The Kontorovich - Lebedev transformation on the Sobolev type spaces

Semyon B. Yakubovich *

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Abstract

The Kontorovich-Lebedev transformation

$$(KLf)(x) = \int_0^\infty K_{i\tau}(x) f(\tau) d\tau, \ x \in \mathbf{R}_+$$

is considered as an operator, which maps the weighted space $L_p(\mathbf{R}_+; \omega(\tau)d\tau), \ 2 \leq p \leq \infty$ into the Sobolev type space $S_p^{N,\alpha}(\mathbf{R}_+)$ with the finite norm

$$||u||_{S_p^{N,\alpha}(\mathbf{R}_+)} = \left(\sum_{k=0}^{N} \int_0^\infty |A_x^k u|^p x^{\alpha_k p - 1} dx \right)^{1/p} < \infty,$$

where $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_N), \alpha_k \in \mathbf{R}, k = 0, \ldots, N$, and A_x is the differential operator of the form

$$A_x u = x^2 u(x) - x \frac{d}{dx} \left[x \frac{du}{dx} \right],$$

and A_x^k means k-th iterate of A_x, $A_x^0 u = u$. Elementary properties for the space $S_p^{N,\alpha}(\mathbf{R}_+)$ are derived. Boundedness and inversion properties for the Kontorovich-Lebedev transform are studied. In the Hilbert case ($p = 2$) the isomorphism between these spaces is established for the special type of weights and Plancherel's type theorem is proved.

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1 Introduction

The object of the present paper is to extend the theory of the important Kontorovich-Lebedev transformation [8], [11]

$$(KLf)(x) = \int_0^\infty K_{i\tau}(x) f(\tau) d\tau, \tag{1.1}$$

on the so-called Sobolev type spaces, which will be defined below. In the following, $x \in \mathbf{R}_+ \equiv (0, \infty)$, $K_{i\tau}(x)$ is the modified Bessel function or the Macdonald function (cf. [1], [8, p. 355]), and the pure imaginary subscript (an index) $i\tau$ is such that τ is restricted to $\mathbf{R}_+$. The function $K_\nu(z)$ satisfies the differential equation

$$z^2 \frac{d^2u}{dz^2} + z\frac{du}{dz} - (z^2 + \nu^2)u = 0, \tag{1.2}$$

for which it is the solution that remains bounded as z tends to infinity on the real line. The modified Bessel function has the asymptotic behaviour (cf. [1], relations (9.6.8), (9.6.9), (9.7.2))

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}[1 + O(1/z)], \qquad z \to \infty, \tag{1.3}$$

and near the origin

$$K_\nu(z) = O\left(z^{-|\mathrm{Re}\nu|}\right), \ z \to 0, \tag{1.4}$$

$$K_0(z) = O(\log z), \ z \to 0. \tag{1.5}$$

Meanwhile, when x is restricted to any compact subset of $\mathbf{R}_+$ and τ tends to infinity we have the following asymptotic [11, p. 20]

$$K_{i\tau}(x) = \left(\frac{2\pi}{\tau}\right)^{1/2} e^{-\pi\tau/2} \sin\left(\frac{\pi}{4} + \tau \log\frac{2\tau}{x} - \tau\right)\left[1 + O(1/\tau)\right], \qquad \tau \to \infty. \tag{1.6}$$

The modified Bessel function can be represented by the integrals of the Fourier and Mellin types [1], [8], [11]

$$K_\nu(x) = \int_0^\infty e^{-x\cosh u} \cosh \nu u du, \tag{1.7}$$

$$K_\nu(x) = \frac{1}{2}\left(\frac{x}{2}\right)^\nu \int_0^\infty e^{-t-\frac{x^2}{4t}} t^{-\nu-1} dt. \tag{1.8}$$

Hence it is not difficult to show that $K_{i\tau}(x)$ is infinitely differentiable with respect to x and τ on $\mathbf{R}_+$ real-valued funtion. We also note that the product of the modified Bessel functions of different arguments can be represented by the Macdonald formula [1], [6], [11]

$$K_{i\tau}(x)K_{i\tau}(y) = \frac{1}{2}\int_0^\infty e^{-\frac{1}{2}\left(u\frac{x^2+y^2}{xy} + \frac{xy}{u}\right)} K_{i\tau}(u)\frac{du}{u}. \tag{1.9}$$

In this paper we deal with the Lebesgue weighted $L_p(\mathbf{R}_+;\omega(x)dx)$ spaces over the measure $\omega(x)dx$ with the norm

$$||f||_p = \left(\int_0^\infty |f(x)|^p \omega(x)dx\right)^{1/p}, \ 1 \le p < \infty, \tag{1.10}$$

$$||f||_\infty = \text{ess sup}|f(x)|. \tag{1.11}$$

In particular, we will use the spaces $L_{\nu,p} \equiv L_p(\mathbf{R}_+;x^{\nu p-1}dx), \ 1 \le p \le \infty, \nu \in \mathbf{R}$, which are related to the Mellin transforms pair [7], [8], [9]

$$f^{\mathcal{M}}(s) = \int_0^\infty f(x)x^{s-1}dx, \tag{1.12}$$

$$f(x) = \frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} f^{\mathcal{M}}(s)x^{-s}ds, \ s = \nu + it, \ x > 0. \tag{1.13}$$

The integrals (1.13)- (1.14) are convergent, in particular, in mean with respect to the norm of the spaces $L_2(\nu - i\infty, \nu + i\infty; ds)$ and $L_2(\mathbf{R}_+;x^{2\nu-1}dx)$, respectively. In addition, the Parseval equality of the form

$$\int_0^\infty |f(x)|^2 x^{2\nu-1}dx = \frac{1}{2\pi}\int_{-\infty}^\infty |f^{\mathcal{M}}(\nu + it)|^2 dt \tag{1.15}$$

holds true.

As it is proved in [12], [13], the Kontorovich-Lebedev operator (1.1) is an isomorphism between the spaces $L_2(\mathbf{R}_+;[\tau \sinh \pi\tau]^{-1}d\tau)$ and $L_2(\mathbf{R}_+;x^{-1}dx)$ with the identity for the square of norms

$$\int_0^\infty |(KLf)(x)|^2 \frac{dx}{x} = \frac{\pi^2}{2}\int_0^\infty |f(\tau)|^2 \frac{d\tau}{\tau \sinh \pi\tau}, \tag{1.16}$$

and the Parseval equality of type

$$\int_0^\infty (KLf)(x)\overline{(KLg(x)}\frac{dx}{x} = \frac{\pi^2}{2}\int_0^\infty f(\tau)\overline{g(\tau)}\frac{d\tau}{\tau \sinh \pi\tau}, \tag{1.17}$$

where $f, g \in L_2(\mathbf{R}_+;[\tau \sinh \pi\tau]^{-1}d\tau)$. We note that the convergence of the integral (1.1) in this case is with respect to the norm (1.10) for the space $L_2(\mathbf{R}_+;x^{-1}dx)$.

However, our goal is to study the Kontorovich-Lebedev transformation in the space $S_p^{N,\alpha}(\mathbf{R}_+), 1 \le p < \infty$, which we call the Sobolev type space with the finite norm

$$||u||_{S_p^{N,\alpha}(\mathbf{R}_+)} = \left(\sum_{k=0}^N \int_0^\infty |A_x^k u|^p x^{\alpha_k p-1}dx\right)^{1/p} < \infty. \tag{1.18}$$

Here $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_N), \alpha_k \in \mathbf{R}, k = 0, \ldots, N$, and A_x is the differential operator (1.2), which is written in the form

$$A_x u = x^2 u(x) - x\frac{d}{dx}\left[x\frac{du}{dx}\right]. \tag{1.19}$$

As usual we denote by A_x^k the k-th iterate of A_x, $A_x^0 u = u$. The differential operator (1.19) was used for instance in [4], [16] in order to construct the spaces of testing functions to consider the Kontorovich-Lebedev transform on distributions (see also in [10]). Recently (see [15]) it is involved to investigate the corresponding class of the Kontorovich-Lebedev convolution integral equations.

In the sequel we will derive imbedding properties for the spaces $S_p^{N,\alpha}(\mathbf{R}_+)$ and we will find integral representations for the functions from $S_p^{N,\alpha}(\mathbf{R}_+)$. Finally we will study the boundedness and inversion properties for the Kontorovich-Lebedev transformation as an operator from the weighted L_p-space $L_p(\mathbf{R}_+; \omega(x)dx)$ into the space $S_p^{N,\alpha}(\mathbf{R}_+)$. When $p = 2, \alpha = 0$ we will prove the Plancherel type theorem and we will establish an isomorphism for the special type of weights between these spaces.

2 Elementary properties for the space $S_p^{N,\alpha}(\mathbf{R}_+)$

Let $\varphi(x)$ belong to the space $C_0^\infty(\mathbf{R}_+)$ of infinitely differentiable functions with a compact support on $\mathbf{R}_+$. Hence taking (1.19), we integrate by parts for any twice continuously differentiable function $u \in C^2(\mathbf{R}_+)$ and we derive the following equality

$$\int_0^\infty u(x) A_x \varphi \frac{dx}{x} = \int_0^\infty A_x u \; \varphi(x) \frac{dx}{x}. \tag{2.1}$$

Now if furthermore we suppose, that for any $\varphi \in C_0^\infty(\mathbf{R}_+)$ and some locally integrable function $v \in L_{loc}(\mathbf{R}_+)$ it satisfies

$$\int_0^\infty u(x) A_x \varphi \frac{dx}{x} = \int_0^\infty v(x) \; \varphi(x) \frac{dx}{x}$$

then subtracting these equalities we immediately obtain

$$\int_0^\infty [A_x u - v(x)] \; \varphi(x) \frac{dx}{x} = 0. \tag{2.2}$$

Consequently, via Du Bois-Reymond lemma we find that $v(x) = A_x u$ almost everywhere in $\mathbf{R}_+$. Thus we use (2.2) to define the so-called generalized derivative $v(x)$ for the function $u(x)$ in terms of the operator A_x. A k-th generalized derivative can be easily defined from (2.1). Indeed, for any $\varphi \in C_0^\infty(\mathbf{R}_+)$ we have that $A_x\varphi \in C_0^\infty(\mathbf{R}_+)$ and we will call $v_k(x) \in L_{loc}(\mathbf{R}_+)$ a k-th generalized derivative for $u \in L_{loc}(\mathbf{R}_+)$ $(v_k(x) \equiv A_x^k u)$ if it satisfies the equality

$$\int_0^\infty u(x) A_x^k \varphi \frac{dx}{x} = \int_0^\infty v_k(x)\ \varphi(x) \frac{dx}{x}. \tag{2.3}$$

Further, from the norm definition (1.18) and elementary inequalities it follows that there are positive constants C_1, C_2 such that

$$C_1 \sum_{k=0}^{n} \left(\int_0^\infty |A_x^k u|^p x^{\alpha_k p - 1} dx \right)^{1/p} \leq \left(\sum_{k=0}^{N} \int_0^\infty |A_x^k u|^p x^{\alpha_k p - 1} dx \right)^{1/p}$$

$$\leq C_2 \sum_{k=0}^{N} \left(\int_0^\infty |A_x^k u|^p x^{\alpha_k p - 1} dx \right)^{1/p}. \tag{2.4}$$

Hence by (1.10) we have the equivalence of norms

$$C_1 \sum_{k=0}^{N} ||A_x^k u||_{L_p(\mathbf{R}_+; x^{\alpha_k p - 1} dx)} \leq ||u||_{S_p^{N,\alpha}(\mathbf{R}_+)} \leq C_2 \sum_{k=0}^{N} ||A_x^k u||_{L_p(\mathbf{R}_+; x^{\alpha_k p - 1} dx)}. \tag{2.5}$$

In order to show that $S_p^{N,\alpha}(\mathbf{R}_+), 1 \leq p < \infty$ is a Banach space we take a fundamental sequence $u_n(x)$, i.e. $||u_n - u_m||_{S_p^{N,\alpha}(\mathbf{R}_+)} \to 0,\ m, n \to \infty$. This will immediately imply that

$$||u_n - u_m||_{L_{\alpha_0, p}} \to 0,$$

$$||A_x^k u_n - A_x^k u_m||_{L_{\alpha_k, p}} \to 0,\ k = 1, \ldots, N,$$

when $m, n \to \infty$. Since spaces $L_{\alpha,p}, k = 0, 1, \ldots, N$ are complete, there are functions $v_0 \in L_{\alpha_0,p},\ v_k \in L_{\alpha_k,p}$ such that

$$||u_n - v_0||_{L_{\alpha_0, p}} \to 0, \tag{2.6}$$

$$||A_x^k u_n - v_k||_{L_{\alpha_k, p}} \to 0,\ k = 1, \ldots, N, \tag{2.7}$$

when $n \to \infty$. If we show that v_k is a k-th generalized derivative of v_0 then we prove that the sequence u_n converges to $v_0 \in S_p^{N,\alpha}(\mathbf{R}_+)$ with respect to the norm (1.18). In fact, from (2.6), (2.7) for any $\varphi \in C_0^\infty(\mathbf{R}_+)$ we have the limit equalities

$$\lim_{n \to \infty} \int_0^\infty u_n(x) \varphi(x) \frac{dx}{x} = \int_0^\infty v_0(x)\ \varphi(x) \frac{dx}{x},$$

$$\lim_{n \to \infty} \int_0^\infty A_x^k u_n\ \varphi(x) \frac{dx}{x} = \int_0^\infty v_k(x)\ \varphi(x) \frac{dx}{x}.$$

But on the other hand,

$$\lim_{n\to\infty}\int_0^\infty A_x^k u_n\ \varphi(x)\frac{dx}{x} = \lim_{n\to\infty}\int_0^\infty u_n(x)\ A_x^k\varphi\frac{dx}{x}$$

$$= \int_0^\infty v_0(x)\ A_x^k\varphi\frac{dx}{x}.$$

Therefore invoking (2.3) we get $v_k(x) = A_x^k v_0$ and we prove that $S_p^{N,\alpha}(\mathbf{R}_+)$ is a Banach space.

For the space $S_p^{1,\alpha}(\mathbf{R}_+)$ we establish an imbedding theorem into Sobolev's weighted space ${}_0W_p^1(\mathbf{R}_+; x^{\gamma p-1}dx)$ with the norm

$$||u||_{{}_0W_p^1(\mathbf{R}_+;x^{\gamma p-1}dx)} = \left(\int_0^\infty |u'(x)|^p x^{\gamma p-1}dx\right)^{1/p}.$$

Indeed, we have the following result.

Theorem 1.*Let* $1 < p < \infty,\ \alpha = (2-\beta, -\beta), \beta > 0$. *The imbedding*

$$S_p^{1,\alpha}(\mathbf{R}_+) \subseteq {}_0W_p^1(\mathbf{R}_+; x^{(1-\beta)p-1}dx)$$

is true.

Proof. Appealing to the classical Hardy's inequality [2]

$$\int_0^\infty x^{-r}\left|\int_0^x f(t)dt\right|^p dx \le \text{const.}\int_0^\infty x^{p-r}\,|f(x)|^p\,dx, \tag{2.8}$$

where $1 < p < \infty,\ r > 1$ we put $f(x) = A_x u/x, r = \beta p + 1,\ \beta > 0$ and we have the estimate

$$\left(\int_0^\infty |A_x u|^p x^{-\beta p-1}dx\right)^{1/p} \ge \text{const.}\left(\int_0^\infty x^{-\beta p-1}\left|\int_0^x \frac{A_t u}{t}dt\right|^p dx\right)^{1/p}$$

$$= \text{const.}\left(\int_0^\infty x^{-\beta p-1}\left|\int_0^x tu(t)dt - xu'(x)\right|^p dx\right)^{1/p} \ge \text{const.}\left[\left(\int_0^\infty x^{p(1-\beta)-1}\,|u'(x)|^p\,dx\right)^{1/p}\right.$$

$$\left.-\left(\int_0^\infty x^{-\beta p-1}\left|\int_0^x tu(t)dt\right|^p dx\right)^{1/p}\right].$$

Thus we get

$$\left(\int_0^\infty x^{p(1-\beta)-1}\,|u'(x)|^p\,dx\right)^{1/p} \le \text{const.}\left[\left(\int_0^\infty |A_x u|^p x^{-\beta p-1}dx\right)^{1/p}\right.$$

$$+\left(\int_0^\infty x^{-\beta p-1}\left|\int_0^x tu(t)dt\right|^p dx\right)^{1/p}\Bigg]. \tag{2.9}$$

Invoking again Hardy's inequality (2.8) to estimate the latter term in (2.9) it becomes

$$\left(\int_0^\infty x^{-\beta p-1}\left|\int_0^x tu(t)dt\right|^p dx\right)^{1/p} \leq \text{const.}\left(\int_0^\infty x^{p(2-\beta)-1}\left|u(x)\right|^p dx\right)^{1/p}.$$

Combining with (2.9) and (1.18) we obtain

$$\left(\int_0^\infty x^{p(1-\beta)-1}\left|u'(x)\right|^p dx\right)^{1/p} \leq \text{const.}\left[\left(\int_0^\infty |A_x u|^p x^{-\beta p-1} dx\right)^{1/p}\right.$$

$$\left.+\left(\int_0^\infty x^{p(2-\beta)-1}\left|u(x)\right|^p dx\right)^{1/p}\right] \leq \text{const.}||u||_{S_p^{1,\alpha}(\mathbf{R}_+)},\ \alpha=(2-\beta,-\beta),\beta>0.$$

Theorem 1 is proved.

Our goal now is to derive integral representations for functions from the space $S_p^{N,\alpha}(\mathbf{R}_+)$. For this we will use a technique from [14]. Precisely, let us introduce for any $u(x) \in L_{\nu,p},\ \nu \in \mathbf{R}$ and $\varepsilon \in (0,\pi)$ the following regularization operator

$$u_\varepsilon(x) = \frac{x\sin\varepsilon}{\pi}\int_0^\infty \frac{K_1((x^2+y^2-2xy\cos\varepsilon)^{1/2})}{(x^2+y^2-2xy\cos\varepsilon)^{1/2}} u(y)dy,\ x>0. \tag{2.10}$$

We are ready to prove the Bochner type representation theorem.

We have

Theorem 2. *Let $u(x) \in L_{\nu,p},\ 0<\nu<1,\ 1\leq p<\infty$. Then*

$$u(x) = \lim_{\varepsilon\to 0} u_\varepsilon(x), \tag{2.11}$$

with respect to the norm in $L_{\nu,p}$. Besides, for $1<p<\infty$ the limit (2.11) *exists for almost all $x>0$.*

Proof. We first show that (2.10) is a bounded operator in $L_{\nu,p}$ under conditions of the theorem. To do this we make the substitution $y = x(\cos\varepsilon + t\sin\varepsilon)$ in the corresponding integral and it becomes

$$u_\varepsilon(x) = \frac{x\sin\varepsilon}{\pi}\int_{-\cot\varepsilon}^\infty \frac{K_1(x\sin\varepsilon\sqrt{t^2+1})}{\sqrt{t^2+1}} u(x(\cos\varepsilon+t\sin\varepsilon))dt. \tag{2.12}$$

Hence owing to the generalized Minkowski inequality and elementary inequality for the modified Bessel function $xK_1(x) \leq 1, x\geq 0$ (see (1.7)) we estimate the norm of the integral (2.12) as follows

$$||u_\varepsilon||_{L_{\nu,p}} \leq \frac{1}{\pi}\int_{-\cot\varepsilon}^\infty \frac{dt}{t^2+1}||u(x(\cos\varepsilon+t\sin\varepsilon))||_{L_{\nu,p}}$$

$$= \frac{1}{\pi}||u||_{L_{\nu,p}} \int_{-\cot\varepsilon}^{\infty} \frac{(\cos\varepsilon + t\sin\varepsilon)^{-\nu}}{t^2+1} dt = ||u||_{L_{\nu,p}}$$

$$\times \frac{\sin\varepsilon}{\pi} \int_0^{\infty} \frac{\cosh\nu\xi}{\cosh\xi - \cos\varepsilon} d\xi, \ 0 < \nu < 1,$$

where we have made the substitution $e^{\xi} = \cos\varepsilon + t\sin\varepsilon$ in the latter integral. However, via formula (2.4.6.6) in [5] we find accordingly,

$$\frac{\sin\varepsilon}{\pi} \int_0^{\infty} \frac{\cosh\nu\xi}{\cosh\xi - \cos\varepsilon} d\xi = \frac{\sin(\nu(\pi-\varepsilon))}{\sin\nu\pi} \leq 1 + \frac{\sin\nu\varepsilon}{\sin\nu\pi}$$

$$\leq 1 + \frac{\pi\nu}{\sin\nu\pi} = C_\nu, \ 0 < \nu < 1.$$

Thus for all $\varepsilon \in (0,\pi)$ we get

$$||u_\varepsilon||_{L_{\nu,p}} \leq C_\nu ||u||_{L_{\nu,p}}. \tag{2.13}$$

Further, by using the identity

$$\frac{1}{\pi} \int_{-\cot\varepsilon}^{\infty} \frac{dt}{t^2+1} = 1 - \frac{\varepsilon}{\pi}$$

and denoting by

$$R(x,t,\varepsilon) = x\sin\varepsilon\sqrt{t^2+1}K_1(x\sin\varepsilon\sqrt{t^2+1}) \tag{2.14}$$

we find that

$$||u_\varepsilon - u||_{L_{\nu,p}} \leq \frac{1}{\pi} \int_{-\cot\varepsilon}^{\infty} \frac{dt}{t^2+1} \, ||u(x(\cos\varepsilon + t\sin\varepsilon))R(x,t,\varepsilon)$$

$$- \left(1 - \frac{\varepsilon}{\pi}\right)^{-1} u(x)\Bigg|\Bigg|_{L_{\nu,p}} \leq \frac{1}{\pi} \int_{-\cot\varepsilon}^{\infty} \frac{dt}{t^2+1} \, ||[u(x(\cos\varepsilon + t\sin\varepsilon))$$

$$- \left(1 - \frac{\varepsilon}{\pi}\right)^{-1} u(x)\Bigg] R(x,t,\varepsilon)\Bigg|\Bigg|_{L_{\nu,p}} + \frac{1}{\pi - \varepsilon} \int_{-\cot\varepsilon}^{\infty} \frac{dt}{t^2+1} \, ||u(x)[R(x,t,\varepsilon)$$

$$-1]||_{L_{\nu,p}} = I_1(\varepsilon) + I_2(\varepsilon).$$

But since [1]

$$\frac{d}{dx}[xK_1(x)] = -xK_0(x),$$

and $xK_1(x) \to 1, \ x \to 0$ we obtain the following representation

$$R(x,t,\varepsilon) - 1 = -\int_0^{x\sin\varepsilon(t^2+1)^{1/2}} yK_0(y)dy.$$

Hence appealing again to the generalized Minkowski inequality we deduce

$$I_2(\varepsilon) = \frac{1}{\pi-\varepsilon}\int_{-\cot\varepsilon}^{\infty}\frac{dt}{t^2+1}\left(\int_0^\infty x^{\nu p-1}\left(\int_0^{x\sin\varepsilon(t^2+1)^{1/2}} yK_0(y)dy\right)^p |u(x)|^p dx\right)^{1/p}$$

$$\leq \frac{1}{\pi-\varepsilon}\int_{-\cot\varepsilon}^{\infty}\frac{dt}{t^2+1}\int_0^\infty yK_0(y)\left(\int_{y/(\sin\varepsilon(t^2+1)^{1/2})}^{\infty} x^{\nu p-1}|u(x)|^p dx\right)^{1/p} dy$$

$$\leq \frac{1}{\pi-\varepsilon}\int_{-\cot\varepsilon}^{\infty}dt\int_0^\infty \xi K_0\left(\xi\sqrt{t^2+1}\right)\left(\int_{\frac{\xi}{\sin\varepsilon}}^{\infty} x^{\nu p-1}|u(x)|^p dx\right)^{1/p} d\xi$$

$$= \frac{1}{\pi-\varepsilon}\int_{-\cot\varepsilon}^{\infty}dt\left(\int_0^{\sqrt{\varepsilon}}+\int_{\sqrt{\varepsilon}}^{\infty}\right)\xi K_0\left(\xi\sqrt{t^2+1}\right)\left(\int_{\frac{\xi}{\sin\varepsilon}}^{\infty} x^{\nu p-1}|u(x)|^p dx\right)^{1/p} d\xi$$

$$\leq \frac{1}{\pi-\varepsilon}\int_{-\cot\varepsilon}^{\infty}dt\int_0^{\sqrt{\varepsilon}} \xi K_0\left(\xi\sqrt{t^2+1}\right)\left(\int_{\frac{\xi}{\sin\varepsilon}}^{\infty} x^{\nu p-1}|u(x)|^p dx\right)^{1/p} d\xi$$

$$+\frac{1}{\pi-\varepsilon}\int_{-\cot\varepsilon}^{\infty}\frac{dt}{t^2+1}\int_0^\infty \xi K_0(\xi)d\xi\left(\int_{\frac{1}{\sqrt{\varepsilon}}}^{\infty} x^{\nu p-1}|u(x)|^p dx\right)^{1/p}$$

$$\leq \frac{\varepsilon^{\nu/2}}{\pi-\varepsilon}||u||_{L_{\nu,p}}\int_{-\infty}^{\infty}(t^2+1)^{\frac{\nu}{2}-1}dt\int_0^\infty \xi^{1-\nu}K_0(\xi)d\xi$$

$$+\frac{\pi}{\pi-\varepsilon}\left(\int_{\frac{1}{\sqrt{\varepsilon}}}^{\infty} x^{\nu p-1}|u(x)|^p dx\right)^{1/p} = \frac{\pi}{\pi-\varepsilon}\left(\varepsilon^{\nu/2}\Gamma(1-\nu)||u||_{L_{\nu,p}}\right.$$

$$\left.+\left(\int_{\frac{1}{\sqrt{\varepsilon}}}^{\infty} x^{\nu p-1}|u(x)|^p dx\right)^{1/p}\right) \to 0,\ \varepsilon\to 0, 0<\nu<1.$$

Concerning the integral I_1 we first approximate $u \in L_{\nu,p}(\mathbf{R}_+)$ by a smooth function $\varphi \in C_0^\infty(\mathbf{R}_+)$. This implies that there exists a function $\varphi \in C_0^\infty(\mathbf{R}_+)$ such that $||f - \varphi||_{L_{\nu,p}} \leq \varepsilon$ for any $\varepsilon > 0$. Hence since the kernel (2.14) $R(x,t,\varepsilon) \leq 1$ then in view of the representation

$$\varphi(x(\cos\varepsilon + t\sin\varepsilon)) - \varphi(x) = \int_1^{\cos\varepsilon+t\sin\varepsilon}\frac{d}{dy}\left[\varphi(xy)\right]dy$$

$$= \int_1^{\cos\varepsilon+t\sin\varepsilon} x\varphi'(xy)dy.$$

In a similar manner we have

$$I_1(\varepsilon) \le \frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty} \frac{dt}{t^2+1}\, ||u(x(\cos\varepsilon + t\sin\varepsilon)) - \varphi(x(\cos\varepsilon + t\sin\varepsilon))||_{L_{\nu,p}}$$

$$+\frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty} \frac{dt}{t^2+1}\left|\left|\varphi(x(\cos\varepsilon + t\sin\varepsilon)) - \left(1-\frac{\varepsilon}{\pi}\right)^{-1} u(x)\right|\right|_{L_{\nu,p}}$$

$$\le ||u-\varphi||_{L_{\nu,p}}\frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty} \frac{(\cos\varepsilon + t\sin\varepsilon)^{-\nu}dt}{t^2+1} + \frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty} \frac{dt}{t^2+1}\left|\left|\varphi(x) - \frac{\pi}{\pi-\varepsilon}u(x)\right|\right|_{L_{\nu,p}}$$

$$+||\varphi'||_{L_{1+\nu,p}}\frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty} \frac{dt}{t^2+1}\left|\int_1^{\cos\varepsilon + t\sin\varepsilon} y^{-\nu-1}dy\right| \le (C_\nu+1)||u-\varphi||_{L_{\nu,p}}$$

$$+\frac{\varepsilon}{\pi}||u||_{\nu,p} + \frac{||\varphi'||_{L_{1+\nu,p}}}{\pi\nu}\int_{-\cot\varepsilon}^{\infty} \frac{|1-(\cos\varepsilon + t\sin\varepsilon)^{-\nu}|}{t^2+1}dt.$$

The latter integral we treat by making the substitution $e^{\xi} = \cos\varepsilon + t\sin\varepsilon$. Then it takes the form

$$\int_{-\cot\varepsilon}^{\infty} \frac{|1-(\cos\varepsilon + t\sin\varepsilon)^{-\nu}|}{t^2+1}dt = \sin\varepsilon\int_0^{\infty}\frac{\sinh\nu\xi}{\cosh\xi - \cos\varepsilon}d\xi$$

$$= \sin\varepsilon\left(\int_0^1 + \int_1^{\infty}\right)\frac{\sinh\nu\xi}{\cosh\xi - \cos\varepsilon}d\xi \le \sin\varepsilon\,\Big(\log(\cosh\xi - \cos\varepsilon)\,|_0^1$$

$$+\int_1^{\infty}\frac{\sinh\nu\xi}{\cosh\xi - 1}d\xi\Big) \le \sin\varepsilon\left[\log\left(2^{-1}\sin^{-2}\frac{\varepsilon}{2}\right) + A_\nu\right],$$

where

$$A_\nu = 1 + \int_1^{\infty}\frac{\sinh\nu\xi}{\cosh\xi - 1}d\xi,\ \ 0<\nu<1.$$

Thus we immediately obtain that $\lim_{\varepsilon\to 0} I_1(\varepsilon) = 0$. Therefore by virtue of the above estimates $\lim_{\varepsilon\to 0}||u_\varepsilon - u||_{L_{\nu,p}} = 0$ and relation (2.11) is proved.

In order to verify the convergence almost everywhere we use the fact that any sequence of functions $\{\varphi_n\} \in C_0^{\infty}(\mathbf{R}_+)$ which converges to u in $L_{\nu,p}$-norm contains a subsequence $\{\varphi_{n_k}\}$ convergent almost everywhere, i.e. $\lim_{k\to\infty}\varphi_{n_k}(x) = u(x)$ for almost all $x>0$. Then we find

$$|u_\varepsilon(x) - u(x)| \le \frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty}\Big|u(x(\cos\varepsilon + t\sin\varepsilon))R(x,t,\varepsilon)$$

$$-\left(1-\frac{\varepsilon}{\pi}\right)^{-1}u(x)\Big|\frac{dt}{t^2+1} \le \frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty}|u(x(\cos\varepsilon + t\sin\varepsilon) - \varphi_{n_k}(x(\cos\varepsilon + t\sin\varepsilon))|\,\frac{dt}{t^2+1}$$

$$+\frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty}|\varphi_{n_k}(x(\cos\varepsilon + t\sin\varepsilon) - \varphi_{n_k}(x)|\,\frac{dt}{t^2+1}$$

$$+\frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty}\left|\varphi_{n_k}(x)R(x,t,\varepsilon)-\left(1-\frac{\varepsilon}{\pi}\right)^{-1}u(x)\right|\frac{dt}{t^2+1}=J_{1\varepsilon}(x)+J_{2\varepsilon}(x)+J_{3\varepsilon}(x).$$

But,

$$J_{3\varepsilon}(x)\le\frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty}\left|\varphi_{n_k}(x)-\left(1-\frac{\varepsilon}{\pi}\right)^{-1}u(x)\right|\frac{dt}{t^2+1}+\frac{1}{\pi-\varepsilon}\int_{-\cot\varepsilon}^{\infty}|u(x)\left[R(x,t,\varepsilon)-1\right]|\frac{dt}{t^2+1}$$

$$\le|\varphi_{n_k}(x)-u(x)|+\frac{\varepsilon}{\pi}|u(x)|+\frac{|u(x)|}{\pi-\varepsilon}\int_{-\cot\varepsilon}^{\infty}\left|\int_0^{x\sin\varepsilon(t^2+1)^{1/2}}yK_0(y)dy\right|\frac{dt}{t^2+1}$$

$$\le|\varphi_{n_k}(x)-u(x)|+\frac{\varepsilon}{\pi}|u(x)|+\frac{|u(x)|\varepsilon^{\nu}x^{\nu}}{\pi-\varepsilon}\int_{-\infty}^{\infty}(t^2+1)^{\nu/2-1}dt\int_0^{\infty}y^{1-\nu}K_0(y)dy$$

$$=|\varphi_{n_k}(x)-u(x)|+\frac{\varepsilon}{\pi}|u(x)|+\frac{\pi\Gamma(1-\nu)\varepsilon^{\nu}x^{\nu}}{\pi-\varepsilon}|u(x)|\to 0,\ 0<\nu<1,$$

when $\varepsilon\to 0,\ k>k_0$ for almost all $x>0$. Similarly,

$$J_{2\varepsilon}(x)=\frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty}\left|\int_1^{\cos\varepsilon+t\sin\varepsilon}x\varphi'_{n_k}(xy)dy\right|\frac{dt}{t^2+1}\le\frac{x}{\pi\nu}\sup_{y\ge 0}y^{1+\nu}|\varphi'_{n_k}(xy)|$$

$$\times\int_{-\cot\varepsilon}^{\infty}\left|1-(\cos\varepsilon+t\sin\varepsilon)^{-\nu}\right|\frac{dt}{t^2+1}\le\sin\varepsilon\left[\log\left(2^{-1}\sin^{-2}\frac{\varepsilon}{2}\right)+A_{\nu}\right]\frac{x}{\pi\nu}\sup_{y\ge 0}y^{1+\nu}|\varphi'_{n_k}(xy)|,$$

which tends to zero almost for all $x>0$ when $\varepsilon\to 0$. Meantime, by taking $1<p<\infty,\ q=\frac{p}{p-1}$ for any $\varepsilon>0$ such that $||u-\varphi_{n_k}||_{L_{\nu,p}}<\varepsilon$ for $k>k_0$ we have

$$J_{1\varepsilon}(x)\le\frac{x^{-\nu}||u-\varphi_{n_k}||_{L_{\nu,p}}}{\pi\sin^{1/p}\varepsilon}\left(\int_{-\cot\varepsilon}^{\infty}\frac{(\cos\varepsilon+t\sin\varepsilon)^{q(1-\nu)-1}dt}{(t^2+1)^q}\right)^{1/q}$$

$$<x^{-\nu}\varepsilon\sin\varepsilon\left(\int_0^{\infty}\frac{\xi^{q(1-\nu)-1}d\xi}{(\xi^2-2\xi\cos\varepsilon+1)^q}\right)^{1/q}.$$

But the latter integral can be treated in terms of the Legendre functions [1] appealing to relation (2.2.9.7) from [5]. This gives the value

$$\int_0^{\infty}\frac{\xi^{q(1-\nu)-1}d\xi}{(\xi^2-2\xi\cos\varepsilon+1)^q}=\left(\frac{\sin\varepsilon}{2}\right)^{1/2-q}\Gamma(q+1/2)\frac{\Gamma(q(1-\nu))\Gamma(q(1+\nu))}{\Gamma(2q)}P^{1/2-q}_{-1/2-q\nu}(-\cos\varepsilon).$$

When $\varepsilon\to 0+$ we have

$$\int_0^{\infty}\frac{\xi^{q(1-\nu)-1}d\xi}{(\xi^2-2\xi\cos\varepsilon+1)^q}\sim\sqrt{\pi}\frac{\Gamma(q-1/2)}{\Gamma(q)}\varepsilon^{1-2q}.$$

Thus

$$J_{1\varepsilon}(x) < \text{const.}\ x^{-\nu}\varepsilon^{1/q} \to 0, \varepsilon \to 0, x > 0$$

and we prove Theorem 2.

Appealing to Theorem 2 we will approximate functions from $S_p^{N,\alpha}(\mathbf{R}_+)$ by singular integral (2.10). Indeed we have

Corollary 1. *Singular integral* (2.10) *is defined on functions from* $S_p^{N,\alpha}(\mathbf{R}_+)$, $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_N)$, $0 < \alpha_k < 1$, $k = 0, 1, \ldots, N$ *and* $1 \le p < \infty$. *Besides*

$$u(x) = \lim_{\varepsilon \to 0} u_\varepsilon(x), \tag{2.15}$$

with respect to the norm in $S_p^{N,\alpha}(\mathbf{R}_+)$.

Proof. Indeed, choosing a fundamental sequence $\{\varphi_n\}$ of $C_0^\infty(\mathbf{R}_+)$- functions, which belongs to $S_p^{N,\alpha}(\mathbf{R}_+)$ we get that it converges to some function $u \in S_p^{N,\alpha}(\mathbf{R}_+)$. This means (see (2.6), (2.8)) that $A_x^k\varphi_n \to A_x^k u$, $n \to \infty$ with respect to the norm in $L_{\alpha_k,p}$, $k = 0, 1, \ldots, N$, respectively.

Defining by

$$\varphi_{\varepsilon,n}(x) = \frac{x \sin\varepsilon}{\pi} \int_0^\infty \frac{K_1((x^2+y^2-2xy\cos\varepsilon)^{1/2})}{(x^2+y^2-2xy\cos\varepsilon)^{1/2}} \varphi_n(y)dy,\ x > 0, \tag{2.16}$$

we employ the relation (2.16.51.8) in [6]

$$\int_0^\infty \tau \sinh((\pi-\varepsilon)\tau) K_{i\tau}(x) K_{i\tau}(y) d\tau$$

$$= \frac{\pi}{2} xy \sin\varepsilon \frac{K_1((x^2+y^2-2xy\cos\varepsilon)^{1/2})}{(x^2+y^2-2xy\cos\varepsilon)^{1/2}}, x, y > 0,\ 0 < \varepsilon \le \pi$$

and we substitute it in (2.16). Changing the order of integration by the Fubini theorem we find

$$\varphi_{\varepsilon,n}(x) = \frac{2}{\pi^2} \int_0^\infty \tau \sinh((\pi-\varepsilon)\tau) K_{i\tau}(x) \int_0^\infty K_{i\tau}(y)\varphi_n(y)\frac{dy}{y}.$$

Meantime, we apply the operator $A_x^k, k = 0, 1 \ldots, N$ (1.19) through both sides of the latter integral. Then via its uniform convergence with respect to $x \in (x_0, X_0) \subset \mathbf{R}_+$ and by using the equalities (see (1.2)) $A_x^k K_{i\tau}(x) = \tau^{2k} K_{i\tau}(x)$, (2.1) we come out with

$$A_x^k \varphi_{\varepsilon,n} = \frac{2}{\pi^2} \int_0^\infty \tau \sinh((\pi-\varepsilon)\tau) K_{i\tau}(x) \int_0^\infty \tau^{2k} K_{i\tau}(y)\varphi_n(y)\frac{dy}{y}$$

$$= \frac{2}{\pi^2} \int_0^\infty \tau \sinh((\pi-\varepsilon)\tau) K_{i\tau}(x) \int_0^\infty K_{i\tau}(y) A_y^k \varphi_n \frac{dy}{y}.$$

This is equivalent to

$$A_x^k \varphi_{\varepsilon,n} = \frac{x \sin \varepsilon}{\pi} \int_0^\infty \frac{K_1((x^2+y^2-2xy\cos\varepsilon)^{1/2})}{(x^2+y^2-2xy\cos\varepsilon)^{1/2}} A_y^k \varphi_n dy. \tag{2.16}$$

Hence

$$A_x^k \varphi_{\varepsilon,n} - (A_x^k u)_\varepsilon = \frac{x \sin \varepsilon}{\pi} \int_0^\infty \frac{K_1((x^2+y^2-2xy\cos\varepsilon)^{1/2})}{(x^2+y^2-2xy\cos\varepsilon)^{1/2}} \left[A_y^k \varphi_n - A_y^k u\right] dy$$

and due to (2.13) we have that $\lim_{n\to\infty} A_x^k \varphi_{\varepsilon,n} = (A_x^k u)_\varepsilon$ with respect to the norm in $L_{\alpha_k,p}$ for each $\varepsilon \in (0,\pi)$. By Theorem 2 we derive that

$$\left|\left|(A_x^k u)_\varepsilon - A_x^k u\right|\right|_{L_{\alpha_k,p}} \to 0, \ \varepsilon \to 0, \ k = 0,1,\ldots,N.$$

If we show that almost for all $x > 0$ $(A_x^k u)_\varepsilon = A_x^k u_\varepsilon, \ k = 0,1,2,\ldots,N$ then via (2.5) we complete the proof of Corollary 1. When $k=0$ it is defined by (2.10). At the same time according to Du Bois-Reymond lemma it is sufficient to show that for any $\psi \in C_0^\infty(\mathbf{R}_+)$

$$\int_0^\infty \left[(A_x^k u)_\varepsilon - A_x^k u_\varepsilon\right] \frac{\psi(x)}{x} dx = 0. \tag{2.17}$$

We have

$$\int_0^\infty \left[(A_x^k u)_\varepsilon - A_x^k u_\varepsilon\right] \frac{\psi(x)}{x} dx = \int_0^\infty \left[(A_x^k u)_\varepsilon - A_x^k \varphi_{\varepsilon,n}\right] \frac{\psi(x)}{x} dx$$

$$+ \int_0^\infty \left[A_x^k \varphi_{\varepsilon,n} - A_x^k u_\varepsilon\right] \frac{\psi(x)}{x} dx = \int_0^\infty \left[(A_x^k u)_\varepsilon - A_x^k \varphi_{\varepsilon,n}\right] \frac{\psi(x)}{x} dx$$

$$+ \int_0^\infty \left[\varphi_{\varepsilon,n} - u_\varepsilon\right] \frac{A_x^k \psi}{x} dx.$$

Now as it is easily seen the right-hand side of the last equality is less than an arbitrary $\delta > 0$ when $n \to \infty$. Thus we prove (2.17) and we complete the proof of Corollary 1.

3 The Kontorovich - Lebedev transformation in $S_2^{N,\alpha}(\mathbf{R}_+)$

Our goal in this section is to establish the boundedness of the Kontorovich-Lebedev transformation (1.1) as an operator $KL : L_2(\mathbf{R}_+; \omega_\alpha(\tau)d\tau) \to S_2^{N,\alpha}(\mathbf{R}_+)$, where the measure $\omega\alpha(\tau)d\tau$ will be defined below. Finally, we will prove the Plancherel theorem and an analog of the Parseval equality (1.17) when $\alpha_k = 0, \ k = 0,1,\ldots,N$.

We begin with the use of the following inequality for the transformation (1.1), which is proved in [13]

$$\int_0^\infty |(KLf)(x)|^2 x^{2\nu-1}dx \le \frac{\pi^{3/2}2^{-2\nu-1}}{\Gamma(2\nu+1/2)}\int_0^\infty |f(\tau)|^2|\Gamma(2\nu+i\tau)|^2 d\tau,\ \nu>0. \tag{3.1}$$

It gives the boundedness for the Kontorovich-Lebedev transformation as an operator $KL: L_2(\mathbf{R}_+; |\Gamma(2\nu+i\tau)|^2 d\tau) \to L_{\nu,2}$. Moreover, when $\nu \to 0+$ it attains equality (1.16) where the measure (see in [1]) $|\Gamma(i\tau)|^2 = \pi\left[\tau \sinh \pi\tau\right]^{-1}$.

Let $f \in C_0^\infty(\mathbf{R}_+)$. Hence since $K_{i\tau}(z)$ is analytic in the right half-plane $\mathrm{Re} z > 0$ (cf. in (1.7)) and integral (1.1) is uniformly convergent on every compact set of $\mathbf{R}_+$, we may repeatedly differentiate under the integral sign to obtain

$$A_x^k KLf = \int_0^\infty A_x^k K_{i\tau}(x) f(\tau)d\tau = \int_0^\infty \tau^{2k} K_{i\tau}(x) f(\tau) d\tau,\ k=0,1,\dots,N. \tag{3.2}$$

Invoking with (3.1), (1.18) we deduce

$$||KLf||_{S_2^{N,\alpha}(\mathbf{R}_+)} = \left(\sum_{k=0}^N \int_0^\infty |A_x^k KLf|^2 x^{2\alpha_k-1}dx\right)^{1/2} \le \left(\int_0^\infty |f(\tau)|^2 \omega_\alpha(\tau)d\tau\right)^{1/2}$$

$$= ||f||_{L_2(\mathbf{R}_+;\omega_\alpha(\tau)d\tau)}, \tag{3.3}$$

where we denoted by

$$\omega_\alpha(\tau) = \pi^{3/2}\sum_{k=0}^N \frac{2^{-2\alpha_k-1}\tau^{4k}|\Gamma(2\alpha_k+i\tau)|^2}{\Gamma(2\alpha_k+1/2)},\ \alpha_k>0,\ k=0,1,\dots,N. \tag{3.4}$$

By virtue of the density of the $C_0^\infty(\mathbf{R}_+)$-functions in L_2 over the measure (3.4) we get that inequality (3.3) is true for any $f(\tau) \in L_2(\mathbf{R}_+;\omega_\alpha(\tau)d\tau)$. The Kontorovich-Lebedev transformation (1.1) in the space $S_2^{N,\alpha}(\mathbf{R}_+)$ we define as follows. Denoting by

$$f_n(\tau) = \begin{cases} f(\tau), & \text{if } \tau \in \left[\frac{1}{n}, n\right], \\ 0, & \text{if } \tau \notin \left[\frac{1}{n}, n\right], \end{cases}$$

we easily see that $||f - f_n||_{L_2(\mathbf{R}_+;\omega_\alpha(\tau)d\tau)} \to 0$, when $n \to \infty$. But with the asymptotic formula (1.6) and Schwarz's inequality we find that integral (1.1) for (KLf_n) exists as a Lebesgue integral for any n. Moreover, from (3.3) we have

$$||KLf_n - KLf_m||_{S_2^{N,\alpha}(\mathbf{R}_+)} \le ||f_n - f_m||_{L_2(\mathbf{R}_+;\omega_\alpha(\tau)d\tau)} \to 0, n, m \to \infty.$$

Therefore the sequence $\{KLf_n\}$ converges in the space $S_2^{N,\alpha}(\mathbf{R}_+)$ and the corresponding integral (1.1) is understood as a limit

$$(KLf)(x) = \lim_{n\to\infty} \int_{1/n}^{n} K_{i\tau}(x) f(\tau) d\tau \tag{3.5}$$

with respect to the norm (1.18). Thus we obtain that the Kontorovich-Lebedev transformation (3.5) is a bounded operator $KL : L_2(\mathbf{R}_+; \omega_\alpha(\tau) d\tau) \to S_2^{N,\alpha}(\mathbf{R}_+)$, where the weighted function $\omega_\alpha(\tau)$ is given by (3.4).

In the case $\alpha = 0$ we can prove the Plancherel type theorem, which will establish an isometric isomorphism between the corresponding L_2- spaces. Indeed, in this case we easily have from (3.4) that

$$\omega_0(\tau) = \frac{\pi^2}{2} \frac{1-\tau^{4(N+1)}}{(1-\tau^4)\tau \sinh \pi\tau}. \tag{3.6}$$

Theorem 3. *Let $f \in L_2(\mathbf{R}_+; \omega_0(\tau) d\tau)$, where the weighted function ω_0 is defined by (3.6). Then the integral (3.5) for the Kontorovich-Lebedev transform converges to $(KLf)(x)$ with respect to the norm in the space $S_2^{N,0}(\mathbf{R}_+)$; and*

$$f_n(\tau) = \frac{2}{\pi^2} \tau \sinh \pi\tau \int_{1/n}^{n} K_{i\tau}(x) (KLf)(x) \frac{dx}{x} \tag{3.7}$$

converges in mean to $f(\tau)$ with respect to the norm in $L_2(\mathbf{R}_+; \omega_0(\tau) d\tau)$. Moreover, the following Parseval equality is true

$$\sum_{k=0}^{N} \int_0^\infty A_x^k KLf \; \overline{A_x^k KLg} \frac{dx}{x} = \frac{\pi^2}{2} \int_0^\infty f(\tau) \overline{g(\tau)} \; \frac{1-\tau^{4(N+1)}}{1-\tau^4} \; \frac{d\tau}{\tau \sinh \pi\tau}, \tag{3.8}$$

where $f, g \in L_2(\mathbf{R}_+; \omega_0(\tau) d\tau)$. In particular,

$$||KLf||_{S_2^{N,0}(\mathbf{R}_+)} = ||f||_{L_2(\mathbf{R}_+; \omega_0(\tau) d\tau)}$$

that is

$$\sum_{k=0}^{N} \int_0^\infty |A_x^k KLf|^2 \frac{dx}{x} = \frac{\pi^2}{2} \int_0^\infty |f(\tau)|^2 \; \frac{1-\tau^{4(N+1)}}{1-\tau^4} \; \frac{d\tau}{\tau \sinh \pi\tau}. \tag{3.9}$$

Finally, for almost all τ and x from $\mathbf{R}_+$ the reciprocal formulas take place

$$(KLf)(x) = \frac{d}{dx} \int_0^\infty \int_0^x K_{i\tau}(y) f(\tau) dy d\tau, \tag{3.10}$$

$$f(\tau) = \frac{2}{\pi^2} \frac{(1-\tau^4)\sinh \pi\tau}{1-\tau^{4(N+1)}} \frac{d}{d\tau} \int_0^\infty \int_0^\tau y K_{iy}(x) \frac{1-y^{4(N+1)}}{1-y^4} \; (KLf)(x) \frac{dy dx}{x}. \tag{3.11}$$

Proof. Let $f \in L_2(\mathbf{R}_+;\omega_0(\tau)d\tau)$. We consider a sequence $\{f_n(\tau)\}$ of $C_0^\infty(\mathbf{R}_+)$-functions, which converges to f. First we find that $\tau^{2k} f_n(\tau) \in C_0^\infty(\mathbf{R}_+)$ for all $k = 0,1,\dots,N$. Hence we invoke (3.2) and we apply the Parseval identity (1.16). As a result we obtain

$$\int_0^\infty |A_x^k KLf_n|^2 \frac{dx}{x} = \frac{\pi^2}{2}\int_0^\infty |f_n(\tau)|^2 \frac{\tau^{4k-1}}{\sinh \pi\tau} d\tau, \ k = 0,1,\dots,N.$$

Making elementary summations we immediately arrive at the equality (3.9). Since $\{(KLf_n)(x)\}$ is a Cauchy sequence in the space $S_2^{N,0}(\mathbf{R}_+)$, then it converges to $(KLf)(x)$ and can be written through the limit (3.5). Moreover passing to the limit we get that (3.9) is true for any $f \in L_2(\mathbf{R}_+;\omega_0(\tau)d\tau)$. Further, taking $x > 0$ we easily have

$$\int_0^x (KLf_n)(y)dy = \int_0^\infty \int_0^x K_{i\tau}(y) f_n(\tau) dy d\tau.$$

Hence we prove that

$$\lim_{n\to\infty} \int_0^x (KLf_n)(y)dy = \int_0^x (KLf)(y)dy = \int_0^\infty \int_0^x K_{i\tau}(y) f(\tau) dy d\tau. \tag{3.12}$$

The latter integral with respect to τ in (3.12) is absolutely convergent and therefore exists in Lebesgue's sense. Indeed, with Schwarz's inequality we derive (cf. in [11], [12], see (1.6))

$$\int_0^\infty \left|\int_0^x K_{i\tau}(y)dy\right| |f(\tau)| d\tau \le ||f||_{L_2(\mathbf{R}_+;\omega_0(\tau)d\tau)} \left(\int_0^\infty \left|\int_0^x K_{i\tau}(y)dy\right|^2 \frac{d\tau}{\omega_0(\tau)}\right)^{1/2} < \infty.$$

Consequently,

$$\left|\int_0^x \left[(KLf_n)(y) - (KLf)(y)\right] dy\right| \le \text{const.} ||f - f_n||_{L_2(\mathbf{R}_+;\omega_0(\tau)d\tau)} \to 0, \ n \to \infty$$

and we prove (3.12). Differentiating with respect to x almost for all $x > 0$ we arrive at (3.10).

Meantime with the parallelogram identity we easily derive (3.9) the Parseval equality (3.8). In particular, putting

$$g(y) = \begin{cases} y, & \text{if } y \in [0,\tau], \\ 0, & \text{if } y \in (\tau,\infty), \end{cases}$$

we find

$$\sum_{k=0}^N \int_0^\infty A_x^k KLf \ A_x^k \int_0^\tau y K_{iy}(x) dy \frac{dx}{x} = \frac{\pi^2}{2}\int_0^\tau f(y) \frac{1 - y^{4(N+1)}}{1 - y^4} \frac{dy}{\sinh \pi y}. \tag{3.13}$$

But the left-hand side of (3.13) can be represented by taking into account (2.1) and the limit equalities (see (1.3), (1.6))

$$\lim_{x\to 0+}\int_0^\tau y^{2k+1}K_{iy}(x)dy = 0,$$

$$\lim_{x\to \infty}\int_0^\tau y^{2k+1}K_{iy}(x)dy = 0,$$

for all $k = 0, 1, \ldots, N$. Thus we obtain

$$\sum_{k=0}^{N}\int_0^\infty A_x^k KLf\ A_x^k \int_0^\tau yK_{iy}(x)dy\frac{dx}{x} = \sum_{k=0}^{N}\int_0^\infty (KLf)(x)\int_0^\tau y^{4k+1}K_{iy}(x)dy\frac{dx}{x}$$

$$= \int_0^\infty (KLf)(x)\int_0^\tau \frac{1-y^{4(N+1)}}{1-y^4}K_{iy}(x)ydy\frac{dx}{x}.$$

Combining with (3.13) and differentiating with respect to τ we arrive at the reciprocal formula (3.11). Finally we prove (3.7). For a sequence $g_n(x) = (KLf)(x),\ x \in [1/n, n], n = 1, 2, \ldots$ of $S_2^{N,0}(\mathbf{R}_+)$- functions, which vanishes outside of the interval $[1/n, n]$ we have

$$f_n(\tau) = \frac{2}{\pi^2}\frac{(1-\tau^4)\sinh \pi\tau}{1-\tau^{4(N+1)}}\frac{d}{d\tau}\int_0^\infty\int_0^\tau yK_{iy}(x)\frac{1-y^{4(N+1)}}{1-y^4}\ g_n(x)\frac{dydx}{x}$$

$$= \frac{2}{\pi^2}\frac{(1-\tau^4)\sinh \pi\tau}{1-\tau^{4(N+1)}}\frac{d}{d\tau}\int_{1/n}^n\int_0^\tau yK_{iy}(x)\frac{1-y^{4(N+1)}}{1-y^4}(KLf)(x)\frac{dydx}{x}. \tag{3.14}$$

Meantime for every n we differentiate under the integral sign in (3.14), which gives

$$f_n(\tau) = \frac{2}{\pi^2}\tau\sinh \pi\tau\int_{1/n}^n K_{i\tau}(x)(KLf)(x)\frac{dx}{x}$$

and it is possible via the uniform convergence with respect to τ of the last integral. If now f is defined by (3.11) then Parseval equality (3.9) implies that

$$||f - f_n||^2_{L_2(\mathbf{R}_+;\omega_0(\tau)d\tau)} = ||KLf - g_n||^2_{S_2^{N,0}(\mathbf{R}_+)} = \sum_{k=0}^{N}\int_n^\infty |A_x^k KLf|^2\frac{dx}{x} \to 0, n\to\infty.$$

Thus we prove (3.7) and we complete the proof of Theorem 3.

4 On the boundedness in $S_p^{N,\alpha}(\mathbf{R}_+),\ p \geq 2$

In this final section we will interpolate the norm of the Kontorovich-Lebedev transformation (1.1) as an operator $KL : L_p(\mathbf{R}_+; \rho_{p,\alpha}(\tau)d\tau) \to S_p^{N,\alpha}(\mathbf{R}_+)$, where $2 \leq p \leq \infty$. The weighted function $\rho_{p,\alpha}(\tau)$ will be indicated below. In the case $p = \infty$ we understand the norm in the space $S_\infty^{N,\alpha}(\mathbf{R}_+)$ as (see (1.18))

$$||u||_{S_\infty^{N,\alpha}(\mathbf{R}_+)} = \lim_{p\to\infty} \left(\sum_{k=0}^{N} \int_0^\infty |A_x^k u|^p x^{\alpha_k p - 1} dx \right)^{1/p}. \tag{4.1}$$

From the equivalence of norms (2.5) we immediately derive that

$$C_1 \sum_{k=0}^{N} ||A_x^k u||_{L_{\alpha_k,\infty}} \leq ||u||_{S_\infty^{N,\alpha}(\mathbf{R}_+)} \leq C_2 \sum_{k=0}^{N} ||A_x^k u||_{L_{\alpha_k,\infty}}, \tag{4.2}$$

where the norm in $L_{\nu,\infty}$ is defined by (see (1.10), (1.11))

$$||f||_{L_{\nu,\infty}} = \text{ess sup}|x^\nu f(x)| = \lim_{p\to\infty} \left(\int_0^\infty |f(x)|^p x^{\nu p - 1} dx \right)^{1/p}. \tag{4.3}$$

We begin to derive an inequality for the modulus of the modified Bessel function $|K_{i\tau}(x)|$. We will apply it below to estimate the $L_{\nu,\infty}$-norm for the $(KLf)(x)$. Indeed, taking the Macdonald formula (1.9) and employing the Schwarz inequality we obtain

$$K_{i\tau}^2(x) = \frac{1}{2} \int_0^\infty e^{-u - \frac{x^2}{2u}} K_{i\tau}(u) \frac{du}{u} \leq \frac{1}{2} \left(\int_0^\infty e^{-2u - \frac{x^2}{u}} u^{-2\nu - 1} du \right)^{1/2}$$

$$\times \left(\int_0^\infty K_{i\tau}^2(u) u^{2\nu - 1} du \right)^{1/2},\ \nu > 0. \tag{4.4}$$

Hence invoking with (1.8) and relation (2.16.33.2) from [6] we calculate the latter product of integrals in (4.4). Thus we get

$$K_{i\tau}^2(x) \leq \pi^{1/4} 2^{(\nu-1)/2} \left(\frac{\Gamma(\nu)}{\Gamma(\nu + 1/2)} \right)^{1/2} |\Gamma(\nu + i\tau)| x^{-\nu} K_{2\nu}^{1/2} \left(2\sqrt{2}x \right),$$

or finally

$$|K_{i\tau}(x)| \leq \pi^{1/8} 2^{(\nu-1)/4} \left(\frac{\Gamma(\nu)}{\Gamma(\nu + 1/2)} \right)^{1/4} |\Gamma(\nu + i\tau)|^{1/2} x^{-\nu/2} K_{2\nu}^{1/4} \left(2\sqrt{2}x \right). \tag{4.5}$$

Invoking with inequality $x^\beta K_\beta(x) \le 2^{\beta-1}\Gamma(\beta), \beta > 0$ (see (1.8)) we find from (1.1), (1.11), (4.5) by straightforward calculations that

$$x^\nu |(KLf)(x)| \le ||f||_\infty \int_0^\infty |K_{i\tau}(x)| d\tau \le 2^{-\nu/2-3/4}\Gamma^{1/2}(\nu)||f||_\infty \int_0^\infty |\Gamma(\nu+i\tau)|^{1/2} d\tau$$
$$= C_\nu ||f||_\infty,$$

where $C_\nu > 0$ is a constant

$$C_\nu = 2^{-\nu/2-3/4}\Gamma^{1/2}(\nu) \int_0^\infty |\Gamma(\nu+i\tau)|^{1/2} d\tau, \quad \nu > 0.$$

Therefore via (4.3) we obtain that the Kontorovich-Lebedev transformation is a bounded operator $KL : L_\infty(\mathbf{R}_+; d\tau) \to L_{\nu,\infty}$ of type (∞,∞) and

$$||KLf||_{L_{\nu,\infty}} \le C_\nu ||f||_\infty. \tag{4.6}$$

But inequality (3.1) says that this operator is of type $(2,2)$ too. Consequently, by the Riesz-Thorin convexity theorem [3] the Kontorovich-Lebedev transformation is of type (p,p), where $2 \le p \le \infty$ i.e. maps the space $L_p(\mathbf{R}_+; |\Gamma(2\nu+i\tau)|^2 d\tau)$ into $L_{\nu,p}$. Moreover for $2 \le p < \infty$ we arrive at the inequality

$$\int_0^\infty |(KLf)(x)|^p \, x^{\nu p-1} dx \le B_{p,\nu} \int_0^\infty |f(\tau)|^p |\Gamma(2\nu+i\tau)|^2 d\tau, \ \nu > 0, \tag{4.7}$$

where we denoted by $B_{p,\nu}$ the constant

$$B_{p,\nu} = \pi^{3/2} 2^{-(p/2+1)\nu-3p/4+1/2} \frac{\Gamma^{p/2-1}(\nu)}{\Gamma(2\nu+1/2)} \left(\int_0^\infty |\Gamma(\nu+i\mu)|^{1/2} d\mu \right)^{p-2}.$$

Hence by the same method as in previous section we prove an analog of the inequality (3.3). Thus we obtain

$$||KLf||_{S_p^{N,\alpha}(\mathbf{R}_+)} \le ||f||_{L_p(\mathbf{R}_+;\rho_{p,\alpha}(\tau)d\tau)}, \tag{4.8}$$

where

$$\rho_{p,\alpha}(\tau) = \sum_{k=0}^N B_{p,\alpha_k} \tau^{4k} |\Gamma(2\alpha_k + i\tau)|^2, \ \alpha_k > 0, k = 0, 1, \ldots, N.$$

In particular, we have $\rho_{2,\alpha}(\tau) = \omega_\alpha(\tau)$ (see (3.4)). So the boundedness of the Kontorovich-Lebedev transformation (1.1) is proved. Finally we show that for all $x > 0$ it exists as a Lebesgue integral for any $f \in L_p(\mathbf{R}_+; \rho_{p,\alpha}(\tau)d\tau), \ p > 2$. Indeed, it will immediately follow from the inequality

$$\int_0^\infty |K_{i\tau}(x) f(\tau)| \, d\tau \le ||f||_{L_p(\mathbf{R}_+;|\Gamma(2\nu+i\tau)|^2 d\tau)}$$

$$\times \left(\int_0^\infty |K_{i\tau}(x)|^q \, |\Gamma(2\nu + i\tau)|^{-2q/p} d\tau \right)^{1/q}, \; q = \frac{p}{p-1},$$

and from the convergence of the latter integral with respect to τ. This is easily seen from (1.6) and the Stirling asymptotic formula for gamma-functions [1] since the integrand behaves as $O\left(e^{\pi\tau q\left(\frac{1}{2}-\frac{1}{p}\right)} \tau^{\frac{q}{p}(1-4\nu)-\frac{q}{2}} \right), \tau \to +\infty$.

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S.B.Yakubovich
Department of Pure Mathematics,
Faculty of Sciences,
University of Porto,
Campo Alegre st., 687
4169-007 Porto
Portugal
E-Mail: syakubov@fc.up.pt