Harmonic Analysis of the Lebedev- Stieltjes integrals

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Abstract

We expand the Bochner technique on the following Lebedev- Stieltjes integrals

$$F(x) = \int_{\mathbb{R}} \frac{K_{i\tau}(e^x)}{|\Gamma(i\tau)|}\, dV(\tau),\ x \in \mathbb{R}$$

which are related to the Kontorovich-Lebedev transformation. Mapping and inversion properties are investigated. The Fourier type series with respect to uncountable orthonormal system of the modified Bessel functions are considered in the Bohr type pre-Hilbert space. The Bessel inequality and Parseval equality are proved.

Keywords: *Harmonic analysis, Stieltjes integrals, Kontorovich-Lebedev transform, modified Bessel functions, Bessel inequality, Fourier series, Parseval equality, almost periodic functions, Bohr space, Fourier integrals*

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1 Introduction and auxiliary results

The aim of this paper is to expand the Bochner technique [2] given for the Fourier- Stieltjes integrals on the following integral

$$F(x) = \int_{\mathbb{R}} \frac{K_{i\tau}(e^x)}{|\Gamma(i\tau)|}\, dV(\tau),\ x \in \mathbb{R}. \tag{1.1}$$

Here i is the imaginary unit, $\Gamma(i\tau)$ is Euler's gamma-function, $V(\tau)$ is a distribution function in the Bochner sense [2], which is bounded and monotone increasing on $\mathbb{R}$, and satisfies everywhere the following equality

$$V(\tau) = \frac{1}{2}[V(\tau+0) + V(\tau-0)].$$

We note that $K_{i\tau}(e^x)$ is the modified Bessel function [3]

$$K_{i\tau}(e^x) = \frac{\pi}{2i\sinh \pi\tau}\left[I_{-i\tau}(e^x) - I_{i\tau}(e^x)\right], \tag{1.2}$$

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where $I_{i\tau}(e^x)$ is, in turn, the modified Bessel function given in terms of the series

$$I_{i\tau}(e^x) = \sum_{m=0}^{\infty} \frac{(e^x/2)^{2m+i\tau}}{m!\Gamma(m+i\tau+1)}. \tag{1.3}$$

Generally $K_\mu(z)$ satisfies the differential equation

$$z^2 \frac{d^2u}{dz^2} + z\frac{du}{dz} - (z^2+\mu^2)u = 0,$$

and has the asymptotic behaviour

$$K_\mu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}[1+O(1/z)], \qquad z \to \infty, \tag{1.4}$$

and near the origin

$$z^{|\mathrm{Re}\mu|} K_\mu(z) = 2^{\mu-1}\Gamma(\mu) + o(1), \ z \to 0, \ \mu \neq 0, \tag{1.5}$$

$$K_0(z) = -\log z + O(1), \ z \to 0. \tag{1.6}$$

Moreover it can be defined by the following integral representations [6]

$$K_\mu(x) = \int_0^\infty e^{-x\cosh u} \cosh \mu u \ du, x > 0, \tag{1.7}$$

$$K_\mu(x) = \frac{1}{2}\left(\frac{x}{2}\right)^\mu \int_0^\infty e^{-t-\frac{x^2}{4t}} t^{-\mu-1} dt, \ x > 0. \tag{1.8}$$

Hence we easily find that $K_{i\tau}(x), \ \tau \in \mathbb{R}$ is real-valued when $\tau \in \mathbb{R}$ and an even function with respect to the index $i\tau$. In the sequel we will appeal to the following integrals (cf. [6], relations 1.12.3.3, 2.16.33.2)

$$\int_x^\infty K_\mu(x)K_\nu(x)\frac{dx}{x} = \frac{x}{\mu^2-\nu^2}\left[K_\mu(x)K'_\nu(x) - K'_\mu(x)K_\nu(x)\right], \tag{1.10}$$

$$\int_0^\infty K_{i\tau}(x)K_{it}(x)x^{\alpha-1}dx = \frac{2^{\alpha-3}}{\Gamma(\alpha)}\left|\Gamma\left(\frac{\alpha+i(\tau+t)}{2}\right)\right|^2 \left|\Gamma\left(\frac{\alpha+i(\tau-t)}{2}\right)\right|^2, \tag{1.11}$$

where $\alpha > 0$ and $'$ denotes a derivative with respect to x.

We show that integral (1.1) is related to the Kontorovich-Lebedev transformation (see [4], [7], [8]). Indeed, putting $V(\tau) = \int_{-\infty}^{\tau} f(t)dt$, where f is nonnegative and belongs to $L_1(\mathbb{R}; dt)$, $y = e^x$, $\varphi(y) = F(\log y)$ and taking into account the value of the modulus of the gamma- function $|\Gamma(i\tau)| = \sqrt{\frac{\pi}{\tau\sinh \pi\tau}}$, integral (1.1) becomes

$$\varphi(y) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \sqrt{\tau \sinh \pi\tau} K_{i\tau}(y) f(\tau) d\tau, \ y > 0, \tag{1.12}$$

which corresponds to the Kontorovich-Lebedev transformation. Finally in this section we note two inequalities for the modified Bessel functions (cf. [4], [5]), which we will use in the sequel. Precisely, for all $x > 0$ and $\tau \in \mathbb{R}$ it satisfies

$$\left| \frac{K_{i\tau}(x)}{\Gamma(i\tau)} \right| \leq e^x, \tag{1.13}$$

$$\left| \frac{K_{i\tau}(x)}{\Gamma(i\tau)} \right| \leq C \frac{\sqrt{|\tau|}}{x^{1/4}}, \tag{1.14}$$

where $C > 0$ is an absolute constant.

The structure of the paper is as follows: in Section 2 we will study mapping properties of functions, which are representable in terms of the Lebedev -Stieltjes integrals and will prove an inversion theorem for the Bochner class of distribution functions $V(\tau)$. In Section 3 we will construct the Fourier type series with respect to an uncountable orthonormal system of the modified Bessel functions in the pre-Hilbert space of the Bohr type [1]. Finally we will establish the Bessel inequality and the Parseval equality for these series.

2 Mapping properties and inversion formula

Appealing to the inequality (1.13) we immediately obtain the estimates

$$\left| \int_{-\infty}^{a} \frac{K_{i\tau}(e^x)}{|\Gamma(i\tau)|} \, dV(\tau) \right| \leq e^{e^x} \left[V(a) - V(-\infty) \right], \tag{2.1}$$

$$\left| \int_{b}^{\infty} \frac{K_{i\tau}(e^x)}{|\Gamma(i\tau)|} \, dV(\tau) \right| \leq e^{e^x} \left[V(\infty) - V(b) \right], \tag{2.2}$$

for any $b \in \mathbb{R}$. This means that for each distribution function $V(\tau)$ integral (1.1) exists for all $x \in \mathbb{R}$. Moreover, we have

$$|F(x)| \leq e^{e^x} \left[V(\infty) - V(-\infty) \right]. \tag{2.3}$$

Consequently, $F(x)$ is a real bounded function on any bounded set of $\mathbb{R}$. If we put

$$F_n(x) = \int_{-n}^{n} \frac{K_{i\tau}(e^x)}{|\Gamma(i\tau)|} \, dV(\tau)$$

then (see (2.1), (2.2))

$$|F(x) - F_n(x)| \leq e^{e^x} \left[V(\infty) - V(n) + V(-n) - V(-\infty) \right],$$

which tends to 0 when $n \to \infty$ on any compact set of $\mathbb{R}$. Thus the sequence $\{F_n(x)\}_{n=1}^{\infty}$ converges uniformly to $F(x)$ on any compact set and represents there a continuous functions.

The main goal of this section is to prove an inversion formula for the integral (1.1). To do this we will modify the inversion theorem for the Lebedev integrals from [9] proving it for absolutely continuous distributions. Therefore $V(\tau)$ can be represented by the indefinite integral of a nonnegative summable function $g(t)$, i.e.

$$V(\tau) = \int_a^\tau g(t)dt, \quad a \in \mathbb{R}. \tag{2.4}$$

Thus we have

Theorem 1. *For a class of distributions* (2.4) *the Lebedev integral* (1.1) *has the following inversion*

$$\frac{1}{2}\left[V(\tau) - V(-\tau) + V(-a)\right] = \lim_{\varepsilon\to 0+} \int_{-\infty}^{\infty} e^{\varepsilon x} \mathcal{K}(\tau, x) F(x) dx, \tag{2.5}$$

where

$$\mathcal{K}(\tau, x) = \frac{1}{\pi} \int_a^\tau \frac{K_{iy}(e^x)}{|\Gamma(iy)|} dy.$$

Proof. Taking into account (2.4) we multiply both sides of (1.1) on $\frac{1}{\pi|\Gamma(i\tau)|} e^{\varepsilon x} K_{i\tau}(e^x)$ for each $\varepsilon > 0$ and $\tau \in \mathbb{R}$ and we integrate with respect to x over $\mathbb{R}$. However asymptotic formulas (1.4), (1.5) and inequalities (2.1), (2.2) allow us to invert the order of integration by virtue of Fubini's theorem. Making elementary substitution $e^x = u$ we appeal to formula (1.11) to calculate the inner integral. As a result we come out with the equality

$$\frac{1}{\pi|\Gamma(i\tau)|} \int_{-\infty}^{\infty} e^{\varepsilon x} K_{i\tau}(e^x) F(x) dx = \frac{2^{\varepsilon-3}}{\pi\Gamma(\varepsilon)} \int_{-\infty}^{\infty} \frac{\left|\Gamma\left(\frac{\varepsilon+i(y+\tau)}{2}\right)\Gamma\left(\frac{\varepsilon+i(\tau-y)}{2}\right)\right|^2}{|\Gamma(i\tau)\Gamma(iy)|}$$
$$\times g(y) dy, \tag{2.7}$$

where $g(y) \in L_1(\mathbb{R}; dy)$. Fixing a small $\delta > 0$ we split the integral in the right-hand side of (2.7) for each $\tau \in \mathbb{R}$ as follows

$$\frac{2^{\varepsilon-3}}{\pi\Gamma(\varepsilon)} \int_{-\infty}^{\infty} \frac{\left|\Gamma\left(\frac{\varepsilon+i(y+\tau)}{2}\right)\Gamma\left(\frac{\varepsilon+i(\tau-y)}{2}\right)\right|^2}{|\Gamma(i\tau)\Gamma(iy)|} g(y) dy$$

$$= \frac{2^{\varepsilon-3}}{\pi\Gamma(\varepsilon)} \left(\int_{|\tau\pm y|\le\delta} + \int_{|\tau\pm y|>\delta}\right) \frac{\left|\Gamma\left(\frac{\varepsilon+i(y+\tau)}{2}\right)\Gamma\left(\frac{\varepsilon+i(\tau-y)}{2}\right)\right|^2}{|\Gamma(i\tau)\Gamma(iy)|} g(y) dy$$

$$= (I_{1\varepsilon}g)(\tau) + (I_{2\varepsilon}g)(\tau). \tag{2.8}$$

But it is easily seen appealing to the Stirling asymptotic formula for gamma-functions [3, Vol. I] that for each $\varepsilon > 0,\ \tau \in \mathbb{R}$

$$\int_{|\tau\pm y|>\delta} \frac{\left|\Gamma\left(\frac{\varepsilon+i(y+\tau)}{2}\right)\Gamma\left(\frac{\varepsilon+i(\tau-y)}{2}\right)\right|^2}{|\Gamma(iy)|} g(y)dy$$

$$= O\left(\int_{|\tau\pm y|>\delta} e^{-\frac{\pi}{2}|y|} g(y)dy\right) < \infty.$$

Hence with the reduction formula for gamma-function $\Gamma(1+z) = z\Gamma(z)$ we have

$$\lim_{\varepsilon\to 0+} (I_{2\varepsilon}g)(\tau) = \lim_{\varepsilon\to 0+} \frac{2^{\varepsilon-3}\varepsilon}{\pi\Gamma(1+\varepsilon)} \int_{|\tau\pm y|>\delta} \frac{\left|\Gamma\left(\frac{\varepsilon+i(y+\tau)}{2}\right)\Gamma\left(\frac{\varepsilon+i(\tau-y)}{2}\right)\right|^2}{|\Gamma(i\tau)\Gamma(iy)|} g(y)dy = 0.$$

Further,

$$(I_{1\varepsilon}g)(\tau) = \frac{2^{\varepsilon-1}}{\pi\Gamma(\varepsilon)} \int_{|\tau\pm y|\le\delta} \frac{\left|\Gamma\left(1+\frac{\varepsilon+i(y+\tau)}{2}\right)\Gamma\left(1+\frac{\varepsilon+i(\tau-y)}{2}\right)\right|^2}{\tau y|\Gamma(i\tau)\Gamma(iy)|(\varepsilon^2+(\tau-y)^2)} g(y)dy$$

$$-\frac{2^{\varepsilon-1}}{\pi\Gamma(\varepsilon)} \int_{|\tau\pm y|\le\delta} \frac{\left|\Gamma\left(1+\frac{\varepsilon+i(y+\tau)}{2}\right)\Gamma\left(1+\frac{\varepsilon+i(\tau-y)}{2}\right)\right|^2}{\tau y|\Gamma(i\tau)\Gamma(iy)|(\varepsilon^2+(\tau+y)^2)} g(y)dy$$

$$= (J^{\pm}_{1\varepsilon}g)(\tau) - (J^{\pm}_{2\varepsilon}g)(\tau). \tag{2.9}$$

Let us treat integral $(J^{\pm}_{1\varepsilon}g)(\tau)$ noting that integral $(J^{\pm}_{2\varepsilon}g)(\tau)$ can be treated in the same manner. Moreover it is easy to verify that integrals $(J^{+}_{1\varepsilon}g)(\tau),\ (J^{-}_{2\varepsilon}g)(\tau)$ tend to zero when $\varepsilon \to 0+$ for all $\tau \in \mathbb{R}$. At the same time we have

$$(J^{-}_{1\varepsilon}g)(\tau) = \frac{2^{\varepsilon-1}}{\pi\Gamma(\varepsilon)} \int_{|\tau-y|\le\delta} \left[\frac{\left|\Gamma\left(1+\frac{\varepsilon+i(y+\tau)}{2}\right)\Gamma\left(1+\frac{\varepsilon+i(\tau-y)}{2}\right)\right|^2}{\tau y|\Gamma(i\tau)\Gamma(iy)|}\right.$$

$$\left. -\Gamma^2\left(1+\frac{\varepsilon}{2}\right)\left|\frac{\Gamma\left(1+\frac{\varepsilon}{2}+i\tau\right)}{\Gamma(1+i\tau)}\right|^2\right] \frac{g(y)}{\varepsilon^2+(\tau-y)^2}dy$$

$$+\frac{2^{\varepsilon-1}}{\pi\Gamma(\varepsilon)}\Gamma^2\left(1+\frac{\varepsilon}{2}\right)\left|\frac{\Gamma\left(1+\frac{\varepsilon}{2}+i\tau\right)}{\Gamma(1+i\tau)}\right|^2 \int_{|\tau-y|\le\delta} \frac{g(y)}{\varepsilon^2+(\tau-y)^2}dy. \tag{2.10}$$

But the modulus of the expression in brackets in the first integral at the right-hand side of (2.10) is less than ε as soon as $|\tau - y| \le \delta$. Therefore from the estimate

$$\frac{2^{\varepsilon-1}}{\pi\Gamma(\varepsilon)} \int_{|\tau-y|\le\delta} \left| \left[\frac{\left|\Gamma\left(1+\frac{\varepsilon+i(y+\tau)}{2}\right)\Gamma\left(1+\frac{\varepsilon+i(\tau-y)}{2}\right)\right|^2}{\tau y|\Gamma(i\tau)\Gamma(iy)|} \right.\right.$$

$$\left.\left. -\Gamma^2\left(1+\frac{\varepsilon}{2}\right)\left|\frac{\Gamma\left(1+\frac{\varepsilon}{2}+i\tau\right)}{\Gamma(1+i\tau)}\right|^2\right]\right| \frac{|g(y)|}{\varepsilon^2+(\tau-y)^2}dy < \frac{2^{\varepsilon-1}}{\pi\Gamma(1+\varepsilon)}\int_{|\tau-y|\le\delta} g(y)dy < \infty$$

and the dominated convergence theorem we conclude that the first term at the right-hand side of (2.10) tends to zero when $\varepsilon \to 0+$. Meanwhile, for any small finite $\delta > \varepsilon$

$$\frac{1}{\Gamma(\varepsilon)}\int_{|\tau-y|\le\delta}\frac{g(y)}{\varepsilon^2+(\tau-y)^2}dy = \frac{1}{\Gamma(\varepsilon)}\int_{|\tau-y|\le\delta}\frac{g(y)-g(\tau)}{\varepsilon^2+(\tau-y)^2}dy$$

$$+\frac{2}{\Gamma(1+\varepsilon)}\arctan\left(\frac{\delta}{\varepsilon}\right)g(\tau), \tag{2.11}$$

and at each Lebesgue point of g we obtain

$$\frac{1}{\Gamma(\varepsilon)}\left|\int_{|\tau-y|\le\delta}\frac{g(y)-g(\tau)}{\varepsilon^2+(\tau-y)^2}dy\right| \le \frac{1}{\Gamma(\varepsilon)}\int_{|\tau-y|\le\delta}\frac{|g(y)-g(\tau)|}{\varepsilon^2+(\tau-y)^2}dy$$

$$= \frac{1}{\Gamma(\varepsilon)}\left[\int_{|\tau-y|<\varepsilon} + \int_{|\tau-y|>\varepsilon}^{|\tau-y|\le\delta}\right]\frac{|g(y)-g(\tau)|}{\varepsilon^2+(\tau-y)^2}dy$$

$$< \frac{1}{\varepsilon\Gamma(1+\varepsilon)}\int_{|\tau-y|<\varepsilon}|g(y)-g(\tau)|dy + \frac{1}{\Gamma(\varepsilon)}\int_\varepsilon^\delta \frac{1}{\rho^2}d\int_0^\rho |g(y+\tau)-g(\tau)|dy$$

$$= \frac{1}{\varepsilon\Gamma(1+\varepsilon)}\int_{|\tau-y|<\varepsilon}|g(y)-g(\tau)|dy + \frac{1}{\Gamma(\varepsilon)}\left[\frac{1}{\delta^2}\int_0^\delta |g(y+\tau)-g(\tau)|dy\right.$$

$$\left. -\frac{1}{\varepsilon^2}\int_0^\varepsilon |g(y+\tau)-g(\tau)|dy + 2\int_\varepsilon^\delta \frac{1}{\rho^3}\int_0^\rho |g(y+\tau)-g(\tau)|dyd\rho\right]$$

$$< \frac{5}{\Gamma(1+\varepsilon)}\max_{0<\rho\le\delta}\left[\frac{1}{\rho}\int_0^\rho |g(y+\tau)-g(\tau)|dy\right] = o(1), \ \delta \to 0.$$

Now we first find a $\delta > 0$ such that the first term at the right-hand side of (2.11) is sufficiently small and then we let ε go to 0. Thus

$$\lim_{\varepsilon\to 0+}\frac{1}{\Gamma(\varepsilon)}\int_{|\tau-y|\le\delta}\frac{g(y)}{\varepsilon^2+(\tau-y)^2}dy = \pi g(\tau)$$

and with (2.10) we find $\lim_{\varepsilon\to 0+}(J^-_{1\varepsilon}g)(\tau)=\frac{g(\tau)}{2}$. Analogously we have $\lim_{\varepsilon\to 0+}(J^+_{2\varepsilon}g)(\tau)=-\frac{g(-\tau)}{2}$. Combining with (2.7), (2.8), (2.9) we obtain finally that at each Lebesgue point of g

$$\lim_{\varepsilon\to 0+}\frac{1}{2\pi|\Gamma(i\tau)|}\int_{-\infty}^{\infty}e^{\varepsilon x}K_{i\tau}(e^x)F(x)dx=\frac{g(\tau)+g(-\tau)}{2}. \tag{2.12}$$

Moreover, from the above estimates it follows that for all $\varepsilon\in(0,1/2]$

$$\frac{1}{\pi|\Gamma(i\tau)|}\left|\int_{-\infty}^{\infty}e^{\varepsilon x}K_{i\tau}(e^x)F(x)dx\right|<C_\tau(1+g(\tau)+g(-\tau)),$$

where $C_\tau>0$ is a constant depending only on τ. Consequently, one can integrate through in (2.12) with respect to τ over any compact $[a,\tau]$ and take out the limit by ε in its left-hand side via the dominated convergence theorem. With simple substitutions and taking into account (2.4) we obtain,

$$\lim_{\varepsilon\to 0+}\frac{1}{\pi}\int_a^{\tau}\frac{1}{|\Gamma(iy)|}\int_{-\infty}^{\infty}e^{\varepsilon x}K_{iy}(e^x)F(x)dxdy=\frac{1}{2}\left[V(\tau)-V(-\tau)+V(-a)\right]. \tag{2.13}$$

Hence formula (2.5) comes immediately after the change of the order of integration in the left-hand side of (2.13) by virtue of the uniform convergence of the inner integral with respect to y for each $0<\varepsilon<1/2$. Indeed, this fact can be achieved by the estimate (see (1.4), (1.6), (1.7), (2.3))

$$\int_{-\infty}^{\infty}e^{\varepsilon x}|K_{iy}(e^x)F(x)|dx<\text{const.}\int_{-\infty}^{\infty}e^{\varepsilon x}K_0(e^x)e^{e^x}dx$$

$$=\text{const.}\int_0^{\infty}t^{\varepsilon-1}K_0(t)e^t dt<\infty,\ 0<\varepsilon<\frac{1}{2}.$$

Theorem 1 is proved.

3 Spectral decompositions of the Lebedev- Stieltjes integrals. Fourier type series.

As it is known (cf. [2]), the distribution function $V(\tau)$ has at most a countable set of discontinuous points, which we denote by $\lambda_0,\ \lambda_1,\ \lambda,\ \dots,\lambda_\nu,\ \dots$, and the corresponding jumps by a_ν. Hence as usual $a_\nu=V(\lambda_\nu+0)-V(\lambda_\nu-0)$ and $\sum_\nu a_\nu\le V(\infty)-V(-\infty)$. Furthermore, let us assume that there exists an equivalent odd distribution function [2], which differs from $V(\tau)$ on a constant and which we will denote again as $V(\tau)$. Then integral (1.1) can be written as

$$F(x)=2\int_{\mathbb{R}_+}\frac{K_{i\tau}(e^x)}{|\Gamma(i\tau)|}\,dV(\tau),\ x\in\mathbb{R}. \tag{3.1}$$

Meanwhile, $V(\tau)$ can be represented as a sum of two distribution functions, namely

$$V(\tau) = S(\tau) + D(\tau), \tag{3.2}$$

where $S(\tau)$ is continuous and $D(\tau)$ is a jump function of the distribution $V(\tau)$. $D(\tau)$ is defined as follows: at each point τ, where $V(\tau)$ is continuous the value of $D(\tau)$ is equal to the sum of jumps of $V(\tau)$ from the left of τ, i.e.

$$D(\tau) = \sum_{\lambda_\nu < \tau} a_\nu. \tag{3.3}$$

Hence we immediately obtain

$$D(\lambda_\nu + 0) - D(\lambda_\nu - 0) = V(\lambda_\nu + 0) - V(\lambda_\nu - 0).$$

When $V(\tau)$ is continuous, then $D(\tau) \equiv 0$. Another least case takes place when $S(\tau) =$ const.

Denoting by $\varphi_\tau(x) = 2\frac{K_{i\tau}(e^x)}{|\Gamma(i\tau)|}$ we easily find that $\varphi_0(x) = 0$ for all $x \in \mathbb{R}$. Taking into account (3.1) we write the Lebedev-Stieltjes integral (1.1) as $F(x) = G(x) + h(x)$, where

$$G(x) = \int_{\mathbb{R}_+} \varphi_\tau(x) \; dD(\tau), \tag{3.4}$$

$$h(x) = \int_{\mathbb{R}_+} \varphi_\tau(x) \; dS(\tau). \tag{3.5}$$

Let us consider function (3.4). By the properties of the Stieltjes integral it can be written in terms of series

$$G(x) = \sum_\nu a_\nu \varphi_{\lambda_\nu}(x). \tag{3.6}$$

Conversely one can show that each series of type (3.6) with positive a_ν such that $\sum_\nu a_\nu$ is convergent, represents a function of type (3.4). Moreover, $D(\tau)$ should be defined as follows: if τ is different from λ_ν for any ν then (3.2) holds. Otherwise we have the equality

$$D(\tau) = \frac{1}{2}\left[D(\tau+0) + D(\tau-0)\right].$$

We will appeal now at the Bohr type mean value [1], [2]

$$\mathcal{M}\{f(x)\} = \lim_{\omega\to\infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f(x)dx. \tag{3.7}$$

First we prove the following lemma.

Lemma 1. *The uncountable system of functions* $\varphi_{\lambda_n}(x) = 2\frac{K_{i\lambda_n}(e^x)}{|\Gamma(i\lambda_n)|},\ \lambda_n \in \mathbb{R}_+,\ n \in \mathbb{N}_0$ *is orthonormal in a sense that*

$$\mathcal{M}\{\varphi_{\lambda_n}(x)\varphi_{\lambda_m}(x)\} = 0, \quad \lambda_n \neq \lambda_m, \tag{3.8}$$

$$\mathcal{M}\{\varphi^2_{\lambda_n}(x)\} = 1, \quad \lambda_n > 0, \tag{3.9}$$

hold true.

Proof. In fact, considering different nonzero positive values $\lambda_n,\ \lambda_m$ we fix some $\delta > 0$ and after simple substitution we split the integral in the left-hand side of (3.7) as follows

$$\mathcal{M}\{\varphi_{\lambda_n}(x)\varphi_{\lambda_m}(x)\} = \lim_{\omega\to\infty}\frac{1}{\omega}\int_{e^{-\omega}}^{e^{-\omega}+\delta}\frac{K_{i\lambda_n}(x)K_{i\lambda_m}(x)}{|\Gamma(i\lambda_n)\Gamma(i\lambda_m)|}\frac{dx}{x}$$

$$+\lim_{\omega\to\infty}\frac{1}{\omega}\int_{e^{-\omega}+\delta}^{e^{\omega}}\frac{K_{i\lambda_n}(x)K_{i\lambda_m}(x)}{|\Gamma(i\lambda_n)\Gamma(i\lambda_m)|}\frac{dx}{x}. \tag{3.10}$$

But the second limit in the right-hand side of (3.10) is zero via asymptotic formula (1.4) for the modified Bessel function and the boundedness of the following integral

$$\int_{e^{-\omega}+\delta}^{e^{\omega}}|K_{i\lambda_n}(x)K_{i\lambda_m}(x)|\frac{dx}{x} < \int_{\delta}^{\infty}K_0^2(x)\frac{dx}{x} < \infty.$$

The first limit one can treat taking (1.2), (1.3) and writing the product of the modified Bessel functions in terms of the of series

$$K_{i\lambda_n}(x)K_{i\lambda_m}(x) = -\frac{\pi^2}{4\sinh\pi\lambda_n\sinh\pi\lambda_m}\left[\frac{x^{i\lambda_n}}{2^{i\lambda_n}\Gamma(1+i\lambda_n)} - \frac{x^{-i\lambda_n}}{2^{-i\lambda_n}\Gamma(1-i\lambda_n)}\right.$$

$$\left.+\sum_{k=1}^{\infty}\frac{(x/2)^{2k+i\lambda_n}}{k!\Gamma(k+i\lambda_n+1)} - \sum_{k=1}^{\infty}\frac{(x/2)^{2k-i\lambda_n}}{k!\Gamma(k-i\lambda_n+1)}\right]\left[\frac{x^{i\lambda_m}}{2^{i\lambda_m}\Gamma(1+i\lambda_m)} - \frac{x^{-i\lambda_m}}{2^{-i\lambda_m}\Gamma(1-i\lambda_m)}\right.$$

$$\left.+\sum_{k=1}^{\infty}\frac{(x/2)^{2k+i\lambda_m}}{k!\Gamma(k+i\lambda_m+1)} - \sum_{k=1}^{\infty}\frac{(x/2)^{2k-i\lambda_m}}{k!\Gamma(k-i\lambda_m+1)}\right].$$

Consequently, when $x \to 0+$ we get

$$K_{i\lambda_n}(x)K_{i\lambda_m}(x) = -\frac{\pi^2}{4\sinh\pi\lambda_n\sinh\pi\lambda_m}\left[\frac{x^{i\lambda_n}}{2^{i\lambda_n}\Gamma(1+i\lambda_n)} - \frac{x^{-i\lambda_n}}{2^{-i\lambda_n}\Gamma(1-i\lambda_n)}\right]$$

$$\times\left[\frac{x^{i\lambda_m}}{2^{i\lambda_m}\Gamma(1+i\lambda_m)} - \frac{x^{-i\lambda_m}}{2^{-i\lambda_m}\Gamma(1-i\lambda_m)}\right] + O(x^2). \tag{3.11}$$

Hence

$$\lim_{\omega\to\infty}\frac{1}{\omega}\int_{e^{-\omega}}^{e^{-\omega}+\delta}K_{i\lambda_n}(x)K_{i\lambda_m}(x)\frac{dx}{x}=-\frac{\pi^2}{4\sinh\pi\lambda_n\sinh\pi\lambda_m}\Big([2^{i(\lambda_n+\lambda_m)}\Gamma(1+i\lambda_n)$$

$$\times\Gamma(1+i\lambda_m)i(\lambda_n+\lambda_m)]^{-1}\lim_{\omega\to\infty}\frac{(e^{-\omega}+\delta)^{i(\lambda_n+\lambda_m)}-e^{-i\omega(\lambda_n+\lambda_m)}}{\omega}$$

$$+[2^{i(\lambda_n-\lambda_m)}\Gamma(1+i\lambda_n)\Gamma(1-i\lambda_m)i(\lambda_n-\lambda_m)]^{-1}\lim_{\omega\to\infty}\frac{(e^{-\omega}+\delta)^{i(\lambda_n-\lambda_m)}-e^{-i\omega(\lambda_n-\lambda_m)}}{\omega}$$

$$+[2^{i(\lambda_m-\lambda_n)}\Gamma(1+i\lambda_m)\Gamma(1-i\lambda_n)i(\lambda_m-\lambda_n)]^{-1}\lim_{\omega\to\infty}\frac{(e^{-\omega}+\delta)^{i(\lambda_m-\lambda_n)}-e^{-i\omega(\lambda_m-\lambda_n)}}{\omega}$$

$$+[2^{-i(\lambda_m+\lambda_n)}\Gamma(1-i\lambda_m)\Gamma(1-i\lambda_n)i(\lambda_m+\lambda_n)]^{-1}$$

$$\times\lim_{\omega\to\infty}\frac{(e^{-\omega}+\delta)^{-i(\lambda_m+\lambda_n)}-e^{i\omega(\lambda_m+\lambda_n)}}{\omega}\Big)=0$$

and we prove (3.8). In the case of (3.9) we come again to (3.11) appealing to the supplement formula for gamma- function $\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$. Therefore we obtain

$$K^2_{i\lambda_n}(x)=-\frac{\pi^2}{4\sinh^2(\pi\lambda_n)}\left[\frac{x^{i\lambda_n}}{2^{i\lambda_n}\Gamma(1+i\lambda_n)}-\frac{x^{-i\lambda_n}}{2^{-i\lambda_n}\Gamma(1-i\lambda_n)}\right]^2+O(x^2)$$

$$=\frac{|\Gamma(i\lambda_n)|^2}{2}-\frac{\pi^2}{4\sinh^2(\pi\lambda_n)}\left[\frac{x^{2i\lambda_n}}{4^{i\lambda_n}\Gamma^2(1+i\lambda_n)}+\frac{x^{-2i\lambda_n}}{4^{-i\lambda_n}\Gamma^2(1-i\lambda_n)}\right]+O(x^2),\ x\to 0+.$$

Thus for any $\lambda_n>0$ and a fix $\delta>0$ (3.9) becomes

$$\mathcal{M}\{\varphi^2_{\lambda_n}(x)\}=\lim_{\omega\to\infty}\frac{2}{\omega|\Gamma(i\lambda_n)|^2}\int_{e^{-\omega}}^{e^{-\omega}+\delta}K^2_{i\lambda_n}(x)\frac{dx}{x}=\lim_{\omega\to\infty}\frac{1}{\omega}\int_{e^{-\omega}}^{e^{-\omega}+\delta}\frac{dx}{x}=1.$$

Lemma 1 is proved.

This lemma allows us to consider Fourier type series (3.6) with respect to the system $\varphi_{\lambda_n}(x)=2\frac{K_{i\lambda_n}(e^x)}{|\Gamma(i\lambda_n)|}$ in the Bohr type pre-Hilbert space (cf. [1], [2]) equipped with the inner product

$$\mathcal{M}\{f(x)g(x)\}=\lim_{\omega\to\infty}\frac{1}{2\omega}\int_{-\omega}^{\omega}f(x)\overline{g(x)}dx. \tag{3.12}$$

As it is known [1], [2], the corresponding Hilbert space contains the space of almost periodic functions. Here below we will establish in a similar manner the fundamental results for the so-called spectral decompositions of the Lebedev- Stieltjes integrals $F(x)$ in terms of the Fourier type series (3.6). However, first we prove the following theorem.

Theorem 2. *Let the sequence* $\{\lambda_\nu\}_0^\infty$ *be with distinct positive numbers and the series* (3.6) *be with non zero coefficients such that* $\sum_\nu a_\nu < \infty$. *Then this series is the Fourier series of its sum* $G(x)$.

Proof. Appealing to the inequality (1.13) we have the estimate

$$|G(x)| \le \sum_\nu a_\nu |\varphi_{\lambda_\nu}(x)| \le 2e^{e^x} \sum_\nu a_\nu,$$

which yields the uniform convergence of the series (3.6) with respect to x from any compact set of $\mathbb{R}$. Hence we multiply (3.6) by $\varphi_{\lambda_n}(x)$ and integrate with respect to x over $[-\omega, \omega]$, $\omega > A > 0$. Changing the order of integration and summation we obtain

$$\frac{1}{2\omega}\int_{-\omega}^{\omega} G(x)\varphi_{\lambda_n}(x)dx = \sum_\nu a_\nu \frac{1}{2\omega}\int_{-\omega}^{\omega} \varphi_{\lambda_n}(x)\varphi_{\lambda_\nu}(x)dx. \tag{3.13}$$

But

$$\frac{1}{2\omega}\int_{-\omega}^{\omega} |\varphi_{\lambda_n}(x)\varphi_{\lambda_\nu}(x)|dx \le \frac{2}{\omega|\Gamma(i\lambda_n)|}\int_{e^{-\omega}}^{e^{\omega}} |K_{i\lambda_n}(x)|e^x \frac{dx}{x}$$

$$= \frac{2}{\omega|\Gamma(i\lambda_n)|}\int_{e^{-\omega}}^{e^{-\omega}+\delta} |K_{i\lambda_n}(x)|e^x \frac{dx}{x} + \frac{2}{\omega|\Gamma(i\lambda_n)|}\int_{e^{-\omega}+\delta}^{e^{\omega}} |K_{i\lambda_n}(x)|e^x \frac{dx}{x}$$

and via asymptotic formula (1.4) we find that

$$\frac{2}{\omega|\Gamma(i\lambda_n)|}\int_{e^{-\omega}+\delta}^{e^{\omega}} |K_{i\lambda_n}(x)|e^x \frac{dx}{x} < \frac{C}{A|\Gamma(i\lambda_n)|}\int_{\delta}^{\infty} \frac{dx}{x^{3/2}} < \infty, \ \delta > 0,$$

where $C > 0$ is an absolute constant. At the same time again with (1.13) we have

$$\frac{2}{\omega|\Gamma(i\lambda_n)|}\int_{e^{-\omega}}^{e^{-\omega}+\delta} |K_{i\lambda_n}(x)|e^x \frac{dx}{x} < \frac{2}{\omega}\int_{e^{-\omega}}^{e^{-\omega}+\delta} e^{2x}\frac{dx}{x}$$

$$< 2e^{2(e^{-A}+\delta)}\frac{\log(e^{-\omega}+\delta)+\omega}{\omega} = 2e^{2(e^{-A}+\delta)}\frac{\log(1+e^{\omega}\delta)}{\omega} < 2e^{2(e^{-A}+\delta)}\left[1+\frac{\log(1+\delta)}{A}\right].$$

Consequently, by virtue of the condition $\sum_\nu a_\nu < \infty$ a series in the right-hand side of (3.13) converges absolutely and uniformly with respect to $\omega > A > 0$ and the mean value process may be taken termwise. Thus passing to the limit through in (3.13) when $\omega \to \infty$ and taking into account relations (3.7), (3.8), (3.9) we get the formula for coefficients of the Fourier type series (3.6)

$$a_\nu \equiv a(\lambda_\nu) = \mathcal{M}\{G(x)\varphi_{\lambda_\nu}(x)\}, \quad \nu \in \mathbb{N}_0, \tag{3.14}$$

and we conclude the proof of Theorem 2.

Now we consider the following formula

$$\mathcal{M}\left\{\left|F(x)-\sum_{n=0}^{N} c_n\varphi_{\lambda_n}(x)\right|^2\right\} = \mathcal{M}\{|F(x)|^2\} - \sum_{n=0}^{N}[a(\lambda_n)]^2 + \sum_{n=0}^{N}|c_n - a(\lambda_n)|^2, \quad (3.15)$$

where $F(x)$ is defined by (3.1), $a(\lambda_n)$ are Fourier coefficients (3.14) and c_n are arbitrary complex numbers. This formula can be easily proved by using Lemma 1, the properties of the inner product (3.12) and the mean value (3.7). If the numbers $a(\lambda_n)$ are chosen for the constants c_n, the there follows the formula

$$\mathcal{M}\left\{\left|F(x)-\sum_{n=0}^{N} a(\lambda_n)\varphi_{\lambda_n}(x)\right|^2\right\} = \mathcal{M}\{|F(x)|^2\} - \sum_{n=0}^{N}[a(\lambda_n)]^2. \quad (3.16)$$

This immediately yields the Bessel type inequality

$$\sum_{n=0}^{N}[a(\lambda_n)]^2 \le \mathcal{M}\{|F(x)|^2\}. \quad (3.17)$$

Assuming that the right-hand side of (3.17) is finite, we observe from (3.17) by similar discussions as for Fourier coefficients of the almost periodic functions (cf. [1]) that $a(\lambda)$ is zero for all $\lambda > 0$ with the exception of an at most enumerable set of values of positive λ, which we denote by $\lambda_0, \lambda_1, \lambda_2, \dots$. Thus the series in (3.17) when $N \to \infty$ is convergent and we have

$$\sum_{n=0}^{\infty}[a(\lambda_n)]^2 \le \mathcal{M}\{|F(x)|^2\}. \quad (3.18)$$

We will prove below that for the class of the Lebedev- Stieltjes integrals $F(x)$ the equality sign always holds in (3.18), i.e. we will establish the Parseval equality for this class of functions

$$\mathcal{M}\{|F(x)|^2\} = \sum_{n=0}^{\infty}[a(\lambda_n)]^2. \quad (3.19)$$

Lemma 2. *Under conditions of Theorem* 2 *each function $F(x)$, which is defined by the Lebedev- Stieltjes integral* (3.1) *satisfies the equality*

$$a(\lambda_\nu) = \mathcal{M}\{F(x)\varphi_{\lambda_\nu}(x)\}, \quad \nu \in \mathbb{N}_0, \quad (3.20)$$

where $a(\lambda_\nu)$ are given by (3.14).

Proof. Taking into account relations (3.4), (3.5) and the equality $F(x) = G(x)+h(x)$ it is sufficient to prove that $\mathcal{M}\{h(x)\varphi_\lambda(x)\} = 0$ for all $\lambda > 0$. Fixing a small $\delta > 0$ we have

$$h(x) = \int_{\mathbb{R}_+} \varphi_\tau(x)\, dS(\tau) = \int_{|\tau-\lambda|\le\delta} \varphi_\tau(x)\, dS(\tau)$$

$$+\int_{|\tau-\lambda|>\delta} \varphi_\tau(x)\ dS(\tau).$$

Hence via the uniform convergence of the latter integrals with respect to $x \in [-\omega, \omega], \omega > A > 0$ we derive

$$\frac{1}{2\omega}\int_{-\omega}^{\omega} h(x)\varphi_\lambda(x)dx = \frac{1}{2\omega}\int_{|\tau-\lambda|\leq\delta}\int_{-\omega}^{\omega} \varphi_\lambda(x)\varphi_\tau(x)dx\ dS(\tau)$$

$$+\frac{1}{2\omega}\int_{|\tau-\lambda|>\delta}\int_{-\omega}^{\omega} \varphi_\lambda(x)\varphi_\tau(x)dx\ dS(\tau). \tag{3.21}$$

The second integral in the right-hand side of (3.21) is zero when $\omega \to \infty$ by virtue of Lemma 1 and the uniform convergence with respect to $\omega > A > 0$. Considering the first integral we use the boundedness of

$$\frac{1}{2\omega}\int_{-\omega}^{\omega} \varphi_\lambda(x)\varphi_\tau(x)dx$$

for all $\lambda,\ \tau > 0,\ \omega > A > 0$ (see the proof of Theorem 2) and the continuity of the distribution $S(\tau)$. Thus we find the estimate

$$\frac{1}{2\omega}\left|\int_{|\tau-\lambda|\leq\delta}\int_{-\omega}^{\omega} \varphi_\lambda(x)\varphi_\tau(x)dx\ dS(\tau)\right| \leq \text{const.}\int_{|\tau-\lambda|\leq\delta} dS(\tau)$$

$$= S(\lambda+\delta) - S(\lambda-\delta),$$

where the latter difference can be made arbitrarily small choosing an appropriate small positive δ. Then passing to the limit in (3.21) when $\omega \to \infty$ we conclude the proof of Lemma 2.

Finally we prove the Parseval equality (3.19). To do this it is sufficient to establish the following limit (see (3.16)) of the convergence in the mean of the partial sums of the Fourier series to $F(x)$

$$\lim_{N\to\infty} \mathcal{M}\left\{\left|F(x) - \sum_{n=0}^{N} a(\lambda_n)\varphi_{\lambda_n}(x)\right|^2\right\} = 0. \tag{3.22}$$

Invoking the properties of the inner product, taking termwise the mean value in the series and using Lemma 2 we deduce the equalities

$$\mathcal{M}\left\{\left|F(x) - \sum_{n=0}^{N} a(\lambda_n)\varphi_{\lambda_n}(x)\right|^2\right\} = \mathcal{M}\left\{\left|G(x) - \sum_{n=0}^{N} a(\lambda_n)\varphi_{\lambda_n}(x) + h(x)\right|^2\right\}$$

$$= \mathcal{M}\left\{\left|\sum_{n=N+1}^{\infty} a(\lambda_n)\varphi_{\lambda_n}(x) + h(x)\right|^2\right\} = \sum_{n=N+1}^{\infty}[a(\lambda_n)]^2 + \mathcal{M}\left\{|h(x)|^2\right\}.$$

But $\sum_{n=N+1}^{\infty}[a(\lambda_n)]^2 \to 0, \ N \to \infty$ since the series $\sum_{n=0}^{\infty} a(\lambda_n)$ with positive terms is convergent. In order to conclude our proof we have to show that $\mathcal{M}\left\{|h(x)|^2\right\} = 0$. Indeed,

$$\mathcal{M}\left\{|h(x)|^2\right\} = \lim_{\omega\to\infty}\int_{\mathbb{R}_+}\frac{1}{2\omega}\int_{-\omega}^{\omega} h(x)\varphi_{\lambda_\tau}(x)dxdS(\tau).$$

But as we see above the latter iterated integral converges absolutely and uniformly with respect to $\omega \in \mathbb{R}_+$. Hence passing to the limit under integral sign and appealing to Lemma 2 we get

$$\lim_{\omega\to\infty}\int_{\mathbb{R}_+}\frac{1}{2\omega}\int_{-\omega}^{\omega} h(x)\varphi_{\lambda_\tau}(x)dxdS(\tau) = \int_{\mathbb{R}_+}\mathcal{M}\left\{h(x)\varphi_{\lambda_\tau}(x)\right\}dS(\tau) = 0.$$

Thus we summarize our results by the following

Theorem 3. *A function $F(x)$, which is represented by the Lebedev- Stieltjes integral* (3.1) *can be decomposed in the Fourier type series in the Bohr type pre-Hilbert space* (3.12)

$$F(x) = 2\sum_{n=0}^{\infty} a_n \frac{K_{i\lambda_n}(e^x)}{|\Gamma(i\lambda_n)|}, \ \lambda_n > 0, \ x \in \mathbb{R}, \tag{3.23}$$

with positive coefficients a_n and the convergent sum $\sum_n a_n$. Series (3.23) *converges in the mean to $F(x)$ and the Parseval equality*

$$\mathcal{M}\{|F(x)|^2\} = \sum_{n=0}^{\infty} a_n^2$$

holds.

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