Generalized Fourier transform associated with the differential operator D_z^n in the complex domain

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Abstract

In this paper, we study an integral transform where the kernel is a solution of the nth differential equation in the complex domain $y^{(n)} + \lambda^n y = 0$, n being an arbitrary positive integer. The case $n = 2$ is reduced to the classical Fourier transform. For the case of a real positive argument an inversion formula is established.

Keywords: *Fourier transforms, Mellin transform, Laplace transform, Post-Widder type formula, Mittag-Leffler function, Hypergeometric functions, Meijer G-function, Bessel functions, Gamma-function, Stirling formula*

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1 Introduction

It is well known that the classical Fourier transform is an integral transform where the kernel is a solution of the second order differential equation $L_2y = y'' + \lambda^2 y = 0$. In this paper, we deal with integral transforms where the kernel is a solution of the nth differential equation

$$L_n y = y^{(n)} + \lambda^n y = 0,$$

n being an arbitrary positive integer. Most of the well known properties of the classical Fourier transform will be generalized in a natural way. Our approach is based on the decomposition with respect to the cyclic group of order n of some complex functions. In Section 2, we recall some preliminary results which we need for our analysis.

2 Definitions and basic properties

2.1 Decomposition with respect to the cyclic group of order n

Let n be an arbitrary fixed positive integer and $\omega_n = exp(\frac{2i\pi}{n})$. A set $U \subset \mathbb{C}$ is called $(n-1)$-symmetic iff $\omega_n U = U$. Let $\mathcal{H}(U) = \mathcal{H}$ be the vector space of complex functions defined on a $(n-1)$-symmetric set U. Define an operator $\sigma : \mathcal{H} \longrightarrow \mathcal{H}$ by

$$\sigma f(z) = f(\omega_n z) \qquad \text{for} \qquad f \in \mathcal{H} \quad \text{and} \quad z \in U.$$

Let $\mathcal{H}_{[n,k]},\ k \in \mathbb{N}_n = \{0, 1, \ldots, n-1\}$, be the vectorial subspace in $\mathcal{H}$ defined by the following symmetry property:

$$f \in \mathcal{H}_{[n,k]} \Longleftrightarrow \sigma f = \omega_n^k f. \tag{2.1}$$

For $n = 2$, the subspaces $\mathcal{H}_{[2,0]}$ and $\mathcal{H}_{[2,1]}$ amount, respectively, to subspaces of even functions and odd functions.
The following decomposition in direct sum holds (cf. [13]):

$$\mathcal{H} = \bigoplus_{k=0}^{n-1} \mathcal{H}_{[n,k]}. \tag{2.2}$$

It follows that for any function f belonging to $\mathcal{H}$ there exists a unique sequence $\left(f_{[n,k]}\right)_{k\in\mathbb{N}_n}$, $f_{[n,k]} \in \mathcal{H}_{[n,k]}$, such that

$$f \;=\; \sum_{k=0}^{n-1} f_{[n,k]}, \tag{2.3}$$

and

$$f_{[n,k]}(z) \;=\; \Pi_{[n,k]}(f)(z) \;=\; \frac{1}{n}\sum_{\ell=0}^{n-1}\omega_n^{-k\ell} f(\omega_n^{\ell} z), \quad k\in\mathbb{N}_n, \tag{2.4}$$

where $\Pi_{[n,k]}$ is the projection operator on $\mathcal{H}_{[n,k]}$ along $\mathcal{H}^{\perp}_{[n,k]} \;=\; \bigoplus_{\substack{\ell=0\\ \ell\neq k}}^{n-1} \mathcal{H}_{[n,\ell]}$.
The identity (2.3) is called the decomposition of the function f with respect to the cyclic group $\{\omega_n^k;\ k\in\mathbb{N}_n\}$ and the functions $f_{[n,k]}$ defined by (2.4) will be referred to as the components with respect to the cyclic group of order n of the function f.

2.2 Trigonometric functions of order n

Put $\mu_n = exp\left(\frac{i\pi}{n}\right)$. The decomposition with respect to the cyclic group of order n of the function $z\longrightarrow exp\,(\mu_n z)$ is given by [8] :

$$exp\,(\mu_n z) \;=\; \sum_{k=0}^{n-1}\mu_n^k g_{n,k}(z), \tag{2.5}$$

with

$$g_{n,k}(z) \;=\; \frac{\mu_n^{-k}}{n}\sum_{\ell=0}^{n-1}\omega_n^{-k\ell} exp\left(\omega_n^{\ell}\mu_n z\right).$$

We refer to $g_{n,k}$ as the trigonometric function of order n and k-th kind. As particular cases expressed by elementary functions, we mention:

$$g_{1.0}(z)=\exp(-z), \quad g_{2,0}(z)=\cos z, \quad g_{2,1}(z)=\sin z,$$

$$g_{4,0}(z) \;=\; \cos\frac{z}{\sqrt{2}}\cosh\frac{z}{\sqrt{2}}, \quad g_{4,1}(z) \;=\; \frac{1}{\sqrt{2}}\left(\sin\frac{z}{\sqrt{2}}\cosh\frac{z}{\sqrt{2}}+\cos\frac{z}{\sqrt{2}}\sinh\frac{z}{\sqrt{2}}\right),$$

$$g_{4,2}(z) \;=\; \sin\frac{z}{\sqrt{2}}\sinh\frac{z}{\sqrt{2}}, \quad g_{4,3}(z) \;=\; \frac{1}{\sqrt{2}}\left(\sin\frac{z}{\sqrt{2}}\cosh\frac{z}{\sqrt{2}}-\cos\frac{z}{\sqrt{2}}\sinh\frac{z}{\sqrt{2}}\right).$$

Put $g_{\lambda,n,k}(z) = g_{n,k}(\lambda z)$, $\lambda \in \mathbb{C}$. For further purposes, we need the following properties of the function $g_{\lambda,n,k}$ [8]:

$$g_{\lambda,n,k}(\omega_n^m z) = \omega_n^{km}\, g_{\lambda,n,k}(z),$$

$$g_{\lambda,n,k}(z) = \sum_{m=0}^{\infty} (-1)^m \frac{(\lambda z)^{nm+k}}{(nm+k)!},$$

$$\frac{d^r g_{\lambda,n,k}}{dz^r}(z) = \begin{cases} \lambda^r g_{\lambda,n,k-r}(z), & \text{if } r \le k, \\ -\lambda^r g_{\lambda,n,n+k-r}(z), & \text{if } r > k. \end{cases}$$

The functions $g_{\lambda,n,k}$ satisfy the differential system

$$\begin{cases} \frac{d^n y(z)}{dz^n} + \lambda^n y(z) = 0, \\ \frac{d^r y}{dz^r}(0) = \lambda^r \delta_{r\,k}. \end{cases}$$

We have the product formulas [13]:

$$g_{\lambda,n,0}(x)\, g_{\lambda,n,0}(y) = \theta_y g_{\lambda,n,0}(x) := \frac{1}{n} \sum_{r=0}^{n-1} g_{\lambda,n,0}(x + \omega_n^r y),$$

$$g_{\lambda,n,k}(x)\, g_{\lambda,n,0}(y) = \theta_y g_{\lambda,n,k}(x) := \frac{1}{n} \sum_{r=0}^{n-1} g_{\lambda,n,k}(x + \omega_n^r y).$$

2.3 Decomposition of the dual

Denote by $\mathcal{H}'$ the dual of $\mathcal{H}$ and by $\langle S, f\rangle$ the effect of $S \in \mathcal{H}'$ on $f \in \mathcal{H}$. The elements of $\mathcal{H}'$ are called *linear functionals* . We have also the following decomposition (cf.[1]):

$$\mathcal{H}' = \overset{n-1}{\underset{k=0}{\oplus}} \mathcal{H}'_{[n,k]}, \tag{2.6}$$

with $\mathcal{H}'_{[n,k]}$ the dual of $\mathcal{H}_{[n,k]}$; that is the set of linear functionals S in $\mathcal{H}'$ satisfying: $\langle S, f\rangle = 0$ for all $f \in \mathcal{H}^{\perp}_{[n,k]} = \oplus_{\substack{j=0\\ j\neq k}}^{n-1} \mathcal{H}'_{[n,j]}$. It follows from (2.6) that for every linear functional $S \in \mathcal{H}'$, there exists a unique sequence $\left(S_{[n,k]}\right)_{k\in\{0,1,\cdots,n-1\}}$, $S_{[n,k]} \in \mathcal{H}'_{[n,k]}$, such that $S = \sum_{k=0}^{n-1} S_{[n,k]}$, with $S_{[n,k]} =$

$\Pi'_{[n,k]}(S)$, where $\Pi'_{[n,k]}$ is the projection operator of $\mathcal{H}'$ onto $\mathcal{H}'_{[n,k]}$ along $\mathcal{H}'^{\perp}_{[n,k]}$.
For every linear functional $S \in \mathcal{H}'$ and for every function $f \in \mathcal{H}$, we have [2]:

$$\langle S, f\rangle \;=\; \sum_{k=0}^{n-1} \langle S_{[n,k]}, f_{[n,k]}\rangle, \quad \text{and} \quad \langle \Pi'_{[n,k]} S\,,\, f\rangle \;=\; \langle S\,,\, \Pi_{[n,k]} f\rangle.$$

2.4 Integration on (n-1)-symmetric stars

Some linear functionals are defined by integration with respect to an arbitrary function μ over a $(n-1)$-symmetric curve $\mathcal{C}$ in the complex plane. That are linear functionals of type

$$\langle S, f\rangle \;=\; \int_{\mathcal{C}} f(z)\mu(z)|dz|, \qquad f \in \mathcal{H},$$

where $f\mu$ is Lebesgue-integrable on $\mathcal{C}$.
Next, we consider the particular case of $\mathcal{C}$ consisting of the $(n-1)$-symmetric star with n rays abutting on the origin O:

$$\mathcal{E}_n(R) \;=\; \bigcup_{k=0}^{n-1} \mathcal{E}_{n,k}(R), \quad \text{where} \quad \mathcal{E}_{n,k}(R) \;=\; \left\{ z \in \mathbb{C} \mid z = re^{\frac{2ik\pi}{n}} \quad , \quad 0 \le r \le R \le +\infty \right\}.$$

The stars $\mathcal{E}_1(R)$ and $\mathcal{E}_2(R)$ are, respectively, the intervals $(0, R)$ and $(-R, R)$.
$\mathcal{E}_n$ denotes the star $\mathcal{E}_n(+\infty)$.
To any function μ defined on the star $\mathcal{E}_n(R)$, we associate a linear functional $S(\mu, \mathcal{E}_n(R))$ as follows (cf. [3]):

$$\langle S(\mu, \mathcal{E}_n(R)), f\rangle \;=\; \int_{\mathcal{E}_n(R)} f(z)\,\mu(z)\,|dz| \;=\; \sum_{k=0}^{n-1} \int_0^R f\left(r\omega_n^k\right)\,\mu(r\omega_n^k)\,dr.$$

We assume that the considered integrals are convergent. This definition may be rewritten, using the projection operators (2.4), as

$$\langle S(\mu, \mathcal{E}_n(R)), f\rangle \;=\; n\int_0^R \Pi_{[n,0]}\,(f\cdot\mu)\,(r)\,dr. \tag{2.7}$$

It is easy to verify the identity: $\Pi_{[n,0]}\left(\Pi_{[n,k]}(f)\cdot\mu\right) = \Pi_{[n,0]}\left(f\cdot\Pi_{[n,n-k]}(\mu)\right)$, which, combining with (2.7) and (2.4) provides the components of the linear functional $S(\mu, \mathcal{E}_n(R))$. That are

$$\Pi'_{[n,k]}\left(S(\mu, \mathcal{E}_n(R))\right) \;=\; S\left(\Pi_{[n,n-k]}(\mu), \mathcal{E}_n(R)\right), \quad k \in \mathbb{N}_n. \tag{2.8}$$

3 Fourier Transform

3.1 Definition

Let's consider the nth order differential equation:

$$\frac{d^n}{dz^n}y(z) = -\lambda^n y(z), \quad \lambda \in \mathbb{C}.$$

The solutions of this equation are combinaisons of the functions

$$e(n,k,\lambda;z) = exp\left(\mu_n^{2k+1}\lambda z\right), \ k \in \mathbb{N}_n.$$

Next, we deal with the linear functional $S\left((e(n,k,\lambda;.), \mathcal{E}_n\right)$. Let f be a function defined on $\mathcal{E}_n$. If $\langle S\left((e(n,k,\lambda;.), \mathcal{E}_n\right), f\rangle$ exists, a new function of the new variable λ is obtained. This function (when it exists) is called the *The generalized Fourier transform* of f. That is the function defined by the sum of integrals:

$$\mathcal{F}_{n,k}(f)(\lambda) \ = \langle S\left((e(n,k,\lambda;.), \mathcal{E}_n\right), f\rangle \ = \sum_{k=0}^{n-1} \int_0^\infty f\left(r\omega_n^k\right)\ exp\left(\mu_n^{2k+1}\lambda r\right)\ dr.$$

over that range of values of λ for which the integrals exist.
For $n = 1$, we have the Laplace transform

$$\mathcal{F}_{1,0}(f)(\lambda) = \int_0^\infty f(x)\exp\left(-\lambda x\right)dx$$

For $n = 2$, we have the classical Fourier transform

$$\mathcal{F}_{2,0}(f)(\lambda) = \int_{-\infty}^\infty f(x)\exp\left(i\lambda x\right)dx \quad \text{and} \quad \mathcal{F}_{2,1}(f)(\lambda) = \int_{-\infty}^\infty f(x)\exp\left(-i\lambda x\right)dx.$$

It's easy to verify that $\sigma \circ \mathcal{F}_{n,k} = \mathcal{F}_{n,k+1}$ and $\mathcal{F}_{n,k} \circ \sigma = \mathcal{F}_{n,k-1}$. So, in the sequel, we limit ourselves to the study of $\mathcal{F}_{n,0} = \mathcal{F}_n$.

3.2 Existence of generalized Fourier transforms

It's easy to prove the existence theorem:
Theorem 3.2: *Let f be a complex function and continuous on the $(n-1)$-symmetric star $\mathcal{E}_n$. Assume that there exist two positive constants M and α such that*

$$|f(r\omega_n)| \leq M \exp(-\alpha r) \quad \textit{for} \quad r \longrightarrow +\infty \quad \textit{and for all} \;\; k \in \{0, 1, \ldots, n-1\}.$$

Then $\mathcal{F}_n(f)(\lambda)$ exists for all λ inside the set $\mathcal{P}_n(\alpha)$ given by

$$\mathcal{P}_n(\alpha) \equiv \left\{\lambda \in \mathbb{C} | \mathrm{Re}\left(\mu_n \omega_n^k \lambda\right) - \alpha < 0, \quad k \in \{0, 1, \ldots, n-1\}\right\}.$$

Note that the transform may also exist in other cases. The set $\mathcal{P}_n(\alpha)$ is expressed as follows:
(i) $\mathcal{P}_1(\alpha)$ is the half plane $\Re\lambda > -\alpha$.
(ii) $\mathcal{P}_2(\alpha)$ is the horizontal band $-\alpha < \Im\lambda < \alpha$.
(iii) $\mathcal{P}_n(\alpha)$, $n \geq 3$, is the smallest polygon with n vertices on the star $\mathcal{E}_n$ and containing the disc $D(0, \alpha)$.
The α given by this theorem is not unique. Indeed, every $\alpha_1 < \alpha$ is also suitable. Let α_0 be the sup of the set of positive α such that $\exp(\alpha r)\left|f(r\omega_n^k)\right|$ is bounded for large values of r and for every $k \in \mathbb{N}_n$. α_0 is called the *convergence region* of the function f. It may be equal to the infinity as for the function $f(z) = \exp(-z^n)$, $n > 1$.

3.3 Properties of $\mathcal{F}_n$

Among the properties of the generalized Fourier transform are the following:
P1. If $\lim_{r \longrightarrow +\infty} f\left(r\omega_n^k\right) \exp\left(\lambda r \mu_n^{2k+1}\right) = 0$ for all $k \in \mathbb{N}_n$, then

$$\mathcal{F}_n(f')(\lambda) = \begin{cases} -\mu_n \lambda \mathcal{F}_n(f)(\lambda), & \text{if } n > 1, \\ -f(0) + \lambda \mathcal{F}_1(f)(\lambda), & \text{if } n = 1. \end{cases}$$

More generally,
P2. Let m be a positive integer.
If $\lim_{r \longrightarrow +\infty} f^{(s)}\left(r\omega_n^k\right) \exp\left(\lambda r \mu_n^{2k+1}\right) = 0$ for all $k \in \mathbb{N}_n$ and for all $s \in \{0, 1, \ldots, m\}$ then

$$\mathcal{F}_n(f^{(m)})(\lambda) = \begin{cases} (-\mu_n \lambda)^m \mathcal{F}_n(f)(\lambda), & \text{if} \quad n > 1, \\ \lambda^m \mathcal{F}_1(f)(\lambda) - \sum_{j=0}^{m-1} \lambda^{m-1-j} f^{(j)}(0), & \text{if} \quad n = 1. \end{cases} \tag{3.1}$$

P3. Let m be a non negative integer, we have

$$\frac{d^m}{d\lambda^m}\left(\mathcal{F}_n(f)\right)(\lambda)=\mu_n^m\mathcal{F}_n\left(X^m f\right)(\lambda). \tag{3.2}$$

where the operator X is given by $Xf(z)=zf(z)$.
Put $(Df)(z)=f'(z)$. From (3.1) and (3.2), we deduce, for $m=n$,

$$\mathcal{F}_n\circ D^n=(-1)^{n+1}X^n\circ\mathcal{F}_n, \quad \text{and} \quad D^n\circ\mathcal{F}_n=-\mathcal{F}_n\circ X^n.$$

Put $H_{2p}=-D^{2p}+X^{2p}$. For $p=1$, H_2 is the harmonic oscillator operator. One easily verifies that $\mathcal{F}_{2p}$ commutate with H_{2p}. It follows then, if f is a eigenfunction of the operator H_{2p} associated with the eigenvalue β, the same is true for $\mathcal{F}_{2p}(f)$. The differential equation $H_n(f)=0$ was considered in [5] and [9].
P4. Let a be a positive real. Define the scaling operator S_a by

$$S_a(f)(z)=\frac{1}{\sqrt{a}}f\left(\frac{z}{a}\right) \qquad \text{and} \quad S_a^{-1}(f)(z)=\sqrt{a}f(az).$$

We have $\mathcal{F}_n\circ S_a=S_a^{-1}\circ\mathcal{F}_n$.

3.4 Decomposition of $\mathcal{F}_n$

By virtue of (2.8), The operator $\mathcal{F}_n$ may be written as $\mathcal{F}_n=\sum_{k=0}^{n-1}\left(\mathcal{F}_n\right)_{[n,k]}$, where

$$\left(\mathcal{F}_n\right)_{[n,k]}(f)=\begin{cases}\left\langle S\left(\Pi_{[n,n-k]}e(n,0,\lambda,.)\right),\mathcal{E}_n\right),f\right\rangle, & \text{if } f\in\mathcal{H}_{[n,k]},\\ 0, & \text{if} \quad f\in\mathcal{H}_{[n,k]}^{\perp}.\end{cases}$$

Using the identities (2.1), (2.3), and (2.5), we deduce, for $f\in\mathcal{H}$ and for a suitable $\lambda\in\mathbb{C}$, that

$$\left(\mathcal{F}_n\right)_{[n,k]}(f)(\lambda)=\begin{cases}n\int_0^{+\infty}g_{n,0}(\lambda r)f_{[n,0]}(r)dr & \text{if } k=0,\\ n\mu_n^{n-k}\int_0^{+\infty}g_{n,n-k}(\lambda r)f_{[n,k]}(r)dr & \text{if } k\in\{1,\ldots,n-1\}.\end{cases}$$

It follows then if a function f belongs to $\mathcal{H}_{[n,k]}$, its transform $\left(\mathcal{F}_n\right)_{[n,k]}(f)$ belongs to $\mathcal{H}_{[n,n-k]}$. In the next section, we deal with the operator $\mathcal{G}_n=\frac{1}{n}\left(\mathcal{F}_n\right)_{[n,0]}$ on the subspace $\mathcal{H}_{[n,0]}$, the component of $\mathcal{H}$ which generalizes, in a natural manner, the component of even functions when $n=2$.

4 The integral transform $\mathcal{G}_n$

The integral transform $\mathcal{G}_n$ may also be considered as the generalized Fourier associated with the differential system

$$\begin{cases} \frac{d^n}{dz^n} y(z) + \lambda^n y(z) = 0, \\ y(0) = 1, \\ \frac{d^j}{dz^j} y(0) = 0, \quad j \in \{1, \ldots, n-1\}. \end{cases}$$

4.1 Generalized translation operator

Definition: We call *generalized translation operators associated with* D_z^n the operators θ_u, $u \in \mathbb{C}$, defined in $\mathcal{H}_{[n,0]}$ by

$$\theta_u f(z) = \frac{1}{n} \sum_{\ell=0}^{n-1} f\left(z + \omega_n^\ell u\right), \quad \text{for all } f \in \mathcal{H}_{[n,0]}.$$

For $n = 1$, we have τ_u , $u \in \mathbb{C}$, the classical translation operator associated with the differential operator D, and defined by

$$\tau_u f(z) = f(z + u) \qquad \text{for all } \ f \in \mathcal{H}$$

The operators θ_u and τ_u are linked by the relations:

$$\theta_u = \frac{1}{n} \sum_{k=0}^{n-1} \tau_{\omega_n^k u}, \quad \text{and} \quad \theta_u f(z) = \Big(\tau_z f\Big)_{[n,0]}(u), \quad \text{for all} \quad f \in \mathcal{H}_{[n,0]}.$$

One can easily verify the following proposition :
Proposition 4.1. *Let f be a function in $\mathcal{H}_{[n,0]}$. The operators θ_u, $u \in \mathbb{C}$, satisfy the following properties:*

(i) $\theta_z f \in \mathcal{H}_{[n,0]}$, *(ii)* $\theta_0 = Id.$, *(iii)* $\theta_z f(u) = \theta_u f(z)$, $(u, z) \in \mathbb{C}^2$,

(iv) $\theta_u \theta_z = \theta_z \theta_u$, *(v)* $\theta_u D_z^n = D_z^n \theta_u$.

(vi) We have the product formula : $\theta_u \, g_{n,o}(\lambda z) = g_{n,o}(\lambda u) \, g_{n,o}(\lambda z)$.

(vii) $\mathcal{G}_n(\theta_z f)(\lambda) = g_{n,0}(-\lambda z) \, \mathcal{G}_n(f)(\lambda)$.

(viii) If $h(z) = f(z) g_{n,0}(\lambda z)$, *then* $\mathcal{G}_n(h)(y) = \theta_\lambda \mathcal{G}_n(f)(y)$.

(ix) The function $\nu(z,u) = \theta_u f(z)$, $f \in \mathcal{H}_{[n,0]}$, is a solution of the differential system :

$$D_u^n \nu = \begin{cases} D_z^n \nu, \\ \nu(z,0) = f(z), \\ D_z^j \nu(z,0) = 0, \; j \in \{1,2,\ldots,n-1\}, \end{cases}$$

where the involved integrals exist.

4.2 Generalized convolution

Let f and g be two functions in $\mathcal{H}_{[n,0]}$. The convolution product of f and g is the function

$$f \star g(x) = \int_0^{+\infty} \theta_x f(y) g(y) dy.$$

We have the following properties:

(i) $f \star g \in \mathcal{H}_{[n,0]}$, (ii) $f \star g(x) = g \star f(-x)$, (iii) $\mathcal{G}_n(f \star g)(\lambda) = \mathcal{G}_n(f)(\lambda).\mathcal{G}_n(g)(-\lambda)$.

5 Generalized Fourier cosine and sine transforms

5.1 Definition and examples

A natural generalization of the Fourier cosine and sine transforms consists to consider the integral transform

$$\mathcal{G}_{n,k}(f)(\lambda) = \int_0^{+\infty} f(t) g_{n,k}(\lambda t) dt, \tag{5.1}$$

where f is a complex-valued function of a real positive variable t and λ is a complex number such that the function $t \longrightarrow f(t) g_{n,k}(\lambda t)$ is Lebesgue-integrable over $[0, +\infty[$. Next, we give the generalized Fourier transforms of some exponential functions and truncated functions. Most of them were established in a separate paper.

The Laplace transform of the generalized trigonometric functions is given by (cf. [8] Vol.III p.216 Eq. (32)):

$$\int_0^{+\infty} e^{-st} g_{n,k}(\lambda t) dt = \frac{s^{n-k-1}}{s^n + \lambda^n}, \qquad \mathrm{Re}\, s > |\lambda|.$$

This also provides the generalized Fourier transform of the exponential function. The eigenfunctions of the Fourier cosine transform are given by the integral representation (cf.[7] Vol.I p.15 Eq.(11)):

$$\int_0^\infty e^{-t^2}\cos(xt)\,dt \;=\; \frac{\sqrt{\pi}}{2}\,e^{-\frac{x^2}{4}}, \tag{5.2}$$

This identity was generalized by (cf. [4])

$$\int_0^\infty e^{-t^2}\,g_{2n,0}(xt)\,dt \;=\; \frac{\sqrt{\pi}}{2}g_{n,0}\big(\frac{x^2}{4}\big).$$

More generally, let m and n be two integers such that $0\le m\le k\le 2n-1$ and $k-m\equiv 0(2)$. We have

$$\int_0^\infty e^{-t^2}\,H_m(t)\,g_{2n,k}(xt)\,dt \;=\; \frac{\sqrt{\pi}}{2}\,x^m\,g_{n,\frac{k-m}{2}}\left(\frac{x^2}{4}\right),$$

where H_m designate Hermite polynomials. Another generalization of (5.2) is given by

$$\int_0^\infty e^{-ut^n}\,g_{n,0}(xt)\,dt \;=\; \frac{\Gamma(\frac{1}{n})}{nu^{\frac{1}{n}}}\;{}_0F_{n-2}\left(\begin{matrix}-;\\ \frac{2}{n},\frac{3}{n},\ldots,\frac{n-1}{n};\end{matrix}\;-\frac{1}{u}\left(\frac{x}{n}\right)^n\right),$$

where $n\ge 2,\ u>0$. This is a special case of the identity:

$$\int_0^\infty e^{-ut^n}\,g_{n,k}(xt)\,dt \;=\; \frac{\Gamma\left(\frac{k+1}{n}\right)x^k}{nk!u^{\frac{k+1}{n}}}\;{}_0F_{n-2}\left(\begin{matrix}-;\\ \Lambda(n,k);\end{matrix}\;-\left(\frac{x}{nu^{\frac{1}{n}}}\right)^n\right),\quad u>0,$$

where $\Lambda(n,k)\equiv\{\frac{k+2}{n},\frac{k+3}{n},\ldots,\frac{k+n}{n},\}\setminus\{\frac{n}{n}\}$. Among the well known integral representations of the Bessel functions, recall the Mehler one (see, e.g. [7] Vol.II p.190 Eq.(34)):

$$\int_0^1(1-t^2)^{\nu-\frac{1}{2}}\cos(tx)\,dt,=\frac{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{2\Gamma(\nu+1)}\;{}_0F_1\left(\begin{matrix}-;\\ \nu+1;\end{matrix}\;-\frac{x^2}{4}\right), \tag{5.3}$$

where $\mathrm{Re}\nu>-\frac{1}{2}$. A first generalization of (5.3) is given by

$$\int_0^1(1-t^2)^{\nu-\frac{1}{2}}\,g_{2n,0}(xt)\,dt=\frac{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{2\Gamma(\nu+1)} \tag{5.4}$$

$$
{}_0F_{2n-1}\left(\begin{matrix} -; \\ \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{\nu+1}{n}, \ldots, \frac{\nu+n}{n}; \end{matrix} - \left(\frac{x}{2n}\right)^{2n} \right), \qquad \mathrm{Re}\nu > -\frac{1}{2}.
$$

More generally,

$$
\int_0^1 (1-t^2)^{\nu-\frac{1}{2}}\, g_{2n,2k}(xt)\, dt = \frac{(-1)^k \Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{2k!\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k}
$$

$$
{}_0F_{2n-1}\left(\begin{matrix} -; \\ \Delta^*(n,k+1), \Delta(n,\nu+1); \end{matrix} - \left(\frac{x}{2n}\right)^{2n} \right), \qquad \mathrm{Re}\nu > -\frac{1}{2},
$$

where, for convenience, $\Delta(n,\alpha)$ (resp. $\Delta^*(n,k+1)$) stands for the set of n (resp. $n-1$) parameters $\dfrac{\alpha}{n}$, $\dfrac{\alpha+1}{n} \ldots, \dfrac{\alpha+n-1}{n}$ (resp. $\Delta(n,k+1)\setminus\{\frac{n}{n}\}$). We also state another generalization of (5.3):

$$
\int_0^1 (1-t^n)^{\mu-1}\, t^{j-1} g_{n,0}(xt)\, dt = \frac{\Gamma(\mu)\Gamma(\frac{j}{n})}{n\Gamma(\mu+\frac{j}{n})}
$$

$$
{}_0F_{n-1}\left(\begin{matrix} -; \\ \frac{1}{n}, \ldots, \frac{j}{n}+\mu, \ldots, \frac{n-1}{n}; \end{matrix} - \left(\frac{x}{n}\right)^{n} \right), \quad 1 \le j \le n-1, \quad \mathrm{Re}\mu > 0.
$$

Notice that (5.4) is a particular case of an interesting integral representation given by Dimovski and Kiryakova (cf.[10] p.32 Eq.(15) or [6] p.34 Eq.(8)):

$$
\int_0^1 t^{n-1} G^{n-1\,0}_{n-1\,n-1}\left(t^n \,\middle|\, \begin{matrix} \nu_1, \ldots, \nu_{n-1} \\ -\frac{1}{n}, \ldots, -\frac{(n-1)}{n} \end{matrix} \right) g_{n,0}(xt)\, dt \ ,
$$

$$
= \frac{(2\pi)^{\frac{n-1}{2}}}{n^{\frac{3}{2}} \ \prod_{\ell=1}^{n-1} \Gamma(\nu_\ell+1)} \ {}_0F_{n-1}\left(\begin{matrix} -; \\ \nu_1, \ldots, \nu_{n-1}; \end{matrix} - \left(\frac{x}{n}\right)^{n} \right),
$$

where $G^{m\,n}_{p\,q}\left(z \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \right)$ designates the Meijer's G-function (see, e.g., [11] p.143, [12], [14] for the definition).

5.2 An inversion formula for the generalized Fourier transform of a real positive argument

Let us consider the transformation (5.1) with $\lambda > 0$, $n \in \mathbb{N}$, $k \in \mathbb{N}_n$ (see Section 2). Our goal is to find an inversion formula for the transform (5.1) in some classes of Lebesgue-integrable functions. Indeed, taking a series representation of the kernel $g_{n,k}$ (see subsection 2.2), which is a particular case of the Mittag-Leffler type function [12], [14]

$$g_{n,k}(z) \;=\; \sum_{m=0}^{\infty} (-1)^m \frac{z^{nm+k}}{(nm+k)!} = z^k E_{1/n}\left(-z^n; 1+k\right), \tag{5.5}$$

we substitute it into (5.1) and invert the order of integration and summation provided with the estimate for each $\lambda > 0$

$$\sum_{m=0}^{\infty} \frac{\lambda^{nm+k}}{(nm+k)!} \int_0^{\infty} |f(t)| t^{mn+k} dt < \int_0^{\infty} |f(t)| e^{\lambda t} dt$$

$$< \int_0^1 |f(t)| e^{\lambda t} t^{-\delta} dt + \int_1^{\infty} |f(t)| e^{\lambda t} dt < \infty, \;\; \delta > 0,$$

i.e. under condition $f \in L_1\left((0,1); t^{-\delta} dt\right) \cap L_1\left((1,\infty); e^{\lambda t} dt\right),\;\; \lambda, \delta > 0$. As a result we obtain the series representation of the transformation (5.1), namely

$$\mathcal{G}_{n,k}(f)(\lambda) \;=\; \sum_{m=0}^{\infty} (-1)^m \frac{\lambda^{mn+k} c_{nm+k}}{(nm+k)!}, \tag{5.6}$$

where we denoted by c_i the i-th moment of the function f

$$c_i = \int_0^{\infty} f(t) t^i dt, \;\; i = 0, 1, \ldots .$$

Generally, denoting by $f^*(s)$ the Mellin transform of the function f [12]

$$f^*(s) = \int_0^{\infty} f(t) t^{s-1} dt, \;\; s \in \mathbb{C} \tag{5.7}$$

we easily find that this function is analytic in the half- plane $\mathrm{Res} \geq 1 - \delta, \; \delta > 0$. Indeed, since $f \in L_1\left((0,1); t^{-\delta} dt\right) \cap L_1\left((1,\infty); e^{\lambda t} dt\right)$ we deduce

(5.8)

$$|f^*(s)| \leq \int_0^{\infty} |f(t)| t^{\mathrm{Res}-1} dt < \int_0^1 |f(t)| t^{-\delta} dt + C_\lambda \int_1^{\infty} |f(t)| e^{\lambda t} dt < \infty,$$

where $C_\lambda = \sup_{t>1} \left(e^{-\lambda t} t^{\mathrm{Res}-1}\right), \;\; \lambda > 0, \;\; \mathrm{Re\ s} \geq 1 - \delta$. Therefore, integral (5.7) converges absolutely and uniformly in the half-plane Re $s \geq 1 - \delta$ and represents there an analytic function.

Let us establish from (5.6) the following integral representation of the transform $\mathcal{G}_{n,k}(f)(\lambda)$

$$\mathcal{G}_{n,k}(f)(\lambda) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^*(1+k-ns)\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1+k-ns)}\lambda^{k-ns}ds, \tag{5.9}$$

where $\Gamma(z)$ is Euler's gamma-function [12] and the integration is over a vertical line $(\gamma-i\infty, \gamma+i\infty)$ in the complex plane s with $0<\gamma<\min\left(\frac{k+\delta}{n},\ 1\right)$, $k\in\mathbb{N}_n, n\in\mathbb{N},\ \delta>0$ and f^* satisfies some integrability conditions, which will be examined below.

First we observe that via estimate (5.8) the function $f^*(1+k-ns)$ is analytic in the left half-plane $\mathrm{Re}\ s<\gamma$. We will also require the absolute integrability of the integrand in (5.9). Appealing to the Stirling asymptotic formula for gamma-functions [12] we find

$$\left|\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1+k-ns)}\right| = O\left(e^{\pi(\frac{n}{2}-1)|\mathrm{Im}s|}|\mathrm{Im}s|^{n\gamma-k-1/2}\right),\ |\mathrm{Im}s|\to\infty.$$

Therefore the integrand in (5.9) belongs to $L_1(\gamma-i\infty,\gamma+i\infty)$ iff

$$\int_{\gamma-i\infty}^{\gamma+i\infty}|f^*(1+k-ns)|\,e^{\pi(\frac{n}{2}-1)|s|}|s|^{n\gamma-k-1/2}|ds|<\infty. \tag{5.10}$$

Now we take any simple closed contour L_{-N} containing a segment of the vertical line $\mathrm{Re}\ s=\gamma,\ 0<\gamma<\min\left(\frac{k+\delta}{n},\ 1\right)$, which surrounds only a certain finite number N of the poles $s=-m,\ m=0,1\ldots,N$ of the gamma-function $\Gamma(s)$. Then by the residue theorem integral over L_{-N} is equal to (see (5.6))

$$\frac{1}{2\pi i}\int_{L_{-N}} f^*(1+k-ns)\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1+k-ns)}\lambda^{k-ns}ds = \sum_{m=0}^{N}(-1)^m\frac{\lambda^{mn+k}c_{nm+k}}{(nm+k)!}. \tag{5.11}$$

We let $N\to\infty$ in (5.11), assuming that the contour L_{-N} being extended with the addition of poles $s=-m$ into the left loop $L_{-\infty}$, is nowhere closer to these poles than some small distance $\varepsilon>0$. Then in the limit we obtain

$$\frac{1}{2\pi i}\int_{L_{-\infty}} f^*(1+k-ns)\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1+k-ns)}\lambda^{k-ns}ds$$

$$= \sum_{m=0}^{\infty}(-1)^m\frac{\lambda^{mn+k}c_{nm+k}}{(nm+k)!} = \mathcal{G}_{n,k}(f)(\lambda),\ \lambda>0. \tag{5.12}$$

Our goal now is to prove that the left-hand loop can be expanded to a vertical line $(\gamma - i\infty,\ \gamma + i\infty)$. This can be done by reasoning similar to that in Jordan's lemma. Indeed, we suppose that for all $\lambda > 0$

$$\max_{s\in C_R}\left| f^*(1+k-ns)\frac{\Gamma(s)\,\Gamma(1-s)}{\Gamma(1+k-ns)}\lambda^{-ns}\right| \to 0,\ R\to\infty, \tag{5.13}$$

where C_R is the left-hand semicircle $C_R = \{s = Re^{i\theta},\ 0<\theta<\pi\}$, then

$$\lim_{R\to\infty}\int_{C_R} f^*(1+k-ns)\frac{\Gamma(s)\,\Gamma(1-s)}{\Gamma(1+k-ns)}\lambda^{k-ns}ds = 0,\ \lambda>0$$

and we prove the representation (5.9). Making a simple change of variable and using the reflection formula for gamma-functions [12], [14], it yields

$$\mathcal{G}_{n,k}(f)(\lambda) = \frac{1}{2n\ i}\int_{\nu-i\infty}^{\nu+i\infty}\frac{f^*(s)\lambda^{s-1}}{\Gamma(s)\sin\left(\frac{\pi}{n}(1+k-s)\right)}ds, \tag{5.14}$$

where $\nu = 1+k-n\gamma \in (\max(1-\delta, 1+k-n), 1+k)$. Hence from the properties of the Mellin transform [14] it follows that $\lambda^{n\gamma-k}\mathcal{G}_{n,k}(f)(\lambda)$ is bounded continuous function on $\mathbb{R}_+$ and tends to zero when $\lambda\to 0+,\ \lambda\to+\infty$.

In order to invert transformation (5.14) we are going to appeal to the Mellin transform theory (cf. [12], [14]). But first, assuming the condition

$$2\ \mathrm{sign}(n-2) + \mathrm{sign}\left(n\gamma - k - \frac{3}{2}\right) > 0, l \tag{5.15}$$

it is not difficult to verify due to the Stirling asymptotic formula for gamma-functions that the following integral converges absolutely for each $x>0$ and we get its value, namely

$$\frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty}\Gamma(s)\sin\left(\frac{\pi}{n}(1+k-s)\right)\ x^{-s}ds = \frac{e^{\frac{\pi i}{n}(1+k)}}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty}\Gamma(s)\left(e^{\frac{\pi i}{n}}x\right)^{-s}\frac{ds}{2i}$$

$$-\frac{e^{-\frac{\pi i}{n}(1+k)}}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty}\Gamma(s)\left(e^{-\frac{\pi i}{n}}x\right)^{-s}\frac{ds}{2i} = \frac{1}{2i}\left[\mu_n^{k+1}e^{-\mu_n x} - \mu_n^{-(k+1)}e^{-x/\mu_n}\right]$$

$$= \mathrm{Im}\left[\mu_n^{k+1}e^{-\mu_n x}\right] = e^{-x\cos(\pi/n)}\sin\left(x\sin\frac{\pi}{n} + \frac{\pi(1+k)}{n}\right),\ x>0. \tag{5.16}$$

The latter kernel can be evidently estimated by $e^{-\cos(\pi/n)x}$. When $n = 2$, then via (5.15) and the definition of γ we take $\gamma \in \left(\frac{k+3/2}{2}, \min\left(\frac{k+\delta}{2},\ 1\right)\right)$. This gives the value $k = 0$ and the corresponding kernel (5.16) is equal to $\cos x$.

Let $n = 3, 4, \ldots,\ k \in \mathbb{N}_n$ or $n = 2,\ k = 0$. Then by the reciprocal Mellin transform (5.7) we get from (5.16) the equality

$$\Gamma(s)\sin\left(\frac{\pi}{n}(1+k-s)\right) = \int_0^\infty e^{-x\cos(\pi/n)}\sin\left(x\sin\frac{\pi}{n} + \frac{\pi(1+k)}{n}\right)$$

$$\times x^{s-1}dx,\ \text{Re } s = \nu, \tag{5.17}$$

where the latter integral converges absolutely and uniformly by s for $\nu > 0$. Furthermore by Fubini's theorem with the use of (5.17) and simple change of variables we obtain from (5.14)

$$\int_0^\infty e^{-x\lambda\cos(\pi/n)}\sin\left(x\lambda\sin\frac{\pi}{n} + \frac{\pi(1+k)}{n}\right)\mathcal{G}_{n,k}(f)(\lambda)d\lambda$$

$$= \frac{1}{2n\ i}\int_{\nu-i\infty}^{\nu+i\infty}\frac{f^*(s)}{\Gamma(s)\sin\left(\frac{\pi}{n}(1+k-s)\right)}\int_0^\infty e^{-x\lambda\cos(\pi/n)}$$

$$\times\sin\left(x\lambda\sin\frac{\pi}{n} + \frac{\pi(1+k)}{n}\right)\lambda^{s-1}d\lambda\ ds$$

$$= \frac{1}{2n\ i}\int_{\nu-i\infty}^{\nu+i\infty} f^*(s)x^{-s}ds = \frac{\pi}{n}f(x),\ x > 0.$$

The latter equality is via the inversion theorem for the Mellin transform [12] since $f^*(s) \in L_1(\nu - i\infty,\ \nu + i\infty)$ (see (5.10)).

Consequently, the following inversion formula of the generalized Fourier transform (5.1) is proved

$$f(x) = \frac{n}{\pi}\int_0^\infty e^{-x\lambda\cos(\pi/n)}\sin\left(x\lambda\sin\frac{\pi}{n} + \frac{\pi(1+k)}{n}\right)\mathcal{G}_{n,k}(f)(\lambda)d\lambda, \tag{5.18}$$

where $x > 0,\ k \in \mathbb{N}_n,\ n = 3, 4, \ldots$ or $n = 2,\ k = 0$ and integral (5.18) is absolutely convergent for any $x > 0$. In the latter case we have a familiar inversion formula for the cosine- Fourier transform.

The result is summarized by

Theorem 5.2. *Let* $f(x) \in L_1\left((0,1); x^{-\delta}dx\right) \cap L_1\left((1,\infty); e^{\lambda x}dx\right)$, $\lambda,\ \delta > 0$, *and* $k \in \mathbb{N}_n, n = 3, 4, \ldots,$ *or* $n = 2, k = 0$. *Let the Mellin transform* $f^*(s)$ (5.7) *satisfy integrability condition* (5.10) *and the limit relation* (5.13). *Then inversion formula* (5.18) *for the generalized Fourier transform* (5.1) $\mathcal{G}_{n,k}(f)(\lambda)$ *holds true for each* $x > 0$ *and the corresponding integral converges absolutely.*

We are going to consider now special cases of the transformation (5.1) when $n = 1$ or $n = 2, k \in \mathbb{N}_n$. Let $n = 1$. Returning to the representation (5.9) and using the reduction formula for gamma-functions we deduce

$$\mathcal{G}_{1,k}(f)(\lambda) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^*(1+k-s)\frac{\Gamma(s)}{(1-s)_k}\lambda^{k-s}ds, \tag{5.19}$$

where $(1-s)_k = (1-s)(2-s)\ldots(k-s)$ is the Pochhammer symbol. In this case we can take $0 < \gamma \leq k+\delta,\ \delta > 0$ and the condition (5.10) is satisfied since $f^*(1+k-s)$ is bounded when Re $s \leq k+\delta$. Moreover, we can differentiate k times with respect to λ under the integral sign in (5.19) via the absolute and uniform convergence, and we arrive at the equality

$$\mathcal{G}_{1,k}(f)^{(k)}(\lambda) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^*(1+k-s)\Gamma(s)\,\lambda^{-s}ds. \tag{5.20}$$

But $f(x) \in L_1(\mathbb{R}_+; x^{k-\gamma}dx)$. Therefore, substituting in (5.20) the value of the Mellin transform $f^*(1+k-s)$ and appealing to the Mellin-Parseval equality [14], the latter representation yields

$$\mathcal{G}_{1,k}(f)^{(k)}(\lambda) = \int_0^\infty e^{-\lambda t}t^k f(t)dt,\ \lambda > 0,\ k \in \mathbb{N}_0, \tag{5.21}$$

which is, in turn, the modified Laplace transform. Its inversion can be done in a similar manner as in [14], Section 3.1 arriving at the Post-Widder type formula. In fact, since (5.10) with $n = 1$ does not guarantee the integrability of $f^*(1+k-s)$ we assume this condition to be valid. By using the limit relation for the gamma-function [14]

$$\frac{1}{\Gamma(s)} = \lim_{m\to\infty} sm^{-s}\prod_{j=1}^{m-1}\left(1+\frac{s}{j}\right) \tag{5.22}$$

we return to (5.20) to write the equality

$$\left(-\lambda\frac{d}{d\lambda}\right)\prod_{j=1}^{m-1}\left(1-\frac{\lambda}{j}\frac{d}{d\lambda}\right)\mathcal{G}_{1,k}(f)^{(k)}(m\lambda)$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} s\prod_{j=1}^{m-1}\left(1+\frac{s}{j}\right) f^*(1+k-s)\Gamma(s)(m\lambda)^{-s}ds. \tag{5.23}$$

Passing to the limit when $m \to \infty$ under the integral sign in (5.23) by the dominated convergence theorem and taking into account that $f^*(1+k-s) \in L_1(\gamma - i\infty, \gamma + i\infty)$, we appeal to (5.22) and deduce the inversion Post- Widder type formula for the modified Laplace transformation (5.21). Precisely, we get

$$\lim_{m\to\infty}\left(-\lambda\frac{d}{d\lambda}\right)\prod_{j=1}^{m-1}\left(1-\frac{\lambda}{j}\frac{d}{d\lambda}\right)\mathcal{G}_{1,k}(f)^{(k)}(m\lambda)$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^*(1+k-s)\lambda^{-s}ds = \lambda^{-1-k}f\left(\frac{1}{\lambda}\right).$$

Finally, let us consider the case $n = 2,\ k \in \mathbb{N}_n$. Returning to (5.6) we write it in the form

$$\mathcal{G}_{2,k}(f)(\lambda) \ = \ \sum_{m=0}^{\infty}(-1)^m\frac{\lambda^{2m+k}c_{2m+k}}{(2m)!(2m+1)_k}. \tag{5.24}$$

Assuming that $f(t) \in L_1(\mathbb{R}_+; e^{\lambda t}dt)$, for any $\lambda > 0$ we can differentiate k times with respect to λ in (5.24) to obtain

$$\mathcal{G}_{2,k}(f)^{(k)}(\lambda) \ = \ \sum_{m=0}^{\infty}(-1)^m\frac{\lambda^{2m}c_{2m+k}}{(2m)!} \ = \ \int_0^\infty f(t)t^k\cos(\lambda t)dt, \tag{5.25}$$

where the latter equality is due to the dominated convergence theorem. So we came again to the cosine Fourier transform, which can be inverted accordingly

$$f(t) = \frac{2}{t^k\pi}\int_0^\infty \cos(\lambda t)\ \mathcal{G}_{2,k}^{(k)}(f)(\lambda)d\lambda,\ t > 0. \tag{5.26}$$

For instance, when $k = 1$ it can be reduced to the case of the sine Fourier transform, permitting integration with respect to λ in (5.25) and integration by parts in (5.26).

6 Generalized Fourier transform of linear functionals

The generalized Fourier transform associated with the differential operator D_z^n of a linear functional $S \in \mathcal{H}'_{[n,k]}$ is the function in $\mathcal{H}_{[n,k]}$ defined by:

$$\mathcal{F}_{n,k}(S)(\lambda) \ = \ \langle S_z \, , \, g_{n,k}(\lambda z) \rangle \quad , \quad \lambda \in \mathbb{C}$$

By means of the translation operators θ_z, we define
1. The *translated of a linear functional* in $\mathcal{H}'_{[n,0]}$ as

$$\langle \theta_z \, S \, , \, f \rangle \ = \ \langle S \, , \, \theta_{-z} \, f \rangle \qquad \text{for all} \quad f \in \mathcal{H}_{[n,0]}.$$

2. The *generalized convolution* of *two linear functionals* T and S in $\mathcal{H}'_{[n,0]}$ as the linear functional $S \times T$ in $\mathcal{H}'_{[n,0]}$) given by

$$\langle S \times T \, , \, f \rangle \ = \ \langle S_\xi \, , \, \langle \theta_{-\xi} T \, , \, f \rangle \rangle, \ = \ \langle S_\xi \, , \, \langle T_z \, , \, \theta_z f(\xi) \rangle \rangle, \quad f \in \mathcal{H}_{[n,0]}$$

We state the following
Proposition 6.1: *Let S and T be two linear functionals in $\mathcal{H}'_{[n,0]}$. We have:*

$$\mathcal{F}_{n,0} \left(\theta_z S \right) (\lambda) \ = g_{n,o}(-\lambda z) \ \mathcal{F}_{n,0} \left(S \right) (\lambda)$$

$$\mathcal{F}_{n,0}(S \times T) \ = \ \mathcal{F}_{n,0}(S) \cdot \mathcal{F}_{n,0}(T)$$

Proof:

$$\begin{aligned}
\mathcal{F}_{n,0} \left(\theta_z S \right) (\lambda) &= \langle (\theta_z \, S)_u \, , \, g_{n,o}(\lambda u) \rangle \\
&= \langle S_u \, , \, \theta_{-z} \, g_{n,o}(\lambda u) \rangle \\
&= \langle S_u \, , \, g_{n,o}(\lambda u) \, g_{n,o}(-\lambda z) \rangle \\
&= g_{n,o}(-\lambda z) \, \mathcal{F}_{n,0} \left(S \right) (\lambda)
\end{aligned}$$

For every $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}
\mathcal{F}_{n,0}(S \times T)(\lambda) &= \langle (T \times S)_z \, , \, g_{n,o}(\lambda z) \rangle \\
&= \langle T_z \, , \, \langle S_u \, , \, \theta_u \, g_{n,o}(\lambda z) \rangle \rangle \\
&= \langle T_z \, , \, \langle S_u \, , \, g_{n,o}(\lambda u) \, g_{n,o}(\lambda z) \rangle \rangle \\
&= \langle T_z \, , \, g_{n,o}(\lambda z) \, \langle S_u \, , \, g_{n,o}(\lambda u) \rangle \rangle \\
&= \langle T_z \, , \, g_{n,o}(\lambda z) \rangle \cdot \langle S_u \, , \, g_{n,o}(\lambda u) \rangle \\
&= \mathcal{F}_{n,0}(T)(\lambda) \cdot \mathcal{F}_{n,0}(S)(\lambda)
\end{aligned}$$

□

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