# ON THE $L_2$- BOUNDEDNESS OF THE OLEVSKII AND THE GAMMA-PRODUCT TRANSFORMS

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**Abstract**

We deal here with a class of integral transformations with respect to parameters of hypergeometric functions or the index transforms. In particular, we treat the familiar Olevskii transform, which is associated with the Gauss hypergeometric function as a kernel. It involves, in turn, as particular cases index transforms of the Mehler-Fock type which are used in the mathematical theory of elasticity. It is shown that boundedness $L_2$- properties for the Olevskii transform are based on the corresponding properties for the so-called Gamma-product transform, which has been introduced recently by the author. Analogs of the Plancherel theorems are proved. It gives that the Olevskii and the Gamma-product transforms are isometric isomorphisms between two weighted $L_2$ - spaces. More examples of such isomorphisms are exhibited for the Mehler- Fock type transforms.

**Keywords**: *Euler Gamma-function, Gauss hypergeometric function, associated Legendre function, Bessel functions, Mellin-Barnes integrals, Kontorovich-Lebedev transform, Mehler-Fock transform*

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## 1 Introduction and Preliminary Results

Let $f : \mathbb{R}_+ \to \mathbb{C}$ be a measurable function. Fixing real positive parameters $c, a$ we will deal with the following Olevskii transformation [15], [21], [22]

$$\mathcal{O}_{c,a}f(x) = \frac{x^{-a}}{\Gamma(c)} \int_0^\infty |\Gamma(a+i\tau)|^2 {}_2F_1\left(a+i\tau, a-i\tau; c; -\frac{1}{x}\right) f(\tau)d\tau, \ x > 0, \tag{1.1}$$

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where integral (1.1) is with respect to parameters of the Gauss hypergeometric function ${}_2F_1$ [1, Chapter 2]. It exists in a definite sense, which will be defined below. In the sequel we will use the weighted Lebesgue spaces $L_p(\Omega;\omega(x)dx)$ with respect to the measure $\omega(x)dx$ equipped with the norm

$$||f||_p = \left(\int_\Omega |f(x)|^p \omega(x)dx\right)^{1/p},\ 1 \le p < \infty,$$

$$||f||_\infty = \text{ess sup}|f(x)|.$$

We note that $\Gamma(z)$ in (1.1) is Euler's Gamma-function [1] and $i$ is the imaginary unit. The operator (1.1) is called also the Jacobi transform, the Fourier-Jacobi transform, the generalized Fourier transform, the index hypergeometric transform, the ${}_2F_1$ - index transform (see [2], [7], [8], [11], [13], [14], [23]). It is not difficult to verify that under conditions on the parameters the Gauss hypergeometric function in (1.1) is represented by the power series for $x \ge 1, \tau \in \mathbb{R}_+$

$${}_2F_1\left(a+i\tau, a-i\tau; c; -\frac{1}{x}\right) = \sum_{n=0}^{\infty} \frac{(a+i\tau)_n (a-i\tau)_n}{(c)_n} \frac{(-1)^n}{x^n n!}. \tag{1.2}$$

When $0 < x < 1$ this function is understood by the relation (cf. [1], [12])

$${}_2F_1\left(a+i\tau, a-i\tau; c; -\frac{1}{x}\right) = \frac{\Gamma(c)\Gamma(-2i\tau)}{\Gamma(a-i\tau)\Gamma(c-a-i\tau)} x^{a+i\tau}$$

$$\times {}_2F_1(a+i\tau, 1-c+a+i\tau; 1+2i\tau; -x)$$

$$+\frac{\Gamma(c)\Gamma(2i\tau)}{\Gamma(a+i\tau)\Gamma(c-a+i\tau)} x^{a-i\tau} {}_2F_1(a-i\tau, 1-c+a-i\tau; 1-2i\tau; -x). \tag{1.3}$$

On the other hand we consider the Gauss function as the following Mellin-Barnes integral [1, Ch. I] (cf. formula (8.4.50.2) from [17])

$$\frac{|\Gamma(a+i\tau)|^2}{\Gamma(c)} x^{-a} {}_2F_1\left(a+i\tau, a-i\tau; c; -\frac{1}{x}\right)$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(s+i\tau)\Gamma(s-i\tau)\frac{\Gamma(a-s)}{\Gamma(c-a+s)} x^{-s} ds,\ x > 0,\ 0 < \gamma < a. \tag{1.4}$$

Series (1.2) can be reobtained if we evaluate integral (1.4) as the sum of residues of the right-hand simple poles $s = a+n,\ n = 0, 1, 2, ...$ of Gamma-functions of the integrand, which are separated from the left-hand ones $s = \pm i\tau - n,\ n = 0, 1, 2, ...$ . However, evaluating the same integral as the sum of residues at the left-hand simple poles we obtain series (1.3). We put down here some of important properties of the Gauss function [1], [20], [22]

$$ {}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z), $$

$$ {}_2F_1(a,b;b;z) = (1-z)^{-a}, $$

$$ {}_2F_1(a,b;c;0) = {}_2F_1(0,b;c;z) = 1, $$

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b) > 0, $$

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a} {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right), \tag{1.5} $$

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z). \tag{1.6} $$

Formula (1.5) is called as the Boltz formula and relation (1.6) is called the self-transformation formula.

One can mention also the integral representation of the Gauss function in terms of the product of Bessel functions (see [16, relation (2.16.21.1)], [22, formula (1.101)])

$$ {}_2F_1\left(a+i\tau, a-i\tau; c; -x^2\right) = \frac{2^{1-2a+c}x^{1-c}\Gamma(c)}{|\Gamma(a+i\tau)|^2} \int_0^\infty y^{2a-c} J_{c-1}(xy) K_{2i\tau}(y) dy, \ x > 0. \tag{1.7} $$

It is easily seen by the asymptotic behavior of the Bessel functions near origin and at the infinity (cf. [12]) that integral (1.7) is absolutely convergent for any $c, a > 0$. We recall that the Gauss function in (1.1) has the following asymptotic behavior for each $\tau \in \mathbb{R}_+$, when $x \to 0+$ (cf. [1], [12], [22])

$$ {}_2F_1\left(a+i\tau, a-i\tau; c; -\frac{1}{x}\right) = O\left(x^a \log x\right), \ x \to 0+. \tag{1.8} $$

We note that kernel (1.8) is a continuous function with respect to $\tau > 0$. Furthermore via [22, Theorem 1.12] we see that when $\tau \to +\infty$ it behaves for each $x > 0$ as

$$ {}_2F_1\left(a+i\tau, a-i\tau; c; -\frac{1}{x}\right) = O\left(\tau^{1/2-c}\right), \ \tau \to +\infty. \tag{1.9} $$

We mention here that the modified Bessel function $K_{2i\tau}(2\sqrt{x})$ is real-valued and it represents the kernel of the Kontorovich-Lebedev transform [18], [19], [21], [22]

$$ [KLf](x) = \int_0^\infty K_{2i\tau}(2\sqrt{x}) f(\tau) d\tau. \tag{1.10} $$

At the same time it can be given by the Mellin-Barnes integral (see [22, relation (1.113)])

$$ K_{2i\tau}(2\sqrt{x}) = \frac{1}{4\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma\left(s+i\tau\right)\Gamma\left(s-i\tau\right) x^{-s} ds, \tag{1.11} $$

where $x > 0, \gamma > 0, \tau \in \mathbb{R}$. As it is known [12], [17], theory of the Mellin - Barnes integrals is based on the Mellin direct and inverse transforms, which are defined by the formulas

$$f^{\mathcal{M}}(s) = \int_0^\infty f(x) x^{s-1} dx, \tag{1.12}$$

$$f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} f^{\mathcal{M}}(s) x^{-s} ds, \ \ s = \gamma + it, \ \ x > 0, \tag{1.13}$$

where integrals (1.12)- (1.13) exist as Lebesgue integrals or, in particular, in mean with respect to the norm of spaces $L_2(\gamma - i\infty, \gamma + i\infty)$ and $L_2(\mathbb{R}_+; x^{2\gamma-1})$, respectively. In the latter case, the Parseval equality holds

$$\int_0^\infty |f(x)|^2 x^{2\gamma-1} dx = \frac{1}{2\pi} \int_{-\infty}^\infty |f^{\mathcal{M}}(\gamma + it)|^2 dt \tag{1.14}$$

The Kontorovich-Lebedev transformation (1.10) (cf. [22], [23]), in turn, is an isomorphism between the spaces $L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$ and $L_2(\mathbb{R}_+; x^{-1}dx)$ with the Parseval equality of the form

$$\int_0^\infty |[KLf](x)|^2 \frac{dx}{x} = \frac{\pi}{2} \int_0^\infty |f(\tau)|^2 |\Gamma(2i\tau)|^2 d\tau. \tag{1.15}$$

The corresponding inverse operator in the mean convergence sense is written in the form

$$f(\tau) = \frac{2}{\pi |\Gamma(2i\tau)|^2} \int_0^\infty K_{2i\tau}(2\sqrt{x}) [KLf](x) \frac{dx}{x}. \tag{1.16}$$

The aim of this paper is to prove the Plancherel type theorem for the Olevskii transformation (1.1). To do this we will study in the sequel the $L_2$- boundedness of the so-called Gamma-product transformation, which was introduced for the first time by the author in [24]

$$[\mathcal{G}f](x) = \text{P.V.} \int_0^\infty \Gamma\left(i(x+\tau)\right) \Gamma\left(i(x-\tau)\right) f(\tau) d\tau, \ \ x \in \mathbb{R}. \tag{1.17}$$

Since $\Gamma(z) \sim \frac{1}{z}, \ z \to 0$, we have that at the point $\tau = |x|$ integral (1.17) is understood in the Cauchy principal value sense. The Gamma-product transform (1.17) is well-defined for instance, when $f \in L_1(\mathbb{R}_+; d\tau)$.

The $L_2$- properties for the Olevskii transform (1.1) and its particular cases we note that the case $c = 2a$ was considered in [21, p. 136]. The Olevskii transformation for some particular values of $c$ has been treated in [13], [14]. About the distributional analog of the Olevskii transform and its particular cases see in [6], [9], [10]. Some mapping properties for these index operators have been investigated also in [3], [4], [5]. Finally, we will exhibit the related results for the Mehler-Fock type transforms (see also in [23], [25]).

# 2 The Gamma-Product Transformation

Let us first consider the Gamma-product transformation (1.17) in the weighted $L_2$-spaces. Our goal is to prove an analog of the Plancherel theorem for this transformation. The result will be used in the sequel to study the $L_2$-properties for the Olevskii transformation (1.1).

**Theorem 1**. *Let $f(\tau) \in L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$. The Gamma-product transform* (1.17), *where the integral converges in mean with respect to the norm in $L_2(\mathbb{R}; dx)$ forms the isomorphism*

$$\mathcal{G} : L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau) \leftrightarrow L_2(\mathbb{R}; dx),$$

*where the reciprocal inverse operator is given by*

$$f(\tau) = l.i.m._{N\to\infty} P.V. \frac{1}{4\pi^2 |\Gamma(2i\tau)|^2} \int_{-N}^{N} \Gamma\left(-i(x+\tau)\right) \Gamma\left(i(\tau - x)\right) [\mathcal{G}f](x) dx, \tag{2.1}$$

*with the convergence with respect to the norm in $L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$. If $f, g \in L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$ then the Plancherel identity is true*

$$\int_{-\infty}^{\infty} [\mathcal{G}f](x) \overline{[\mathcal{G}g](x)} dx = 4\pi^2 \int_0^\infty f(\tau) \overline{g(\tau)} |\Gamma(2i\tau)|^2 d\tau. \tag{2.2}$$

*In particular, the Parseval equality holds*

$$\int_{-\infty}^{\infty} |[\mathcal{G}f](x)|^2 \, dx = 4\pi^2 \int_0^\infty |f(\tau)|^2 |\Gamma(2i\tau)|^2 d\tau. \tag{2.3}$$

**Proof.** We suppose first that $f \in C_0^\infty(\mathbb{R}_+)$, i.e belongs to the space of smooth functions with compact support on $\mathbb{R}_+$. Hence taking $\alpha > 0$ we introduce the following function

$$\Phi_f(z) = \int_0^\infty \Gamma\left(z + i\tau\right) \Gamma\left(z - i\tau\right) f(\tau) d\tau, \tag{2.4}$$

where $z = \alpha + ix$. By the elementary inequality for Euler's Gamma-function [1] $|\Gamma(z)| \leq |\Gamma(\text{Re} z)|$ we get that integral (2.4) converges uniformly with respect to $z \in \mathbb{C}, 0 < \alpha_0 \leq \alpha \leq A,\ A > 0$ since

$$\int_0^\infty |\Gamma\left(z + i\tau\right) \Gamma\left(z - i\tau\right) f(\tau)| \, d\tau \leq \Gamma^2(\alpha) \int_{\text{supp f}} |f(\tau)| d\tau < C_A \int_{\text{supp f}} |f(\tau)| d\tau < \infty,$$

where $C_A > 0$ is a constant. But the integrand is analytic with respect to $z$ and $A > 0$ is arbitrary. Consequently, we immediately conclude that $\Phi_f(z)$ is analytic in the right half-plane. Further, we invoke integral (1.11) for the modified Bessel function $K_{2i\tau}(2\sqrt{x})$

and the inverse Mellin transform (1.13) to represent the product of Gamma-functions as (see (1.12))

$$\Gamma(z+i\tau)\,\Gamma(z-i\tau) = 2\int_0^\infty K_{2i\tau}(2\sqrt{x})x^{z-1}dx. \tag{2.5}$$

Substituting (2.5) into (2.4), we change the order of integration via Fubini's theorem and we write $\Phi_f(z)$ as a composition of the Kontorovich-Lebedev transform (1.10) and the Mellin transform (1.12)

$$\Phi_f(z) = 2\int_0^\infty [KLf](x)x^{z-1}dx. \tag{2.6}$$

Hence by the Parseval equality (1.14) for the Mellin transform we obtain

$$\int_{-\infty}^\infty |\Phi_f(\alpha+ix)|^2dx = 8\pi\int_0^\infty |[KLf](t)|^2t^{2\alpha-1}dt. \tag{2.7}$$

However, letting $\alpha \in (0,A)$ we appeal to the Parseval equality (1.15) and to the inequality $|K_{2i\tau}(2\sqrt{t})| \le K_0(2\sqrt{t})$ (cf. [1], [22]) for the modified Bessel functions in order to majorize the integral at the right-hand side of (2.7) as

$$\int_0^\infty |[KLf](t)|^2t^{2\alpha-1}dt = \left(\int_0^1 + \int_1^\infty\right)|[KLf](t)|^2t^{2\alpha-1}dt$$

$$\le \int_0^1 |[KLf](t)|^2\frac{dt}{t} + \int_1^\infty K_0^2(2\sqrt{t})t^{2A-1}dt\left(\int_{\text{supp}f}|f(\tau)|d\tau\right)^2 < \infty. \tag{2.8}$$

Consequently, the left-hand side of (2.7) is bounded with respect to $\alpha \in (0,A)$ and $\Phi_f(z) \in \mathbb{H}_2^{(0,A)}$, where $\mathbb{H}_2^{(0,A)}$ denotes the Hardy space [19] of analytic functions in the strip $\text{Re}z \in (0,A)$ such that

$$\sup_{\alpha\in(0,A)}\int_{-\infty}^\infty |\Phi_f(\alpha+ix)|^2dx < \infty.$$

Moreover, invoking again (2.7) we have

$$\int_{-\infty}^\infty |\Phi_f(\alpha_1+ix) - \Phi_f(\alpha_2+ix)|^2dx = 8\pi\int_0^\infty |[KLf](t)|^2\left(t^{2\alpha_1}-t^{2\alpha_2}\right)\frac{dt}{t}. \tag{2.9}$$

The right-hand side of the equality (2.9) tends to zero when $\alpha_i \to 0+,\ i=1,2$ since via (2.8) and the dominated convergence theorem we can pass to the limit under the integral sign. Thus $\Phi_f(\alpha+ix)$ when $\alpha\to 0+$ converges in mean to some function $\Phi(ix)$. Hence denoting its inverse Mellin transform by $\varphi(t) \in L_2(\mathbb{R}_+;t^{-1}dt)$ we find from (1.14) that

$$\int_{-\infty}^\infty |\Phi_f(ix) - \Phi_f(\alpha+ix)|^2dx = 2\pi\int_0^\infty |\varphi(t) - 2[KLf](t)t^\alpha|^2\frac{dt}{t} \to 0, \alpha\to 0+. \tag{2.10}$$

By Fatou's lemma this implies the equality

$$\int_0^\infty |\varphi(t) - 2[KLf](t)|^2 \frac{dt}{t} = 0,$$

which means that $\varphi(t) = 2[KLf](t)$ almost for all $t \in \mathbb{R}_+$. Taking into account equality (1.15) we arrive at the following Parseval identities

$$\int_{-\infty}^\infty |\Phi_f(ix)|^2 dx = 8\pi \int_0^\infty |[KLf](t)|^2 \frac{dt}{t} = 4\pi^2 \int_0^\infty |f(\tau)|^2 |\Gamma(2i\tau)|^2 d\tau. \tag{2.11}$$

If we prove that $\Phi_f(ix) = [\mathcal{G}f](x)$ almost for all $x \in \mathbb{R}$ then we establish (2.3) for any $f \in C_0^\infty(\mathbb{R}_+)$. Indeed, let $|x| \notin \mathrm{supp} f$. Then appealing to the dominated convergence theorem we immediately pass to the limit under the integral sign in (2.4) when $\alpha \to 0+$ to obtain

$$\Phi_f(ix) = \lim_{\alpha\to 0+} \int_{\mathrm{supp} f} \Gamma(\alpha + i(x+\tau))\, \Gamma(\alpha + i(x-\tau))\, f(\tau) d\tau$$

$$= \int_{\mathrm{supp} f} \Gamma(i(x+\tau))\, \Gamma(i(x-\tau))\, f(\tau) d\tau = [\mathcal{G}f](x). \tag{2.12}$$

Otherwise let $x > 0$ be such that $x \in \mathrm{supp} f$ (the case $x < 0$ can be considered in the same manner). By definition of the Cauchy principal value we write integral (1.17) in the form

$$[\mathcal{G}f](x) = \lim_{\delta\to 0+} \left( \int_0^{x-\delta} + \int_{x+\delta}^\infty \right) \Gamma(i(x+\tau))\, \Gamma(i(x-\tau))\, f(\tau) d\tau. \tag{2.13}$$

Our goal is to prove that equality (2.12) holds in this case. In fact, it is easily seen that one can write (2.13) as the iterated limit

$$[\mathcal{G}f](x) = \lim_{\delta\to 0+} \lim_{\alpha\to 0+} \left( \int_0^{x-\delta} + \int_{x+\delta}^\infty \right) \Gamma(\alpha + i(x+\tau))\, \Gamma(\alpha + i(x-\tau))\, f(\tau) d\tau$$

$$= \lim_{\alpha\to 0+} \Phi(\alpha + ix) - \lim_{\delta\to 0+} \lim_{\alpha\to 0+} \int_{x-\delta}^{x+\delta} \Gamma(\alpha + i(x+\tau))\, \Gamma(\alpha + i(x-\tau))\, f(\tau) d\tau$$

and equality (2.12) will be true if we show that for all $x > 0$

$$I(x) = \lim_{\delta\to 0+} \lim_{\alpha\to 0+} \int_{x-\delta}^{x+\delta} \Gamma(\alpha + i(x+\tau))\, \Gamma(\alpha + i(x-\tau))\, f(\tau) d\tau = 0.$$

Indeed, making use the mean value theorem we have $f(x+t) - f(x) = tf'(x + t\xi(t))$, where $0 \le \xi \le 1$. Therefore,

$$I(x) = \lim_{\delta\to 0+} \lim_{\alpha\to 0+} \int_{-\delta}^{\delta} \Gamma(\alpha + i(2x+t))\, \Gamma(\alpha - it)\, f(x+t) dt$$

$$= \lim_{\delta\to 0+}\lim_{\alpha\to 0+}\int_{-\delta}^{\delta}\Gamma\left(\alpha+i(2x+t)\right)\Gamma\left(\alpha-it\right)tf'(x+t\xi(t))dt$$

$$+f(x)\lim_{\delta\to 0+}\lim_{\alpha\to 0+}\int_{-\delta}^{\delta}\Gamma\left(\alpha+i(2x+t)\right)\Gamma\left(\alpha-it\right)dt = I_1(x)+I_2(x).$$

It is clear that $I_1(x)=0$ for all $x>0$ since invoking the relation $\Gamma(1+z)=z\Gamma(z)$ we find

$$\int_{-\delta}^{\delta}\left|\Gamma\left(\alpha+i(2x+t)\right)\Gamma\left(\alpha-it\right)tf'(x+t\xi(t))\right|dt = \int_{-\delta}^{\delta}\left|\Gamma\left(\alpha+i(2x+t)\right)\frac{\Gamma\left(1+\alpha-it\right)}{\sqrt{\alpha^2+t^2}}\right.$$

$$\left.\times\, tf'(x+t\xi(t))\right|dt \le C_x\delta\to 0,\ \delta\to 0+,$$

where $C_x>0$ is a constant, which depends only on $x$. Considering $I_2(x)$ we use a representation for the product of Gamma-functions as the following cosine Fourier integral (cf. [22, relation (1.104)]

$$\Gamma\left(\alpha+i(x+\tau)\right)\Gamma\left(\alpha+i(x-\tau)\right) = \frac{\Gamma\left(2(\alpha+ix)\right)}{2^{2(\alpha+ix)-1}}\int_0^\infty\frac{\cos\tau y}{\cosh^{2(\alpha+ix)}(y/2)}dy. \tag{2.14}$$

Hence we obtain

$$I_2(x) = f(x)\lim_{\delta\to 0+}\lim_{\alpha\to 0+}\int_{x-\delta}^{x+\delta}\Gamma\left(\alpha+i(x-\tau)\right)\Gamma\left(\alpha+i(x+\tau)\right)d\tau$$

$$= f(x)\frac{\Gamma\left(2ix\right)}{2^{2(ix-1)}}\lim_{\delta\to 0+}\lim_{\alpha\to 0+}\int_0^\infty\frac{\sin\delta y\cos xy}{y\cosh^{2(\alpha+ix)}(y/2)}dy,$$

where the change of the order of integration is allowed by virtue of the absolute and uniform convergence of the integral with respect to $y$ for each $\alpha>0$. However, the latter integral is uniformly convergent with respect to $\alpha\ge 0$ via the Abel test. Thus, passing to the limit through the integral sign when $\alpha\to 0+$ we have

$$I_2(x) = f(x)\frac{\Gamma\left(2ix\right)}{2^{2(ix-1)}}\lim_{\delta\to 0+}\int_0^\infty\frac{\sin\delta y\cos xy}{y\cosh^{2ix}(y/2)}dy.$$

It remains to show that the latter limit is zero. Indeed, splitting it on two integrals over $(0,1)$ and $(1,\infty)$ we find that in the first integral we can easily pass to the limit by $\delta\to 0+$ via the absolute and uniform convergence and this gives zero. Appealing again to the Abel test we conclude that the second integral

$$\int_1^\infty\frac{\cos xy}{y}\frac{\sin\delta y}{\cosh^{2ix}(y/2)}dy$$

converges uniformly by $\delta\ge 0$ for each $x>0$. Thus it tends to zero when $\delta\to 0+$ and $I_2(x)=0$ for all $x>0$. Combining with the above arguments and returning to (2.12) we establish that $\Phi_f(ix) = [\mathcal{G}f](x)$ almost for all $x \in \mathbb{R}$. Consequently, the Parseval equality (2.3) holds for any $f \in C_0^\infty(\mathbb{R}_+)$. Since this set of functions is dense in $L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$, we extend the equality (2.3) for the whole space. Hence with the parallelogram identity we immediately establish the Plancherel equality (2.2) and we continuously extend the Gamma-product transform (1.17) on the whole space $L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$. As in [24] we show that this transformation forms the isomorphism

$$\mathcal{G} : L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau) \leftrightarrow L_2(\mathbb{R}; dx)$$

and its extension can be represented by the following limit with respect to the norm in $L_2(\mathbb{R}; dx)$

$$[\mathcal{G}f](x) = \text{l.i.m.}_{N\to\infty}\text{P.V.} \int_0^N \Gamma\left(i(x+\tau)\right)\Gamma\left(i(x-\tau)\right) f(\tau)d\tau.$$

We show that reciprocal formula (2.1) follows from the Plancherel equality (2.2). Let us fix $\xi > 0$ and take

$$g(\tau) = \begin{cases} \tau, & \text{if } \tau \in [0,\xi], \\ 0, & \text{if } \tau \in (\xi,\infty), \end{cases}$$

which evidently belongs to the space $L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$. Hence from (2.2) we arrive at the equality

$$4\pi^2 \int_0^\xi \tau f(\tau)|\Gamma(2i\tau)|^2 d\tau = \int_{-\infty}^\infty [\mathcal{G}f](x) \int_0^\xi \tau\Gamma\left(-i(x+\tau)\right)\Gamma\left(i(\tau-x)\right) d\tau dx. \tag{2.15}$$

Choosing sufficiently big $N \in \mathbb{N}$ and denoting by

$$[\mathcal{G}f]_N(x) = \begin{cases} [\mathcal{G}f](x), & \text{if } x \in [-N,N], \\ 0, & \text{if } x \notin [-N,N], \end{cases}$$

we observe that $||[\mathcal{G}f] - [\mathcal{G}f]_N||_{L_2(\mathbb{R})} \to 0,\ N \to \infty$. Moreover the corresponding sequence $\{f_N\}$ by Parseval equality (2.3) converges to $f$ and from (2.15) we obtain

$$4\pi^2 \int_0^\xi \tau f_N(\tau)|\Gamma(2i\tau)|^2 d\tau = \int_{-N}^N [\mathcal{G}f](x) \int_0^\xi \tau\Gamma\left(-i(x+\tau)\right)\Gamma\left(i(\tau-x)\right) d\tau dx.$$

Since by the Cauchy- Schwarz inequality $\tau f_N(\tau)|\Gamma(2i\tau)|^2 \in L_1(0,\xi)$ it implies

$$f_N(\tau) = \frac{1}{4\pi^2\tau|\Gamma(2i\tau)|^2} \frac{d}{d\xi} \int_{-N}^N [\mathcal{G}f](x) \int_0^\xi \tau\Gamma\left(-i(x+\tau)\right)\Gamma\left(i(\tau-x)\right) d\tau dx. \tag{2.16}$$

and we found that $f(\tau) = \text{l.i.m.}_{N\to\infty} f_N(\tau)$ with respect to the norm in $L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$.

Meanwhile, fixing $N \in \mathbb{N}$ the order of integration in (2.16) may be inverted due to the familiar Poincaré - Bertrand formula. Hence we derive

$$f_N(\tau) = \frac{1}{4\pi^2\tau|\Gamma(2i\tau)|^2}\frac{d}{d\xi}\int_0^{\xi}\tau\int_{-N}^{N}[\mathcal{G}f](x)\Gamma\left(-i(x+\tau)\right)\Gamma\left(i(\tau-x)\right)dxd\tau. \tag{2.17}$$

But

$$\tau\int_{-N}^{N}[\mathcal{G}f](x)\Gamma\left(-i(x+\tau)\right)\Gamma\left(i(\tau-x)\right)dx = \tau\int_{-N}^{N}[\mathcal{G}f](x)\frac{\Gamma\left(1-i(x+\tau)\right)\Gamma\left(1+i(\tau-x)\right)}{\tau^2-x^2}dx$$

$$= \frac{1}{2}\int_{-N}^{N}[\mathcal{G}f](x)\frac{\Gamma\left(1-i(x+\tau)\right)\Gamma\left(1+i(\tau-x)\right)}{\tau+x}dx - \frac{1}{2}\int_{-N}^{N}[\mathcal{G}f](x)$$

$$\times\frac{\Gamma\left(1-i(x+\tau)\right)\Gamma\left(1+i(\tau-x)\right)}{x-\tau}dx = \frac{1}{2}\int_{-N}^{N}[\mathcal{G}f](x)$$

$$\times\frac{\Gamma\left(1-i(x+\tau)\right)\Gamma\left(1+i(\tau-x)\right)-\Gamma\left(1+2i\tau\right)}{\tau+x}dx + \frac{\Gamma\left(1+2i\tau\right)}{2}\int_{-N}^{N}\frac{[\mathcal{G}f](x)}{\tau+x}dx$$

$$-\frac{1}{2}\int_{-N}^{N}[\mathcal{G}f](x)\frac{\Gamma\left(1-i(x+\tau)\right)\Gamma\left(1+i(\tau-x)\right)-\Gamma\left(1-2i\tau\right)}{x-\tau}dx$$

$$-\frac{\Gamma\left(1-2i\tau\right)}{2}\int_{-N}^{N}\frac{[\mathcal{G}f](x)}{x-\tau}dx = J_1(\tau)+J_2(\tau)-J_3(\tau)-J_4(\tau).$$

Appealing to the M.Riesz theorem about the $L_p$-boundedness of the Hilbert transform (cf. Theorem 101 in [19]) we conclude that functions $J_2(\tau), J_4(\tau) \in L_1((0,\xi);d\tau)$. At the same time by the mean value theorem we easily get that $J_1(\tau), J_3(\tau)$ are bounded, i.e. belong to $L_1((0,\xi);d\tau)$ for each $\xi > 0$. Thus differentiating in (2.17) with respect to $\xi$ and passing to the limit by the norm in $L_2(\mathbb{R}_+;|\Gamma(2i\tau)|^2d\tau)$ when $N\to\infty$ we establish (2.1). This ends the proof of Theorem 1.

**Remark 1.** Via the asymptotic by $\tau\to+\infty$ of the ratio of Gamma-functions (see the Stirling formula, for instance in [1], [12], [22]) we have for each $x\in\mathbb{R}$

$$\frac{\Gamma\left(i(x+\tau)\right)\Gamma\left(i(x-\tau)\right)}{\Gamma\left(2i\tau\right)} = O\left(\frac{1}{\sqrt{\tau}}\right),\ \tau\to+\infty.$$

Hence one can show that for $f\in L_2(\mathbb{R}_+;|\Gamma(2i\tau)|^2d\tau)$ integral (1.17) generally does not exist. Nevertheless according to Theorem 1 it converges in mean with respect to the norm in $L_2(\mathbb{R};dx)$.

Now we describe the class of functions $f\in L_2(\mathbb{R}_+;|\Gamma(2i\tau)|^2d\tau)$, having the corresponding Gamma-product transform $[\mathcal{G}f](x)\in L_2(\mathbb{R};dx)$ as a limit almost everywhere when $\alpha \to 0+$ of the function (2.4) $\Phi_f(z) \in \mathbb{H}_2,\ z = \alpha + ix$. This means that $\Phi_f$ is analytic in the right half-plane $\alpha > 0$ and satisfies the condition

$$\sup_{\alpha>0} \int_{-\infty}^{\infty} |\Phi_f(\alpha + ix)|^2 dx < \infty. \tag{2.18}$$

**Theorem 2**. *Let $f \in L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$. The Gamma-product transform $[\mathcal{G}f](x)$ is the limit of $\Phi_f(\alpha + ix) \in \mathbb{H}_2$ almost everywhere when $\alpha \to 0+$ if and only if the Kontorovich-Lebedev transform* (1.10) *$[KLf](x)$ is equal to zero almost for all $x > 1$.*

**Proof.** *Necessity.* The necessity follows immediately from equality (2.7). Indeed, since its left-hand side keeps bounded when $\alpha \to \infty$, we have for any $A > 1$

$$\int_A^{\infty} |[KLf](t)|^2 \frac{dt}{t} \le A^{-2\alpha} \int_A^{\infty} |[KLf](t)|^2 t^{2\alpha - 1} dt$$

$$\le \frac{A^{-2\alpha}}{8\pi} \sup_{\alpha>0} \int_{-\infty}^{\infty} |\Phi_f(\alpha + ix)|^2 dx = \text{const.}\ A^{-2\alpha} \to 0,\ \alpha \to \infty.$$

Therefore,

$$\int_A^{\infty} |[KLf](t)|^2 \frac{dt}{t} = 0$$

and $[KLf](x) = 0$ almost for all $x > 1$. Since when $\alpha \to 0+$ the left-hand side of (2.7) is bounded we find via Fatou's lemma that $[KLf](x) \in L_2((0,1); x^{-1}dx)$.

*Sufficiency.* Conversely, let $f \in L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$ be such that $[KLf](x) = 0$ almost for all $x > 1$. Hence by (2.6) we put

$$\Phi_f(\alpha + ix) = 2 \int_0^1 [KLf](t) t^{\alpha + ix - 1} dt,\ \alpha > 0.$$

Since by the Cauchy- Schwarz inequality

$$\int_0^1 |[KLf](t)| t^{\alpha-1} dt \le \left( \int_0^1 |[KLf](t)|^2 \frac{dt}{t} \right)^{1/2} \left( \int_0^1 t^{2\alpha-1} dt \right)^{1/2}$$

$$= \frac{1}{\sqrt{2\alpha}} \left( \int_0^1 |[KLf](t)|^2 \frac{dt}{t} \right)^{1/2} < \infty$$

we have that $\Phi_f(\alpha + ix)$ is analytic in the right half-plane and invoking equality (2.7) we obtain

$$\int_{-\infty}^{\infty} |\Phi_f(\alpha + ix)|^2 dx = 8\pi \int_0^1 |[KLf](t)|^2 t^{2\alpha-1} dt \le 8\pi \int_0^1 |[KLf](t)|^2 \frac{dt}{t} < \infty.$$

Thus it satisfies the condition (2.18) and according to the theory of Hardy's spaces $\Phi_f(\alpha+ix)$ attains in the mean sense and almost for all $x \in \mathbb{R}$ its limit value in $\mathbb{H}_2$ $\Theta_f(x)$ when $\alpha \to 0+$.

However, if we take a sequence $\{f_n\}_{n=1}^{\infty}$ of $C_0^{\infty}(\mathbb{R}_+)$-functions, which converges to $f$ with respect to the norm in $L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$, then (see (2.6))

$$\Phi_{f_n}(\alpha+ix) = 2\int_0^{\infty} [KLf_n](t) t^{\alpha+ix-1} dt,$$

and analogously we derive that almost for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$ $\Phi_{f_n}(\alpha+ix) \to \Theta_{f_n}(x)$ when $\alpha \to 0+$. Consequently, owing to Theorem 1 we get that $\Theta_{f_n}(x) = [\mathcal{G}f_n](x)$ almost for all $x \in \mathbb{R}$. Furthermore, by (2.3) we have

$$\int_{-\infty}^{\infty} |[\mathcal{G}f](x) - [\mathcal{G}f_n](x)|^2 dx = \int_{-\infty}^{\infty} |[\mathcal{G}(f-f_n)](x)|^2 dx$$

$$= 4\pi^2 \int_0^{\infty} |f(\tau) - f_n(\tau)|^2 |\Gamma(2i\tau)|^2 d\tau \to 0, n \to \infty. \tag{2.19}$$

On the other hand invoking (1.15) we find

$$\int_{-\infty}^{\infty} |\Theta_f(x) - \Theta_{f_n}(x)|^2 dx = \lim_{\alpha \to 0+} \int_{-\infty}^{\infty} |\Phi_f(\alpha+ix) - \Phi_{f_n}(\alpha+ix)|^2 dx$$

$$= \lim_{\alpha \to 0+} 8\pi \int_0^{\infty} |[KLf](t) - [KLf_n](t)|^2 t^{2\alpha-1} dt = \lim_{\alpha \to 0+} 8\pi \left[ \int_0^1 |[KL(f-f_n)](t)|^2 t^{2\alpha-1} dt \right.$$

$$\left. + \int_1^{\infty} |[KLf_n](t)|^2 t^{2\alpha-1} dt \right] = 8\pi \left[ \int_0^1 |[KL(f-f_n)](t)|^2 \frac{dt}{t} \right.$$

$$\left. + \int_1^{\infty} |[KLf_n](t)|^2 \frac{dt}{t} \right] \to 8\pi \int_1^{\infty} |[KLf](t)|^2 \frac{dt}{t}, \ n \to \infty.$$

But under conditions of the theorem the latter integral is zero. Thus, combining with (2.19) we obtain $\Theta_f(x) = [\mathcal{G}f](x)$ almost for all $x \in \mathbb{R}$. This completes the proof of Theorem 2.

We will establish now a generalization of Theorem 2.

**Theorem 3**. *Let $f \in L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$. Let also $\Phi_f(\alpha+ix)$ is analytic in the right half-plane $\alpha > 0$ and satisfies the condition*

$$\int_{-\infty}^{\infty} |\Phi_f(\alpha+ix)|^2 dx = O\left(y^{2\alpha}\right), \ y > 0. \tag{2.20}$$

*The Gamma-product transform* $[\mathcal{G}f](x)$ *is the limit of* $\Phi_f(\alpha+ix)$ *almost everywhere when* $\alpha \to 0+$ *if and only if the Kontorovich-Lebedev transform* $[KLf](x)$ *is equal to zero almost for all* $x > y$*. Besides, if* $y$ *is a least of such numbers, then*

$$\lim_{\alpha\to\infty} \frac{1}{2\alpha} \log \int_{-\infty}^{\infty} |\Phi_f(\alpha+ix)|^2 dx = \log y. \tag{2.21}$$

**Proof.** The first part of the theorem can be proved by previous Theorem 2. Indeed, denoting by $\Psi_f(z) = y^{-z}\Phi_f(z)$ we invoke condition (2.20) to derive

$$\int_{-\infty}^{\infty} |\Psi_f(\alpha+ix)|^2 dx = y^{-2\alpha} \int_{-\infty}^{\infty} |\Phi_f(\alpha+ix)|^2 dx = O(1).$$

Hence $\Psi_f(z) \to \Psi_f(ix),\ \alpha \to 0+$ almost for all $x \in \mathbb{R}$. Moreover, since (see (2.7))

$$y^{-2\alpha} \int_{-\infty}^{\infty} |\Phi_f(\alpha+ix)|^2 dx = 8\pi y^{-2\alpha} \int_0^{\infty} |[KLf](t)|^2 t^{2\alpha-1} dt = 8\pi \int_0^{\infty} |[KLf](ty)|^2 t^{2\alpha-1} dt,$$

it follows again that for $A > 1$

$$\int_A^{\infty} |[KLf](ty)|^2 \frac{dt}{t} = \int_{Ay}^{\infty} |[KLf](t)|^2 \frac{dt}{t} \le A^{-2\alpha} \int_A^{\infty} |[KLf](ty)|^2 t^{2\alpha-1} dt$$
$$\le \text{const.} A^{-2\alpha} \to 0,\ \alpha \to \infty.$$

Therefore $[KLf](x)$ is zero almost for all $x > y$. In the same manner as in Theorem 2 we prove the sufficiency of this condition.

Further,

$$\int_{-\infty}^{\infty} |\Phi_f(\alpha+ix)|^2 dx = 8\pi \int_0^{y} |[KLf](t)|^2 t^{2\alpha-1} dt \le 8\pi y^{2\alpha} \int_0^{\infty} |[KLf](t)|^2 \frac{dt}{t}. \tag{2.22}$$

Thus

$$\frac{1}{2\alpha} \log \int_{-\infty}^{\infty} |\Phi_f(\alpha+ix)|^2 dx \le \log y + \frac{1}{2\alpha} \log \left( 8\pi \int_0^{\infty} |[KLf](t)|^2 \frac{dt}{t} \right).$$

Passing to the limit when $\alpha \to \infty$ in the latter inequality we get

$$\lim_{\alpha\to\infty} \frac{1}{2\alpha} \log \int_{-\infty}^{\infty} |\Phi_f(\alpha+ix)|^2 dx \le \log y. \tag{2.23}$$

On the other hand letting

$$\rho(t) = \int_t^{y} |[KLf](u)|^2 \frac{du}{u}$$

and taking sufficiently small positive $\delta$ after integration by parts we arrive at the estimate

$$8\pi \int_0^y |[KLf](t)|^2 t^{2\alpha-1} dt = 16\pi\alpha \int_0^y \rho(t) t^{2\alpha-1} dt$$

$$\geq 16\pi\alpha \ \rho(y-\delta) \int_0^{y-\delta} t^{2\alpha-1} dt = 8\pi\rho(y-\delta) \ (y-\delta)^{2\alpha}$$

Thus combining with (2.22) we take a logarithm from the both sides of the latter inequality and we divide it by $2\alpha$. Hence

$$\frac{1}{2\alpha} \log \int_{-\infty}^{\infty} |\Phi_f(\alpha+ix)|^2 dx \geq \log(y-\delta) + \frac{1}{2\alpha} \log(8\pi\rho(y-\delta)).$$

Passing to the limit when $\alpha \to \infty$ we obtain

$$\lim_{\alpha\to\infty} \frac{1}{2\alpha} \log \int_{-\infty}^{\infty} |\Phi_f(\alpha+ix)|^2 dx \geq \log(y-\delta).$$

Since $\delta > 0$ is an arbitrary small number it gives

$$\lim_{\alpha\to\infty} \frac{1}{2\alpha} \log \int_{-\infty}^{\infty} |\Phi_f(\alpha+ix)|^2 dx \geq \log y.$$

Taking into account inequality (2.23) we get (2.21). This proves the second part of the theorem. Theorem 3 is proved.

# 3 The Olevskii Transformation

We return to the Olevskii transformation (1.1) in order to prove the following Plancherel type theorem. We have

**Theorem 4**. *Let $c > a > 0$. The Olevskii transformation* (1.1) *is the isomorphism*

$$\mathcal{O}_{c,a} : L_2\left(\mathbb{R}_+; \left|\frac{\Gamma(2i\tau)\Gamma(a+i\tau)}{\Gamma(c-a+i\tau)}\right|^2 d\tau\right) \leftrightarrow L_2\left(\mathbb{R}_+; (1+x)^{2a-c}\frac{dx}{x}\right), \tag{3.1}$$

*where integral* (1.1) *converges in mean with respect to the norm in* $L_2\left(\mathbb{R}_+; (1+x)^{2a-c}\frac{dx}{x}\right)$. *The inverse operator is given by the formula*

$$f(\tau) = l.i.m._{N\to\infty} \frac{|\Gamma(c-a+i\tau)|^2}{2\pi\Gamma(c)|\Gamma(2i\tau)|^2} \int_{1/N}^{N} (1+x)^{2a-c} x^{-a-1} {}_2F_1\left(a+i\tau, a-i\tau; c; -\frac{1}{x}\right) \mathcal{O}_{c,a} f(x) dx, \tag{3.2}$$

*where the limit is in mean square with respect to the norm in the space* $L_2\left(\mathbb{R}_+;\left|\frac{\Gamma(2i\tau)\Gamma(a+i\tau)}{\Gamma(c-a+i\tau)}\right|^2 d\tau\right)$. *Besides, if* $f,g\in L_2\left(\mathbb{R}_+;\left|\frac{\Gamma(2i\tau)\Gamma(a+i\tau)}{\Gamma(c-a+i\tau)}\right|^2 d\tau\right)$ *then the Plancherel formula holds*

$$\int_0^\infty \mathcal{O}_{c,a}f(x)\overline{\mathcal{O}_{c,a}g(x)}(1+x)^{2a-c}\frac{dx}{x} = 2\pi\int_0^\infty \left|\frac{\Gamma(2i\tau)\Gamma(a+i\tau)}{\Gamma(c-a+i\tau)}\right|^2 f(\tau)\overline{g(\tau)}d\tau \tag{3.3}$$

*with the Parseval equality*

$$\int_0^\infty \left|\mathcal{O}_{c,a}f(x)\right|^2(1+x)^{2a-c}\frac{dx}{x} = 2\pi\int_0^\infty \left|\frac{\Gamma(2i\tau)\Gamma(a+i\tau)}{\Gamma(c-a+i\tau)}\right|^2 |f(\tau)|^2 d\tau. \tag{3.4}$$

**Proof.** Let $f\in C_0^\infty(\mathbb{R}_+)$. Then we use integral representation (1.4) to substitute it in (1.1) and to invert the order of integration via Fubini's theorem. This is indeed possible due to the absolute and uniform convergence of the integral (1.4) with respect to $\tau\in\mathbb{R}_+$. Thus taking into account equality (2.4) we arrive at the representation

$$\mathcal{O}_{c,a}f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \Phi_f(\alpha+iy)\frac{\Gamma(a-\alpha-iy)}{\Gamma(c-a+\alpha+iy)}x^{-\alpha-iy}dy, x>0,\ 0<\alpha<a. \tag{3.5}$$

On the other hand, employing the self-transformation formula (1.6) for the Gauss function we represent the Olevskii transform in the form

$$\mathcal{O}_{c,a}f\left(x\right) = \frac{x^{a-c}(1+x)^{c-2a}}{\Gamma(c)}\int_0^\infty |\Gamma(a+i\tau)|^2 {}_2F_1\left(c-a+i\tau, c-a-i\tau; c; -\frac{1}{x}\right)f(\tau)d\tau,$$

which gives the following operational relation

$$\mathcal{O}_{c,a}f\left(x\right) = (1+x)^{c-2a}\mathcal{O}_{c,c-a}h\left(x\right), \tag{3.6}$$

with $h(\tau)=\left|\frac{\Gamma(a+i\tau)}{\Gamma(c-a+i\tau)}\right|^2 f(\tau)$. Hence taking into account (3.5), (3.6), as the consequence of the Parseval equality for the Mellin transform (1.14) with the parallelogram identity we obtain

$$\int_0^\infty \left|\mathcal{O}_{c,a}f(x)\right|^2(1+x)^{2a-c}x^{2\alpha-1}dx = \int_0^\infty \mathcal{O}_{c,c-a}h(x)\overline{\mathcal{O}_{c,a}f(x)}x^{2\alpha-1}dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty \Phi_h(\alpha+iy)\overline{\Phi_f(\alpha+iy)}\frac{\Gamma(a-\alpha+iy)}{\Gamma(c-a+\alpha-iy)}\frac{\Gamma(c-a-\alpha-iy)}{\Gamma(a+\alpha+iy)}dy. \tag{3.7}$$

Equality (3.5) yields (see (1.13)) that

$$\mathcal{O}_{c,a}f(x)\leftrightarrow \Phi_f(\alpha+iy)\frac{\Gamma(a-\alpha-iy)}{\Gamma(c-a+\alpha+iy)},$$

$$\mathcal{O}_{c,c-a}h\,(x) \leftrightarrow \Phi_h(\alpha+iy)\frac{\Gamma(c-a-\alpha-iy)}{\Gamma(a+\alpha+iy)},$$

where $0<\alpha\le b<\min(a,c-a)$ are Mellin's $L_2$-pairs and all integrals in (3.7) are finite. In fact, we will show that for any $f\in C_0^\infty(\mathbb{R}_+)$

$$\sup_{0<\alpha\le b}\int_{-\infty}^{\infty}\left|\Phi_f(\alpha+iy)\frac{\Gamma(a-\alpha+iy)}{\Gamma(c-a+\alpha-iy)}\right|^2 dy<\infty, \tag{3.8}$$

$$\sup_{0<\alpha\le b}\int_{-\infty}^{\infty}\left|\Phi_h(\alpha+iy)\frac{\Gamma(c-a-\alpha-iy)}{\Gamma(a+\alpha+iy)}\right|^2 dy<\infty. \tag{3.9}$$

Then since the integrands in (3.8), (3.9) are analytic in the strip $0<\alpha<\min(a,c-a)$ we will get immediately that each one belongs to the Hardy space $\mathbb{H}_2^{0<\alpha\le b}$. Thus almost everywhere one admits the limit $L_2$- values, which by Theorem 3 are equal correspondingly,

$$[\mathcal{G}f](y)\frac{\Gamma(a+iy)}{\Gamma(c-a-iy)}, \tag{3.10}$$

and

$$[\mathcal{G}h](y)\frac{\Gamma(c-a-iy)}{\Gamma(a+iy)}. \tag{3.11}$$

Furthermore, in the same manner as in (2.10) we conclude that $\mathcal{O}_{c,a}f(x)$, $\mathcal{O}_{c,c-a}h\,(x)$ are reciprocal Mellin's transforms (1.13) from $L_2(\mathbb{R}_+;x^{-1}dx)$. Moreover, by (1.14) we have the Parseval equalities

$$\int_0^\infty |\mathcal{O}_{c,a}f(x)|^2\frac{dx}{x}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|[\mathcal{G}f](y)\frac{\Gamma(a+iy)}{\Gamma(c-a-iy)}\right|^2 dy, \tag{3.12}$$

$$\int_0^\infty |\mathcal{O}_{c,c-a}h(x)|^2\frac{dx}{x}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|[\mathcal{G}h](y)\frac{\Gamma(c-a-iy)}{\Gamma(a+iy)}\right|^2 dy. \tag{3.13}$$

Hence returning to (3.7) and employing the Cauchy-Schwarz inequality we find by using Fatou's lemma that

$$\int_0^\infty |\mathcal{O}_{c,a}f(x)|^2(1+x)^{2a-c}\frac{dx}{x}\le \liminf_{\alpha\to 0+}\int_0^\infty |\mathcal{O}_{c,a}f(x)|^2(1+x)^{2a-c}x^{2\alpha-1}dx$$

$$=\frac{1}{2\pi}\liminf_{\alpha\to 0+}\int_{-\infty}^{\infty}\Phi_h(\alpha+iy)\overline{\Phi_f(\alpha+iy)}\frac{\Gamma(a-\alpha+iy)}{\Gamma(c-a+\alpha-iy)}\frac{\Gamma(c-a-\alpha-iy)}{\Gamma(a+\alpha+iy)}dy$$

$$\le \sup_{0<\alpha\le b}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\Phi_h(\alpha+iy)\Phi_f(\alpha+iy)\frac{\Gamma(a-\alpha+iy)}{\Gamma(c-a+\alpha-iy)}\frac{\Gamma(c-a-\alpha-iy)}{\Gamma(a+\alpha+iy)}\right|dy$$

$$\leq \sup_{0<\alpha\leq b}\frac{1}{2\pi}\left[\left(\int_{-\infty}^{\infty}\left|\Phi_f(\alpha+iy)\frac{\Gamma(a-\alpha+iy)}{\Gamma(c-a+\alpha-iy)}\right|^2 dy\right)^{1/2}\right.$$

$$\left.\times\left(\int_{-\infty}^{\infty}\left|\Phi_h(\alpha+iy)\frac{\Gamma(c-a-\alpha-iy)}{\Gamma(a+\alpha+iy)}\right|^2 dy\right)^{1/2}\right] < \infty.$$

So in order to prove (3.8), (3.9) we appeal again to integral representation (2.14) for the product of Gamma-functions. We substitute it into (2.4), change the order of integration and the result we write in the form

$$\Phi_f(\alpha+iy) = \sqrt{\frac{\pi}{2}}\frac{\Gamma(2(\alpha+iy))}{2^{2(\alpha+iy)-1}}\int_0^{\infty}\frac{d\hat{f}}{dt}\frac{dt}{\cosh^{2(\alpha+iy)}t},\ 0<\alpha\leq b<\min(a,c-a), \quad (3.14)$$

where $\hat{f}(t)$ denotes the following Fourier sine integral

$$\hat{f}(t) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(\tau)\frac{\sin\tau t}{\tau}d\tau.$$

After integration by parts and elimination of the outintegrated terms in (3.14) we use the relation $\Gamma(2z)2z = \Gamma(1+2z)$ and the substitution $e^{\xi} = \cosh^2 t$. Thus we arrive at the following Fourier integral

$$\Phi_f(\alpha+iy) = \sqrt{\frac{\pi}{2}}\frac{\Gamma(1+2(\alpha+iy))}{2^{2(\alpha+iy)}}\int_0^{\infty} e^{-(\alpha+iy)\xi}\hat{f}(\text{arccosh } e^{\xi/2})d\xi.$$

Hence

$$\int_{-\infty}^{\infty}\left|\Phi_f(\alpha+iy)\frac{\Gamma(a-\alpha+iy)}{\Gamma(c-a+\alpha-iy)}\right|^2 dy = \frac{\pi}{2^{2\alpha+1}}\int_{-\infty}^{\infty}\left|\frac{\Gamma(1+2(\alpha+iy))\Gamma(a-\alpha+iy)}{\Gamma(c-a+\alpha-iy)}\right|^2$$

$$\times\left|\int_0^{\infty} e^{-(\alpha+iy)\xi}\hat{f}(\text{arccosh } e^{\xi/2})d\xi\right|^2 dy. \quad (3.15)$$

However, the Gamma-ratio in (3.15) is bounded on $(\alpha-i\infty,\alpha+i\infty),\ 0\leq\alpha\leq b<\min(a,c-a)$ since via Stirling's asymptotic formula [1] we have

$$\left|\frac{\Gamma(1+2(\alpha+iy))\Gamma(a-\alpha+iy)}{\Gamma(c-a+\alpha-iy)}\right| = O(e^{-\pi|y|}|y|^{2a-c+1/2}),\ |y|\to\infty.$$

Consequently, applying twice the Parseval equality for the Fourier transform and making elementary substitutions we obtain from (3.15)

$$\int_{-\infty}^{\infty}\left|\Phi_f(\alpha+iy)\frac{\Gamma(a-\alpha+iy)}{\Gamma(c-a+\alpha-iy)}\right|^2 dy \leq C_1\int_{-\infty}^{\infty}\left|\int_0^{\infty} e^{-(\alpha+iy)\xi}\hat{f}(\text{arccosh } e^{\xi/2})d\xi\right|^2 dy$$

$$= C_2 \int_0^\infty e^{-2\gamma\xi} |\hat{f}(\text{arccosh } e^{\xi/2})|^2 d\xi \le C_2 \int_0^\infty |\hat{f}(\text{arccosh } e^{\xi/2})|^2 d\xi$$

$$= 2C_2 \int_0^\infty |\hat{f}(y)|^2 \tanh y \; dy \le 2C_2 \int_{\text{supp} f} |f(\tau)|^2 \frac{d\tau}{\tau^2} < \infty,$$

where $C_1, C_2$ are absolute positive constants. Thus we have proved (3.8). In the same manner we establish (3.9). Combining now with (3.6), (3.12), (3.13) and applying (1.14), (2.2) as a consequence of (3.7) we derive the chain of equalities

$$\int_0^\infty |\mathcal{O}_{c,a} f(x)|^2 (1+x)^{2a-c} \frac{dx}{x} = \int_0^\infty \mathcal{O}_{c,c-a} h(x) \overline{\mathcal{O}_{c,a} f(x)} \frac{dx}{x}$$

$$= \frac{1}{2\pi} \int_{-\infty}^\infty [\mathcal{G}h](y) \overline{[\mathcal{G}f](y)} dy = 2\pi \int_0^\infty \left| \frac{\Gamma(2i\tau)\Gamma(a+i\tau)}{\Gamma(c-a+i\tau)} \right|^2 |f(\tau)|^2 d\tau,$$

which prove (3.4) for any $f \in C_0^\infty(\mathbb{R}_+)$. Moreover, it gives the validity of the Plancherel identity (3.3). As in Theorem 1 we continuously extend these equalities from the dense set of smooth functions with compact support on the whole weighted $L_2$-spaces to obtain the desired isomorphism (3.1). The Olevskii transform (1.1) is understood as a limit in the mean square with respect to the norm in the space $L_2\left(\mathbb{R}_+; (1+x)^{2a-c} \frac{dx}{x}\right)$. The reciprocal formula (3.2) can be proved as follows. From the Plancherel identity (3.3) it is not difficult to arrive at the equality

$$f(\tau) = \left| \frac{\Gamma(c-a+i\tau)}{\Gamma(a+i\tau)\Gamma(2i\tau)} \right|^2 \frac{1}{2\pi\Gamma(c)\tau} \frac{d}{d\tau} \int_0^\infty \mathcal{O}_{c,a} f(x) \int_0^\tau t\, |\Gamma(a+it)|^2$$

$$\times {}_2F_1\left(a+it, a-it; c; -\frac{1}{x}\right) (1+x)^{2a-c} x^{-a-1} dt dx.$$

Hence $f(\tau) = \text{l.i.m.}_{N\to\infty} f_N(\tau)$, where the limit is in the mean square sense with respect to the norm in the space $L_2\left(\mathbb{R}_+; \left|\frac{\Gamma(2i\tau)\Gamma(a+i\tau)}{\Gamma(c-a+i\tau)}\right|^2\right)$ and

$$f_N(\tau) = \left| \frac{\Gamma(c-a+i\tau)}{\Gamma(a+i\tau)\Gamma(2i\tau)} \right|^2 \frac{1}{2\pi\Gamma(c)\tau} \frac{d}{d\tau} \int_{1/N}^N \mathcal{O}_{c,a} f(x) \int_0^\tau t\, |\Gamma(a+it)|^2$$

$$\times {}_2F_1\left(a+it, a-it; c; -\frac{1}{x}\right) (1+x)^{2a-c} x^{-a-1} dt dx$$

$$= \frac{|\Gamma(c-a+i\tau)|^2}{2\pi\Gamma(c)|\Gamma(2i\tau)|^2} \int_{1/N}^N (1+x)^{2a-c} x^{-a-1} {}_2F_1\left(a+i\tau, a-i\tau; c; -\frac{1}{x}\right) \mathcal{O}_{c,a} f(x) dx,$$

since we can put the derivative under the sign of the latter integral via its uniform convergence with respect to $\tau$. Theorem 4 is proved.

Let us consider particular cases of the Olevskii transform (1.1), which are associated with the Mehler-Fock integrals [18], [21], [22]. Precisely, putting in (1.4) $a = \frac{1}{2}, c = 1 - \mu,\ \mu < \frac{1}{2}$ we employ relation (8.4.41.12) in [17] to obtain

$$ {}_2F_1\left(\frac{1}{2} + i\tau, \frac{1}{2} - i\tau; 1-\mu; -\frac{1}{x}\right) = \Gamma(1-\mu)(1+x)^{-\mu/2} P^{\mu}_{-1/2+i\tau}\left(\frac{2}{x}+1\right), $$

where $P^{\mu}_{\nu}(z)$ is associated Legendre function of the first kind [20]. Thus we arrive at the formula of the generalized Mehler-Fock transform

$$ [P_\mu f](x) = x^{-1/2}(1+x)^{-\mu/2} \int_0^\infty |\Gamma(1/2+i\tau)|^2 P^{\mu}_{i\tau-1/2}\left(1+\frac{2}{x}\right) f(\tau)d\tau, \tag{3.16} $$

where integral (3.16) is convergent with respect to the norm in $L_2\left(\mathbb{R}_+; (1+x)^{\mu}\frac{dx}{x}\right)$. According to Theorem 4 it forms the isometric isomorphism

$$ [P_\mu f] : L_2\left(\mathbb{R}_+; \left|\frac{\Gamma(2i\tau)\Gamma(1/2+i\tau)}{\Gamma(1/2-\mu+i\tau)}\right|^2 d\tau\right) \leftrightarrow L_2\left(\mathbb{R}_+; (1+x)^{\mu}\frac{dx}{x}\right) $$

with the Parseval equality

$$ \int_0^\infty |[P_\mu f](x)|^2 (1+x)^{\mu}\frac{dx}{x} = 2\pi \int_0^\infty \left|\frac{\Gamma(2i\tau)\Gamma(1/2+i\tau)}{\Gamma(1/2-\mu+i\tau)}\right|^2 |f(\tau)|^2 d\tau. $$

The reciprocal inverse operator is written in the form

$$ f(\tau) = \frac{1}{2\pi}\left|\frac{\Gamma(1/2-\mu+i\tau)}{\Gamma(2i\tau)}\right|^2 \int_0^\infty (1+x)^{\mu/2} x^{-3/2} P^{\mu}_{-1/2+i\tau}\left(\frac{2}{x}+1\right)[P_\mu f](x)dx, $$

where the latter integral converges with respect to the norm in $L_2\left(\mathbb{R}_+; \left|\frac{\Gamma(2i\tau)\Gamma(1/2+i\tau)}{\Gamma(1/2-\mu+i\tau)}\right|^2\right)$.

Finally, if we set $c = a+1$, then by virtue of the formula (7.3.1.52) in [17] we have

$$ {}_2F_1\left(a+i\tau, a-i\tau; a+1; -\frac{1}{x}\right) = \frac{\Gamma(a+1)x^a(1+x)^{(1-a)/2}}{2i\tau} \times \left[P^{1-a}_{i\tau}\left(1+\frac{2}{x}\right) - P^{1-a}_{i\tau-1}\left(1+\frac{2}{x}\right)\right]. $$

Thus we obtain the following transformation of the Mehler-Fock type

$$ [P^a f](x) = \frac{i}{2}(1+x)^{(1-a)/2}\int_0^\infty |\Gamma(a+i\tau)|^2 \left[P^{1-a}_{i\tau-1}\left(1+\frac{2}{x}\right) - P^{1-a}_{i\tau}\left(1+\frac{2}{x}\right)\right] f(\tau)\frac{d\tau}{\tau}. $$

It isomorphically maps the space $L_2\left(\mathbb{R}_+; \left|\frac{\Gamma(2i\tau)\Gamma(a+i\tau)}{\Gamma(1+i\tau)}\right|^2 d\tau\right)$ onto the space $L_2\left(\mathbb{R}_+; (1+x)^{a-1}\frac{dx}{x}\right)$. Moreover, the Parseval equality

$$\int_0^\infty |[P^a f](x)|^2 (1+x)^{a-1}\frac{dx}{x} = 2\pi \int_0^\infty \left|\frac{\Gamma(2i\tau)\Gamma(a+i\tau)}{\Gamma(1+i\tau)}\right|^2 |f(\tau)|^2 d\tau$$

holds. The inverse operator is given by the formula

$$f(\tau) = \frac{|\Gamma(1+i\tau)|^2}{4\pi i\tau |\Gamma(2i\tau)|^2} \int_0^\infty (1+x)^{(a-1)/2} \left[P_{i\tau}^{1-a}\left(1+\frac{2}{x}\right) - P_{i\tau-1}^{1-a}\left(1+\frac{2}{x}\right)\right] [P^a f](x)\frac{dx}{x},$$

where the convergence is with respect to the norm in $L_2\left(\mathbb{R}_+; \left|\frac{\Gamma(2i\tau)\Gamma(a+i\tau)}{\Gamma(1+i\tau)}\right|^2 d\tau\right)$.

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