Fixed points of endomorphisms over special confluent rewriting systems

Pedro V. Silva

Centro de Matemática, Faculdade de Ciências,
Universidade do Porto, R. Campo Alegre 687,
4169-007 Porto, Portugal
e-mail: pvsilva@fc.up.pt

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ABSTRACT

Cayley graphs of monoids defined through special confluent rewriting systems are known to be hyperbolic metric spaces which admit a compact completion given by irreducible finite and infinite words. In this paper, we prove that the fixed point submonoids for endomorphisms of these monoids which are boundary injective (or have bounded length decrease) are rational, with similar results holding for infinite fixed points. Decidability of these properties is proved, and constructibility is proved for the case of bounded length decrease. These results are applied to free products of cyclic groups, providing a new generalization for the case of infinite fixed points.

1 Introduction

Gersten proved in the eighties that the fixed point subgroup of a free group automorphism φ is finitely generated [11]. Using a different approach, Cooper gave an alternative proof, proving also that the fixed points of the continuous extension of φ to the boundary of the free group is in some sense finitely generated [9]. Bestvina and Handel achieved in 1992 a major breakthrough through their innovative train track techniques, bounding the rank of the fixed point subgroup and the generating set for the infinite fixed points [4]. Their approach was pursued by Maslakova in 2003 to prove that the fixed point subgroup can be effectively computed [17].

Gersten's result was generalized to further classes of groups and endomorphisms in subsequent years. Goldstein and Turner extended it to monomorphisms of free groups [12], and later to arbitrary endomorphisms [13]. Collins and Turner extended it to automorphisms of free products of freely indecomposable groups [8], and recently Sykiotis to monomorphisms [21]. The interested reader can find more information in Ventura's excellent survey [22].

Infinite fixed points of automorphisms of free groups were also discussed by Bestvina and Handel in [4]. Gaboriau, Jaeger, Levitt and Lustig remarked in [10] that some of the results on infinite fixed points would hold for virtually free groups with some adaptations.

Cassaigne and the author developped in [6] an approach to the study of monoids defined by special confluent rewriting systems that preserves some of the features of the free group case and contains free products of cyclic groups as a particular case, as well as the partially reversible monoids introduced in [20]. In fact, the undirected Cayley graph of these monoids is hyperbolic and has a compact completion for the prefix metric. Uniformly continuous endomorphisms, algorithmically characterized in [6], admit a continuous extension to the boundary. In [7], the same authors used this approach to study the dynamics of infinite periodic points for two classes of endomorphisms of the monoids in question.

The present paper intends to pursue this same approach to prove finite generation properties for both finite and infinite fixed points. This is achieved through a combination of automata theoretical, combinatorial and topological techniques. Two classes of endomorphisms are studied: boundary-injective endomorphisms and endomorphisms with bounded length decrease. The first class provides new proofs for the already known results for monomorphisms of free groups and more generally free products of cyclic groups. The second class provides constructibility results that are reminiscent of those of Maslakova. Moreover, both classes are recursive and algorithms to test the corresponding properties are provided.

The paper is organized as follows. Section 2 is devoted to preliminaries and Section 3 presents the basic results that lie behind the full approach through the introduction of a certain automata-theoretic property for endomorphisms: the finite-splitting property. This theoretical property, which we ignore to be recursive or not, is explored in Sections 4 and 5 through two important subclasses: boundary-injective endomorphisms and endomorphisms with bounded length decrease. In Section 6 we apply our results to the case of groups to recover the known result on finite fixed points and obtain what we believe to be a new one on the infinite fixed points. We end the paper by presenting some open problems in Section 7.

2 Preliminaries

Given a monoid M, we denote by Rat$M$ the set of all *rational* subsets of M, i.e., the smallest family of subsets of M containing the finite sets and closed under union, product and the star operator (X^* denotes the submonoid of M generated by $X \subseteq M$). For details on rational languages, the reader is referred to [2, 19].

In the particular case of a free monoid $M = A^*$, a combinatorial description is terms of finite automata is usually preferred. We define a (finite) A-automaton to be a quadruple $\mathcal{A} = (Q, q_0, T, \delta)$ where Q is a (finite) set, $q_0 \in Q$ is the initial vertex, $T \subseteq Q$ are the terminal vertices and $E \subseteq Q \times A \times Q$. The language recognized by $\mathcal{A}$ is

$$L(\mathcal{A}) = \{w \in A^* \mid \text{ there exists a path } q_0 \xrightarrow{w} t \in T \text{ in } \mathcal{A}\}.$$

Such a path is called *successful*. Note that the empty word labels a (trivial) path $q_0 \xrightarrow{1} q_0$.

The classical Kleene's Theorem states that $L \subseteq A^*$ is rational if and only if $L = L(\mathcal{A})$ for some finite A-automaton $\mathcal{A}$.

Let $\mathcal{A} = (Q, q_0, T, E)$ be an A-automaton. We say that $\mathcal{A}$ is

- *deterministic* if the implication
$$(p, a, q), (p, a, r) \in E \Rightarrow q = r$$
holds;

- *trim* if every vertex $q \in Q$ lies in some successful path.

If $L(\mathcal{A}) \neq \emptyset$, the *trim subautomaton* of $\mathcal{A}$ is defined by
$$\operatorname{tr}(\mathcal{A}) = (Q \cap S, q_0, T \cap S, E \cap (S \times A \times S)),$$
where S consists of all vertices of $\mathcal{A}$ that lie in some successful path. Trivially, $L(\operatorname{tr}(\mathcal{A})) = L(\mathcal{A})$.

Another case that will be relevant for us is the case of M being a group, when we have the following result of Anisimov and Seifert:

Proposition 2.1 *[19, Prop. II.6.2] Let H be a subgroup of a group G. Then $H \in RatG$ if and only if H is finitely generated.*

A finite *rewriting system* is a formal expression $\langle A \mid R \rangle$, where A is a finite alphabet and R a finite subset of $A^* \times A^*$. The elements of R are called *rules*. Given $u, v \in A^*$, we write $u \longrightarrow_R v$ if
$$u = xry, \quad v = xsy$$
for some $x, y \in A^*$ and $(r, s) \in R$. We denote by $\overset{*}{\longrightarrow}$ the reflexive and transitive closure of the relation $\longrightarrow$. The subscript R will be usually omitted. The *congruence* on A^* generated by R will be denoted by $R^\sharp$. Note that $R^\sharp = \overset{*}{\longrightarrow}_{R \cup R^{-1}}$. The quotient $M = A^*/R^\sharp$ is said to be the monoid defined by the rewriting system R.

A rewriting system $\langle A \mid R \rangle$ is said to be

- *special* if $R \subseteq A^+ \times \{1\}$;

- *confluent* if, whenever $u \overset{*}{\longrightarrow} v$ and $u \overset{*}{\longrightarrow} w$, there exists $z \in A^*$ such that $v \overset{*}{\longrightarrow} z$ and $w \overset{*}{\longrightarrow} z$:

$$\begin{tikzcd} u \arrow[r, "*"] \arrow[d, "*"'] & v \arrow[d, dashed, "*"] \\ w \arrow[r, dashed, "*"] & z \end{tikzcd}$$

It is known (see [5, Section 2.2]) that every monoid defined by a finite special confluent rewriting system can be defined by a finite *normalized* length-reducing confluent rewriting system, i.e., satisfying the two conditions:

- for every $(r, s) \in R$, $|r| > 1$;

- if $(r, s), (arb, s') \in R$, then $ab = 1$ and $s' = s$.

Therefore, we are entitled to assume whenever convenient that our special confluent rewriting systems are normalized.

Let $\langle A \mid R \rangle$ be a special confluent rewriting system. We say that $w \in A^*$ is *irreducible* (with respect to R) if $w \notin \cup_{(r,s) \in R} A^* r A^*$. For every $u \in A^*$, there is exactly one irreducible

$v \in A^*$ such that $u \xrightarrow{*} v$: existence follows from R being length-reducing, and uniqueness from confluence. We denote this unique irreducible word by $\overline{u}$. It is well known (see [5]) that the equivalence

$$uR^\sharp v \Leftrightarrow \overline{u} = \overline{v}$$

holds for all $u, v \in A^*$, hence $\overline{A^*} = \{\overline{u} \mid u \in A^*\}$ constitutes a set of normal forms for the monoid $M = A^*/R^\sharp$. Moreover,

$$M \cong (\overline{A^*}, \cdot),$$

where $\cdot$ denotes the binary operation on $\overline{A^*}$ defined by $u \cdot v = \overline{uv}$. We denote the monoid $(\overline{A^*}, \cdot)$ by A_R^*. We shall often abuse notation and identify A_R^* with $\overline{A^*}$. We write also $A_R^+ = \overline{A^*} \setminus \{1\}$.

We denote by A^ω the set of all infinite words of the form $a_1a_2a_3\ldots$, with $a_n \in A$ for every $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. For details on infinite words, see [18]. Write

$$A^\infty = A^* \cup A^\omega.$$

Given $\alpha \in A^\infty$ and $n \in \mathbb{N}$, we denote by $\alpha^{(n)}$ the n-th letter of α (if $\alpha \in A^*$ and $n > |\alpha|$, we set $\alpha^{(n)} = 1$). We write also

$$\alpha^{[n]} = \alpha^{(1)}\alpha^{(2)} \ldots \alpha^{(n)}, \quad \alpha^{[k,n]} = \alpha^{(k)}\alpha^{(k+1)} \ldots \alpha^{(n)}.$$

An infinite word $\alpha \in A^\omega$ is said to be *irreducible* (with respect to R) if $\alpha^{[n]}$ is irreducible for every $n \in \mathbb{N}$. We denote the set of all irreducible infinite words (with respect to R) by A_R^ω and we write

$$A_R^\infty = A_R^* \cup A_R^\omega.$$

For all $\alpha, \beta \in A^\infty$, we define

$$r(\alpha, \beta) = \begin{cases} \min\{n \in \mathbb{N} \mid \alpha^{(n)} \neq \beta^{(n)}\} & \text{if } \alpha \neq \beta \\ \infty & \text{if } \alpha = \beta \end{cases}$$

and we write

$$d(\alpha, \beta) = 2^{-r(\alpha,\beta)},$$

using the convention $2^{-\infty} = 0$. It follows easily from the definition that d is an ultrametric on A^∞, satisfying in particular the axiom

$$d(\alpha, \beta) \leq \max\{d(\alpha, \gamma), d(\gamma, \beta)\}.$$

We shall identify A^∞ with the metric space (A^∞, d). It is well known that the metric space A^∞ is compact (and therefore complete) [18, Chapter III]. Note that $\lim_{n\to\infty} \alpha_n = \alpha$ if and only if

$$\forall k \in \mathbb{N}\, \exists m \in \mathbb{N}\, \forall n \in \mathbb{N}\, (n \geq m \Rightarrow \alpha_n^{[k]} = \alpha^{[k]}).$$

The following result is straightforward:

Proposition 2.2 *[6, Prop. 2.2 and Cor. 2.3] If $\langle A \mid R \rangle$ is a special confluent rewriting system, then A_R^∞ is a closed subspace of (A^∞, d), being therefore compact and complete.*

We remark that, since $\alpha = \lim_{n\to\infty} \alpha^{[n]}$ for every $\alpha \in A^\infty$, (A^∞, d) (respectively (A_R^∞, d)) is the completion of (A^*, d) (respectively (A_R^*, d)).

Note also that d induces the discrete topology on A^* since $B_{2^{-(n+1)}}(u) = \{u\}$ for every $u \in A^n$. In particular, the product of finite words is continuous when we consider the product topology on $A_R^* \times A_R^*$.

It is important to remark that rational languages are preserved by reduction:

Theorem 2.3 *[1] Let $\langle A \mid R\rangle$ be a finite special confluent rewriting system and let $L \subseteq A^*$ be rational. Then $\overline{L}$ is rational and effectively constructible from L.*

It is easy to see that the amount of reduction a word can cause on another depends on its length. We denote by t_R the maximum length of a relator r for $(r, 1) \in R$.

Lemma 2.4 *[6, Lemma 4.2] For all $u, v \in A_R^*$,*

(i) There exist $u', u'', v', v'' \in A_R^*$ such that:

$$u = u'u'', \quad v = v'v'', \quad \overline{uv} = u'v'',$$

$$|u''v'| \leq min\{|u|, |v|\} \cdot t_R, \quad |u''| \leq (t_R - 1)|v|.$$

(ii) $|\overline{uv}| \geq max\{|v| - (t_R - 1)|u|, |u| - (t_R - 1)|v|\}$.

This allows us to define the *mixed product*

$$\begin{array}{rcl} A_R^* \times A_R^\omega & \to & A_R^\omega \\ (u, \alpha) & \mapsto & \overline{u\alpha} \end{array}$$

that turns out to be continuous when we consider the product topology on $A_R^* \times A_R^\omega$ (see [6, Theorem 4.4]).

The next result determines which endomorphisms of A_R^* admit a continuous extension to the completion A_R^∞. Such an extension is said to be *proper* if $(A_R^\omega)\Phi \subseteq A_R^\omega$. We note that the equivalence between (i) and (ii) follows immediately from topological arguments.

Theorem 2.5 *[6, Theor. 8.4] Let $\langle A \mid R\rangle$ be a finite special confluent rewriting system and let φ be a nontrivial endomorphism of A_R^*. Then the following conditions are equivalent:*

(i) φ can be extended to a continuous mapping $\Phi : A_R^\infty \to A_R^\infty$;

(ii) φ is uniformly continuous;

(iii) $w\varphi^{-1}$ is finite for every $w \in A_R^*$.

Moreover, if these conditions hold the continuous mapping Φ is unique, proper and given by $\alpha\Phi = \lim_{n\to\infty} \alpha^{[n]}\varphi$.

The following is a straightforward consequence:

Corollary 2.6 *[6, Cor. 8.5] If A_R^* is a group with no finite nontrivial normal subgroups and φ is an endomorphism of A_R^*, the following conditions are equivalent:*

(i) φ is uniformly continuous;

(ii) φ is either trivial or injective.

The decidability issue is less straightforward. However, the answer is positive:

Theorem 2.7 *[6, Theor. 8.7] Given a finite special confluent rewriting system $\langle A \mid R\rangle$ and an endomorphism φ of A_R^*, it is decidable whether or not φ is uniformly continuous.*

Finally, let

$$\text{Fix}\varphi = \{u \in A_R^* \mid u\varphi = u\}.$$

Clearly, Fixφ is always a submonoid of A_R^* (a subgroup if A_R^* is a group).

3 Finite-splitting endomorphisms

We start this section by proving the Bounded Reduction Property for special confluent rewriting systems. This is a classical result for free group automorphisms [9].

Given an endomorphism φ of A_R^*, we define

$$h_\varphi = \max\{|a\varphi|;\ a \in A\}.$$

Proposition 3.1 *Let $\langle A \mid R\rangle$ be a finite special confluent rewriting system and let φ be a uniformly continuous endomorphism of A_R^*. Then there exists a constant $M_\varphi \in \mathbb{N}$ such that, whenever*

$$uv \in R_A, \quad u\varphi = xy, \quad v\varphi = zt, \quad \overline{yz} = 1, \quad (uv)\varphi = xt,$$

then $|y|, |z| \leq M_\varphi$.

Proof. Write $u = a_1 \dots a_n$ and $v = b_1 \dots b_m$ with $a_i, b_j \in A$. We have $xy = u\varphi = \overline{(a_1\varphi)\dots(a_n\varphi)}$. If $x \neq 1$, then the last letter of x originated from some $a_i\varphi$. We define then $u' = a_1 \dots a_i$ and $u'' = a_{i+1}\dots a_n$. If $x = 1$, we take $u' = 1$ and $u'' = u$. Similarly, $zt = v\varphi = \overline{(b_1\varphi)\dots(b_m\varphi)}$. If $t \neq 1$, then the first letter of t originated from some $b_j\varphi$. We define then $v' = b_1 \dots b_{j-1}$ and $v'' = b_j \dots b_m$. If $t = 1$, we take $v' = v$ and $v'' = 1$. Write $u'\varphi = xx'$ and $v''\varphi = t't$. Then $y = \overline{x'(u''\varphi)}$, $z = \overline{(v'\varphi)t'}$ and so

$$\overline{x'(u''v')\varphi\, t'} = \overline{x'(u''\varphi)(v'\varphi)t'} = \overline{yz} = 1.$$

By Lemma 2.4, we get

$$0 = |\overline{x'(u''v')\varphi\, t'}| \geq |\overline{x'(u''v')\varphi}| - (t_R - 1)|t'| \geq |(u''v')\varphi| - (t_R - 1)|x't'|,$$

hence

$$|(u''v')\varphi| \leq (t_R - 1)|x't'| \ < 2(t_R - 1)h_\varphi.$$

Since φ is uniformly continuous, it follows from Theorem 2.5 that there exist only finitely words $w \in A_R^*$ such that $|w\varphi| \ < 2(t_R - 1)h_\varphi$. Thus $|u''v'|$ can be bounded by some $M \in \mathbb{N}$. Since $y = \overline{x'(u''\varphi)}$, we get $|y| \leq (M+1)h_\varphi$. Similarly, $|z| \leq (M+1)h_\varphi$ and so the claim holds for $M_\varphi = (M+1)h_\varphi$. $\square$

Given $u, v \in A_R^*$, we write $u \leq v$ if u is a prefix of v and denote by $u \wedge v$ the longest common prefix of u and v. We also write $u \leq_s v$ if u is a suffix of v.

Next we introduce notation used in [14] in the study of free group automorphisms. Given $u \in A_R^*$, let $\sigma(u) = u \wedge (u\varphi)$ and write

$$u = \sigma(u)\,\tau(u), \quad u\varphi = \sigma(u)\,\rho(u).$$

Define also

$$\sigma'(u) = \wedge\{\sigma(v);\ v \in A_R^*,\ u \leq v\}$$

and write $\sigma(u) = \sigma'(u)\sigma''(u)$.

Lemma 3.2 *If $\varphi \in A_R^*$ is uniformly continuous, then $|\sigma''(u)| \leq M_\varphi$ for every $u \in A_R^*$.*

Proof. Suppose that $|\sigma''(u)| > M_\varphi$ for some $u \in A_R^*$. Since the set of prefixes of u is well ordered by $\leq$, there exists some $v \in A_R^*$ such that $u \leq v$ and $\sigma'(u) = \sigma(u) \wedge \sigma(v)$: otherwise, if a would be the first letter of $\sigma''(u)$, then $\sigma'(u)a$ would be a prefix of $\sigma(v)$ for every v, a contradiction. Write $v = uw$. Then $v\varphi = \overline{(u\varphi)(w\varphi)}$ and by Proposition 3.1 we get

$$|(v\varphi) \wedge (u\varphi)| \geq |u\varphi| - M_\varphi. \tag{1}$$

Now let a be the first letter of $\sigma''(u)$. We have $u\varphi = \sigma'(u)\sigma''(u)\rho(u)$ and $|\sigma''(u)| > M_\varphi$, hence $\sigma'(u)a \leq v\varphi$ by (1). On the other hand, $\sigma'(u)a \leq u \leq v$ and so $\sigma'(u)a \leq \sigma(v)$. Together with $\sigma'(u)a \leq \sigma(u)$, this contradicts $\sigma'(u) = \sigma(u) \wedge \sigma(v)$. $\square$

For every $u \in A_R^*$, $\sigma''(u)\tau(u)$ is a suffix of u. We define a suffix $\lambda(u)$ of $\sigma'(u)$ as follows: $\lambda(u)\sigma''(u)\tau(u)$ is the shortest suffix of u (having $\sigma''(u)\tau(u)$ as a suffix) satisfying

$$|\lambda(u)\sigma''(u)\tau(u)| \geq t_R - 1 \quad \text{or} \quad \lambda(u)\sigma''(u)\tau(u) = u.$$

We define also

$$C(u) = (\lambda(u), \sigma''(u), \tau(u), \rho(u)).$$

Lemma 3.3 *Let $u, v \in A_R^*$ be such that $C(u) = C(v)$ and $ua \in A_R^*$. Then $va \in A_R^*$ and $C(ua) = C(va)$.*

Proof. If $|\lambda(u)\sigma''(u)\tau(u)| < t_R - 1$, then $v = \lambda(v)\sigma''(v)\tau(v) = \lambda(u)\sigma''(u)\tau(u) = u$ and the lemma holds trivially. Hence we may assume that $|\lambda(u)\sigma''(u)\tau(u)| \geq t_R - 1$. Now u and v share a common suffix of length $t_R - 1$ and so $ua \in A_R^*$ yields $va \in A_R^*$.

Moreover, if $uw \in A_R^*$, we have $\sigma'(u) \leq \sigma'(uw)$ and

$$uw = \sigma'(u)\sigma''(u)\tau(u)w, \quad (uw)\varphi = \sigma'(u)\overline{\sigma''(u)\rho(u)(w\varphi)}.$$

Hence the configuration $(\sigma''(ua), \tau(ua), \rho(ua))$ will depend only on $(\sigma''(u), \tau(u), \rho(u))$. Thus $C(u) = C(v)$ yields $(\sigma''(ua), \tau(ua), \rho(ua)) = (\sigma''(va), \tau(va), \rho(va))$.

Finally, $|\lambda(u)\sigma''(u)\tau(u)| \geq t_R - 1$ implies $|ua| \geq t_R$ and so $\lambda(ua)\sigma''(ua)\tau(ua)$ is the shortest suffix of ua such that $|\lambda(ua)\sigma''(ua)\tau(ua)| \geq t_R - 1$. Now $\sigma'(u) \leq \sigma'(ua)$ yields

$$\sigma''(ua)\tau(ua) \leq_s \sigma''(u)\tau(u)a$$

and so

$$\lambda(ua)\sigma''(ua)\tau(ua) \leq_s \lambda(u)\sigma''(u)\tau(u)a.$$

Therefore $\lambda(ua)\sigma''(ua)\tau(ua)$ is the shortest suffix of $\lambda(u)\sigma''(u)\tau(u)a$ such that $|\lambda(ua)\sigma''(ua)\tau(ua)| \geq t_R - 1$.

Since $C(u) = C(v)$, we conclude that $\lambda(ua)\sigma''(ua)\tau(ua) = \lambda(va)\sigma''(va)\tau(va)$. Since we had already proved that $\sigma''(ua)\tau(ua) = \sigma''(va)\tau(va)$, we get $\lambda(ua) = \lambda(va)$ and therefore $C(ua) = C(va)$ as required. $\square$

We build now a deterministic A-automaton $\mathcal{A}'_\varphi = (Q', q_0, T', E')$ by taking

- $Q' = \{C(u) \mid u \leq \alpha \text{ for some } \alpha \in \text{Fix}\Phi\}$;
- $q_0 = C(1)$;
- $T' = \{C(u) \in Q' \mid \tau(u) = \rho(u) = 1\}$;
- $E' = \{(C(u), a, C(v)) \in Q' \times A \times Q' \mid v = ua\}$.

In view of Lemma 3.3, E' and therefore $\mathcal{A}'_\varphi$ are well defined.

Let S denote the set of all vertices $q \in Q'$ such that there exist at least two edges in $\mathcal{A}'_\varphi$ leaving q. Let Q denote the set of all vertices $q \in Q'$ such that there exists some path

$$q_0 \longrightarrow q \longrightarrow p \in S \cup T'.$$

We define $\mathcal{A}_\varphi = (Q, q_0, T, E)$ by taking

$$T = T' \cap Q, \quad E = E' \cap (Q \times A \times Q).$$

Lemma 3.4 *Let p be a uniformly continuous endomorphism of A_R^*. Then* $\text{Fix}\varphi = L(\mathcal{A}_\varphi)$.

Proof. Let $u = a_1 \ldots a_n \in A_R^*$ with $a_1, \ldots, a_n \in A$. Assume first that $u \in \text{Fix}\varphi$. Then there exists a path

$$q_0 = C(1) \xrightarrow{a_1} C(u^{[1]}) \xrightarrow{a_2} C(u^{[2]}) \xrightarrow{a_3} \ldots \xrightarrow{a_n} C(u) \in T' \tag{2}$$

in $\mathcal{A}'_\varphi$. Indeed, $u \in \text{Fix}\varphi$ yields $\tau(u) = \rho(u) = 1$ and so $C(u) \in T'$. The latter implies that (2) is a path in $\mathcal{A}_\varphi$. Thus $u \in L(\mathcal{A}_\varphi)$.

Conversely, assume that $u \in L(\mathcal{A}_\varphi)$. Then we have a path of the form (2) in $L(\mathcal{A}_\varphi)$ and so in particular $\tau(u) = \rho(u) = 1$. Therefore $u \in \text{Fix}\varphi$. $\square$

We say that φ is *finite-splitting* if S is finite.

Theorem 3.5 *Let p be a finite-splitting uniformly continuous endomorphism of A_R^*. Then* $\text{Fix}\varphi \in \mathit{Rat}A^*$.

Proof. In view of Lemma 3.4, it suffices to prove that Q is finite. Call a path

$$p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} \ldots \xrightarrow{a_n} p_n$$

in $\mathcal{A}_\varphi$ a *bridge* if $p_1, \ldots, p_{n-1} \notin S \cup T$. Clearly, if we fix $p \in Q$ and $p' \in S \cup T$, there are only finitely many bridges $p \longrightarrow p'$: we might have a choice for the first edge (if $p \in S$), but the rest of the path would be fully determined (if there exists a path at all).

We claim now that every $q \in Q$ must occur in a bridge $p \longrightarrow p'$ for $p \in \{q_0\} \cup S \cup T$ and $p' \in S \cup T$. Indeed, any path $q_0 \xrightarrow{u} t \in T$ may be factored as

$$q_0 \xrightarrow{u_1} q_1 \xrightarrow{u_2} \dots \xrightarrow{u_n} q_n = t,$$

where $q_1, \dots, q_{n-1}$ denote the unique intermediate ocurrences of vertices in $S \cup T$. Thus, if q occurs in the path $q_0 \xrightarrow{u} t \in T$, it must occur in some bridge $q_{i-1} \xrightarrow{u_i} q_i$, which happens to be of the required form. Since there are only finitely many such bridges by the first part of the proof, Q is finite and so $\mathrm{Fix}\varphi \in \mathrm{Rat}A^*$. $\square$

We note that $\mathrm{Fix}\varphi$ being a rational submonoid of A^* does not imply that $\mathrm{Fix}\varphi$ is a finitely generated submonoid of A^*, as the next example shows:

Example 3.6 *Let $A = \{a, b, b^{-1}, c\}$, $R = \{(bb^{-1}, 1)\}$ and $\varphi : A_R^* \to A_R^*$ be the endomorphism defined by*

$$a \mapsto ab, \quad b \mapsto b, \quad c \mapsto b^{-1}c.$$

Then $\mathrm{Fix}\varphi \in \mathrm{Rat}A^*$ but it is not finitely generated.

Proof. Let $u \in A_R^*$. Then we may write

$$u = b^{-i_0} b^{j_0} x_1 b^{-i_1} b^{j_1} \dots x_n b^{-i_n} b^{j_n}$$

for some unique $n \geq 0$, $i_r, j_r \in \mathbb{N}$ and $x_r \in \{a, c\}$. Hence

$$u\varphi = \overline{b^{-i_0} b^{j_0} (x_1\varphi) b^{-i_1} b^{j_1} \dots (x_n\varphi) b^{-i_n} b^{j_n}}.$$

Clearly, for $k, l \in \{0, 1\}$ and $i, j \in \mathbb{N}$, we have $\overline{b^k b^{-i} b^j b^{-l}} = b^{-i} b^j$ if and only if $k = l = 0$ or

$$k = l = 1, \quad ij = 0.$$

It follows easily that $u\varphi = u$ if and only if $u \in (\{b, b^{-1}\} \cup ab^*c \cup a(b^{-1})^*c)^*$. Hence $\mathrm{Fix}\varphi \in \mathrm{Rat}A^*$ but it is not finitely generated since no element in $ab^*c \cup a(b^{-1})^*c)^*$ can be nontrivially decomposed as a product of other fixed points. $\square$

We start now the discussion of infinite fixed points, beginning with a few useful lemmas.

Lemma 3.7 *Let p be a uniformly continuous endomorphism of A_R^* and let Φ be its continuous extension to A_R^∞. For every $\alpha \in A_R^\omega$, we have $\alpha \in \mathrm{Fix}\Phi$ if and only if $\lim_{n\to+\infty} |\sigma(\alpha^{[n]})| = +\infty$.*

Proof. Assume that $\alpha \in \mathrm{Fix}\Phi$ and let $k \in \mathbb{N}$. Since $\alpha = \alpha\Phi = \lim_{n\to+\infty} \alpha^{[n]}\varphi$ by continuity, there exists some $m \in \mathbb{N}$ such that

$$n \geq m \Rightarrow r(\alpha^{[n]}\varphi, \alpha) > k.$$

We can assume that $m \geq k$, yielding $r(\alpha^{[n]}\varphi, \alpha^{[n]}) > k$ and thus $|\sigma(\alpha^{[n]})| \geq k$ for every $n \geq m$. Therefore $\lim_{n\to+\infty} |\sigma(\alpha^{[n]})| = +\infty$.

Conversely, assume that $\lim_{n\to+\infty} |\sigma(\alpha^{[n]})| = +\infty$. Suppose that $\alpha\Phi \neq \alpha$. Then $r(\alpha\Phi, \alpha) = k$ for some $k \in \mathbb{N}$. Since $\lim_{n\to+\infty} |\sigma(\alpha^{[n]})| = +\infty$, there exists some $m \in \mathbb{N}$ such that

$$n \geq m \Rightarrow |\sigma(\alpha^{[n]})| > k$$

and so

$$n \geq m \Rightarrow r(\alpha^{[n]}\varphi, \alpha) > k.$$

By continuity of Φ, we get $r(\alpha\Phi, \alpha) > k$, a contradiction. Therefore $\alpha\Phi = \alpha$ as required. $\square$

Lemma 3.8 *Let $u, v \in A_R^*$ be such that $C(u) = C(v)$ and let $\alpha \in A_R^\infty$. Then $u\alpha \in \mathrm{Fix}\Phi$ if and only if $v\alpha \in \mathrm{Fix}\Phi$.*

Proof. The case $\alpha \in A_R^*$ follows from Lemma 3.3, hence we may assume that $\alpha \in A_R^\omega$. Assume that $u\alpha \in \mathrm{Fix}\Phi$. By Lemma 3.7, we get $\lim_{n\to+\infty} |\sigma(u\alpha^{[n]})| = +\infty$. Since $C(u) = C(v)$, it follows easily that $\lim_{n\to+\infty} |\sigma(v\alpha^{[n]})| = +\infty$ as well, and so $v\alpha \in \mathrm{Fix}\Phi$ as required. $\square$

Lemma 3.9 *Let p be a uniformly continuous endomorphism of A_R^*. Then there exist constants $K_1, K_2 > 0$ such that*

$$|\sigma(u)| \geq \frac{|u|}{K_1} - K_2$$

for every $C(u) \in Q'$.

Proof. Since φ is uniformly continuous, it follows from Theorem 2.5 that there exists some $K_1 > 0$ such that

$$|u| \geq K_1 \Rightarrow |u\varphi| > 2M_\varphi$$

holds for every $u \in A_R^*$. Let $C(u) \in Q'$. We may write $u = u_1 \dots u_{s+1}$ with $|u_1| = \dots |u_s| = K_1$ and $|u_{s+1}| < K_1$. Now $|u_i\varphi| \geq 2M_\varphi + 1$ for $i = 1, \dots, s$ and Proposition 3.1 yields

$$|u\varphi| = |\overline{(u_1\varphi)\dots(u_{s+1}\varphi)}| \geq (2M_\varphi + 1)s - 2M_\varphi s = s \geq \frac{|u|}{K_1} - 1. \tag{3}$$

We claim that

$$|\sigma(u)| \geq \frac{|u|}{K_1} - M_\varphi - 1. \tag{4}$$

Suppose not. Then (3) yields $|\rho(u)| > M_\varphi$. On the other hand, $|u| > \frac{|u|}{K_1} - M_\varphi - 1$ yields $\tau(u) \neq 1$. This contradicts $C(u) \in Q'$ and so (4) holds. $\square$

Given a finite automaton $\mathcal{A} = (Q, q_0, T, E)$, we denote by $L_\omega(\mathcal{A})$ the set of words $\alpha \in A_R^\omega$ such that there is an infinite path $q_0 \xrightarrow{\alpha} \dots$ in $\mathcal{A}$. We write $L_\infty(\mathcal{A}) = L(\mathcal{A}) \cup L_\omega(\mathcal{A})$.

Theorem 3.10 *Let p be a finite-splitting uniformly continuous endomorphism of A_R^* and let Φ be its continuous extension to A_R^∞. Then there exist $L_1, \dots, L_s \in RatA^*$ and $\alpha_1, \dots, \alpha_s \in A_R^\omega$ such that*

$$\mathrm{Fix}\Phi = L_\infty(\mathcal{A}_\varphi) \cup L_1\alpha_1 \cup \dots \cup L_s\alpha_s. \tag{5}$$

Proof. By Lemma 3.4, we have $L(\mathcal{A}_\varphi) \subseteq \mathrm{Fix}\Phi$. Next let $\alpha \in L_\omega(\mathcal{A}_\varphi)$. Since the vertex set Q is finite, we can bound $|\tau(u)|$ and $|\rho(u)|$ for $C(u) \in Q$ by some $N \in \mathbb{N}$. In particular, we have $|\tau(\alpha^{[n]})|, |\rho(\alpha^{[n]})| \leq N$ for every $n \in \mathbb{N}$ and so $|\sigma(\alpha^{[n]})| \geq n - N$ for every $n \in \mathbb{N}$. Hence, for every $k \in \mathbb{N}$,

$$n \geq k + N \Rightarrow |\sigma(\alpha^{[n]})| \geq k$$

and so $\lim_{n\to+\infty}|\sigma(\alpha^{[n]})| = +\infty$. Thus $\alpha \in \mathrm{Fix}\Phi$ by Lemma 3.7 and so $L_\omega(\mathcal{A}_\varphi) \subseteq \mathrm{Fix}\Phi$.

Now we remark that in $Q' \setminus Q$ there are only finitely many vertices $C(v)$ such that $(C(u), a, C(v)) \in E'$ for some $a \in A$ and some $C(u) \in Q$. Let P be the subset of all such vertices $C(v)$. Write $P = \{p_1, \ldots, p_s\}$. For $j = 1, \ldots, s$, let L_j denote the set of all reduced words ua such that there exists a path $q_0 \xrightarrow{u} q$ in $\mathcal{A}_\varphi$ and $(q, a, p_j) \in E'$. Clearly, L_j is a rational language. We claim that there exists an infinite path $p_j \xrightarrow{\alpha_j} \ldots$ in $\mathcal{A}'_\varphi$. Indeed, we have $p_j = C(v)$ for some $v \leq \alpha \in \mathrm{Fix}\Phi$. If $\alpha \in A_R^*$, then $vw \in \mathrm{Fix}\varphi = L(\mathcal{A}_\varphi)$ for some $w \in A_R^*$ and so $p_j = C(v) \in Q$, a contradiction. Hence $\alpha \in A_R^\omega$ and so we can fix an infinite path $p_j \xrightarrow{\alpha_j} \ldots$ in $\mathcal{A}'_\varphi$ for $\alpha_j = \alpha^{[|v|+1]}\alpha^{[|v|+2]} \ldots$. We show that $z\alpha_j \in \mathrm{Fix}\Phi$ for every $z \in L_j$.

Indeed, by Lemma 3.7 we only need to show that $\lim_{n\to+\infty}|\sigma(z\alpha_j^{[n]})| = +\infty$. Let $k \in \mathbb{N}$. Taking the constants $K_1, K_2 > 0$ from Lemma 3.9, choose $m \geq k$ such that $\frac{|z|+m}{K_1} - K_2 \geq k$. We show that

$$n \geq m \Rightarrow |\sigma(z\alpha_j^{[n]})| \geq k. \tag{6}$$

In fact, $n \geq m$ yields

$$|\sigma(z\alpha_j^{[n]})| \geq \frac{|z\alpha_j^{[n]}|}{K_1} - K_2 \geq \frac{|z| + m}{K_1} - K_2 \geq k$$

by Lemma 3.9. Hence (6) holds and so $\lim_{n\to+\infty}|\sigma(z\alpha_j^{[n]})| = +\infty$ as required.

Therefore

$$L_\infty(\mathcal{A}_\varphi) \cup L_1\alpha_1 \cup \ldots \cup L_s\alpha_s \subseteq \mathrm{Fix}\Phi.$$

It remains to prove the opposite inclusion. Let $\beta \in \mathrm{Fix}\Phi$. By Lemma 3.4, we may assume that $\beta \in A_R^\omega$.

Clearly, there exists an infinite path $C(1) \xrightarrow{\beta} \ldots$ in $\mathcal{A}'_\varphi$. If all vertices in this path lie in Q, then $\beta \in L_\omega(\mathcal{A}_\varphi)$. Hence we may assume that $C(\beta^{[m]})$ is the first vertex in $Q' \setminus Q$. Now $C(\beta^{[m-1]}) \in Q$ by minimality of m and so $C(\beta^{[m]}) \in P$, say $C(\beta^{[m]}) = p_j$. It follows that $\beta^{[m]} \in L_j$. Since $C(\beta^{[m]}) \notin Q$, no vertex in the infinite path $p_j \xrightarrow{\alpha_j} \ldots$ may belong to S. Hence $p_j \xrightarrow{\alpha_j} \ldots$ is the unique infinite path leaving p_j. Thus $\alpha_j = \beta^{(m+1)}\beta^{(m+2)} \ldots$ and so $\beta = \beta^{[m]}\alpha_j \in L_j\alpha_j$. Therefore (5) holds. $\square$

4 Boundary-injective endomorphisms

Given a uniformly continuous endomorphism φ of A_R^*, let $\Phi : A_R^\infty \to A_R^\infty$ denote its (unique) continuous extension. We say that φ is *boundary-injective* if Φ is injective. Note that if φ is injective, then Φ is proper by Theorem 2.5. Hence Φ is injective if both φ and the restriction $\Phi|_{A_R^\omega}$ are injective.

The next example shows that an injective endomorphism is not always boundary-injective:

Example 4.1 *Let $A = \{a, b, c, d\}$ and $\varphi : A^* \to A^*$ be the endomorphism defined by*

$$a \mapsto ab, \quad b \mapsto a, \quad c \mapsto cb, \quad d \mapsto bc.$$

Then φ is injective but not boundary-injective.

Proof. Since $\{ab, a, cb, bc\}$ is a suffix code (no word is a suffix of another), then $\{ab, a, cb, bc\}^*$ is free on $\{ab, a, cb, bc\}$ [3] and so φ is injective. However, the continuous extension Φ fails to be injective since

$$(ac^\omega)\Phi = a(bc)^\omega = (bd^\omega)\Phi.$$

$\square$

Given $w \in A^*$ and $n \in \mathbb{N}$, we denote by $\mathrm{Suff}_n(w)$ the suffix of length n of w if $|w| > n$, or w otherwise.

We produce next an alternative characterization of boundary-injectivity avoiding infinite words:

Lemma 4.2 *Let φ be a uniformly continuous endomorphism of A_R^*. Then the following conditions are equivalent:*

(i) φ is boundary-injective;

(ii) there exists some $p_\varphi \in \mathbb{N}$ such that, whenever $uv, uw \in A_R^*$,

$$r((uv)\varphi, (uw)\varphi) > p_\varphi + |u\varphi| \Rightarrow r(v, w) > 1.$$

Proof. Clearly, condition (ii) fails if and only if

$$\forall p \in \mathbb{N}\ \exists u_pv_p, u_pw_p \in A_R^*\ (r((u_pv_p)\varphi, (u_pw_p)\varphi) > p + |u_p\varphi| \wedge r(v_p, w_p) = 1). \qquad (7)$$

We show that we can bound $|u_p|$. Indeed, let $z = \mathrm{Suff}_{M_\varphi}(u_p\varphi)$ and let u'_p be the shortest suffix of u_p such that $z \leq_s u'_p\varphi$. We claim that we can replace u_p by u'_p in (7). Indeed, we may assume that $|u_p\varphi| \geq M_\varphi$, otherwise $|u_p|$ can be bounded by uniform continuity of φ. If $u\varphi = xz$ and $u'_p\varphi = yz$, it follows from Proposition 3.1 that x and y remain untouched in the reduction of both $(u_p\varphi)(v_p\varphi)$ and $(u'_p\varphi)(v_p\varphi)$. Hence

$$\begin{aligned} r((u_pv_p)\varphi, (u_pw_p)\varphi) > p + |u_p\varphi| &\Leftrightarrow r(\overline{z(v_p\varphi)}, \overline{z(w_p\varphi)}) > p + M_\varphi \\ &\Leftrightarrow r((u'_pv_p)\varphi, (u'_pw_p)\varphi) > p + |u'_p\varphi| \end{aligned}$$

and we may indeed replace u_p by u'_p in (7). Proceeding letter by letter in u_p from right to left, it is easy to see that $|u'_p\varphi| < M_\varphi + h_\varphi$. Since φ is uniformly continuous, it follows from Theorem 2.5 that $|u_p|$ can be bounded. Since we can refine the sequence (u_p, v_p, w_p) at our convenience, a simple application of the pigeonhole principle allows us to assume that u_p is constant. Therefore (7) is equivalent to

$$\exists u \in A_R^*\ \forall p \in \mathbb{N}\ \exists uv_p, uw_p \in A_R^*\ (r((uv_p)\varphi, (uw_p)\varphi) > p + |u\varphi| \wedge r(v_p, w_p) = 1)$$

and therefore to

$$\exists u \in A_R^*\ \forall p \in \mathbb{N}\ \exists uv_p, uw_p \in A_R^*\ (r((uv_p)\varphi, (uw_p)\varphi) > p \wedge r(v_p, w_p) = 1). \qquad (8)$$

We complete our proof by showing that (8) holds if and only if Φ is not injective.

Assume that (8) holds. Since A_R^∞ is compact, we can refine the sequence to make $(v_p)_p$ converge to some $\alpha \in A_R^\infty$, and further refinement can make $(w_p)_p$ converge to some $\beta \in A_R^\infty$. Clearly, $r(\alpha, \beta) = 1$ and so $u\alpha \neq u\beta$.

Since the mixed product is continuous and so is Φ, we have

$$(u\alpha)\Phi = \lim_{p\to+\infty}(uv_p)\varphi, \quad (u\beta)\Phi = \lim_{p\to+\infty}(uw_p)\varphi.$$

Now $r((uv_p)\varphi,(uw_p)\varphi) > p$ implies that the sequences $(uv_p)\varphi$ and $(uw_p)\varphi$ share the same limit, hence $(u\alpha)\Phi = (u\beta)\Phi$. Thus Φ is not injective.

Conversely, assume that Φ is not injective. Then there exist $u \in A_R^*$ and distinct $\alpha,\beta \in A_R^\infty$ such that $u\alpha, u\beta$ are reduced, $r(\alpha,\beta) = 1$ and $(u\alpha)\Phi = (u\beta)\Phi$. Let $\gamma = (u\alpha)\Phi$. For each $p \in \mathbb{N}$, since $\gamma = (u \lim_{n\to+\infty}\alpha^{[n]})\Phi = \lim_{n\to+\infty}(u\alpha^{[n]})\varphi$ by continuity, we can choose v_p to be a prefix of α such that $\gamma^{[p]} \leq (uv_p)\varphi$, with $v_p = \alpha$ if $\alpha \in A_R^*$. Similarly, let w_p be a prefix of β such that $\gamma^{[p]} \leq (uw_p)\varphi$, with $w_p = \beta$ if $\beta \in A_R^*$. Clearly, uv_p and uw_p are reduced.

Since Φ is proper by Theorem 2.5, $\alpha \in A_R^*$ if and only if $\gamma \in A_R^*$ if and only if $\beta \in A_R^*$. In that case, we get $(uv_p)\varphi = (uw_p)\varphi$. So in any case we get $r((uv_p)\varphi,(uw_p)\varphi) > p$. On the other hand, $r(\alpha,\beta) = 1$ yields $r(v_p,w_p) = 1$. Therefore (8) holds as required. □

The following lemmas will be helpful:

Lemma 4.3 *Let p be a uniformly continuous endomorphism of A_R^*.*

(i) If $uv \in R_A$, $\tau(u) \neq 1$ and $|\rho(u)| > M_\varphi$, then $\sigma(uv) = \sigma(u)$.

(ii) If $C(u) \in Q'$, then $\tau(u) = 1$ or $|\rho(u)| \leq M_\varphi$.

Proof. (i) Let a and b denote the first letter of $\tau(u)$ and $\rho(u)$, respectively. By definition of $\sigma(u)$, we have $a \neq b$. Now Proposition 3.1 yields

$$uv = \sigma(u)\tau(u)v \in \sigma(u)aA_R^*, \quad (uv)\varphi = \overline{\sigma(u)\rho(u)(v\varphi)} = \sigma(u)\overline{\rho(u)(v\varphi)} \in \sigma(u)bA_R^*$$

and so $\sigma(uv) = \sigma(u)$.

(ii) Suppose that $\tau(u) \neq 1$ and $|\rho(u)| > M_\varphi$. Since $C(u) \in Q'$, Lemma 3.8 implies $u \leq \alpha$ for some $\alpha \in \mathrm{Fix}\Phi$. Suppose first that $\alpha = uv \in A_R^*$. Then there is a path $C(u)\xrightarrow{v}C(\alpha) \in T'$ in $\mathcal{A}'_\varphi$. It follows from part (i) that $\sigma(uv) = \sigma(u) < uv$, contradicting $C(\alpha) \in T$.

Hence we may assume that $\alpha \in A_R^\omega$. By part (i), we get $\lim_{n\to+\infty}\sigma(\alpha^{[n]}) = \sigma(u)$, contradicting Lemma 3.7. Therefore the lemma is proved. □

Lemma 4.4 *Let p be a uniformly continuous endomorphism of A_R^*. If $C(u) \in Q'$, $\tau(u) = 1$ and $|\rho(u)| > 2M_\varphi$, then there is at most one edge out of $C(u)$ in $\mathcal{A}'_\varphi$. If a is its label, then $\tau(ua) = 1$.*

Proof. Write $\rho(u) = axy$ with $a \in A$ and $|x| = M_\varphi$. Suppose that there is an edge leaving $C(u)$ in $\mathcal{A}'_\varphi$ with label $b \neq a$. Then $C(ub) \in Q$ and $(ub)\varphi = \overline{\sigma(u)\rho(u)(b\varphi)} = \overline{u\rho(u)(b\varphi)}$. Since $|\rho(u)| > 2M_\varphi$, it follows from Proposition 3.1 that $uax \leq (ub)\varphi$. Thus $\sigma(ub) = u$, $\tau(ub) = b$ and $ax \leq \rho(ub)$ yields $|\rho(ub)| > M_\varphi$, contradicting $C(u) \in Q$. Hence $C(u)\xrightarrow{a}C(ua)$ is the only edge that can possibly leave $C(u)$. Since $uax \leq (ua)\varphi$, we get $\tau(ua) = 1$ as well. □

We can now prove:

Theorem 4.5 *Let p be a boundary-injective endomorphism of A_R^*. Then φ is uniformly continuous and finite-splitting.*

Proof. Since φ is injective, it is uniformly continuous by Theorem 2.5. We prove next that

(U) If $C(u) \in Q'$ and $|\tau(u)| > p_\varphi + M_\varphi$, then there is at most one edge out of $C(u)$ in $\mathcal{A}'_\varphi$.

Suppose there is more than one edge leaving $C(u)$. Then there exist distinct $a, b \in A$ labelling edges out of $C(u)$. By Lemma 3.8, we have $ua\alpha, ub\beta \in \mathrm{Fix}\Phi$ for some $\alpha, \beta \in A_R^\infty$. Since $r(a\alpha, b\beta) = 1$, it follows easily through continuity from the definition of p_φ that $r((ua\alpha)\Phi, (ub\beta)\Phi) \leq p_\varphi + |u\varphi|$. Thus $r((ua\alpha)\Phi, u) \leq p_\varphi + |u\varphi|$ or $r((ub\beta)\Phi, u) \leq p_\varphi + |u\varphi|$. Without loss of generality, we may assume that $r((ua\alpha)\Phi, u) \leq p_\varphi + |u\varphi|$.

Since $C(u) \in Q'$ and $|\tau(u)| \neq 1$, it follows from Lemma 4.3(ii) that $|\rho(u)| \leq M_\varphi$. Thus $|\tau(u)| > p_\varphi + M_\varphi$ yields $|\tau(u)| > p_\varphi + |\rho(u)|$ and thus $|u| > p_\varphi + |u\varphi|$. Together with $r((ua\alpha)\Phi, u) \leq p_\varphi + |u\varphi|$, this implies $r((ua\alpha)\Phi, ua\alpha) \leq p_\varphi + |u\varphi|$, contradicting $ua\alpha \in \mathrm{Fix}\Phi$. Therefore there is exactly one edge out of $C(u)$. Therefore (U) holds.

Let $C(u) \in S$. By Lemma 4.3(ii), we have $\tau(u) = 1$ or $|\rho(u)| \leq 2M_\varphi$. By (U), we get $|\tau(u)| \leq p_\varphi + M_\varphi$. Finally, by Lemma 4.4, it follows that $\tau(u) = 1$ implies $|\rho(u)| \leq 2M_\varphi$. Therefore we must have

$$|\tau(u)| \leq p_\varphi + M_\varphi, \quad |\rho(u)| \leq 2M_\varphi$$

in any case. By Lemma 3.2, $|\sigma''(u)|$ is bounded and it follows from the definition that $|\lambda(u)|$ is bounded as well. Hence S is finite and so φ is finite-splitting. □

Now we get from Theorems 3.5, 3.10 and 4.5:

Corollary 4.6 *Let p be a boundary-injective endomorphism of A_R^*. Then* $\mathrm{Fix}\varphi \in \mathit{Rat}A^*$.

Corollary 4.7 *Let p be a boundary-injective endomorphism of A_R^* and let Φ be its continuous extension to A_R^∞. Then there exist $L_1, \ldots, L_s \in \mathit{Rat}A^*$ and $\alpha_1, \ldots, \alpha_s \in A_R^\omega$ such that*

$$\mathrm{Fix}\Phi = L_\infty(\mathcal{A}_\varphi) \cup L_1\alpha_1 \cup \ldots \cup L_s\alpha_s.$$

Properties such as injectivity or boundary-injectivity turn out to be decidable for uniformly continuous endomorphisms as we prove next.

Theorem 4.8 *Let φ be a uniformly continuous endomorphism of A_R^*. Then it is decidable whether or not*

(i) φ is injective;

(ii) φ is boundary-injective.

Proof. (i) Since φ is uniformly continuous, it follows from Theorem 2.5 that there exists $r \in \mathbb{N}$ such that

$$\forall w \in A_R^* \, (|w| \geq r \Rightarrow |w\varphi| \ > 2M_\varphi). \tag{9}$$

Moreover, it follows from the proof of [6, Theor. 8.4] that r can be effectively computed. Suppose that φ is not injective. Among all distinct $u, v \in A_R^*$ such that $u\varphi = v\varphi$, let $|u|$ be minimal, and let $|v|$ be minimal for such u. We show that $|u|$ and $|v|$ can be bounded.

We prove first that

$$r(u,v) \leq r. \tag{10}$$

Indeed, suppose that $r(u,v) > r$. Then we may write $u = awu'$, $v = awv'$ with $a \in A$ and $|w| \geq r$. Since $u \neq v$, we have $wu' \neq wv'$. Write $w\varphi = z_1z_2z_3$ with $|z_1| = |z_3| = M_\varphi$. It follows from Proposition 3.1 that

$$u\varphi = \overline{(a\varphi)z_1z_2z_3(u'\varphi)} = \overline{(a\varphi)z_1}z_2\overline{z_3(u'\varphi)}.$$

Similarly, $v\varphi = \overline{(a\varphi)z_1}z_2\overline{z_3(v'\varphi)}$. Hence $u\varphi = v\varphi$ yields

$$(wu')\varphi = z_1z_2\overline{z_3(u'\varphi)} = z_1z_2\overline{z_3(v'\varphi)} = (wv')\varphi,$$

contradicting the minimality of $|u|$. Thus (10) holds.

Assume that $|u| = m$ and $|v| = n$. For all $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n\}$, let

$$u'_i = \mathrm{Suff}_{t_R-1}(u^{[i]}), \quad v'_j = \mathrm{Suff}_{t_R-1}(v^{[j]}).$$

Let

$$P = \{(i,j) \in \{r,\ldots,m\} \times \{r,\ldots,n\} : 0 \leq |u^{[i]}\varphi| - |v^{[j]}\varphi| \leq h_\varphi\}.$$

For every $(i,j) \in P$, we define $\mu(i,j) \in A_R^* \times A_R^*$ as follows: if $d = |u^{[i]}\varphi| - |v^{[j]}\varphi|$, let

$$\mu(i,j) = (\mathrm{Suff}_{M_\varphi+d}(u^{[i]}\varphi), \mathrm{Suff}_{M_\varphi}(v^{[j]}\varphi)).$$

We show that, for all $(i,j),(k,l) \in P$,

$$(u'_i, v'_j, \mu(i,j)) = (u'_k, v'_l, \mu(k,l)) \Rightarrow (i,j) = (k,l). \tag{11}$$

Assume that $(u'_i, v'_j, \mu(i,j)) = (u'_k, v'_l, \mu(k,l))$. By symmetry, it suffices to show that the following two cases are impossible:

- Case I: $i = k$ and $j < l$;
- Case II: $i < k$.

Case I: Let $w = v^{[j]}v^{[l+1,n]}$. Since $v'_l v^{[l+1,n]}$ is irreducible and $v'_j = v'_l$, we have $w \in A_R^*$. It follows from (10) that $r(u,v) \leq r$ and so $j \geq r$ yields $r(u,w) = r(u,v) \leq r$. Thus $u \neq w$. We show that $w\varphi = u\varphi$. Indeed, $\mu(i,j) = \mu(i,l)$ implies $|u^{[i]}\varphi| - |v^{[l]}\varphi| = |u^{[i]}\varphi| - |v^{[j]}\varphi| = d$ and so $|v^{[j]}\varphi| = |v^{[l]}\varphi|$. By Proposition 3.1, $v^{[j]} < v^{[l]}$ implies that $v^{[j]}\varphi$ and $v^{[l]}\varphi$ can only differ in its suffix of length M_φ. Hence $\mathrm{Suff}_{M_\varphi}(v^{[j]}\varphi) = \mathrm{Suff}_{M_\varphi}(v^{[l]}\varphi)$ yields $v^{[j]}\varphi = v^{[l]}\varphi$, contradicting the minimality of $|u|$.

Case II: Let $w = u^{[i]}u^{[k+1,m]}$ and $z = v^{[j]}v^{[l+1,n]}$. Similarly to Case I, we have $w,z \in A_R^*$. It follows from (10) that $r(u,v) \leq r$ and so $i,j \geq r$ yields $r(w,z) = r(u,v) \leq r$. Thus $w \neq z$. We show that $w\varphi = z\varphi$. Indeed, $\mu(i,j) = \mu(k,l)$ implies $|u^{[k]}\varphi| - |v^{[l]}\varphi| = |u^{[i]}\varphi| - |v^{[j]}\varphi| = d$. Moreover, we may write

$$u^{[i]}\varphi = x_ix, \quad u^{[k]}\varphi = x_kx, \quad v^{[j]}\varphi = y_jy, \quad v^{[l]}\varphi = y_ly$$

with $|x| = M_\varphi + d$ and $|y| = M_\varphi$ (since $j,l \geq r$ and in view of (9)). Now Proposition 3.1 yields

$$u\varphi = \overline{(u^{[k]}\varphi)(u^{[k+1,m]}\varphi)} = \overline{x_kx(u^{[k+1,m]}\varphi)} = x_k\overline{x(u^{[k+1,m]}\varphi)},$$

$$v\varphi = \overline{(v^{[l]}\varphi)(v^{[l+1,n]}\varphi)} = \overline{y_l y(v^{[l+1,n]}\varphi)} = y_l \overline{y(v^{[l+1,n]}\varphi)}.$$

Since $|x_k x| = |y_l y| + d$, we get $|x_k| = |y_l|$, hence $x_k = y_l$ and

$$\overline{x(u^{[k+1,m]}\varphi)} = \overline{y(v^{[l+1,n]}\varphi)}.$$

Similarly, we have $x_i = y_j$ and so

$$\begin{aligned} w\varphi &= \overline{(u^{[i]}\varphi)(u^{[k+1,m]}\varphi)} = \overline{x_i x(u^{[k+1,m]}\varphi)} = x_i \overline{x(u^{[k+1,m]}\varphi)} \\ &= y_j \overline{y(v^{[l+1,n]}\varphi)} = \overline{y_j y(v^{[l+1,n]}\varphi)} = \overline{(v^{[j]}\varphi)(v^{[l+1,n]}\varphi)} = z\varphi, \end{aligned}$$

contradicting the minimality of $|u|$. Therefore (10) holds.

Clearly, we can bound the number of possibilities for $(u'_i, v'_j, \mu(i,j))$, hence we can bound $|P|$ by (11). Next we show that

$$|u\varphi| \; < \; (|P| + r + 1)h_\varphi. \tag{12}$$

Indeed, assume that q is the integer quotient of the division of $|u\varphi|$ by h_φ. For $s = 1, \ldots, q$, let i_s be the least integer $i \in \{1, \ldots, m\}$ such that $|u^{[i]}\varphi| \geq sh_\varphi$; let j_s be the least integer $j \in \{1, \ldots, n\}$ such that $0 \leq |u^{[i]}\varphi| - |v^{[j]}\varphi| \leq h_\varphi$. It is easy to see that i_s and j_s are well defined and

$$i_s, j_s \geq r \Rightarrow (i_s, j_s) \in P.$$

Moreover, $i_1 < \ldots < i_q$. Now $i_s h_\varphi \geq |u^{[i_s]}\varphi| \geq sh_\varphi$ and so $s \geq r$ implies $i_s \geq r$. Similarly, $j_s h_\varphi \geq |v^{[j_s]}\varphi| \geq (s-1)h_\varphi$ and so $s \geq r+1$ implies $j_s \geq r$. Hence $(i_s, j_s) \in P$ whenever $s \geq r+1$ and so

$$|P| \geq q - r > \frac{|u\varphi|}{h_\varphi} - 1 - r$$

and so (12) holds.

Thus we can bound $|u\varphi|$, and since φ is uniformly continuous, we can bound $|u|$ and $|v|$ as well. Therefore we can effectively compute some $N \in \mathbb{N}$ such that φ is not injective if and only if $u\varphi = v\varphi$ holds for some distinct $u, v \in A_R^*$ of length $\leq N$. This is certainly decidable.

(ii) We may assume that φ is injective. Since Φ is proper by Theorem 2.5, we must decide whether or not $\alpha\Phi = \beta\Phi$ for some distinct $\alpha, \beta \in A_R^\omega$.

Clearly, we can determine an upper bound K for the number of distinct values that $(u'_i, v'_j, \mu(i,j))$ can take for arbitrary words u and v. Since φ is uniformly continuous, there exists some $r' \in \mathbb{N}$ and $N' > r$ such that

$$|w| \; > r' \Rightarrow |w\varphi| \; > 3M_\varphi \tag{13}$$

$$|w| \geq N' \Rightarrow |w\varphi| \geq (Kr(r'+r+2) + r + 1)h_\varphi + M_\varphi \tag{14}$$

hold for every $w \in A_R^*$. Moreover, it follows from the proof of [6, Theor. 8.4] that r' and N' can be effectively computed. We show that φ is not boundary-injective if and only if there exist $u, v \in A_R^*$ satisfying

$$r(u,v) \leq r, \quad |u|, |v| = N', \quad r(u\varphi, v\varphi) \geq |v\varphi| - M_\varphi. \tag{15}$$

Assume first that φ is not boundary-injective. Since φ is injective, then $\alpha\Phi = \beta\Phi$ for some distinct $\alpha, \beta \in A_R^\omega$. Similarly to the proof of (10), we may assume that $r(\alpha, \beta) \leq r$. Let $u = \alpha^{[N']}$ and $v = \beta^{[N']}$. Since $r(\alpha, \beta) \leq r$ and $N' > r$, we have $r(u, v) = r(\alpha, \beta) \leq r$. Since $\alpha\Phi = \beta\Phi$, it follows from Proposition 3.1 and continuity that

$$r(u\varphi, \alpha\Phi) \geq |u\varphi| - M_\varphi, \quad r(v\varphi, \beta\Phi) \geq |v\varphi| - M_\varphi.$$

Thus

$$r(u\varphi, v\varphi) \geq \min\{|u\varphi|, |v\varphi|\} - M_\varphi.$$

Exchanging the roles of u and v if necessary, we obtain $r(u\varphi, v\varphi) \geq |v\varphi| - M_\varphi$. Therefore (15) holds.

Conversely, assume that there exist $u, v \in A_R^*$ satisfying (15). We consider P as in the proof of part (i). The proof of (12) can be easily adapted to show that

$$r(u\varphi, v\varphi) < (|P| + 1 + r)h_\varphi,$$

hence (14) and (15) yield

$$(|P| + r + 1)h_\varphi + M_\varphi > \ |v\varphi| \geq (Kr(r' + r + 2) + r + 1)h_\varphi + M_\varphi$$

and so $|P| \geq Kr(r' + r + 2)$. Thus there exist $r(r' + r + 2)$ distinct elements $(i, j) \in P$ such that $(u'_i, v'_j, \mu(i, j))$ remains constant.

Assume that $(i, j) \neq (k, l)$ and $(u'_i, v'_j, \mu(i, j)) = (u'_k, v'_l, \mu(k, l))$. We claim that

$$i \neq k \qquad \text{and} \qquad j \neq l. \tag{16}$$

It suffices to exclude the case $i = k$ and $j < l$, the others being analogous. Suppose that $i = k$ and $j < l$. Since $\mu(i, j) = \mu(i, l)$, we have $|v^{[j]}\varphi| = |v^{[l]}\varphi|$. Write $v^{[j]}\varphi = x_j y_j$ and $v^{[l]}\varphi = x_l y_l$ with $|y_j| = |y_l| = M_\varphi$. Then $|x_j| = |x_l|$ and $v^{[j]} < v^{[l]}$ implies $x_j = x_l$ by Proposition 3.1. On the other hand, $\mu(i, j) = \mu(i, l)$ also yields $y_j = y_l$ and so $v^{[j]}\varphi = x_j y_j = x_l y_l = v^{[l]}\varphi$. Since φ is injective, we get $v^{[j]} = v^{[l]}$ and so $j = l$, a contradiction.

Hence (16) holds and we may assume that $i < k$. We show that

$$j \leq l + r'. \tag{17}$$

Indeed, suppose that $j > l + r'$. Since $\mu(i, j) = \mu(k, l)$, we have

$$|v^{[l]}\varphi| - |v^{[j]}\varphi| = |u^{[k]}\varphi| - |u^{[i]}\varphi|.$$

By Proposition 3.1, we have

$$|u^{[k]}\varphi| \geq |u^{[i]}\varphi| - M_\varphi,$$

$$|v^{[j]}\varphi| \geq |v^{[l]}\varphi| + |v^{[l+1,j]}\varphi| - 2M_\varphi.$$

Hence

$$|v^{[l+1,j]}\varphi| \leq |v^{[j]}\varphi| - |v^{[l]}\varphi| + 2M_\varphi = |u^{[i]}\varphi| - |u^{[k]}\varphi| + 2M_\varphi \leq 3M_\varphi$$

and so $|v^{[l+1,j]}| \leq r'$ by (13). Thus $j - l \leq r'$ and (17) holds.

Let $t = r(r'+r+2)$ and let $(i_1,j_1),\ldots,(i_t,j_t) \in P$ be distinct and satisfy

$$(u'_{i_1}, v'_{j_1}, \mu(i_1,j_1)) = \ldots = (u'_{i_t}, v'_{j_t}, \mu(i_t,j_t)).$$

By (16), we may assume $i_1 < \ldots < i_t$. Since $j_r, j_{2r}, \ldots, j_{r(r'+r+2)}$ are all distinct by (16), there exist $1 \leq q_1 < \ldots < q_{r'+2} \leq r'+r+2$ such that $j_{q_e} \leq |v| - r$ for $e = 1,\ldots,r'+2$. Suppose that

$$j_{rq_1} > j_{rq_2} > \ldots > j_{rq_{r'+2}}.$$

Then $j_{rq_1} > j_{rq_{r'+2}} + r'$, contradicting (17). Hence $j_{rq_e} < j_{rq_f}$ for some $1 \leq e < f \leq r'+2$. Hence there exist some $(i,j),(k,l) \in P$ such that

$$(u'_i, v'_j, \mu(i,j)) = (u'_k, v'_l, \mu(k,l)), \quad i+r \leq k, \quad j < l \leq |v| - r.$$

Let

$$\alpha = u^{[i]}(u^{[i+1,k]})^\omega, \quad \beta = v^{[j]}(v^{[j+1,l]})^\omega.$$

Since $i,j \geq r$, we have $r(\alpha,\beta) = r(u,v) \leq r$ and so $\alpha \neq \beta$. Moreover, since $u'_i u^{[i+1,k]}$ is irreducible and $u'_i = u'_k$, we have $u^{[i]}(u^{[i+1,k]})^2 \in A_R^*$. A straightforward induction shows that the $t_R - 2$ letters preceding each $u^{[k]}$ in $u^{[i]}(u^{[i+1,k-1]}u^{[k]})^\omega$ are the same, hence $\alpha \in A_R^\omega$. Similarly, $\beta \in A_R^\omega$.

Since $k - i \geq r$, we have $|u^{[i+1,k]}\varphi| > 2M_\varphi$ by (9) and so $|u^{[k]}\varphi| > |u^{[i]}\varphi|$ by Proposition 3.1. Write $v^{[j]}\varphi = xy$ with $|y| = M_\varphi$. Then x remains untouched in the reduction $\overline{(v^{[j]}\varphi)(v^{[j+1,l]}\varphi)}$. Since $\mu(i,j) = \mu(k,l)$ and $|u^{[i]}\varphi| < |u^{[k]}\varphi|$, it follows that $v^{[l]}\varphi = xzy$ for some $z \in A_R^*$. Write $u^{[i]}\varphi = x'w$ with $|x'| = |x|$. Since $\mu(i,j) = \mu(k,l)$, we may write $u^{[k]}\varphi = x''w$ with $|x''| = |xz|$. Now $xz \leq v\varphi$ and $x', x'' \leq u\varphi$ by Proposition 3.1. We claim that $|v^{[l]}\varphi| < |v\varphi|$. Indeed, we have $v = v^{[l]}w$ with $|w| \geq r$. By (9), we get $|w\varphi| > 2M_\varphi$ and so Proposition 3.1 yields

$$|v\varphi| \geq |v^{[l]}\varphi| + |w\varphi| - 2M_\varphi > |v^{[l]}\varphi|.$$

Since $r(u\varphi, v\varphi) \geq |v\varphi| - M_\varphi > |v^{[l]}\varphi| - M_\varphi$, it follows that $xz \leq u\varphi$ and so $x' = x$, $x'' = xz$. Thus

$$u^{[i]}\varphi = xw, \quad u^{[k]}\varphi = xzw, \quad v^{[j]}\varphi = xy, \quad v^{[l]}\varphi = xzy.$$

Let $g = u^{[i+1,k]}\varphi$ and $h = v^{[j+1,l]}\varphi$. It follows easily that $\overline{wg} = zw$ and $\overline{yh} = zy$ since x remains untouched in the reduction of both $\overline{xwg}$ and $\overline{xyh}$. Hence

$$\alpha'\Phi = \lim_{n\to+\infty} \overline{xwg^n} = \lim_{n\to+\infty} \overline{xz^n w}.$$

Since Φ is proper by Theorem 2.5, z has not finite order and so

$$\alpha'\Phi = \lim_{n\to+\infty} \overline{xz^n}.$$

Similarly, $\beta'\Phi = \lim_{n\to+\infty} \overline{xz^n}$ and so Φ is not injective.

Therefore (15) holds and decidability follows. $\square$

5 Bounded length decrease

It should be clear that boundary-injectivity is not a necessary condition for a uniformly continuous endomorphism to produce a rational fixed point set: it suffices to take some non-injective nonerasing endomorphism of a free monoid. We introduce now a class of endomorphisms that covers in particular such cases.

We say that an endomorphism φ of A_R^* has *bounded length decrease* if

$$\exists d_\varphi \in \mathbb{N}\ \forall u \in A_R^*\ |u| - |u\varphi| \leq d_\varphi.$$

We note that d_φ can be arbitrarily large, as the next example shows.

Example 5.1 *Let $n \geq 2$, $A = \{a, b\}$, $R = \{(a^n, 1)\}$ and $\varphi : A_R^* \to A_R^*$ be the endomorphism defined by*

$$a \mapsto a^{n-1}, \quad b \mapsto b^n.$$

Then φ has bounded length decrease and $d_\varphi = n - 2$.

Proof. It is immediate that φ has bounded length decrease. Since $a^{n-1}\varphi = a$, it follows easily that $d_\varphi = n - 2$. □

We have:

Theorem 5.2 *Let p be an endomorphism of A_R^* with bounded length decrease. Then φ is uniformly continuous and finite-splitting.*

Proof. It follows from Theorem 2.5 that φ is uniformly continuous. We adapt the proof of Theorem 4.5 and prove that S is finite also in this case. We replace (U) by

$$\forall u \in A_R^*\ (|\rho(u)| \leq M_\varphi \Rightarrow |\tau(u)| \leq d_\varphi + M_\varphi). \tag{18}$$

This follows immediately from $|u| - |u\varphi| = |\tau(u)| - |\rho(u)|$.

Let $C(u) \in S$. By Lemma 4.3(ii), we have $\tau(u) = 1$ or $|\rho(u)| \leq 2M_\varphi$. By (18), we get $|\tau(u)| \leq d_\varphi + M_\varphi$. Finally, by Lemma 4.4, it follows that $\tau(u) = 1$ implies $|\rho(u)| \leq 2M_\varphi$. Therefore we must have

$$|\tau(u)| \leq d_\varphi + M_\varphi, \quad |\rho(u)| \leq 2M_\varphi$$

in any case and so φ is finite-splitting as in the proof of Theorem 4.5. □

Now we get from Theorems 3.5, 3.10 and 5.2:

Corollary 5.3 *Let p be an endomorphism of A_R^* having bounded length decrease. Then* Fix$\varphi \in RatA^*$.

Corollary 5.4 *Let p be an endomorphism of A_R^* having bounded length decrease and let Φ be its continuous extension to A_R^∞. Then there exist $L_1, \ldots, L_s \in RatA^*$ and $\alpha_1, \ldots, \alpha_s \in A_R^\omega$ such that*

$$\text{Fix}\Phi = L_\infty(\mathcal{A}_\varphi) \cup L_1\alpha_1 \cup \ldots \cup L_s\alpha_s. \tag{19}$$

We can prove that both Fixφ and the decomposition in Corollary 5.4 can be made effectively constructible. Given $\alpha \in A_R^\omega$, we say that α is effectively constructible if $\alpha^{(n)}$ can be computed for every $n \geq 1$.

Theorem 5.5 *Let p be an endomorphism of A_R^* having bounded length decrease. Then we can effectively construct finite A-automata $\mathcal{A}, \mathcal{A}_1, \mathcal{A}_s$ and $\alpha_1, \ldots, \alpha_s \in A_R^\omega$ such that*

$$\mathrm{Fix}\Phi = L_\infty(\mathcal{A}) \cup L(\mathcal{A}_1)\alpha_1 \cup \ldots \cup L(\mathcal{A}_s)\alpha_s.$$

In particular, $\mathrm{Fix}\varphi = L(\mathcal{A})$.

Proof. Let $\mathcal{A}'' = (Q'', q_0, T'', E'')$ be the deterministic A-automaton defined by

- $Q'' = \{C(u) \mid u \in A_R^*,\ \tau(u) = 1\} \cup \{C(u) \mid u \in A_R^*,\ |\rho(u)| \leq M_\varphi\}$;
- $q_0 = C(1)$;
- $T'' = \{C(u) \in Q'' \mid \tau(u) = \rho(u) = 1\}$;
- $E'' = \{(C(u), a, C(v)) \in Q'' \times A \times Q'' \mid v = ua\}$.

In view of Lemma 3.3, E'' and therefore $\mathcal{A}''$ are well defined. By Lemma 4.3(ii), $Q' \subseteq Q''$ and so $\mathcal{A}'_\varphi$ is a subautomaton of $\mathcal{A}''$. In fact, it follows from Lemma 3.4 that $\mathcal{A}'_\varphi$ is a subautomaton of $\mathrm{acc}(\mathcal{A}'')$, the accessible part of $\mathcal{A}''$ (consisting of all those vertices and edges that are accessible from the initial vertex). It is straightforward to check that we can replace $\mathcal{A}'_\varphi$ by $\mathcal{A}''$ in the constructions and proofs of Section 3 and the present one, with one simple adaptation: in the proof of Theorem 3.10, we cannot assume anymore the existence of an infinite path out of every vertex of P, so we only keep in P those vertices having that property. We remark that we did not define $\mathcal{A}'_\varphi$ this way because this adaptation does not hold for boundary-injective endomorphisms. While (18) remains valid and able to fulfill its role, (U) does not hold anymore.

Now, to simplify notation, we keep the notation $\mathcal{A}_\varphi$, P, and so on, but assumed to be built upon $\mathcal{A}''$. We show that $\mathcal{A}_\varphi$ is effectively constructible.

Since the proof of Lemma 4.4 still holds when we replace Q' by Q'', we have

$$\forall C(u) \in S \qquad |\rho(u)| \leq 2M_\varphi. \tag{20}$$

We can also prove that

$$|\rho(u)| > d_\varphi + 4M_\varphi \Rightarrow |\rho(uv)| > 2M_\varphi \tag{21}$$

holds for every $C(uv) \in Q''$.

Indeed, we have $\tau(u) = 1$ and so $(uv)\varphi = \overline{u\rho(u)(v\varphi)} = u\overline{\rho(u)(v\varphi)}$. Now Proposition 3.1 yields

$$|\overline{\rho(u)(v\varphi)}| \geq |\rho(u)| + |v\varphi| - 2M_\varphi > d_\varphi + 2M_\varphi + |v| - d_\varphi = |v| + 2M_\varphi$$

and so $|(uv)\varphi| - |uv| > 2M_\varphi$. Thus (21) holds.

In view of (18), (20) and (21), it follows that to compute S we only have to care about finitely many $C(u)$: those satisfying

$$|\tau(u)| \leq d_\varphi + M_\varphi, \quad |\rho(u)| \leq d_\varphi + 4M_\varphi.$$

Hence S can be effectively determined. Once again, it follows from (18), (20) and (21) that Q and therefore $\mathcal{A}_\varphi$ can be effectively constructed.

Clearly, we can determine all $C(v) \in Q'' \setminus Q$ which admit an edge $C(u) \xrightarrow{a} C(v)$ for some $C(u) \in Q$ and $a \in A$. If we can determine whether such a $C(v)$ belongs to P, i.e., whether it admits an infinite path $C(v) \xrightarrow{\alpha} \ldots$ in $\mathcal{A}''$, then we can construct both finite automata recognizing the languages L_j and the infinite words α_j (we can compute prefixes of arbitrary length since there is a unique (infinite) path leaving each vertex of P).

Consider then such a $C(v)$, a potential member of P. Since $C(v) \notin Q$, there is a unique path leaving $C(v)$: we cannot reach any vertex in S. In view of Lemma 4.3(ii) and (18), any vertex $C(w)$ accessible from $C(v)$ in $\mathcal{A}''$ must satisfy $|\tau(u)| \leq d_\varphi + M_\varphi$. Thus, if we extend our path sufficiently, we will end up either coming to a dead end (and then $C(v) \notin P$), or entering a loop (in which case $C(v) \in P$ and the corresponding infinite word α is eventually periodic) or reaching a vertex $C(w)$ with $|\rho(w)| \ > d_\varphi + 4M_\varphi$. Note that in that case $\tau(w) = 1$ by Lemma 4.3(ii) and the combination of (21) and Lemma 4.4 make sure that the path can always be extended. Thus $C(v) \in P$ in that case and the proof is completed. $\square$

We consider now the problem of deciding bounded length decrease. We say that $u \in A_R^*$ is *cyclically reduced* if $u^\omega \in A_R^\omega$.

Lemma 5.6 *Let φ be a uniformly continuous endomorphism of A_R^*. The following conditions are equivalent:*

(i) φ has bounded length decrease;

(ii) $\forall u \in A_R^* \ (u^\omega \in A_R^\omega \Rightarrow |u\varphi| \geq |u|)$.

Proof. (i) $\Rightarrow$ (ii). Suppose that u is cyclically reduced and $|u| - |u\varphi| = k > 0$. Then $|u^n| - |u^n\varphi| \geq nk$ for every $n > 0$ and so φ has not bounded length decrease.

(ii) $\Rightarrow$ (i). Suppose that φ has not bounded length decrease. Let

$$K = |\{u \in A_R^* : |u| \leq t_R - 1\}|, \quad N = (K+1)|A| + 4KM_\varphi. \tag{22}$$

Then $|u| - |u\varphi| \ > N$ for some $u \in A_R^*$, which we may assume to have minimum length n. For $i = 0, \ldots, n$, let

$$s_i = \mathrm{Suff}_{t_R}(u^{[i]}), \quad S = \{s_i \mid i = 0, \ldots, n\}.$$

We define a sequence of integers

$$1 \leq i_1 < j_1 < i_2 < j_2 < \ldots < i_k < j_k \leq n$$

satisfying

$$s_{i_l} = s_{j_l}, \tag{23}$$

$$\forall m > j_l \ s_m \neq s_{j_l} \tag{24}$$

for every $l \in \{1, \ldots, k\}$ as follows. Let $r \geq 1$ and assume that i_l, j_l satisfying (23) and (24) are defined for $l = 1, \ldots, r-1$. Taking $j_0 = 0$ for our convenience, let

$$X = \{i > j_{r-1} \mid \exists j > i : s_i = s_j\}.$$

If $X = \emptyset$, then $k = r-1$ and the sequence is completed. Otherwise, let $i_r = \min X$ and

$$j_r = \max\{j > i \mid s_i = s_j\}.$$

It is immediate that (23) and (24) hold for $l = r$, hence the sequence is well defined. Moreover, it follows from (24) that $k \leq |S|$ and so the sequence is finite. Taking $i_{k+1} = n$, we define

$$v_l = u^{[j_l+1, i_{l+1}]}, \quad w_m = u^{[i_m+1, j_m]}$$

for $l = 0, \ldots, k$ and $m = 1, \ldots, k$. Clearly, $u = v_0 w_1 v_1 \ldots w_k v_k$. By minimality of the i_l, we have $|v_l| \leq |A|$ for every l. By Proposition 3.1, we get

$$|u\varphi| \geq |v_0\varphi| + \ldots + |v_k\varphi| + |w_1\varphi| + \ldots + |w_k\varphi| - 4KM_\varphi$$

and so

$$\begin{aligned} N < |u| - |u\varphi| &\leq (K+1)|A| + |w_1| + \ldots + |w_k| - |w_1\varphi| - \ldots - |w_k\varphi| + 4KM_\varphi \\ &\leq (|S|+1)|A| + 4KM_\varphi + (|w_1| - |w_1\varphi|) + \ldots (|w_k| - |w_k\varphi|). \end{aligned}$$

Hence $|w_m| > |w_m\varphi|$ for some $m \in \{1, \ldots, k\}$. We claim that w_m is cyclically reduced.

Let $s = s_{j_m}$. Since $s_{j_m} = s_{i_m}$, we have $|s| = t_R$. Moreover, $sw_m = zs$ for some $z \in A_R^*$ and so by [15, Proposition 1.3.4] there exist $x, y \in A_R^*$ and $d, e, f \in \mathbb{N}$ such that

$$s = (xy)^d x, \quad w_m = (yx)^e, \quad z = (xy)^f.$$

Hence $(xy)^{d+e}x = sw_m \in A_R^*$. Since $w_m^\omega = (yx)^\omega$, this yields $w_m^\omega \in A_R^\omega$: indeed, $|s| = t_R$ implies that any factor of $(yx)^\omega$ of length $\leq t_R$ must be a factor of $(xy)^{d+e}x$.

Therefore (ii) fails as required. □

Theorem 5.7 *Let φ be a uniformly continuous endomorphism of A_R^*. Then it is decidable whether or not φ has bounded length decrease.*

Proof. By Lemma 5.6, we only need to decide whether or not

$$\exists u \in A_R^* \; (u^\omega \in A_R^\omega \;\wedge\; |u\varphi| \;<\; |u|). \tag{25}$$

holds. We do so by bounding the minimum length required for such a word u.

Assume then that $u \in A_R^*$ satisfies (25) and has minimum length n. For $i = 0, \ldots, n$, write

$$\theta_i = (\mathrm{Suff}_{t_R-1}(u^{[i]}), \mathrm{Suff}_{M_\varphi}(u^{[i]}\varphi)).$$

Let K be an upper bound for the number of possible values of the θ_i. We prove that

$$\theta_i = \theta_j \Rightarrow |u^{[j]}\varphi| - |u^{[i]}\varphi| \; < j - i \tag{26}$$

holds for $0 \leq i < j \leq n$.

Indeed, suppose that $\theta_i = \theta_j$ and $|u^{[j]}\varphi| - |u^{[i]}\varphi| \geq j - i$ for some $0 \leq i < j \leq n$. Let $v = u^{[i]}u^{[j+1,n]}$. Since $\mathrm{Suff}_{t_R-1}(u^{[i]}) = \mathrm{Suff}_{t_R-1}(u^{[j]})$, we have $i \geq t_R - 1$ and $v \in A_R^*$. We claim that v is cyclically reduced. Since $|v| \geq t_R - 1$, it is enough to show that $vu^{[i]} \in A_R^*$, which follows from $u^\omega \in A_R^\omega$ and $\mathrm{Suff}_{t_R-1}(u^{[i]}) = \mathrm{Suff}_{t_R-1}(u^{[j]})$. Thus v is cyclically reduced.

Hence $|v\varphi| \geq |v|$ by minimality of $|u|$. Let $s = \mathrm{Suff}_{M_\varphi}(u^{[i]}\varphi)$. Then we may write

$$u^{[i]}\varphi = xs, \quad u^{[j]}\varphi = ys, \quad u^{[j+1,n]}\varphi = z$$

for some $x, y, z \in A_R^*$. If $|s| = M_\varphi$, it follows from Proposition 3.1 that

$$u\varphi = y\overline{sz}, \quad v\varphi = x\overline{sz}.$$

If $|s| < M_\varphi$, then $x = y = 1$ and the same equalities hold. Thus

$$\begin{aligned} |u| - |u\varphi| &\leq |u| - |u\varphi| - (j - i) - |u^{[i]}\varphi| + |u^{[j]}\varphi| \\ &= |u| - (j - i) - |x| + |y| - |u\varphi| \\ &= |v| - |v\varphi| \leq 0, \end{aligned}$$

a contradiction. Therefore (26) holds.

Let N be as in (22), and let v be an arbitrary prefix of u. Write $u = vw$ and suppose that $|w| - |w\varphi| > N$. Taking $z \in A_R^*$ minimal for the property $|z| - |z\varphi| > N$, it follows from the proof of Lemma 5.6 that there exists some cyclically reduced word x such that $|x\varphi| < |x| < |z|$. Since $|z| \leq |w| \leq |u|$, this contradicts the minimality of $|u|$. Hence $|w| - |w\varphi| \leq N$. Since $|u\varphi| \geq |v\varphi| + |w\varphi| - 2M_\varphi$ by Proposition 3.1, we get

$$|v\varphi| - |v| \leq |u\varphi| - |w\varphi| + 2M_\varphi - |u| + |w| < N + 2M_\varphi. \tag{27}$$

Suppose now that $|u| > (2N + 2M_\varphi + 1)K$. Let $r = 2N + 2M_\varphi + 1$. Then there exist $0 \leq i_1 < i_2 < \ldots < i_r < n$ such that $\theta_{i_1} = \ldots = \theta_{i_r}$. By (27), we have $|u^{[i_1]}\varphi| - i_1 < N + 2M_\varphi$. By (26), we get

$$|u^{[i_r]}\varphi| - i_r < \ldots < |u^{[i_2]}\varphi| - i_2 < |u^{[i_1]}\varphi| - i_1.$$

Hence $|u^{[i_r]}\varphi| - i_r < N + 2M_\varphi - (r-1) = -N$ and so $|u^{[i_r]}| - |u^{[i_r]}\varphi| > N$. Once again, this implies the existence of some cyclically reduced word x such that $|x\varphi| < |x| < i_r < n = |u|$, contradicting the minimality of $|u|$. Thus $|u| \leq (2N+2M_\varphi+1)K$ and (25) becomes decidable as required. $\square$

6 The group case

We start by identifying those groups which can be defined through our type of rewriting system:

Proposition 6.1 *A group G can be defined by a finite special confluent rewriting system if and only if G is a free product of finitely many cyclic groups.*

Proof. If G is isomorphic to the free product of a free group FG_A and cyclic groups $C_{n_1}, \ldots, C_{n_k}$, then it follows from basic facts on presentations of free products [16, Section IV.1] that G can be defined through the rewriting system

$$\langle A \cup A^{-1}, b_1, \ldots, b_k \mid aa^{-1} \to 1,\ a^{-1}a \to 1,\ b_i^{n_i} \to 1\ (a \in A,\ i = 1, \ldots, k)\rangle.$$

It is immediate that this rewriting system has the required properties.

Conversely, assume that G is a group defined by a finite special confluent rewriting system $\langle A \mid R\rangle$. We assume R to be normalized.

Let $r \to 1$ be a rule in R, and write $r = as$ with $a \in A$. Since G is a group, a must be left invertible and so we must have some rule $ta \to 1$ in R. Hence tas reduces to both t and s. Since R is normalized, both s and t are irreducible and so $s = t$ by confluency. Iterating this procedure, we conclude that

$$(uv \to 1) \in R \Rightarrow (vu \to 1) \in R \tag{28}$$

holds for all $u, v \in A_R^*$. We say that $vu \to 1$ is a cyclic conjugate of $uv \to 1$.

Next we prove that

$$(au \to 1), (av \to 1) \in R \Rightarrow u = v \tag{29}$$

holds for all $a \in A$ and $u, v \in A_R^*$. Indeed, we have $(ua \to 1) \in R$ by (28) and so uav reduces to both u and v. Since R is normalized, both u and v are irreducible and so $u = v$ by confluency. Thus (29) holds.

It follows that each $a \in A$ appears in at most one rule and all its cyclic conjugates. Moreover, a rule $aua \ldots \to 1$ must be of the form $(au)^n \to 1$. On the other hand, since G is a group, each $a \in A$ must appear in at least in one rule. Partitioning A with repect to the content of the rules, it follows that G must be a free product of groups H defined through rewriting systems of the form

$$\langle a_1, \ldots, a_k \mid (a_1 \ldots a_k)^n \to 1,\ (a_2 \ldots a_k a_1)^n \to 1,\ (a_k a_1 \ldots a_{k-1})^n \to 1 \rangle.$$

It is routine to check that H can be defined through the group presentation

$$\mathrm{Gp}\langle a_1, \ldots, a_k \mid (a_1 \ldots a_k)^n = 1 \rangle.$$

Replacing $a_1, \ldots, a_k$ by the alternative generating set $a_1, \ldots, a_{k-1}, b = a_1 \ldots a_k$, it follows that H can be presented by

$$\mathrm{Gp}\langle a_1, \ldots, a_{k-1}, b \mid b^n = 1 \rangle$$

and so H is a free product of a free group by the cyclic group C_n. $\square$

We can apply Theorem 3.5 to the particular case of groups. For the particular case of free groups, this was first proved by Goldstein and Turner [12]. The present case is covered by the work of Sykiotis:

Theorem 6.2 *[21] Let G be a free product of finitely many cyclic groups and let φ be a monomorphism of G. Then* $\mathrm{Fix}\varphi$ *is finitely generated.*

Proof. We may assume that G is infinite. By Proposition 6.1, G can be defined by a finite special confluent rewriting system $\langle A \mid R \rangle$. By Corollary 2.6, φ is uniformly continuous. Moreover, φ is boundary-injective:

Indeed, suppose that φ is not boundary-injective. Then there exist $\alpha, \beta \in A_R^\omega$ such that $\alpha\Phi = \beta\Phi$ and $r(\alpha, \beta) = r < \infty$. Let

$$P = \{(i, j) \in \mathbb{N} \times \mathbb{N} : 0 \leq |\alpha^{[i]}\varphi| - |\beta^{[j]}\varphi| \ < h_\varphi\}$$

and define $\mu(i, j)$ for every $(i, j) \in P$ as in the proof of Theorem 4.8. Since $\alpha\Phi$ is an infinite word, P is infinite and we can take $(i, j) \in P$ with

$$|\{(k, l) \in P \mid \mu(k, l) = \mu(i, j)\}| = \infty.$$

Moreover, we may assume that $|\alpha^{[i]}\varphi|, |\beta^{[j]}\varphi| \geq M_\varphi$. Let $g, h \in A_R^*$ be such that $\overline{\alpha^{[i]}g} = \overline{\beta^{[j]}h} = 1$. Now take $(k,l) \in P$ such that

$$\mu(k,l) = \mu(i,j), \quad k \geq r + (t_R - 1)|g|, \quad l \geq r + (t_R - 1)|h|. \tag{30}$$

In view of Proposition 3.1, we may write

$$\alpha^{[i]}\varphi = xw, \quad \alpha^{[k]}\varphi = xzw, \quad \beta^{[j]}\varphi = xy, \quad \beta^{[l]}\varphi = xzy$$

for some $x, y, z, w \in A_R^*$ with $|y| = M_\varphi$. Let $u = \overline{\alpha^{[k]}g}$ and $v = \overline{\beta^{[l]}h}$. By (30) and Lemma 2.4, we have $r(u,v) = r(\alpha,\beta) = r$ and $u \neq v$. Let $x' \in A_R^*$ be such that $\overline{x'x} = 1$. Then

$$\begin{aligned} u\varphi &= \overline{(\alpha^{[k]}\varphi)(g\varphi)} = \overline{xzx'xw(g\varphi)} = \overline{xzx'(\alpha^{[i]}g)\varphi} = \overline{xzx'} \\ &= \overline{xzx'(\beta^{[j]}h)\varphi} = \overline{xzx'xy(h\varphi)} = \overline{(\beta^{[l]}\varphi)(h\varphi)} = v\varphi \end{aligned}$$

and φ would not be injective, a contradiction. Thus φ is boundary-injective and so Fixφ is a rational subset of the group G by Theorem 3.5. Since a subgroup of a group is rational if and only if it finitely generated by Proposition 2.1, Fixφ is finitely generated. □

We consider now infinite fixed points in the group case. We denote by $(\mathrm{Fix}\varphi)^c$ the topological closure of Fixφ in the completion. As far as we know, this result is new in its full generality, the automorphism case following from results and remarks in [10].

Theorem 6.3 *Let G be a free product of finitely many cyclic groups and let φ be a monomorphism of G. Let Φ be its continuous extension to the completion of G. Then there exist infinite fixed points $\beta_1, \ldots, \beta_s$ such that*

$$\mathrm{Fix}\Phi = (\mathrm{Fix}\varphi)^c \cup (\mathrm{Fix}\varphi)\beta_1 \cup \ldots \cup (\mathrm{Fix}\varphi)\beta_s.$$

Proof. We may assume that G is infinite. By Proposition 6.1, G can be defined by a finite special confluent rewriting system $\langle A \mid R\rangle$. As observed in the proof of Theorem 6.2, φ is uniformly continuous and boundary-injective. By Theorem 3.10, there exist $L_1, \ldots, L_s \in \mathrm{Rat}A^*$ and $\alpha_1, \ldots, \alpha_s \in A_R^\omega$ such that

$$\mathrm{Fix}\Phi = L_\infty(\mathcal{C}_\varphi) \cup L_1\alpha_1 \cup \ldots \cup L_s\alpha_s.$$

Take $\beta_i \in L_i\alpha_i$ for $i = 1, \ldots, s$. Clearly, $\overline{(\mathrm{Fix}\varphi)\beta_i} \subseteq \mathrm{Fix}\Phi$ for every i. By continuity, we have $(\mathrm{Fix}\varphi)^c \subseteq \mathrm{Fix}\Phi$ as well. Thus

$$(\mathrm{Fix}\varphi)^c \cup \overline{(\mathrm{Fix}\varphi)\beta_1} \cup \ldots \cup \overline{(\mathrm{Fix}\varphi)\beta_s} \subseteq \mathrm{Fix}\Phi.$$

Conversely, let $\alpha \in \mathrm{Fix}\Phi$. We may assume that $\alpha \in A_R^\omega$. Suppose first that $\alpha \in L_\omega(\mathcal{C}_\varphi)$. Then some vertex q of $\mathcal{C}_\varphi$ is visited infinitely often in the path $q_0 \xrightarrow{\alpha} \ldots$ and we may factor this path as

$$q_0 \xrightarrow{u_1} q \xrightarrow{u_2} q \xrightarrow{u_3} \ldots$$

Let $v = u_1$ and $w \in A_R^*$ be such that $\overline{vw} = 1$. We show that

$$\overline{vu_iw} \in \mathrm{Fix}\varphi \text{ for every } i > 1 \tag{31}$$

and

$$\alpha = \lim_{n\to+\infty} \overline{vu_2wvu_3w\dots vu_nw}. \tag{32}$$

Indeed, it follows from the definition of $\mathcal{C}_\varphi$ that $\tau(vu_i) = \tau(v)$ and $\rho(vu_i) = \rho(v)$. Let $z \in A_R^*$ be such that $\overline{z\sigma(v)} = 1$. Then we have

$$\begin{aligned}(\overline{vu_iw})\varphi &= \overline{\sigma(vu_i)\rho(vu_i)(w\varphi)} = \overline{\sigma(vu_i)z\sigma(v)\rho(v)(w\varphi)} = \overline{\sigma(vu_i)z(vw)\varphi} = \overline{\sigma(vu_i)z}\\ &= \overline{\sigma(vu_i)zvw} = \overline{\sigma(vu_i)z\sigma(v)\tau(v)w} = \overline{\sigma(vu_i)\tau(v)w} = \overline{\sigma(vu_i)\tau(vu_i)w}\\ &= \overline{vu_iw}.\end{aligned}$$

Thus (31) holds.

Now, since G is a group, $\overline{vw} = 1$ implies $\overline{wv} = 1$ and so

$$\overline{vu_2wvu_3w\dots vu_nw} = \overline{u_1u_2\dots u_nw}.$$

Since $u_1\dots u_n < \alpha$ is reduced and $\lim_{n\to+\infty}|u_1u_2\dots u_n| = +\infty$, we get

$$\lim_{n\to+\infty} \overline{vu_2wvu_3w\dots vu_nw} = \lim_{n\to+\infty} \overline{u_1u_2\dots u_nw} = \lim_{n\to+\infty} \overline{u_1u_2\dots u_n} = \alpha$$

and so (32) holds. Now $\alpha \in (\mathrm{Fix}\varphi)^c$ by (31) and (32).

Thus we may assume that $\alpha \in L_i\alpha_i$ for some $i \in \{1,\dots,s\}$. By the proof of Theorem 3.10, we may assume that $L_i = L(Q_1, q_0, p_i, E_1)$ for some $p_i \in P_1$.

Write $\alpha = u\alpha_i$ and $\beta_i = v\alpha_i$ with $u, v \in L_i$. Let $w \in A_R^*$ be such that $\overline{vw} = 1$. Then $\overline{wv} = 1$ and so

$$\alpha = \overline{uwv\alpha_i} = \overline{uw\beta_i}.$$

Thus it suffices to prove that $\overline{uw} \in \mathrm{Fix}\varphi$. Indeed, it follows from the definition of $\mathcal{B}_\varphi$ that $\tau(u) = \tau(v)$ and $\rho(u) = \rho(v)$. Let $z \in A_R^*$ be such that $\overline{z\sigma(v)} = 1$. Then we have

$$\begin{aligned}(\overline{uw})\varphi &= \overline{\sigma(u)\rho(u)(w\varphi)} = \overline{\sigma(u)z\sigma(v)\rho(v)(w\varphi)} = \overline{\sigma(u)z(vw)\varphi} = \overline{\sigma(u)z}\\ &= \overline{\sigma(u)zvw} = \overline{\sigma(u)z\sigma(v)\tau(v)w} = \overline{\sigma(u)\tau(v)w} = \overline{\sigma(u)\tau(u)w}\\ &= \overline{uw}.\end{aligned}$$

This completes the proof of the theorem. □

7 Conclusion

We hope to have provided some further evidence for the potential of automata-theoretic techniques in the study of dynamical problems, for monoids and for groups as well. We list now some open problems that arise naturally from this work:

We have no examples of uniformly continuous endomorphisms which are not finite-splitting, neither do we know whether or not this property is decidable.

We would love to have a proof that $\mathrm{Fix}\varphi$ is effectively constructible when φ is boundary-injective, providing in particular an alternative (combinatorial?) proof for Maslakova's Theorem, but it has eluded us so far.

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