \int \varphi d\mu \int \psi d\mu = \int (\bar{\varphi} \circ T^n)\bar{\psi} d\nu - \int \bar{\varphi} d\nu \int \bar{\psi} d\nu.$$

This implies the result on the decay of correlations. We similarly obtain CLT.

8. Stochastic stability

The goal of this section is to study the statistical behavior of orbits evolving under small random perturbations of a fixed dynamical system, and to understand how the statistical behavior of this random system is related to the original system, when we consider small random perturbations (stochastic stability). The main reference for our approach is [AA1].

Consider a dynamical system $f\colon M \to M$ and a continuous map

$$\begin{array}{rccl} F: & T & \longrightarrow & C^2(M,M) \\ & t & \longmapsto & f_t \end{array}$$

defined in a metric space T and ranging in the space of C^2 maps from M into itself, with $f = f_{t^*}$ for some fixed $t^* \in T$. Given $\underline{t} = (t_1, t_2, t_3, \dots)$ in the product space $T^{\mathbb{N}}$ we define

$$f^0_{\underline{t}} = \mathrm{id}_M, \quad \text{and} \quad f^n_{\underline{t}} = f_{t_n} \circ \cdots \circ f_{t_1}, \quad \text{for each } n \geq 1.$$

We say that $\left(f^n_{\underline{t}}(x)\right)_{n\geq 1}$ is a *random orbit* of $x \in M$. In the presence of a critical set $\mathcal{C}$ for f we shall assume that the transformations f_t have a common critical set, and impose that

$$Df_t(x) = Df(x), \quad \text{for all } x \in M \setminus \mathcal{C} \text{ and all } t \in T.$$

Perturbations of this type may be implemented for instance in Lie groups by considering additive noise.

Consider also a family of probability measures $(\theta_\epsilon)_{\epsilon>0}$ in T such their supports $\mathrm{supp}(\theta_\epsilon)$ form a nested decreasing sequence of compact connected sets, and $\mathrm{supp}(\theta_\epsilon) \to \{t^*\}$ when $\epsilon \to 0$. Given an integer $n \geq 1$ and $x \in M$, we consider the function $\tau^n_x\colon T^{\mathbb{N}} \to M$ defined as $\tau^n_x(\underline{t}) = f^n_{\underline{t}}(x)$.

Definition 8.1. We shall refer to $\{F, (\theta_\epsilon)_{\epsilon>0}\}$ with the above properties as a *random perturbation* of f. We say that a random perturbation $\{F, (\theta_\epsilon)_{\epsilon>0}\}$ is *non-degenerate*, if for small $\epsilon > 0$ we may choose $\xi = \xi(\epsilon) > 0$ such that for all $x \in M$:

(1) $\{f_t(x)\colon t \in \mathrm{supp}(\theta_\epsilon)\}$ contains a ball of radius ξ around $f(x)$;
(2) $(\tau_x)_*\theta^{\mathbb{N}}_\epsilon$ is absolutely continuous with respect to m.

Definition 8.2. Let $\{F, (\theta_\epsilon)_{\epsilon>0}\}$ be a random perturbation of $f\colon M \to M$. Given $\epsilon > 0$ we say that a probability measure μ^ϵ on the Borel sets of M is a *physical measure* if, for a positive Lebesgue measure subset of points $x \in M$,

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\big(f^n_{\underline{t}}(x)\big) = \int \varphi \, d\mu^\epsilon \tag{10}$$

for every continuous $\varphi\colon M \to \mathbb{R}$ and $\theta^{\mathbb{N}}_\epsilon$ almost every $\underline{t} \in T^{\mathbb{N}}$. The set of points $x \in M$ for which (10) holds for every continuous φ and $\theta^{\mathbb{N}}_\epsilon$ almost every $\underline{t} \in T^{\mathbb{N}}$ is denoted by $B(\mu^\epsilon)$ and called the *basin* of μ^ϵ.

Theorem 8.3. [AA1] *Let $\{F, (\theta_\epsilon)_{\epsilon>0}\}$ be a non-degenerate random perturbation of a non-uniformly expanding C^2 map f. Then there is $l \geq 1$ such that for small $\epsilon > 0$ there exist physical measures $\mu^\epsilon_1, \dots, \mu^\epsilon_l$ for which:*

(1) *given $x \in M$ there is $T_1(x), \dots, T_l(x)$ a partition $\theta^{\mathbb{N}}_\epsilon$ mod 0 of $T^{\mathbb{N}}$ such that*

$$\mu^\epsilon_i = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n-1} \delta_{f^j_{\underline{t}}(x)}, \quad \textit{for all } \underline{t} \in T_i(x);$$

(2) *if* $m|B(\mu_i^\epsilon)$ *denotes the normalized restriction of* m *to the basin of* μ_i^ϵ, *then*

$$\mu_i^\epsilon = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \int (f_{\underline{t}}^j)_*\big(m|B(\mu_i^\epsilon)\big)\, d\theta_\epsilon^{\mathbb{N}}(\underline{t}).$$

Both limits are taken in the weak topology.*

The number of physical measures may depend on the noise level $\epsilon > 0$, being however a decreasing function of ϵ. The previous theorem shows that the number of physical measures stabilizes for sufficiently small noise level. Letting l be the number of physical measures for small noise level and p be the number of SRB measures for f, Proposition 4.4 in [AA1] shows that for sufficiently small $\epsilon > 0$ we have $l \le p$. Examples for which $l < p$ and $l = p$ are given in [AA1].

Definition 8.4. We say that $f\colon M \to M$ is *stochastically stable* if, for every non-degenerate random perturbation of f, the accumulation points in the weak* topology of the physical measures (when $\epsilon \to 0$) are convex linear combinations of the SRB measures of f.

Theorem 8.5. [AA1] *Let* $f\colon M \to M$ *be a non-uniformly expanding* C^2 *local diffeomorphism. If* f *is stochastically stable, then there is* $c > 0$ *such that for* $\theta_\epsilon^{\mathbb{N}} \times m$ *almost every* $(\underline{t}, x) \in T^{\mathbb{N}} \times M$ *one has*

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log \|Df(f_{\underline{t}}^j(x))^{-1}\| < -c.$$

This result motivates the next definition.

Definition 8.6. Let $f\colon M \to M$ be a C^2 local diffeomorphism except, possibly, in a null Lebesgue measure set $\mathcal{C}$, and let $\{F, (\theta_\epsilon)_{\epsilon>0}\}$ be a random perturbation of f. We say that f is *non-uniformly expanding on random orbits* if:

(1) there is $c > 0$ such that for $\theta_\epsilon^{\mathbb{N}} \times m$ almost all $(\underline{t}, x) \in T^{\mathbb{N}} \times M$

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log \|Df(f_{\underline{t}}^j(x))^{-1}\| < -c;$$

(2) given $\gamma > 0$ small, there is $\delta > 0$ such that for $\theta_\epsilon^{\mathbb{N}} \times m$ almost all $(\underline{t}, x) \in T^{\mathbb{N}} \times M$

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta(f_{\underline{t}}^j(x), \mathcal{C}) < \gamma.$$

Given $0 < \alpha < 1$ and $\delta > 0$, we say that $n \ge 1$ is a (α, δ)-*hyperbolic time* for $(\underline{t}, x) \in T^{\mathbb{N}} \times M$ if for every $1 \le k \le n$

$$\prod_{j=n-k}^{n-1} \|Df_{t_{j+1}}(f_{\underline{t}}^j(x))^{-1}\| \le \alpha^k \quad \text{and} \quad \operatorname{dist}_\delta(f_{\underline{t}}^{n-k}(x), \mathcal{C}) \ge \alpha^{bk}, \tag{11}$$

where $b > 0$ is a small constant. The positive frequency of hyperbolic times and bounded distortion may also be obtained for non-uniformly expanding maps on random orbits, thus arriving at results similar to Proposition 2.4 and Proposition 2.5 in this context. In particular, we may define a first hyperbolic time function $h_\epsilon\colon T^{\mathbb{N}} \times M \to \mathbb{Z}^+$. Assuming $h_\epsilon \in L^1(\theta_\epsilon^{\mathbb{N}} \times m)$, we have

$$\|h_\epsilon\|_1 = \sum_{k=0}^{\infty} k\, (\theta_\epsilon^{\mathbb{N}} \times m)\{(\underline{t}, x)\colon h_\epsilon(\underline{t}, x) = k\} < \infty.$$

We say that the family of first hyperbolic time functions $(h_\epsilon)_{\epsilon>0}$ has *uniform L^1 tail* if the above series converges uniformly, as a series of functions in the variable $\epsilon > 0$.

Theorem 8.7. [AA1] *Let $f\colon M \to M$ be a non-uniformly expanding C^2 map. If f is non-uniformly expanding on random orbits and $(h_\epsilon)_\epsilon$ has uniform L^1 tail, then f is stochastically stable.*

It is shown in [AA1] that the local diffeomorphisms from §1.2 and Viana maps satisfy the assumptions of Theorem 8.7, thus being stochastically stable.

9. Partially hyperbolic diffeomorphisms

In the previous sections we have presented several results on the statistical properties of non-uniformly expanding endomorphisms of finite dimensional Riemannian manifolds. Some of these results have versions for diffeomorphisms with attractors having a partially hyperbolic structure.

Let $f : M \to M$ be a C^1 diffeomorphism of a Riemannian manifold M. We say that a positively invariant compact set $K \subset M$ has a *dominated decomposition* if there is a Df-invariant splitting of the tangent bundle $T_K M = E^{cs} \oplus E^{cu}$ over K and there is a constant $\lambda < 1$ such that for some choice of a Riemannian metric

$$\|Df \mid E^{cs}_x\| \cdot \|Df^{-1} \mid E^{cu}_{f(x)}\| \le \lambda, \quad \text{for all } x \in K.$$

We call E^{cs} the *centre-stable* sub-bundle and E^{cu} the *centre-unstable* sub-bundle of the decomposition. We say that E^{cs} is *uniformly contractive* in K, and in such case we denote it by E^{ss}, if there is $\lambda < 1$ such that $\|Df \mid E^{ss}_x\| \le \lambda$ for all $x \in K$; analogously we say that E^{cu} is *uniformly expanding* in K, and in such case we denote it by E^{uu}, if there is $\lambda < 1$ such that $\|(Df \mid E^{cu}_x)^{-1}\| \le \lambda$ for all $x \in K$.

Non-uniform contraction and non-uniform expansion, respectively in the centre-stable and centre-unstable directions, are defined in average along orbits. We say that the centre-stable sub-bundle is *non-uniformly contractive* in K if there is $c_s > 0$ such that for all $x \in K$

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df | E^{cs}_{f^j(x)}\| < -c_s.$$

The centre-unstable sub-bundle is said to be *non-uniformly expanding* in K if there is $c_u > 0$ such that for all $x \in K$

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^{-1} \mid E^{cu}_{f^j(x)}\| < -c_u.$$

We say that f is *partially hyperbolic* in K if one of the sub-bundles has uniform behavior and the other one has non-uniform behavior. That is, we consider either a situation of the type $E^{ss} \oplus E^{cu}$ with non-uniform expansion along the E^{cu} direction, or a situation of the type $E^{cs} \oplus E^{uu}$ with non-uniform contraction along the E^{cs} direction.

Let us now enumerate some recent results on the statistical properties of partially hyperbolic diffeomorphisms.

(1) The existence of SRB measures for partially hyperbolic diffeomorphisms of the type $E^{cs} \oplus E^{uu}$ was studied in [Car] and in [BoV]. These SRB measures are precisely the Gibbs states obtained in [PS].
(2) The existence of SRB for partially hyperbolic diffeomorphisms of the type $E^{ss} \oplus E^{cu}$ is proved in [ABV]. Also in that work some sufficient conditions for the existence

of SRB measures for diffeomorphisms with dominated splitting and simultaneous non-uniform expansion and non-uniform contraction are given.

(3) Results on the decay of correlations for for partially hyperbolic diffeomorphisms of the type $E^{cs} \oplus E^{uu}$ were obtained in [Cas].

(4) The study of the decay of correlations for partially hyperbolic diffeomorphisms of the type $E^{ss} \oplus E^{cu}$ is being done in [AP].

(5) The stochastic stability of diffeomorphisms with dominated splitting is studied in [AAV], where sufficient conditions for its validity are given.

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