SEMI-COMPLETE FOLIATIONS ASSOCIATED TO HAMILTONIAN VECTOR FIELDS

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ABSTRACT. We classify the foliations associated to hamiltonian vector fields in $\mathbb{C}^2$ with an isolated singularity. In particular we classify the foliations associated to vector fields obtained by the differential equation $\ddot{x} + f(x) = 0$.

1. INTRODUCTION

The definition of a semi-complete vector field relatively to a (relatively compact) open set U is introduced in [6]. The importance of that definition is that:

Proposition 1. [6] *Let X be a complete holomorphic vector field on M. The restriction of X to any connected, (relatively compact) open set U ($U \subseteq M$) is a semi-complete vector field relatively to U.*

Therefore, if a holomorphic vector field in an open set U is not semi-complete it cannot be extended to a compact manifold containing U.

There exist no hamiltonian holomorphic vector fields defined in a compact manifold. However there exist sympletic vector fields defined in compact manifolds (for example, the linear flow in the complex torous). As sympletic vector fields are locally hamiltonian, it is important to study the semi-completude of hamiltonian vector fields in a neighbourhood of an isolated singularity, or better, the foliations associated to hamiltonian vector fields in a neighbourhood of the singularity.

Without loss of generality we can assume that the isolated singularity is the origin.

In this paper we prove the following results:

Theorem. *Let $\mathcal{F}$ be the foliation associated to a non nilpotent hamiltonian vector field $X : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ with an isolated singularity at the origin and such that $J_0^2 X \neq 0$. Let H be the hamiltonian function associated to X. Then $\mathcal{F}$ admits a semi-complete representative, in a neighbourhood of the singularity, if and only if $(\frac{\partial^2 H}{\partial x \partial y}|_0)^2 - \frac{\partial^2 H}{\partial x^2}|_0 \frac{\partial^2 H}{\partial y^2}|_0 \neq 0$ or if H can be written in the form $H = u f_1 f_2 f_3$, where u is a holomorphic function verifying $u(0) \neq 0$ and f_1, f_2 and f_3 are irreducible holomorphic functions such that $f_i(0) = 0$ and $J_0^1 f_i$ is non zero for all $i = 1, 2, 3$, and the level curves of order zero of f_1, f_2 and f_3 have distinct tangents at the origin.*

Theorem. *Let $\mathcal{F}$ be the foliation associated to a nilpotent hamiltonian vector field $X : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ with an isolated singularity at the origin. Then $\mathcal{F}$ admits a semi-complete representative, in a neighbourhood of the singularity, if and only if X is analytically equivalent to $(2y - x^2)\partial/\partial x + 2xy\partial/\partial y$ or to $2y\partial/\partial x - 3x^2\partial/\partial y$.*

The differential equations of type $\ddot{x} + f(x) = 0$ model mechanical and physical problems. A differential equation of that type can be written as a system of differential equations associated to the vector field:

$$(1) \qquad \begin{cases} \dot{x} = y \\ \dot{y} = -f(x) \end{cases}$$

Many problems in physics are defined in compact manifolds. We usually want to know if the solution of the system for a given initial condition is defined for all the time. Therefore, by proposition 1, if the vector field is not semi-complete the solution canot be defined for all the time.

In this artical we also prove:

Theorem. *Let f be a function such that $f(0) = 0$. Then the foliation associated to the holomorphic vector field (1) admits a semi-complete vector field as representative in a neighbourhood of the origin if and only if $f'(0) \neq 0$ or (1) is analytically equivalent to $2y\partial/\partial x - 3x^2\partial/\partial y$.*

Real vector fields are always semi-complete (this is an immediate consequence of a result in [8]). It is obvious that complete vector fields in complex time implies complete in real time which also implies complete in positive time. However:

Theorem. [1] *Let M be a complex manifold such that any bounded plurisubharmonic function on M is constant. If M has a plurisubharmonic exhaustion function then any $\mathbb{R}^+$ complete vector field is $\mathbb{C}$ complete.*

Therefore, if X is not semi-complete then it is not ($\mathbb{C}$) complete neither $\mathbb{R}^+$ nor $\mathbb{R}$ complete.

It is important to remark that $\mathbb{C}^n$ verifies the hypothesis of the theorem.

Definition. A function $s : G \to \mathbb{R} \cup \{\infty\}$, where $G \subseteq \mathbb{C}$, is called subharmonic if s is upper semicontinuous on G and if $D \subseteq G$ is a disk, $h : \bar{D} \to \mathbb{R}$ is continuous, $h|_D$ is harmonic and $h \geq s$ on ∂D, then $h \geq s$ on D.

Definition. A function $p : G \to \mathbb{R} \cup \{\infty\}$, where $G \subseteq \mathbb{C}^n$ is called plurisubharmonic on G if for every tangent vector (a, w) at $a \in G$ the function

$$p_{a,w}(t) = p(a + tw)$$

is subharmonic on the connected component of the set $\alpha_{a,w}^{-1}(G)$ containing $0 \in \mathbb{C}$, where $\alpha_{a,w}(t) = a + tw$.

Finally

Definition. A nonconstant continuous function $f : G \to \mathbb{R}$ is called an exhaustion function if for all $c < \sup_G(f)$ the set

$$\{z \in G : f(z) < c\}$$

is relatively compact in G.

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2. Premilinaries - Definitions and Basic Results

In this section we introduce the definitions and the basic and most important results related and necessary to the problem.

Definition 1. Let X be a holomorphic vector field defined in a complex manifold M. We say that X is complete if there is a holomorphic application

$$\Phi : \mathbb{C} \times M \to M$$

such that

a) $\Phi(0, x) = x \qquad \forall x \in M$

b) $\Phi(T_1 + T_2, x) = \Phi(T_2, \Phi(T_1, x)) \qquad \forall x \in M, \forall T_1, T_2 \in \mathbb{C}$

c) $X(x) = \frac{d}{dT}|_{T=0}\Phi(T, x)$

We can also define $\mathbb{R}$ ($\mathbb{R}^+$) complete vector fields substituing $\mathbb{R}$ ($\mathbb{R}^+$) for $\mathbb{C}$ in the last definition.

Definition 2. Let X be a holomorphic vector field defined in an open set U, $U \subseteq M$, where M is a complex manifold. We say that X is semi-complete relatively to U if there exists a holomorphic application

$$\Phi : \Omega \subseteq \mathbb{C} \times U \to U$$

where Ω is an open set containing $\{0\} \times U$ and such that

a) $X(x) = \frac{d}{dT}|_{T=0}\Phi(T, x)$

b) $\Phi(T_1 + T_2, x) = \Phi(T_2, \Phi(T_1, x))$, when the two members are defined

c) $(T_i, x) \in \Omega$ and $(T_i, x) \to \partial\Omega \quad \Rightarrow \quad \Phi(T_i, x) \to \partial U$

We call Φ the semi-complete flow associated to the vector field X.

In [6] and [7], Rebelo presents sufficient and necessary conditions for a vector field to be semi-complete relatively to an open set U. The regular orbits of a vector field X ($X \not\equiv 0$) are Riemann Surfaces. To each one of its orbits (leafs), L, we can associate a holomorphic differential 1-form, dT_L, such that $dT_L(X) = 1$.

Proposition 2. [7] *Let X be a holomorphic vector field defined in a neighbourhood U of the origin of $\mathbb{C}^n$. Suppose that for all regular orbits L of X and every $c : [0,1] \to L$ such that $c(0) \neq c(1)$ the integral of dT_L over c is non zero. Then the vector field X is semi-complete relatively to U.*

We will only consider the foliations associated to hamiltonian vector fields with an isolated singularity at the origin.We present here the three most important classifications of semi-complete vector fields, in $\mathbb{C}^2$, at an isolated singularity, which will be important to prove our main result.

In [7] it is proved:

Theorem 1. [7] *Let $\mathcal{F}$ be the foliation associated to a holomorphic vector field in $\mathbb{C}^2$ with an isolated singularity at the origin and such that its linear part at the origin is diagonalizable and non singular. Then $\mathcal{F}$ admits a representative which is semi-complete in a neighbourhood of the origin.*

Rebelo and Ghys, in [2], classified the semi-complete vector fields X, in $\mathbb{C}^2$, with an isolated singularity at p and such that $J^1_p X = 0$. They obtained:

Theorem 2. [2] *Let X be a semi-complete vector field defined in a complex surface. Let p be an isolated singularity of X such that $J^1_p X = 0$. Then there exists a neighbourhood U of p such that, in local coordinates, X is analytically conjugated to one of the following vector fields:*

1. $f[x^2 \frac{\partial}{\partial x} - y(nx-(n+1)y)\frac{\partial}{\partial y}]$, *where* $n \in \mathbb{N}_0$
2. $f[x(x-2y)\frac{\partial}{\partial x} + y(y-2x)\frac{\partial}{\partial y}]$
3. $f[x(x-3y)\frac{\partial}{\partial x} + y(y-3x)\frac{\partial}{\partial y}]$
4. $f[x(2x-5y)\frac{\partial}{\partial x} + y(y-4x)\frac{\partial}{\partial y}]$

where f is a holomorphic function in U and such that $f(0,0) \neq 0$.

The singularity can always be assumed to be the origin.

In the same paper, semi-complete nilpotent hamiltonian vector fields, with an isolated singularity, were classified:

Theorem 3. [2] *Let $X : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a nilpotent vector field with an isolated singularity at the origin. If X is semi-complete in some neighbourhood U of the origin then X is analytically conjugated to one of the following vector fields:*

1. $f[(2y-x^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}]$

2. $f[(3y-x^2)\dfrac{\partial}{\partial x}+4xy\dfrac{\partial}{\partial y}]$
3. $f[2y\dfrac{\partial}{\partial x}-3x^2\dfrac{\partial}{\partial y}]$
4. $f[(y-2x^2)\dfrac{\partial}{\partial x}-2xy\dfrac{\partial}{\partial y}]$

where f is a holomorphic function in U and such that $f(0,0)\neq 0$.

It is important to remember that:

Theorem 4. [6] *If X is a holomorphic vector field in $\mathbb{C}^2$ with an isolated singularity at p and such that $J^2_p X = 0$, then X is not semi-complete in any neighbourhood of the singularity.*

3. Hamiltonian Vector Fields

From now on we assume that $X : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is a hamiltonian vector field with an isolated singularity at the origin.

By Theorem 4 it only makes sense to consider foliations defined by holomorphic hamiltonian vector fields satisfying $J^2_0 X \neq 0$. Equivalently, it only makes sense to consider holomorphic hamiltonian functions H such that $J^3_0 H \neq 0$.

Lemma 1. *Let X be a hamiltonian vector field, in $\mathbb{C}^2$, with an isolated singularity at the origin. Then fX is a hamiltonian vector field iff f is a first integral of X.*

Proof. Let X be a hamiltonian vector field in $\mathbb{C}^2$ and $Y = fX$, where f is a holomorphic function. Suppose that Y is also a hamiltonian vector field. Let F and G be the hamiltonian functions associated to X and Y, respectively. Then

$$\frac{\partial G}{\partial y} = f\frac{\partial F}{\partial y}$$

$$\Rightarrow G(x,y) = fF - \int \frac{\partial f}{\partial y} F dy + h(x)$$

$$\Rightarrow \frac{\partial G}{\partial x} = \frac{\partial f}{\partial x}F + f\frac{\partial F}{\partial x} - \int \left(\frac{\partial^2 f}{\partial x \partial y}F + \frac{\partial f}{\partial y}\frac{\partial F}{\partial x}\right) dy + h'(x)$$

Thus Y is hamiltonian iff

$$\frac{\partial f}{\partial x}F - \int \left(\frac{\partial^2 f}{\partial x \partial y}F + \frac{\partial f}{\partial y}\frac{\partial F}{\partial x}\right) dy = -h'(x)$$

Deriving in order to y we obtain

$$\frac{\partial f}{\partial x}\frac{\partial F}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial F}{\partial x} = 0$$

i.e., f is a first integral of X. □

The last Lemma alows us to say that if X is a semi-complete hamiltonian vector field then every hamiltonian vector field with the same foliation is also semi-complete. However, we are interested in foliations associated to hamiltonian vector fields instead of vector fields themselves.

The foliations associated to hamiltonian vector fields have an interesting property which will simplify the classification of foliations defined by hamiltonian vector fields X such that $J_0^1 X = 0$ but $J_0^2 X \neq 0$.

Proposition 3. *Suppose that X_1 and X_2 are two holomorphic vector fields analytically conjugated in a neighbourhood of the origin. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the foliations associated to X_1 and X_2, respectively. Then, if $\mathcal{F}_1$ admits a hamiltonian vector field as its representative, so does $\mathcal{F}_2$. Moroever, if $\mathcal{F}_1$ admits a hamiltonian vector field as its representative, the first non zero jet of X_1 is hamiltonian.*

Proof. As X_1 and X_2 are analytically conjugated in a neighbourhood of the origin, there exists a holomorphic diffeomorphism H such that

$$X_2 = (DH)^{-1}(X_1 \circ H)$$

As $\mathcal{F}_1$ admits a hamiltonian vector field as its representative, there exists a holomorphic function F such that

$$\nabla F . X_1 = 0$$

A hamiltonian vector field whose foliation is $\mathcal{F}_1$ is given by

$$\begin{cases} \dot{x} = \dfrac{\partial F}{\partial y} \\ \dot{y} = -\dfrac{\partial F}{\partial x} \end{cases}$$

Consider the function $G = F \circ H$. Thus

$$\nabla G = ((\nabla F) \circ H).DH$$

and, consequently

$$\begin{aligned} \nabla G . X_2 =& ((\nabla F) \circ H)(DH)(DH)^{-1}.(X_1 \circ H) \\ =& ((\nabla F) \circ H).(X_1 \circ H) \\ =& 0 \end{aligned}$$

This means that G is a first integral of X_2 and the foliation associated to the hamiltonian vector field

$$\begin{cases} \dot{x} = \dfrac{\partial G}{\partial y} \\ \dot{y} = -\dfrac{\partial G}{\partial x} \end{cases}$$

is $\mathcal{F}_2$.

Suppose that $\mathcal{F}_1$ admits a hamiltonian vector field X as its representative. As X and X_1 have the same foliation in a neighbourhood of the origin, there exists a holomorphic function f such that $f(0,0) \neq 0$ and $X_1 = fX$. In particular, the first non zero jet of X_1, $J^k X_1$, is a multiple of the first non zero jet of X, $J^k X$. As X is hamiltonian, so is each of its homogeneous components. In particular, $J^k X$ is hamiltonian and so is $J^k X_1$.

□

This property is important for the classification of foliations associated to hamiltonian vector fields in the following sense: hamiltonian vector fields are not well behaved relatively to conjugation, because if X is a hamiltonian vector field and H a holomorphic diffeomorphism, then $Y = (DH)^{-1}(X \circ H)$ is not in general a hamiltonian vector field. However, the foliation associated to Y admits a hamiltonian vector field as its representative, i.e., there exists a holomorphic function f, such that $f(0,0) \neq 0$ and fY is a hamiltonian vector field.

Lemma 2. *Let X and Y be two holomorphic vector fields analytically conjugated in a neighbourhood of the origin. Then the first non zero jet of X and Y at the origin have the same degree and are analytically conjugated by a linear holomorphic diffeomorphism.*

Proof. Let H be the analytic diffeomorphism conjugating X and Y.

As the linear part of H verifies $|DH(0)| \neq 0$, the first non zero jet of $X \circ H$ has the same degree of the first non zero jet of X and the first non zero jet of $DH.Y$ has the same degree of the first non zero jet of Y. As

$$DH.Y = X \circ H$$

the degrees of the first non zero jet of X and Y, at the origin, are equal. Suppose that this degree is k. Then

$$J_0^k(DH.Y) = J_0^k(X \circ H)$$

or, equivalently

$$DH(0).J_0^k Y = J_0^k X \circ J_0^1 H$$

because, as $J_0^1 H \not\equiv 0$, $J_0^k(DH.Y) = DH(0).J_0^k Y$ and $J_0^k(X \circ H) = J_0^k X \circ J_0^1 H$

As $D(J_0^1 H)(0) = DH(0)$, $J_0^1 H$ is a linear analytic diffeomorphism that conjugating $J_0^k Y$ and $J_0^k X$.

□

Let X be a hamiltonian vector field with an isolated singularity at the origin and such that $J_0^2 X \neq 0$. As X is of type

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial y} \\ \dot{y} = -\dfrac{\partial H}{\partial x} \end{cases}$$

for some holomorphic function H, we can easily verify that the eigenvalues of the matrix associated to its linear part are symmetrical.

If $Y = fX$ for a holomorphic function f ($f(0,0) \neq 0$) and a holomorphic hamiltonian vector field X, Lemma 2 guarantees that the eigenvalues of $DY(0)$ are symmetrical. Consequently, if they are non zero then $DY(0)$ is diagonalizable.

In this way, we have to separate the classification of the hamiltonian vector fields in three distint cases:

a) The eigenvalues are non zero
b) The eigenvalues are both zero, but $J_0^1 X \neq 0$
c) $J_0^1 X = 0$, but $J_0^2 X \neq 0$

3.1. **The case that the eigenvalues are non zero.** In this case we have nothing to do. The existence of a semi-complete representative of the foliation $\mathcal{F}$, associated to a hamiltonian vector field, is an immediate consequence of proposition 1:

Proposition 4. *Let X be a hamiltonian vector field with an isolated singularity at the origin of $\mathbb{C}^2$ and such that $DX(0)$ has non zero eigenvalues. Let $\mathcal{F}$ be the foliation associated to X. Then $\mathcal{F}$ admits a representative that is semi-complete.*

Remark 1. The eigenvalues of $DX(0)$ are given by the solutions of the equation $s^2 - (\frac{\partial^2 H}{\partial x \partial y}|_0)^2 + \frac{\partial^2 H}{\partial x^2}|_0 \frac{\partial^2 H}{\partial y^2}|_0 = 0$. So $DX(0)$ has non zero eigenvalues iff $(\frac{\partial^2 H}{\partial x \partial y}|_0)^2 + \frac{\partial^2 H}{\partial x^2}|_0 \frac{\partial^2 H}{\partial y^2}|_0 \neq 0$.

The representative of $\mathcal{F}$ that is semi-complete does not have necessarily to be a hamiltonian vector field.

Suppose that both eigenvalues are non zero. Then their quocient is equal to -1, i.e., belongs to $\mathbb{R}^-$. Then X admits two separatrices with different tangents at the origin [4]. The next result give us a sufficient condition, based in the holonomy relatively to the separatrices, to the vector field be semi-complete.

Proposition 5. *Let X be a vector field, with an isolated singularity at the origin and such that the eigenvalues of its linear part are non zero. If the holonomy relatively to one of its separatrices is the identity then X is semi-complete.*

Proof. We can suppose that the separatrices are given by the x and y-axis. By hypothesis, X can be written in the form

$$\begin{cases} \dot{x} = \alpha x g(x,y) \\ \dot{y} = -\alpha y h(x,y) \end{cases}$$

where g and h are holomorphic functions such that $g(0,0) \neq 0$ and $h(0,0) \neq 0$.

Suppose that the holonomy relatively to the x-axis is the identity. This means that, for each leaf different from $\{x = 0\} \setminus \{(0,0)\}$, we can write $y = y(x)$.

Let c be a curve such that $\int_c dT_L = 0$ for a given leaf L. Then

$$0 = \int_c \frac{dx}{\alpha x g(x,y)} = \int_{p(c)} \frac{dx}{\alpha x g(x,y(x))}$$

where $p(x,y) = x$. As the 1-dimensional vector field $\alpha x g(x,y(x))\frac{\partial}{\partial x}$ is analytically conjugated to $\alpha x \frac{\partial}{\partial x}$, by a holomorphic diffeomorphism H, we have that

$$\int_{H(p(c))} \frac{dx}{\alpha x} = 0$$

which means that $H(p(c))$, and consequently $p(c)$, are closed and homotopic to a point. In this way, as $y = y(x)$, we have that c is also closed.

By Proposition 2, X is semi-complete in a neighbourhood of the origin. □

As the holonomy is independent of the representative of the foliation we have as an immediate consequence:

Corollary 1. *Let X be a hamiltonian vector field with an isolated singularity at the origin such that $|DX(0)| \neq 0$ and the holonomy relatively to one of its separatrices is the identity. Let $\mathcal{F}$ be the foliation associated to X. Then any representative of $\mathcal{F}$ is a semi-complete vector field in some neighbourhood of the origin.*

3.2. **The case that the eigenvalues are both equal to zero, but $J_0^1 X \neq 0$.** Let X be a hamiltonian holomorphic vector field, with an isolated singularity at the origin, such that the eigenvalues of its linear part are both equal to zero but $J_0^1 X \neq 0$. By a linear change of coordinates, X is analytically conjugated to a vector field of type

$$\begin{cases} \dot{x} = y + f(x,y) \\ \dot{y} = g(x,y) \end{cases} \tag{2}$$

where f and g are holomorphic functions such that $J_0^1 f = 0 = J_0^1 g$, i.e., X is analytically conjugated to a nilpotent vector field.

For foliations associated to hamiltonian nilpotent vector fields we obtain:

Theorem 5. *Let $\mathcal{F}$ be the foliation associated to a nilpotent hamiltonian vector field $X : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ with an isolated singularity at the origin. Then $\mathcal{F}$ admits a semi-complete representative, in a neighbourhood of the singularity, if and only if X is analytically equivalent to $(2y - x^2)\partial/\partial x + 2xy\partial/\partial y$ or to $2y\partial/\partial x - 3x^2\partial/\partial y$.*

Proof. Suppose that X is written in its normal form (2). Then

$$DX(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3}$$

By Theorem 3 we know that gX is a semi-complete vector field, for some holomorphic function g with $g(0,0) \neq 0$ iff X is analytically equivalent to one of the 4 vector fields:

1. $Y_1 = [(2y - x^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}]$
2. $Y_2 = [(3y - x^2)\frac{\partial}{\partial x} + 4xy\frac{\partial}{\partial y}]$
3. $Y_3 = [2y\frac{\partial}{\partial x} - 3x^2\frac{\partial}{\partial y}]$
4. $Y_4 = [(y - 2x^2)\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y}]$

so, it is sufficient to consider hamiltonian vector fields.

As X is a hamiltonian vector field, if X is analytically conjugated to Y_i, for some $i = 1, 2, 3$ or 4, then there exists a holomorphic function g, with $g(0,0) \neq 0$, such that gY_i is a hamiltonian vector field (Proposition 3).

Y_1 and Y_3 are hamiltonioan vector fields with hamiltonian functions $H_1(x,y) = y^2 - x^2y$ and $H_3(x,y) = y^2 - x^3$, respectively. So, if X is analytically equivalent to Y_1 or to Y_3, the foliation associated to X admits a semi-complete hamiltonian vector field as its representative.

The vector field Y_2 is not a hamiltonian vector field. We pretend to know if there exists a holomorphic function g, $g(0,0) \neq 0$, such that gY_2 is a hamiltonian vector field.

The holomorphic function $H_2(x,y) = y(y - x^2)^2$ is a first integral of Y_2 and, if F is another first integral of Y_2 then $F = H(y(y - x^2)^2)$ for some holomorphic function H.

Suppose that there exists a holomorphic function g, with $g(0,0) \neq 0$, such that gY_2 is a hamiltonian vector field. Let K be the hamiltonian function associated to gY_2. As K is also a first integral of Y_2, $K = H(y(y - x^2)^2)$ for some holomorphic function H. Then:

$$\frac{\partial K}{\partial x} = \frac{dH}{dt}|_{t=y(y-x^2)^2}\frac{\partial(y(y-x^2)^2)}{\partial x} = -2xy(y - x^2)\frac{dH}{dt}|_{t=y(y-x^2)^2}$$

and

$$\frac{\partial K}{\partial y} = \frac{dH}{dt}|_{t=y(y-x^2)^2}\frac{\partial(y(y-x^2)^2)}{\partial y} = (y - x^2)(3y - x^2)\frac{dH}{dt}|_{t=y(y-x^2)^2}$$

i.e.,

$$gY_2 = (y - x^2)\frac{dH}{dt}|_{t=y(y-x^2)^2}Y_2$$

contradicting the fact that gY_2 has an isolated singularity. This means that the foliation associated to Y_2 does not addmit a hamiltonian vector field as its representative.

Finally, let us consider the vector field Y_4. Y_4 is also not a hamiltonian vector field. We can easily verify that the curves $y = 0$ and $y = x^2$ are holomorphic invariant curves of the foliation. With this property we prove that $H(x, y) = y^{-2}(y - x^2)$ is a meromorphic first integral of Y_4. In this way all curves in the set $\{y - x^2 - ky^2 = 0 : k \in \mathbb{C}\}$ are separatrices of Y_4. As we have an infinity number of separatrices, the foliation associated to Y_4 does not addmit any hamiltonian vector field as its representative. □

The last Theorem can be expressed in terms of the hamiltonian functions in the following way:

Corollary 2. *The set of holomorphic functions, defined in a neighbourhood of the origin, whose foliation associated to the correspondent hamiltonian vector field addmits a semi-complete representative are given by $F \circ H_1$ or $F \circ H_3$, where F is a holomorphic diffeomorphism in a neighbourhood of the origin.*

3.3. **The case $J_0^1 X = 0$ but $J_0^2 X \neq 0$.** Consider now a hamiltonian vector field X such that $J_0^1 X = 0$ but $J_0^2 X \neq 0$.

Let $Y = fX$ where f is a holomorphic function such that $f(0, 0) \neq 0$. By Theorem 2, Y is semi-complete in some neighbourhood of the origin if and only if it is analytically equivalent to X_i, for some $i = 1, 2, 3$ or 4. As Y is analytically equivalent to a given vector field Z iff so is X, the foliation associated to X admits a semi-complete representative iff all its representatives are semi-complete. So it is sufficient to analyse the hamiltonian vector fields.

Let $X : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a hamiltonian vector field with an isolated singularity at the origin and such that $J_0^1 X = 0$ but $J_0^2 X \neq 0$.

Suppose that X is analytically equivalent to X_i, for some $i = 1, \ldots, 4$. By Proposition 3, there exists a holomorphic function g, with $g(0, 0) \neq 0$, such that gX_i is hamiltonian. In particular, $J_0^2(gX_i)$ is hamiltonian. But $J_0^2(gX_i)$ is a multiple of X_i, and only X_2 is a hamiltonian vector field. Thus X is semi-complete if and only if is analytically equivalent to

$$x(x - 2y)\partial/\partial x + y(y - 2x)\partial/\partial y$$

It allows us to classify all foliations associated to semi-complete hamiltonian vector fields X such that $J_0^1 X = 0$ but $J_0^2 X \neq 0$.

Proposition 6. *Let X be a hamiltonian vector field in $\mathbb{C}^2$ with an isolated singularity at the origin and such that $J_0^1 X = 0$ but $J_0^2 X \neq 0$. Then the foliation associated to X admits a semi-complete vector field as its representative if and only if it has exactly 3 holomorphic separatrices with distinct tangents at the origin.*

Proof. As we said before, fX is semi-complete in some neighbourhood of the origin, for some holomorphic function satisfying $f(0, 0) \neq 0$, iff X is analytically equivalent to X_2.

Let H be a holomorphic diffeomorphism in a neighbourhood of the origin. Then

$$(DH)^{-1}.(fX_2 \circ H) = (f \circ H)((DH)^{-1}(X_2 \circ H))$$

represents all vector fields analitically equivalent to X_2.

The leaves of the vector fields fX_2 ($f(0,0) \neq 0$) are given by the level curves of the hamiltonian function associated to the hamiltonian vector field X_2: $F(x,y) = xy(x-y)$. Their separatrices are given by $\{x=0\}$, $\{y=0\}$ and $\{x=y\}$. So, if X is a semi-complete hamiltonian vector field then X admits 3 holomorphic separatrices with different tangents at the origin.

Suppose now that X admits 3 holomorphic separatrices with different tangents at the origin and let G be the corresponding hamiltonian function. As $\mathbb{C}\{x,y\}$, the set of holomorphic functions in $\mathbb{C}^2$, is a unique factorization domain, G can be written in the form $G = uf_1f_2f_3$, where u is a unit, i.e., verifies $u(0,0) \neq 0$, and f_1, f_2 and f_3 are holomorphic irreducible functions such that $\{f_1 = 0\}$, $\{f_2 = 0\}$ and $\{f_3 = 0\}$ are the separatrices of X.

As the tangents of the separatrices at the origin are distinct, the foliation associated to the homogeneous vector field $J_0^2 X$ has 3 distinct straight lines as separatrices, or equivalently, $J_0^2 X$ has an isolated singularity at the origin: as the tangents at the origin are distinct, the hamiltonian function associated to X is of type $G(x,y) = k(x-\alpha y + g_1(x,y))(x-\beta y + g_2(x,y))(x-\gamma y + g_3(x,y,z))$, up to a change of coordinates, where $k \in \mathbb{C}$, α, β, γ are distinct constants in $\mathbb{C}$ and g_1, g_2 and g_3 are holomorphic function of order greater or equal to 2; then the hamiltonian function associated to $J_0^2 X$ is equal to $k(x-\alpha y)(x-\beta y)(x-\gamma y)$ and, consequently, $J_0^2 X$ has three distinct separatrices and an isolated singularity at the origin.

We can easily prove:

Lemma 3. *Let Y be a homogeneous hamiltonian vector field in $\mathbb{C}^2$ of degree 2. Then Y is analytically equivalent to X_2 iff the origin is an isolated singularity of Y (geometrically, Y has 3 distinct straight lines as separatrices).*

By Lemma 3, we can suppose that $J^1 f_1 = x$, $J^1 f_2 = y$ and $J^1 f_3 = x - y$.

Lemma 4. *There exists holomorphic functions g and h such that $g(0) = 1 = h(0)$ and $gf_1 - hf_2 = f_3$.*

Denote gf_1 and hf_2 by h_1 and h_2, respectively. Then G is written in the form

$$G = vh_1h_2(h_1 - h_2)$$

where $v = \frac{u}{gh}$ is such that $v(0,0) = u(0,0) \neq 0$.

We have that $(w,z) = K(x,y) = (v^{1/3}h_1, v^{1/3}h_2)$ represents a holomorphic diffeomorphism because $DK(0,0) = (v(0,0))^{1/3} Id$. Finally, in these coordinates G is written as

$$G(w,z) = wz(w-z) = F(w,z)$$

which means that X is analytically equivalently to X_2 and, consequently, semi-complete in some neighbourhood of the origin.

□

Corollary 3. *All homogeneous vector fields of degree 2 with an isolated singularity are semi-complete in a neighbourhood of the singularity.*

This result is an immediat consequence of Lemma 3.

Proof of Lemma 3. Let P be the hamiltonian function associated to Y. P is a homogeneous polynomial of degree 3. Any homogeneous polynomial can be decomposed in a product of linear terms: $P(x,y) = (x-\alpha_1 y)(x-\alpha_2 y)(x-\alpha_3 y)$ up to a change of coordinates.

Suppose that $\alpha_i = \alpha_j$ for some $i \neq j$. Then $\partial P/\partial x$ and $\partial P/\partial y$ have $x - \alpha_i y$ as common factor. As the origin is not an isolated singularity, Y is not analytically equivalent to X_2.

Suppose now that $\alpha_i \neq \alpha_j$. In this case, making the change of coordinates

$$\begin{cases} u = x - \alpha_1 y \\ v = x - \alpha_2 y \end{cases}$$

P is written in the form

$$P = uv\left(\frac{\alpha_3-\alpha_2}{\alpha_1-\alpha_2}u + \frac{\alpha_1-\alpha_3}{\alpha_1-\alpha_2}v\right)$$

where, by supposition, $\frac{\alpha_3-\alpha_2}{\alpha_1-\alpha_2}$ and $\frac{\alpha_1-\alpha_3}{\alpha_1-\alpha_2}$ are both well defined and different from zero. Putting $\frac{\alpha_3-\alpha_2}{\alpha_1-\alpha_2}$ in evidence we obtain

$$P = \frac{\alpha_3-\alpha_2}{\alpha_1-\alpha_2}uv(u - \frac{\alpha_3-\alpha_1}{\alpha_3-\alpha_2}v)$$

Making the change of variable

$$\tilde{v} = \frac{\alpha_3-\alpha_1}{\alpha_3-\alpha_2}v$$

P turns into a multiple of $F(x,y) = xy(x-y)$:

$$P = \frac{(\alpha_3-\alpha_2)^2}{(\alpha_1-\alpha_2)(\alpha_3-\alpha_1)}u\tilde{v}(u-\tilde{v})$$

and, without dificult, we can transform this multiple of F into F. It is sufficient to make the change of variables

$$\begin{cases} \bar{u} = \left(\frac{(\alpha_3-\alpha_2)^2}{(\alpha_1-\alpha_2)(\alpha_3-\alpha_1)}\right)^{1/3} u \\ \\ \bar{v} = \left(\frac{(\alpha_3-\alpha_2)^2}{(\alpha_1-\alpha_2)(\alpha_3-\alpha_1)}\right)^{1/3} \tilde{v} \end{cases}$$

The holomorphic diffeomorphism between the foliations associated to Y and X_2 is given by the composition of all those linear transformations. □

Proof of Lemma 4. Let f_1, f_2 and f_3 be such that $J^1 f_1 = x$, $J^1 f_2 = y$ and $J^1 f_3 = x - y$.

By the Weierstrass Preparation Theorem ([3]), f_1, f_2 and f_3 can be written in the form:

$$\begin{aligned} f_1(x,y) &= (x + y^n h_1(y))F_1(x,y) \\ f_2(x,y) &= (y + x^m h_2(x))F_2(x,y) \\ f_3(x,y) &= (x - y h_3(y))F_3(x,y) \end{aligned}$$

where h_i and F_i are holomorphic in a neighbourhood of the origin, for all $i = 1,2,3$, $n, m \geq 2$, $F_1(0,0) = F_2(0,0) = F_3(0,0) = h_3(0) = 1$, $h_1(0) \neq 0$ and $h_2(0) \neq 0$.

We want to prove that there exists holomorphic functions g and h such that $g(0,0) = 1 = h(0,0)$ and $gf_1 - hf_2 = f_3$. This is equivalent to solve the equation:

$$g(x + y^n h_1)F_1 - h(y + x^m h_2)F_2 = (x - y h_3)F_3$$

$$\Leftrightarrow x(gF_1 - x^{m-1} h h_2 F_2) - y(hF_2 - y^{n-1} h_1 g F_1) = xF_3 - y h_3 F_3$$

or, equivalently, the system

$$\begin{cases} gF_1 - x^{m-1} h h_2 F_2 = F_3 \\ hF_2 - y^{n-1} h_1 g F_1 = h_3 F_3 \end{cases}$$

Its solution is given by

$$\begin{cases} g(x,y) = \dfrac{F_3(x,y)(1 + x^{m-1} h_2(x) h_3(y))}{F_1(1 - x^{m-1} y^{n-1} h_1(y) h_2(x))} \\ h(x,y) = \dfrac{F_3(x,y)(h_3(y) + y^{n-1} h_1(y))}{F_2(x,y)(1 - x^{m-1} y^{n-1} h_1(y) h_2(x))} \end{cases}$$

The functions g and h are obviously holomorphic in a neighbourhood of the origin and are such that $g(0,0) = \frac{F_3(0,0)}{F_1(0,0)} = 1$ and $h(0,0) = \frac{F_3(0,0) h_3(0)}{F_2(0,0)} = 1$, as we pretend to prove. □

As Corollary of Propositions 4 and 6 we obtain:

Theorem 6. *Let $\mathcal{F}$ be the foliation associated to a non nilpotent hamiltonian vector field $X : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ with an isolated singularity at the origin and such that $J_0^2 X \neq 0$. Let H be the hamiltonian function associated to X. Then $\mathcal{F}$ admits a semi-complete representative, in a neighbourhood of the singularity, if and only if $(\frac{\partial^2 H}{\partial x \partial y}|_0)^2 - \frac{\partial^2 H}{\partial x^2}|_0 \frac{\partial^2 H}{\partial y^2}|_0 \neq 0$ or if H can be written in the form $H = u f_1 f_2 f_3$, where u is a holomorphic function verifying $u(0) \neq 0$ and f_1, f_2 and f_3 are irreducible holomorphic functions such that $f_i(0) = 0$ and $J_0^1 f_i$ is non zero for all $i = 1,2,3$, and the level curves of order zero of f_1, f_2 and f_3 have distinct tangents at the origin.*

4. The differential equations of type $\ddot{x} + f(x) = 0$

Consider the differential equation given by

$$\ddot{x} + f(x) = 0, \quad x \in \mathbb{C} \tag{4}$$

As we said before, the equation (4) can be written as the differential system of the first order in $\mathbb{C}^2$ (1):

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x) \end{cases}$$

Our objective is to classify the semi-complete vector fields of type (1) in a neighbourhood of the origin. In this way we assume that the origin is a singular point (otherwise the system would be analytically conjugated to the vector field $\frac{\partial}{\partial x}$, in a neighbourhood of the origin, which is obviously semi-complete). On the other hand, $f \not\equiv 0$ guarantees that the origin is an isolated singularity.

Our main result in this section is:

Theorem 7. *Let f be a non zero function such that $f(0) = 0$. Then the foliation associated to the holomorphic vector field (1) admits a semi-complete vector field as representative in a neighbourhood of the origin if and only if $f'(0) \neq 0$ or (1) is analytically equivalent to $2y\partial/\partial x - 3x^2\partial/\partial y$.*

Proof. Let X be the holomorphic vector field

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x) \end{cases}$$

such that $f(0) = 0$ and $f \not\equiv 0$. Then the origin is an isolated singularity.

Suppose that $f'(0) \neq 0$. Then the origin is an isolated singularity of X which can be written in the form:

$$\begin{cases} \dot{x} = y \\ \dot{y} = ax + h(x) \end{cases}$$

where $a \in \mathbb{C} \setminus \{0\}$ and h is a holomorphic function such that $h(0) = 0 = h'(0)$.

As $a \neq 0$ the eigenvalues of $DX(0)$ are non zero and symmetrical: b and $-b$ such that $b^2 = -a$. By Theorem 1 the foliation associated to X admits a semi-complete vector field as its representative.

Suppose now that $f'(0) = 0$. Then X is a nilpotent hamiltonian vector field. The vector field gX is semi-complete in some neighbourhood of the origin, for some holomorphic function g with $g(0, 0) \neq 0$, iff X is analytically equivalent to Y_1 or Y_3 (Theorem 5).

The vector field Y_3 is a hamiltonian vector field of type (1), contrary to Y_1.

If X is analytically equivalent to Y_1 then $\tilde{X}$, the blow-up of X at the singularity, is analytically equivalent to $\tilde{Y_1}$, the blow-up of Y_1. In particular, $J_0^2\tilde{X}$ is analytically conjugated to a multiple of X_3 by a linear change of coordinates ($\tilde{Y_1}$ and X_3 are analytically conjugated homogeneous polynomials of degree 2).

As $h'(0) = 0$, X is of type $X = y\frac{\partial}{\partial x} + x^2p(x)\frac{\partial}{\partial y}$, and so, $\tilde{X}$ is given by:

$$\tilde{X} = tx\frac{\partial}{\partial x} + (xp(x) - t^2)\frac{\partial}{\partial y}$$

If $p(0) \neq 0$, $\tilde{X}$ is still a nilpotente hamiltonian vector field and, consequently, not analytically equivalent to X_3 (Lemma 2).

So let $p(0) = 0$. Then $J_0^2\tilde{X}$ is given by

$$J_0^2\tilde{X} = tx\frac{\partial}{\partial x} + (ax^2 - t^2)\frac{\partial}{\partial y}$$

If $a = 0$ the origin is not an isolated singularity of $J_0^2\tilde{X}$, but as the origin is an isolated singularity of X_3 we conclude that $J_0^2\tilde{X}$ and X_3 are not analytically equivalent in any neighbourhood of the origin.

Suppose that $a \neq 0$. Then

$$\omega_{J_0^2\tilde{X}} = txdt + (t^2 - ax^2)dx$$

If $J_0^2\tilde{X}$ is analytically conjugated to a multiple of X_3 by a holomorphic diffeomorphism $F = (F_1, F_2)$ then $F^*\omega_{J_0^2\tilde{X}} \wedge \omega_{X_3} \equiv 0$. We have:

$$F^*\omega_{J_0^2\tilde{X}} = \left[F_1F_2\frac{\partial F_2}{\partial x} + (F_2^2 - aF_1^2)\frac{\partial F_1}{\partial y}\right]dx + \left[F_1F_2\frac{\partial F_2}{\partial y} + (F_2^2 - aF_1^2)\frac{\partial F_1}{\partial y}\right]dy$$

We take $F(x,t) = (\alpha x + \beta y, \gamma x + \eta y)$. All solutions, except one, of $F^*\omega_{J_0^2\tilde{X}} \wedge \omega_{X_3} \equiv 0$ verifies $\alpha\eta - \beta\gamma = 0$, which means that H is not a diffeomorphism.

The solution not verifying $\alpha\eta - \beta\gamma = 0$ is given by:

$$\{\gamma = \eta, a = 2\eta^2/\beta^2, \alpha = -\beta\}$$

Solving in order to η we obtain:

$$\{\gamma = \eta, \beta = \sqrt{\frac{2}{a}}\eta, \alpha = -\sqrt{\frac{2}{a}}\eta\} \cup \{\gamma = \eta, \beta = -\sqrt{\frac{2}{a}}\eta, \alpha = \sqrt{\frac{2}{a}}\eta\}$$

The determinant of DH is equal to $\pm 2\sqrt{\frac{2}{a}}\eta$ and so not equal to zero if we take $\eta \neq 0$.

As $F^*\omega_{J_0^2\tilde{X}} \wedge \omega_{X_3} \equiv 0$ is a necessary but not a sufficient condition for the equivalence of vector fields we now have to test if there really can exists conjugation between $J_0^2\tilde{X}$ and a multiple of X_3.

We calculate the expression of $(DH)^{-1}(J_0^2\tilde{X} \circ H)$, which has also a big expression, and we can easily verify that $(DH)^{-1}(J_0^2\tilde{X} \circ H) = rX_3$, for some $r \in \mathbb{C}$, implies that $\eta = 0$. As H is not a diffeomorphism, $J_0^2\tilde{X}$ is not analytically equivalent to X_3 and, consequently, X is not semi-complete in any neighbourhood of the origin. □

We should make an important remark. When we blow-up a vector field we shoud take two charts: $y = tx$ and $x = uy$. In the last proof we took only one chart. There is one reason for this: it is easy to verify that if $J_0^1 X = y\frac{\partial}{\partial x}$ then if we consider the variables (u, y), i.e., if we take $x = uy$, the vector field in these new coordinates do not have any singularity along the divisor $\{y = 0\}$.

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