# On periodic points of free inverse monoid endomorphisms

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December 17, 2012

**Abstract**

It is proved that the periodic point submonoid of a free inverse monoid endomorphism is always finitely generated. Using Chomsky's hierarchy of languages, we prove that the fixed point submonoid of an endomorphism of a free inverse monoid can be represented by a context-sensitive language but, in general, it cannot be represented by a context-free language.

## 1 Introduction

The dynamical study of the automorphisms of a free group and their space of ends is a well established subject in discrete Dynamical Systems. Hence it is a natural issue to explore these problems in the more general setting of semigroups. In [3], Cassaigne and the second author studied finiteness conditions for the infinite fixed points of (uniformly continuous) endomorphisms of monoids defined by special confluent rewriting systems, extending results known for free monoids [11]. This line of research was pursued by the second author in subsequent papers [17, 18, 19]. Recently, in a joint paper with Sykiotis [13], we considered the case of graph groups, studying the periodic and fixed points of endomorphisms. Inspired by this work, we studied in [14] the case of endomorphisms of trace monoids and their extensions to real traces.

In this paper we consider the case of endomorphisms of free inverse monoids. Inverse monoids are a natural generalization of groups. Indeed, while groups are represented by bijective functions, inverse semigroups are represented by partial injective functions. Although these structures seem to be close, the behavior of their respective free objects can be radically different. In this paper we provide further evidence to this viewpoint. In fact, we show that the inverse submonoid of the points fixed by an endomorphism of a free inverse monoid is not in general finitely generated as in the group case. However, we can still frame the fixed point submonoid in Chomsky's hierarchy, at the context-sensitive level. A better outcome is achieved for the periodic point submonoid (finitely generated, hence rational) and for some related submonoids called radicals (context-free).

The paper is organized as follows. Section 2 is devoted to preliminaries. In Section 3 we show that the inverse submonoid of periodic points is finitely generated, while the inverse submonoid of fixed points is not in general context-free. This is achieved by proving a gap theorem for endomorphisms whose induced endomorphisms in the free group has no nontrivial fixed points in the boundary. In Section 4, we introduce a family of particular inverse submonoids called radicals, useful in describing the fixed points submonoid. We show that the radical behaves better than the fixed points submonoids by proving that the $N$th radical is context-free for suitable $N$. Using this result we finally frame the fixed point submonoid in the context-sensitive class. Finally, we raise some open problems in the last section.

## 2 Preliminaries

The reader is assumed to have some familiarity with the basics of formal language theory. For references, see [1, 6].

### 2.1 Free groups

Let $A$ be a finite alphabet and let $\tilde{A} = A \cup A^{-1}$ be the *involutive alphabet* where $A^{-1}$ is the set of *formal* inverses of $A$. The operator $^{-1} : A \to A^{-1} : a \mapsto a^{-1}$ is extended to an involution on the free monoid $\widetilde{A}^*$ through

$$1^{-1} = 1, \;\; (a^{-1})^{-1} = a, \;\; (uv)^{-1} = v^{-1}u^{-1} \quad (a \in A;\; u, v \in \widetilde{A}^*).$$

Let $\sim$ be the congruence on $\widetilde{A}^*$ generated by the relation $\{(aa^{-1}, 1) \mid a \in \widetilde{A}\}$. The quotient $F_A = \widetilde{A}^*/\sim$ is the *free group* on $A$, and $\sigma : \widetilde{A}^* \to F_A$ denotes the canonical homomorphism.

We denote by $R_A$ the set of all reduced words on $\widetilde{A}^*$, i.e.

$$R_A = \tilde{A}^* \setminus \bigcup_{a \in \tilde{A}} \tilde{A}^* a a^{-1} \tilde{A}^*$$

For each $u \in \tilde{A}^*$, $\overline{u} \in R_A$ is the (unique) reduced word $\sim$-equivalent to $u$. We write also $\overline{u\sigma} = \overline{u}$. As usual, we often identify the elements of $F_A$ with their reduced representatives. For $B \subseteq \widetilde{A}^*$, $\overline{B}$ denotes the set of reduced words of $B$.

We describe now the *boundary* of $F_A$. Given $u, v \in F_A$, written in reduced form, we denote by $u \wedge v$ the longest common prefix of $u$ and $v$. Let

$$r(u,v) = \begin{cases} |u \wedge v| & \text{if } u \neq v \\ +\infty & \text{otherwise} \end{cases}$$

and define $d(u,v) = 2^{-r(u,v)}$ (using the convention $2^{-\infty} = 0$). Then $d$ is know as the *prefix metric* on $F_A$. In fact, $d$ is an ultrametric since

$$d(u,w) \leq \max\{d(u,v), d(v,w)\}$$

holds for all $u, v, w \in F_A$.

The metric space $(F_A, d)$ is not complete, but its completion admits a simple description: we add to $F_A$ all the (right) infinite reduced words $a_1a_2a_3\ldots$ on $\widetilde{A}$. These new elements are called the *boundary* of $F_A$ and the completion (which is indeed compact) is denoted by $\widehat{F_A}$. The metric on $\widehat{F_A}$ admits the same description as $d$ and is also denoted by $d$.

By standard topology results (see [4, Section XIV.6]), any uniformly continuous endomorphism $\theta$ of $F_A$ admits a (unique) continuous extension $\widehat{\theta} : \widehat{F_A} \to \widehat{F_A}$. Thus, if $(u_n)_n$ is a sequence on $F_A$ converging in $\widehat{F_A}$, then

$$(\lim_{n\to\infty} u_n)\widehat{\theta} = \lim_{n\to\infty} u_n\theta$$

by continuity. The uniformly continuous endomorphisms of $F_A$ turn out to be the monomorphisms. For details on the boundary and endomorphism extensions on a general context (containing free groups as a particular case), the reader is referred to [2].

## 2.2 Free inverse monoids

A monoid $M$ is said to be *inverse* if

$$\forall u \in M \; \exists! u^{-1} \in M : (uu^{-1}u = u \;\wedge\; u^{-1}uu^{-1} = u^{-1}).$$

For details on inverse monoids, see [12].

Let $E(M)$ denote the subset of idempotents of $M$. Then $M$ is inverse if and only if

$$\forall u \in M \; \exists v \in M : uvu = u$$

and $ef = fe$ for all $e, f \in E(M)$. Furthermore, by the well-known Vagner-Preston representation theorem, inverse monoids are represented by partially injective transformations as groups are represented by permutations.

We recall also that there is a multiplicative partial order (the *natural partial order*) defined on $M$ by

$$s \leq t \quad \text{if } s = et \text{ for some } e \in E(M).$$

Let $\rho$ be the congruence on $\widetilde{A}^*$ generated by the relation

$$\{(uu^{-1}u, u) \mid u \in \widetilde{A}^*\} \cup \{(uu^{-1}vv^{-1}, vv^{-1}uu^{-1}) \mid u, v \in \widetilde{A}^*\},$$

known as the *Vagner congruence* on $\widetilde{A}^*$. The quotient $M_A = \widetilde{A}^*/\rho$ is the *free inverse monoid* on $A$, and $\pi : \widetilde{A}^* \to M_A$ denotes the canonical homomorphism. Note that $\pi$ is *matched*, since $u^{-1}\pi = (u\pi)^{-1}$ for every $u \in \widetilde{A}^*$. Moreover, there exists a canonical homomorphism $\sigma' : M_A \to F_A$ such that $\sigma = \pi\sigma'$. In fact, $(u\rho)\sigma' = \overline{u}$ for every $u \in \widetilde{A}^*$.

Two normal forms for $M_A$ were introduced independently by Scheiblich [16] and Munn [10], following respectively an algebraic and a geometric approach. We can make them converge with the help of the *Cayley graph* $C_A$ of $F_A$. Indeed, let $C_A$ have vertex set $F_A$ and edges $g \xrightarrow{a} ga$ for all $g \in F_A$ and $a \in \widetilde{A}$. Note that $C_A$ is an *inverse graph*: $(p, a, q)$ is an edge of $C_A$, so is $(q, a^{-1}, p)$. Such edges are said to be the inverse of each other. It is well known that, if $A \neq \emptyset$, the Cayley graph $C_A$ is an infinite tree (if we view pairs of inverse edges as a single undirected edge).

We can identify $M_A$ with the set $M'_A$ of all ordered pairs $(\Gamma, g)$, where $\Gamma$ is a finite connected inverse subgraph of $C_A$ having both 1 and $g$ as vertices. If we view $\Gamma$ as a finite birooted tree (for roots 1 and $g$), we have the Munn representation. If we focus on its set of vertices (a finite prefix closed subset of $F_A$), we have the Scheiblich representation. Now $F_A$ acts on the left of $C_A$ in the obvious way and the multiplication on $M'_A$ can be given through

$$(\Gamma, g) \cdot (\Gamma', g') = (\Gamma \cup g\Gamma', gg'),$$

and inversion through

$$(\Gamma, g)^{-1} = (g^{-1}\Gamma, g^{-1}).$$

Finally, the homomorphism $\pi$ translates into $\pi' : \widetilde{A}^* \to M'_A$ given by $u\pi = (\mathrm{MT}(u), u\sigma)$, where $\mathrm{MT}(u)$ (the *Munn tree* of $u$) is the finite inverse subgraph of $C_A$ defined by reading the path $1 \xrightarrow{u} g$ in $C_A$. We shall usually identify $u\sigma$ with $\overline{u}$ and $\mathrm{MT}(u)$ with the prefix-closed set $T(u)$ of $R_A$ having as elements the reduced forms of the vertices of $\mathrm{MT}(u)$.

Note that
$$T(uu^{-1}) = T(u)$$
holds for every $u \in \widetilde{A}^*$. Moreover, for all $u, v \in \widetilde{A}^*$, we have

$$u\rho \leq v\rho \qquad \Leftrightarrow \qquad T(u) \supseteq T(v) \ \wedge \ \overline{u} = \overline{v}.$$

Also, if we view $\mathrm{MT}(u)$ as an automaton $\mathcal{A}_u$ with initial vertex 1 and terminal vertex $u$, then

$$L(\mathcal{A}_u) = \{v \in \widetilde{A}^* \mid v \geq u \text{ in } M_A\}.$$

Such basic properties of $M_A$ will be used throughout the paper without further reference.

Finally, we define the norm $||u|| = \max\{|\overline{v}| : v \in T(u)\}$ for every $u \in M_A$.

### 2.3 Chomsky's hierarchy in $M_A$

Following [1], we can define *rational subsets* of $M_A$ with the help of the homomorphism $\pi$: we say that $X \subseteq M_A$ is rational if $X = L\pi$ for some rational $L \subseteq \widetilde{A}^*$. This idea can be extended to higher classes of languages in the Chomsky's hierarchy, for instance we say that $X \subseteq M_A$ is *context-free* (respectively *context-sensitive*) whenever $X = L\pi$ for some context-free (respectively context-sensitive) $L \subseteq \widetilde{A}^*$.

Actually, in the case of rational subsets, the concept is independent from the homomorphism considered, and this property holds for arbitrary monoids.

Anisimov and Seifert's Theorem (see [1]) states that a subgroup of a group $G$ is rational if and only if it is finitely generated. We note that there is an analogous of this theorem for inverse semigroups [9, Lemma 3.6] stating that a closed inverse subsemigroup of an inverse semigroup is rational if and only if it is finitely generated (here closed meaning upper set for the natural partial order).

## 3 Fixed points and periodic points

Given a monoid $M$, we denote by $\mathrm{End}(M)$ the monoid of endomorphisms of $M$. Given $\varphi \in \mathrm{End}(M)$,

$$\mathrm{Fix}(\varphi) = \{x \in M : x\varphi = x\}$$

is the submonoid of *fixed points* of $\varphi$, and

$$\mathrm{Per}(\varphi) = \cup_{n\geq 1} \mathrm{Fix}(\varphi^n)$$

is the submonoid of *periodic points* of $\varphi$ (note that $\mathrm{Fix}(\varphi^m) \subseteq \mathrm{Fix}(\varphi^n)$ whenever $m|n$, hence $(\mathrm{Fix}(\varphi^k))(\mathrm{Fix}(\varphi^\ell)) \subseteq \mathrm{Fix}(\varphi^{k\ell})$ and so $\mathrm{Per}(\varphi)$ is indeed a submonoid). Moreover, if $M$ is an inverse monoid (a group), then both $\mathrm{Fix}(\varphi)$ and $\mathrm{Per}(\varphi)$ are inverse submonoids (subgroups).

Let $\theta \in \mathrm{End}(F_A)$. By [5], $\mathrm{Fix}(\theta)$ is finitely generated. Now, by Nielsen's Theorem (see [8]), every subgroup of a free group is free. Recall that the *rank* of a free group $F$ is the cardinality of a basis of $F$. By [7], every subgroup $\mathrm{Fix}(\theta^n)$ has actually rank $\leq |A|$. According to Takahasi's Theorem [20], every ascending chain of subgroups of bounded rank of $F_A$ must be stationary, and so must be

$$\mathrm{Fix}(\theta^{1!}) \subseteq \mathrm{Fix}(\theta^{2!}) \subseteq \mathrm{Fix}(\theta^{3!}) \subseteq \ldots$$

Since $\mathrm{Per}(\theta) = \cup_{n\geq 1} \mathrm{Fix}(\theta^{n!})$, this provides a proof for the following well-known result:

**Proposition 3.1.** *Let* $\theta \in \mathrm{End}(F_A)$*. Then* $\mathrm{Per}(\theta) = \mathrm{Fix}(\theta^N)$ *for some* $N \geq 1$*.*

The minimum integer $N$ such that $\mathrm{Per}(\theta) = \mathrm{Fix}(\theta^N)$ is called the *curl* of $\theta$ and is denoted by $\mathrm{Curl}(\theta)$.

**Lemma 3.2.** *Let* $\varphi \in \mathrm{End}(M_A)$ *and let* $N = \mathrm{Curl}(\overline{\varphi})$*. Let* $k \geq 1$*. Then* $\mathrm{Curl}(\overline{\varphi}^{Nk}) = 1$*.*

*Proof.* We have

$$\mathrm{Fix}(\overline{\varphi}^{Nk}) \subseteq \mathrm{Per}(\overline{\varphi}^{Nk}) \subseteq \mathrm{Per}(\overline{\varphi}) = \mathrm{Fix}(\overline{\varphi}^{N}) \subseteq \mathrm{Fix}(\overline{\varphi}^{Nk}),$$

hence $\mathrm{Fix}(\overline{\varphi}^{Nk}) = \mathrm{Per}(\overline{\varphi}^{Nk})$ and so $\mathrm{Curl}(\overline{\varphi}^{Nk}) = 1$. □

Assume now that $\varphi \in \mathrm{End}(M_A)$. Then $\varphi$ induces an endomorphism $\overline{\varphi} \in \mathrm{End}(F_A)$ defined by $\overline{\varphi} = (\sigma')^{-1}\varphi\sigma'$. It is straightforward to check that $\overline{\varphi}$ is a well defined homomorphism using the fact that $\varphi$ sends idempotents into idempotents.

**Lemma 3.3.** *Let* $\varphi \in \mathrm{End}(M_A)$ *and let* $u \in M_A$*. Then the following conditions are equivalent:*

*(i)* $u \in \mathrm{Fix}(\varphi)$*;*

*(ii)* $uu^{-1} \in \mathrm{Fix}(\varphi)$ *and* $\overline{u} \in \mathrm{Fix}(\overline{\varphi})$.

*Moreover, in this case we have* $(T(u))\overline{\varphi} \subseteq T(u)$.

*Proof.* (i) $\Rightarrow$ (ii). The first part follows from $\mathrm{Fix}(\varphi)$ being an inverse submonoid of $M_A$. On the other hand, we have

$$\overline{u}\,\overline{\varphi} = \overline{u}(\sigma')^{-1}\varphi\sigma' = u\varphi\sigma' = u\sigma' = \overline{u}.$$

(ii) $\Rightarrow$ (i). We have $\overline{u}\,\overline{u}^{-1} \geq uu^{-1}$, hence $(\overline{u}\,\overline{u}^{-1})\varphi \geq (uu^{-1})\varphi = uu^{-1}$. It follows that $\overline{u}\varphi = e\overline{u}$ for some idempotent $e \geq uu^{-1}$ and so

$$u\varphi = (uu^{-1}\overline{u})\varphi = uu^{-1}e\overline{u} = uu^{-1}\overline{u} = u.$$

Finally, let $v \in T(u)$. Then $vv^{-1} \geq uu^{-1}$ and $uu^{-1} \in \mathrm{Fix}(\varphi)$ yields $(vv^{-1})\varphi \geq (uu^{-1})\varphi = uu^{-1}$. Hence $T(v\varphi) = T((vv^{-1})\varphi) \subseteq T(uu^{-1}) = T(u)$. Thus

$$v\overline{\varphi} = v(\sigma')^{-1}\varphi\sigma' = v\varphi\sigma' \in T(v\varphi) \subseteq T(u)$$

as required. $\square$

Let $\varphi \in \mathrm{End}(M_A)$. An idempotent $e \in E(M_A)$ is said to be $\varphi$-*stable* (or simply stable when the endomorphism $\varphi$ is clear from the context) whenever the *orbit* $\{e\varphi^n \mid n \geq 0\}$ is finite. It follows that $\{e\varphi^n \mid n \geq 0\} = \{e, e\varphi, \ldots, e\varphi^{p-1}\}$ for some $p \geq 1$. We call the smallest such $p$ the *period* of $e$ with respect to $\varphi$. By minimality of $p$, the idempotents $e, e\varphi, e\varphi^2, \ldots, e\varphi^{p-1}$ are all distinct. Finally, we define

$$\mathcal{K}_\varphi(e) = \prod_{i=0}^{p-1} e\varphi^i,$$

$$\mathrm{St}_\varphi = \{a \in \widetilde{A} \mid aa^{-1} \text{ is } \varphi\text{-stable}\}.$$

We show that $\mathrm{St}_\varphi \neq \emptyset$ if there exist nontrivial fixed points, but first we prove a simple lemma. Given $u, v \in \widetilde{A}^*$ and a vertex $p$ of $\mathrm{MT}(u)$, we say that $\mathrm{MT}(v)$ embeds in $\mathrm{MT}(u)$ at $p$ if $\overline{pT(v)} \subseteq T(u)$.

**Lemma 3.4.** *Let* $\varphi \in \mathrm{End}(M_A)$ *and* $u \in M_A$. *Let* $p \xrightarrow{v} q$ *be a path in* $\mathrm{MT}(u)$. *Then* $\mathrm{MT}(v\varphi)$ *embeds in* $\mathrm{MT}(u\varphi)$ *at* $p\overline{\varphi}$.

*Proof.* Considering $p$ in reduced form, there exists a path $1 \xrightarrow{p} p \xrightarrow{v} q \xrightarrow{w} \overline{u}$ in $\mathrm{MT}(u)$ and so $pvw \geq u$ in $M_A$. Hence $(pvw)\varphi \geq u\varphi$ and so $T((pvw)\varphi) \subseteq T(u\varphi)$. It follows that

$$\overline{(p\overline{\varphi})T(v\varphi)} = \overline{(p\varphi)T(v\varphi)} \subseteq T((p\varphi)(v\varphi)(w\varphi)) = T((pvw)\varphi) \subseteq T(u\varphi)$$

as claimed. $\square$

**Proposition 3.5.** *Let $\varphi \in \mathrm{End}(M_A)$ and let $u \in \mathrm{Fix}(M_A)$. If $a \in \widetilde{A}$ labels some edge in* $\mathrm{MT}(u)$*, then $a \in \mathrm{St}_\varphi$.*

*Proof.* Let $p \xrightarrow{a} q$ be an edge of $\mathrm{MT}(u)$. By Lemma 3.4, $\mathrm{MT}((aa^{-1})\varphi^n)$ embeds in $\mathrm{MT}(u\varphi^n) = \mathrm{MT}(u)$ at $p\overline{\varphi^n}$ for every $n \geq 0$. This bounds the size of $\mathrm{MT}((aa^{-1})\varphi^n)$, hence the orbit of $aa^{-1}$ is finite and so $a \in \mathrm{St}_\varphi$. □

We prove yet another simple lemma:

**Lemma 3.6.** *Let $\varphi \in \mathrm{End}(M_A)$ and let $n \geq 1$. Then $\mathrm{St}_{\varphi^n} = \mathrm{St}_\varphi$.*

*Proof.* Let $a \in \mathrm{St}_{\varphi^n}$. Then $(aa^{-1})\varphi^{nk} = (aa^{-1})\varphi^{n(k+p)}$ for some $k \geq 0$ and $p \geq 1$. Hence the $\varphi$-orbit of $aa^{-1}$ is finite and so $a \in \mathrm{St}_\varphi$. Since the $\varphi$-orbit of $aa^{-1}$ contains the $\varphi^n$-orbit, the converse implication follows. □

The set of $\varphi$-tiles is defined by

$$\mathcal{T}_\varphi = \{\mathcal{K}_\varphi(aa^{-1})a \mid a \in \mathrm{St}_\varphi\}.$$

**Theorem 3.7.** *Let $\varphi \in \mathrm{End}(M_A)$ be such that* $\mathrm{Curl}(\overline{\varphi}) = 1$*. Then* $\mathrm{Fix}(\varphi) = \mathcal{T}_\varphi^*$.

*Proof.* Let $a \in St_\varphi$ and $t = \mathcal{K}_\varphi(aa^{-1})a$. Let $p$ denote the period of $aa^{-1}$. Since $aa^{-1} \geq \mathcal{K}_\varphi(aa^{-1})$, we have $tt^{-1} = \mathcal{K}_\varphi(aa^{-1})$. By Lemma 3.4 applied to $\varphi^n$ we know that $\mathrm{MT}(a\varphi^n)$ embeds in $\mathrm{MT}((aa^{-1})\varphi^n)$ and therefore in $\mathrm{MT}(t)$. Thus $a\overline{\varphi^n}$ labels a path in $\mathrm{MT}(t)$ for every $n$. Since $\mathrm{MT}(t)$ is a finite tree, it admits only finitely many paths of reduced label, hence the orbit $\{a\overline{\varphi^n} \mid n \geq 0\}$ must be finite. Since $\overline{\varphi^n} = \overline{\varphi}^n$, it follows that $a \in \mathrm{Per}(\overline{\varphi})$ and so $a \in \mathrm{Fix}(\overline{\varphi})$ since $\mathrm{Curl}(\overline{\varphi}) = 1$. Hence $aa^{-1} \geq (aa^{-1})\varphi$ and so

$$(tt^{-1})\varphi = (\prod_{i=0}^{p-1}(aa^{-1})\varphi^i)\varphi = \prod_{i=1}^{p}(aa^{-1})\varphi^i = \prod_{i=0}^{p-1}(aa^{-1})\varphi^i = tt^{-1}.$$

Since $\overline{t} = a \in \mathrm{Fix}(\overline{\varphi})$, it follows from Lemma 3.3 that $t \in \mathrm{Fix}(\varphi)$. Hence $\mathcal{T}_\varphi \subseteq \mathrm{Fix}(\varphi)$ and so $\mathcal{T}_\varphi^* \subseteq \mathrm{Fix}(\varphi)$.

Conversely, let $u \in \mathrm{Fix}(\varphi)$. Write $u = a_1 \ldots a_m$ with $a_i \in \widetilde{A}$. For $i = 1, \ldots, m$, let $t_i = \mathcal{K}_\varphi(a_i a_i^{-1})a_i$. By Proposition 3.5, we have $a_i \in \mathrm{St}_\varphi$ and so $t_i \in \mathcal{T}_\varphi$ for every $i$.

We show that $u = t_1 \ldots t_m$ in $M_A$. Indeed, by Lemma 3.4, $\mathrm{MT}((a_i a_i^{-1})\varphi^n)$ embeds in $\mathrm{MT}(u\varphi^n) = \mathrm{MT}(u)$ at $(\overline{a_1 \ldots a_{i-1}})\overline{\varphi}^n$ for every $n \geq 0$. Hence the orbit $\{(\overline{a_1 \ldots a_{i-1}})\overline{\varphi}^n \mid n \geq 0\}$ must be finite and since $\mathrm{Curl}(\overline{\varphi}) = 1$ we get $\overline{a_1 \ldots a_{i-1}} \in \mathrm{Per}(\overline{\varphi}) = \mathrm{Fix}(\overline{\varphi})$. Thus $(\overline{a_1 \ldots a_{i-1}})\overline{\varphi}^n = \overline{a_1 \ldots a_{i-1}}$

and it follows easily that $\mathrm{MT}(t_i)$ embeds in $\mathrm{MT}(u)$ at $\overline{t_1 \dots t_{i-1}}$ and so $T(t_1 \dots t_m) \subseteq T(u)$.

Now, since $a_i = \overline{t_i}$, it is not difficult to check that $t_1 ... t_m \leq u$, from which the converse inclusion follows as well. Together with $\overline{t_1 \dots t_m} = \overline{a_1 \dots a_m} = \overline{u}$, this implies $u = t_1 \dots t_m$ in $M_A$. Therefore $\mathrm{Fix}(\varphi) = \mathcal{T}_\varphi^*$. □

We can now solve completely the case of periodic points:

**Theorem 3.8.** *Let* $\varphi \in \mathrm{End}(M_A)$. *Then* $\mathrm{Per}(\varphi)$ *is finitely generated.*

*Proof.* Let $N = \mathrm{Curl}(\overline{\varphi})$. We show that

$$\mathrm{Per}(\varphi) = (\bigcup_{k \geq 1} \mathcal{T}_{\varphi^{Nk}})^*. \tag{1}$$

Indeed, let $u \in \mathrm{Per}(\varphi)$. Then $u = u\varphi^k$ for some $k \geq 1$ and so $u = u\varphi^{Nk}$. By Lemma 3.2, we have $\mathrm{Curl}(\overline{\varphi^{Nk}}) = 1$, hence $u \in \mathrm{Fix}(\varphi^{Nk}) = \mathcal{T}_{\varphi^{Nk}}^*$ by Theorem 3.7.

Conversely, we have $\mathcal{T}_{\varphi^{Nk}} \subseteq \mathrm{Fix}(\varphi^{Nk}) \subseteq \mathrm{Per}(\varphi)$ for every $k$ by Theorem 3.7. Since $\mathrm{Per}(\varphi)$ is a submonoid, (1) holds.

Now it suffices to show that $B = \bigcup_{k \geq 1} \mathcal{T}_{\varphi^{Nk}}$ is finite. Indeed, if $t \in \mathcal{T}_{\varphi^{Nk}}$, then $t = \mathcal{K}_{\varphi^{Nk}}(aa^{-1})a$ for some $a \in \mathrm{St}_{\varphi^{Nk}}$. Since there are only finitely many choices for $a \in \widetilde{A}$ and the $\varphi$-orbit of $aa^{-1}$ is finite for every such $a$, it follows that $B$ is finite and so $\mathrm{Per}(\varphi)$ is finitely generated. □

**Corollary 3.9.** *Let* $\varphi \in \mathrm{End}(M_A)$. *The following conditions are equivalent:*

*(i)* $\mathrm{Fix}(\varphi)$ *is infinite;*

*(ii)* $\mathrm{Per}(\varphi)$ *is infinite;*

*(iii)* $\mathrm{Per}(\varphi) \nsubseteq E(M_A)$.

*Proof.* (i) ⇒ (ii). Trivial.

(ii) ⇒ (i). We build an infinite sequence $(e_n)_n$ of (distinct) elements of $\mathrm{Fix}(\varphi)$ as follows. Let $n \geq 1$ and assume that $e_1, \dots, e_{n-1}$ are already defined. Since $\mathrm{Per}(\varphi)$ is infinite, there exists some $u \in \mathrm{Per}(\varphi)$ such that $||u|| > ||e_i||$ for $i = 1, \dots, n-1$. We have $uu^{-1} \in \mathrm{Per}(\varphi)$, say $(uu^{-1})\varphi^m = uu^{-1}$. Clearly, we can take $e_n = \mathcal{K}_\varphi(uu^{-1}) = \prod_{i=0}^{m-1}(uu^{-1})\varphi^i \in \mathrm{Fix}(\varphi)$. Since $||e_n|| \geq ||u|| > ||e_i||$, then $e_n \neq e_i$ for $i = 1, \dots, n-1$. Thus we build an infinite sequence $(e_n)_n$ and so $\mathrm{Fix}(\varphi)$ is infinite.

(ii) ⇒ (iii). By Theorem 3.8, $\mathrm{Per}(\varphi)$ is finitely generated, and every finitely generated submonoid of $E(M_A)$ is finite.

(iii) $\Rightarrow$ (ii). If $u \in \mathrm{Per}(\varphi) \setminus E(M_A)$, then $u^*$ is an infinite submonoid of $\mathrm{Per}(\varphi)$. □

Given $\varphi \in \mathrm{End}(M_A)$ with $\overline{\varphi}$ injective, we may write $\widehat{\varphi} = \widehat{\overline{\varphi}}$. Recall that we need injectivity to extend $\overline{\varphi}$ to the boundary of $F_A$. If there are no nontrivial fixed points in $\widehat{\varphi}$, a good deal of the hierarchy collapses:

**Theorem 3.10.** *Let $\varphi \in \mathrm{End}(M_A)$ be such that $\overline{\varphi}$ is injective and $\mathrm{Fix}(\widehat{\varphi}) = 1$. Then the following conditions are equivalent:*

*(i)* $\mathrm{Fix}(\varphi)$ *is context-free;*

*(ii)* $\mathrm{Fix}(\varphi)$ *is rational;*

*(iii)* $\mathrm{Fix}(\varphi)$ *is finitely generated;*

*(iv)* $\mathrm{Fix}(\varphi)$ *is finite;*

*(v)* $\mathrm{Per}(\varphi)$ *is finite;*

*(vi)* $\mathrm{Per}(\varphi) \subseteq E(M_A)$.

*Proof.* (i) $\Rightarrow$ (iv). Assume that $\mathrm{Fix}(\varphi) = K\pi$ for some $K \subseteq \widetilde{A}^*$ context-free. By the Pumping Lemma for context-free languages, there exists a constant $p \geq 1$ such that every $w \in K$ of length $> p$ admits a factorization $w = w_1w_2w_3w_4w_5$ such that:

- $|w_2w_3w_4| \leq p$;
- $w_2w_4 \neq 1$;
- $w_1w_2^nw_3w_4^nw_5 \in K$ for every $n \geq 0$.

For every $m \geq 0$, let

$$\begin{aligned}\Lambda_m = \{ \quad & (u,v) \in \mathrm{Fix}(\varphi) \times \mathrm{Fix}(\varphi) \mid u \neq v \text{ and } \exists q \in T(u) \\ & \exists w \in M_A : T(v) = T(u) \cup \overline{qT(w)},\ |q| \geq m,\ ||w|| \leq p\}.\end{aligned}$$

We show that

$$\text{only finitely many } \Lambda_m \text{ are nonempty.} \tag{2}$$

Indeed, suppose that (2) fails. This amounts to say that $\Lambda_m \neq \emptyset$ for every $m \geq 0$. Fix $(u_m, v_m) \in \Lambda_m$ and let $q_m, w_m$ be as in the definition of $\Lambda_m$. Let

$$M = \max\{||a\varphi|| : a \in \widetilde{A}\}.$$

We claim that

$$d(q_m, q_m\overline{\varphi}) < 2^{pM-m}. \tag{3}$$

Indeed, we have $T(v_m) = T(u_m) \cup \overline{q_m T(w_m)}$. Hence

$$v_m v_m^{-1} = u_m u_m^{-1} q_m w_m w_m^{-1} q_m^{-1}$$

and so

$$\begin{array}{rcl} v_m v_m^{-1} & = & (v_m v_m^{-1})\varphi = (u_m u_m^{-1} q_m w_m w_m^{-1} q_m^{-1})\varphi \\ & = & u_m u_m^{-1} (q_m w_m w_m^{-1} q_m^{-1})\varphi. \end{array} \tag{4}$$

Since $q_m q_m^{-1} \geq u_m u_m^{-1}$, we have $(q_m q_m^{-1})\varphi \geq (u_m u_m^{-1})\varphi = u_m u_m^{-1}$ and so $T(q_m\varphi) \subseteq T(u_m)$. Hence (4) yields

$$T(v_m) = T(u_m) \cup \overline{(q_m\varphi)T(w_m\varphi)} = T(u_m) \cup \overline{(q_m\overline{\varphi})T(w_m\varphi)}.$$

Now it is easy to see that $||w_m|| \leq p$ yields $||w_m\varphi|| \leq pM$. Let $a \in \widetilde{A}$ be such that $q_m a \in T(v_m) \setminus T(u_m)$. Then $q_m a = \overline{(q_m\overline{\varphi})z}$ for some reduced word $z$ of length $\leq pM$ and so $q_m\overline{\varphi} = \overline{q_m a z^{-1}}$. Since $|q_m| \geq m$ and $z^{-1}$ can cancel at most $pM$ letters from the reduced word $q_m a$, we get $r(q_m, q_m\overline{\varphi}) > m - pM$. Therefore (3) holds.

Now, since the completion $\widehat{F_A}$ is compact, the sequence $(q_m)_m$ must admit a convergent subsequence $(q_{i_m})_m$. Let $\alpha = \lim_{m\to\infty} q_{i_m}$. We claim that $\alpha \in \mathrm{Fix}(\widehat{\varphi})$. Indeed, by continuity we have $\alpha\widehat{\varphi} = \lim_{m\to\infty} q_{i_m}\overline{\varphi}$. Let $\varepsilon > 0$. Since $\alpha = \lim_{m\to\infty} q_{i_m}$, there exists some $t \geq 1$ such that

$$m \geq t \;\Rightarrow\; d(q_{i_m}, \alpha) < \varepsilon.$$

Moreover, we may assume that $2^{pM-t} < \varepsilon$. Thus, if $m \geq t$, and since $d$ is an ultrametric, we get

$$\begin{array}{rcl} d(q_{i_m}\overline{\varphi}, \alpha) & \leq & \max\{d(q_{i_m}\overline{\varphi}, q_{i_m}), d(q_{i_m}, \alpha)\} < \max\{2^{pM-m}, \varepsilon\} \\ & \leq & \max\{2^{pM-t}, \varepsilon\} < \varepsilon. \end{array}$$

Hence $\alpha\widehat{\varphi} = \alpha$ and so $\alpha \in \mathrm{Fix}(\widehat{\varphi})$. Since $|q_m| \geq m$ for every $m$, we have $\alpha \neq 1$, a contradiction. Therefore (2) holds.

Suppose now that $\mathrm{Fix}(\varphi)$ is infinite. Since $\mathrm{Fix}(\overline{\varphi}) = 1$, it follows from Lemma 3.3 that $\mathrm{Fix}(\varphi) \subseteq E(M_A)$. Let $m > p$. Since $\mathrm{Fix}(\varphi)$ is infinite, there exists some $w \in K$ with $||w|| \geq m$. We may assume that $w$ has minimal length among all such words.

Since $|w| \geq ||w|| \geq m > p$, there exists a factorization $w = w_1 w_2 w_3 w_4 w_5$ such that:

- $|w_2 w_3 w_4| \leq p$;

- $w_2w_4 \neq 1$;
- $w_1w_2^nw_3w_4^nw_5 \in K$ for every $n \geq 0$.

Let $u = w_1w_3w_5 \in K$. By minimality of $w$, we have $||u|| < m$, hence $u \neq uw$ in $M_A$. We claim that

$$(u, uw) \in \Lambda_{m-p}.$$

First of all, we note that $\overline{u} = 1 = \overline{w}$ yields $\overline{w_2w_3w_4} = \overline{w_3}$. There exists a path

$$1 \xrightarrow{w_1} \overline{w_1} \xrightarrow{w_3} \overline{w_1w_3} \xrightarrow{w_5} 1$$

in $\mathrm{MT}(u)$. We have

$$T(u) = T(w_1) \cup \overline{w_1T(w_3)} \cup \overline{w_1w_3T(w_5)},$$

$$T(uw) = T(u) \cup T(w_1) \cup \overline{w_1T(w_2w_3w_4)} \cup \overline{w_1w_2w_3w_4T(w_5)}.$$

Since $\overline{w_2w_3w_4} = \overline{w_3}$, we get

$$T(uw) = T(u) \cup \overline{w_1T(w_2w_3w_4)}. \tag{5}$$

Note that $||w_2w_3w_4|| \leq |w_2w_3w_4| \leq p$. Suppose that $|\overline{w_1}| < m-p$. Then the maximal length of a word in $\overline{w_1T(w_2w_3w_4)}$ is $< m$, and in view of $||u|| < m$ and (5) we get $||w|| < m$, a contradiction. Hence $|\overline{w_1}| \geq m - p$. Together with (5), this yields $(u, uw) \in \Lambda_{m-p}$ as claimed. Since $m$ is arbitrary large, this contradicts (2). Therefore $\mathrm{Fix}(\varphi)$ is not context-free.

(iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). Trivial.

(iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi). By Corollary 3.9. □

**Corollary 3.11.** *Let $\varphi \in \mathrm{End}(M_A)$ be such that $\varphi|_{\widetilde{A}}$ is a permutation without fixed points. Then $\mathrm{Fix}(\varphi)$ is not context-free.*

*Proof.* Clearly, $\overline{\varphi}$ is an automorphism and $\mathrm{Fix}(\widehat{\varphi}) = 1$. Moreover, $\mathrm{Per}(\varphi) = M_A$. By Theorem 3.10, $\mathrm{Fix}(\varphi)$ is not context-free. □

The above corollary provides an infinite class of examples, for arbitrary curl $> 1$, where $\mathrm{Fix}(\varphi)$ is not context-free.

## 4 Radicals

We introduce now the concept of *radical*, inspired by analogous definitions in other contexts. Radicals occupy an intermediate position between the submonoids of fixed points and periodic points.

Given $\varphi \in \mathrm{End}(M_A)$ and $n \geq 1$, we define

$$\mathrm{Rad}_n(\varphi) = \{u \in \mathrm{Fix}(\varphi^n) \mid \overline{u} \in \mathrm{Fix}(\overline{\varphi})\}.$$

The following result summarizes some of the basic properties of radicals:

**Lemma 4.1.** *Let* $\varphi \in \mathrm{End}(M_A)$ *and* $m, n \geq 1$. *Then:*

*(i)* $\mathrm{Rad}_n(\varphi)$ *is an inverse submonoid of* $M_A$*;*

*(ii)* $\mathrm{Fix}(\varphi) = \mathrm{Rad}_1(\varphi) \leq \mathrm{Rad}_n(\varphi) \leq \mathrm{Per}(\varphi)$*;*

*(iii) if* $m|n$*, then* $\mathrm{Rad}_m(\varphi) \leq \mathrm{Rad}_n(\varphi)$*;*

*(iv)* $\mathrm{Fix}(\varphi) = \{\mathcal{K}_\varphi(uu^{-1})u \mid u \in \mathrm{Rad}_n(\varphi)\}$.

*Proof.* (i) – (iii) Immediate.

(iv) Let $u \in \mathrm{Rad}_n(\varphi)$ and $v = \mathcal{K}_\varphi(uu^{-1})u$. Then $(uu^{-1})\varphi^n = uu^{-1}$ and it follows easily that $\mathcal{K}_\varphi(uu^{-1}) \in \mathrm{Fix}(\varphi)$. Hence $vv^{-1} = \mathcal{K}_\varphi(uu^{-1}) \in \mathrm{Fix}(\varphi)$. On the other hand, $\overline{v} = \overline{u} \in \mathrm{Fix}(\overline{\varphi})$, and Lemma 3.3 yields $u \in \mathrm{Fix}(\varphi)$.

Conversely, let $u \in \mathrm{Fix}(\varphi) \subseteq \mathrm{Rad}_n(\varphi)$. Then $u = \mathcal{K}_\varphi(uu^{-1})u$ and we are done. □

Radicals may behave better than submonoids of fixed points:

**Theorem 4.2.** *Let* $\varphi \in \mathrm{End}(M_A)$ *and* $N = \mathrm{Curl}(\overline{\varphi})$. *Then* $\mathrm{Rad}_N(\varphi)$ *is context-free.*

*Proof.* By Lemma 3.2, we have $\mathrm{Curl}(\overline{\varphi}^N) = 1$, hence $\mathrm{Fix}(\varphi^N)$ is finitely generated by Theorem 3.7. Hence there exists a rational language $L \subseteq \widetilde{A}^*$ such that $L\pi = \mathrm{Fix}(\varphi^N)$.

On the other hand, since $H = \mathrm{Fix}(\overline{\varphi})$ is finitely generated by [5], its pre-image $H\sigma^{-1}$ is context-free by [15] (see also [1, Ex. III.2.5]). We claim that

$$\mathrm{Rad}_N(\varphi) = (L \cap H\sigma^{-1})\pi.$$

Indeed, if $u \in \mathrm{Rad}_N(\varphi)$, then $u = v\pi$ for some $v \in L$. Since $v\sigma = v\pi\sigma' = u\sigma' = \overline{u} \in H$, we get $v \in L \cap H\sigma^{-1}$ and so $u \in (L \cap H\sigma^{-1})\pi$. The opposite inclusion is similar.

Since $L \cap H\sigma^{-1}$ is the intersection of a context-free language with a rational language, it is context-free and so $\mathrm{Rad}_N(\varphi)$ is context-free. □

**Theorem 4.3.** *Let* $\varphi \in \mathrm{End}(M_A)$ *be such that* $\mathrm{Fix}(\overline{\varphi}) = 1$*. Let* $n \geq 1$*. Then the following conditions are equivalent:*

*(i)* $\mathrm{Rad}_n(\varphi)$ *is rational;*

*(ii)* $\mathrm{Rad}_n(\varphi)$ *is finite.*

*Proof.* (i) ⇒ (ii). Assume that $\mathrm{Rad}_n(\varphi) = L\pi$ for some $L \subseteq \widetilde{A}^*$ rational. Then $L = L(\mathcal{A})$ for some finite deterministic trim automaton $\mathcal{A} = (Q, q_0, T, E)$. Since $\mathrm{Fix}(\overline{\varphi}) = 1$, we have $\overline{\mathrm{Rad}_n(\varphi)} = 1$ and so $\overline{L} = 1$. Since $\mathcal{A}$ is trim, it follows that $\overline{z} = 1$ whenever $z$ labels a cycle in $\mathcal{A}$.

Suppose now that $\mathrm{Rad}_n(\varphi)$ is infinite and write $m = |Q|$. There exists some $e \in \mathrm{Rad}_n(\varphi)$ with $||e|| \geq m$. Take $u \in L \cap e\pi^{-1}$. Then $u$ must have some prefix $v$ such that $|\overline{v}| = ||e||$, hence there exists some path in $\mathcal{A}$ of the form

$$q_0 \xrightarrow{v} q \xrightarrow{w} t \in T.$$

If we successively remove all nontrivial cycles from the path $q_0 \xrightarrow{v} q$, we get a path $q_0 \xrightarrow{v'} q$ of length $< m$. However, since $\overline{z} = 1$ whenever $z$ labels a cycle in $\mathcal{A}$, we have $\overline{v'} = \overline{v}$ and so $m \leq ||e|| = |\overline{v}| = |\overline{v'}| \leq |v'| < m$, a contradiction. Therefore $\mathrm{Rad}_n(\varphi)$ is finite.

(ii) ⇒ (i). Trivial. □

The next corollary provides an infinite class of examples, for arbitrary curl $> 1$, where $\mathrm{Rad}_N(\varphi)$ is not rational:

**Corollary 4.4.** *Let* $\varphi \in \mathrm{End}(M_A)$ *be such that* $\varphi|_{\widetilde{A}}$ *is a permutation without fixed points. Let* $N = \mathrm{Curl}(\overline{\varphi})$*. Then* $\mathrm{Rad}_N(\varphi)$ *is context-free but not rational.*

*Proof.* By Theorem 4.2, $\mathrm{Rad}_N(\varphi)$ is context-free. Since $\overline{\varphi}|_{\widetilde{A}}$ is also a permutation without fixed points, we have $\mathrm{Per}(\overline{\varphi}) = F_A$, hence $\mathrm{Fix}(\overline{\varphi}^N) = F_A$ and so $\overline{\varphi}^N = 1_{F_A}$. It follows that $\varphi^N = 1_{M_A}$. Since $\mathrm{Fix}(\overline{\varphi}) = 1$, we get $\mathrm{Rad}_N(\varphi) = E(M_A)$ and so $\mathrm{Rad}_N(\varphi)$ is infinite. Since $\mathrm{Fix}(\overline{\varphi}) = 1$, we may apply Theorem 4.3, thus $\mathrm{Rad}_N(\varphi)$ is not rational. □

Finally, we use Lemma 4.1(iv) to show that $\mathrm{Fix}(\varphi)$ is always context-sensitive:

**Theorem 4.5.** *Let* $\varphi \in \mathrm{End}(M_A)$*. Then* $\mathrm{Fix}(\varphi)$ *is context-sensitive.*

*Proof.* Context-sensitive languages can be characterized as languages accepted by *linear-bounded Turing machines*, i.e. Turing machines $\mathcal{T}$ for which there exists some constant $K \geq 1$ such that any word $u \in L(\mathcal{T})$ can be accepted by some computation using only the first $K|u|$ cells of the tape.

By Theorem 3.8, we have $\mathrm{Per}(\varphi) = \mathrm{Fix}(\varphi^m)$ for some $m \geq 1$. Let $A_1, \ldots, A_{m-1}$ be disjoint copies of $A$, inducing bijections $\alpha_i : \widetilde{A} \to \widetilde{A_i}$ for $i = 1, \ldots, m-1$. Let

$$B_i = \{a \in A \mid a\varphi^i \neq 1\}$$

for $i = 1, \ldots, m-1$. We define a matched homomorphism $\beta_i : \widetilde{A}^* \to \widetilde{B_i\alpha_i}^*$ by

$$a\beta_i = \begin{cases} a\alpha_i & \text{if } a \in B_i \\ 1 & \text{if } a \in A \setminus B_i \end{cases}$$

Let $N = \mathrm{Curl}(\overline{\varphi})$. By Theorem 4.2, $\mathrm{Rad}_N(\varphi)$ is context-free. Hence we have $\mathrm{Rad}_N(\varphi) = C\pi$ for some context-free language $C \subseteq \widetilde{A}^*$. We define

$$L = \{uu^{-1}(u\beta_1)(u^{-1}\beta_1)(u\beta_2)(u^{-1}\beta_2)\ldots(u\beta_{m-1})(u^{-1}\beta_{m-1})u \mid u \in C\}.$$

It is easy to see that $L$ is accepted by a linear-bounded Turing machine $\mathcal{T}$. Indeed, since $C$ is context-free, $C$ is itself accepted by a linear-bounded Turing machine $\mathcal{T}'$. We can make $\mathcal{T}$ consider all the possible factorizations of the input potentially leading to some word of the form

$$uu^{-1}(u\beta_1)(u^{-1}\beta_1)\ldots(u\beta_{m-1})(u^{-1}\beta_{m-1})u$$

(using multiple tracks to help), then using $\mathcal{T}'$ as a subroutine and checking each one of the presumed factors using the definitions of the $B_i$ and $\beta_i$. Therefore $L$ is context-sensitive.

For $i = 1, \ldots, m-1$, fix a matched endomorphism $\psi_i$ of $\widetilde{A}^*$ satisfying $\psi_i\pi = \pi\varphi^i$ (it suffices to have $a\psi_i\pi = a\pi\varphi^i$ for every $a \in A$, so it really exists). Let $X = A \cup B_1\alpha_1 \cup \ldots \cup B_{m-1}\alpha_{m-1}$ and define a matched homomorphism $\gamma : \widetilde{X}^* \to \widetilde{A}^*$ as follows. Given $a \in A$, let $a\gamma = a$. Given $a \in B_i$, let $a\alpha_i\gamma = a\psi_i$. Note that $a\alpha_i\gamma \neq 1$ since $a\varphi^i \neq 1$ by definition of $B_i$.

Since context-sensitive languages are closed under $\varepsilon$-free homomorphisms, it follows that $L\gamma$ is context-sensitive. Note that $a\alpha_i\gamma\pi = a\psi_i\pi = a\pi\varphi^i$ whenever $a \in B_i$, so $a\beta_i\gamma\pi = a\pi\varphi^i$ for every $a \in A$ (if $a \notin B_i$, then $a\beta_i = 1 = a\pi\varphi^i$). Thus we get

$$\begin{aligned} L\gamma\pi &= \{uu^{-1}(u\beta_1)(u^{-1}\beta_1)\ldots(u\beta_{m-1})(u^{-1}\beta_{m-1})u \mid u \in C\}\gamma\pi \\ &= \{vv^{-1}(v\varphi)(v^{-1}\varphi)\ldots(v\varphi^{m-1})(v^{-1}\varphi^{m-1})v \mid v \in C\pi\}. \end{aligned}$$

Recall that $C\pi = \mathrm{Rad}_N(\varphi)$. Moreover, we claim that

$$vv^{-1}(v\varphi)(v^{-1}\varphi)\ldots(v\varphi^{m-1})(v^{-1}\varphi^{m-1}) = \mathcal{K}_\varphi(vv^{-1}). \tag{6}$$

Indeed, if $p$ is the period of $vv^{-1}$, then $\mathcal{K}_\varphi(vv^{-1}) = vv^{-1}(vv^{-1})\varphi \ldots (vv^{-1})\varphi^{p-1}$. Since $vv^{-1} \in \mathrm{Per}(\varphi) = \mathrm{Fix}(\varphi^m)$, we have $p \leq m$ and so (6) holds. Therefore, by Lemma 4.1(iv), we get

$$\mathrm{Fix}(\varphi) = \{\mathcal{K}_\varphi(vv^{-1})v \mid v \in \mathrm{Rad}_N(\varphi)\} = L\gamma\pi.$$

Since $L\gamma$ is context-sensitive, so is $\mathrm{Fix}(\varphi)$. □

# 5 Open Problems

We give a list of natural open problems originated by the previous results.

**Problem 5.1.** *To find examples of* $\varphi \in \mathrm{End}(M_A)$ *such that:*

*(i)* $\mathrm{Fix}(\varphi)$ *is context-free but not rational;*

*(ii)* $\mathrm{Fix}(\varphi)$ *is rational but not finitely generated.*

**Problem 5.2.** *To show that a basis of* $\mathrm{Per}(\theta)$ *can be computed for every* $\theta \in \mathrm{End}(F_A)$.

**Problem 5.3.** *To show that* $\mathrm{Curl}(\theta)$ *can be computed for every* $\theta \in \mathrm{End}(F_A)$.

If Problem 5.2 is solved, then Problem 5.3 is solved as well: we just compute basis of $\mathrm{Per}(\theta)$, $\mathrm{Per}(\theta^{2!})$, $\mathrm{Per}(\theta^{3!})$ until we reach $\mathrm{Per}(\theta^{n!}) = \mathrm{Per}(\theta^{(n+1)!})$ for some $n$ (and we know we eventually will) by Proposition 3.1.

**Problem 5.4.** *To compute* $\mathrm{St}_\varphi$ *for an arbitrary* $\varphi \in \mathrm{End}(M_A)$.

If we compute the $\varphi$-stable letters, we can compute the $\varphi$-tiles in $\mathcal{T}_\varphi$.

## Acknowledgments

The authors acknowledge support from the European Regional Development Fund through the programme COMPETE and by the Portuguese Government through the FCT – Fundação para a Ciência e a Tecnologia under the project PEst-C/MAT/UI0144/2011. The first author also acknowledges the support of the FCT project SFRH/BPD/65428/2009.

## References


[1] J. Berstel, *Transductions and Context-free Languages*, Teubner, 1979.

[2] J. Cassaigne and P. V. Silva, Infinite words and confluent rewriting systems: endomorphism extensions, *Internat. J. Algebra Comput.* 19(4) (2009), 443–490.

[3] J. Cassaigne and P. V. Silva, Infinite periodic points of endomorphisms over special confluent rewriting systems, *Ann. Inst. Fourier* 59(2) (2009), 769–810.

[4] J. Dugundji, *Topology*, Allyn and Bacon, 1966.

[5] R. Z. Goldstein and E. C. Turner, Fixed subgroups of homomorphisms of free groups, *Bull. London Math. Soc.* 18 (1986), 468–470.

[6] J. E. Hopcroft, R. Motwani and J.- D. Ullman, *Introduction to Automata Theory, Languages and Computation*, 3rd ed., Addison-Wesley 2006.

[7] W. Imrich and E. C. Turner, Endomorphisms of free groups and their fixed points, *Math. Proc. Cambridge Philos. Soc.* 105 (1989), 421–422.

[8] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Springer-Verlag 1977.

[9] S. W. Margolis and J. C. Meakin, Free inverse monoids and graph immersions, *Internat. J. Algebra Comput.* 3(1) (1993), 79–99.

[10] W. D. Munn, Free inverse semigroups, *Proc. London Math. Soc.* 30 (1974), 385–404.

[11] D. Perrin and J.-E. Pin, *Infinite Words: Automata, Semigroups, Logic and games*, Pure and Applied Mathematics Series 141, Elsevier Academic Press, Amsterdam, 2004.

[12] M. Petrich, *Inverse Semigroups*, Wiley, New York, 1984.

[13] E. Rodaro, P. V. Silva and M. Sykiotis, Fixed points of endomorphisms of graph groups, *J. Group Theory* (to appear).

[14] E. Rodaro and P. V. Silva Fixed points of endomorphisms of endomorphisms of trace monoids, arXiv:1211.4517

[15] J. Sakarovitch, Sur les groupes infinis, considérés comme mono¨des syntaxiques de langages formels, In: *Séminaire dAlgèbre Paul Dubreil, 29ème année (Paris, 19751976)*, pages 168–179, Lecture Notes in Math. 586, Springer, Berlin, 1977.

[16] H. E. Scheiblich, Free inverse semigroups, *Proc. Amer. Math. Soc.* 38 (1973), 1–7.

[17] P. V. Silva, Fixed points of endomorphisms over special confluent rewriting systems, *Monatsh. Math.* 161(4) (2010), 417–447.

[18] P. V. Silva, Fixed points of endomorphisms of certain free products, *RAIRO Theoret. Inf. Appl.* 46 (2012), 165–179.

[19] P. V. Silva, Fixed points of endomorphisms of virtually free groups, *Pacific J. Math.* (to appear).

[20] M. Takahasi, Note on chain conditions in free groups, *Osaka Math. J.* 3 (1951) 221–225.