# Infinite words and confluent rewriting systems: endomorphism extensions

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ABSTRACT

Infinite words over a finite special confluent rewriting system $R$ are considered and endowed with natural algebraic and topological structures. Their geometric significance is explored in the context of Gromov hyperbolic spaces. Given an endomorphism $\varphi$ of the monoid generated by $R$, existence and uniqueness of several types of extensions of $\varphi$ to infinite words (endomorphism extensions, weak endomorphism extensions, continuous extensions) are discussed. Characterization theorems and positive decidability results are proved for most cases.

## 1 Introduction

In view of the possibilities offered to language theory by the study of free groups [14, 15] and more general structures such as PR-monoids [16], it is a natural idea to extend some of the theory on infinite words to the more general setting of monoids defined by finite special confluent rewriting systems (in fact, some of our results hold for monadic confluent and even length-reducing confluent rewriting systems). We recall that a rewriting system $\{(r_1, s_1), \ldots, (r_n, s_n)\}$ is said to be *special* if $s_1 = \ldots = s_n = 1$.

Monoids defined through finite special confluent rewriting systems allow normal forms consisting of irreducible elements, hence they can be viewed as proper subsets of a free monoid with a particular binary operation (concatenation followed by total reduction, such as in the free group case). This approach can, up to some extent, be generalized to infinite words that are endowed with algebraic and topological structures that constitute natural generalizations of their free monoid counterparts. The fact that we can view infinite words as the space of ends of the undirected Cayley graph of the original monoid gives geometric significance to this topology.

We should note that infinite iteration of a (finite) word can no longer be assumed in every case due to the existence of periodic elements, thus our approach involves a partial version of the usual concept of $\omega$-monoid [12].

This paper is specifically devoted to the basic problem of endomorphism extensions: under which circumstances can an endomorphism $\varphi$ of the monoid of finite words be extended to an endomorphism (continuous map, weak endomorphism) on the partial $\omega$-monoid of infinite words? We introduce also the concept of extendable endomorphism, where attention is focused on the "natural" extension to infinite words. Characterization theorems leading to positive decidability results are obtained in most cases. To obtain them, various results on rational languages over finite special confluent rewriting systems had to be proved. We should also mention that a related paper (on the existence of infinite periodic points for endomorphisms) [4] is being written by the authors.

The paper is organized as follows: Section 2 is devoted to preliminaries, Section 3 to convergence of powers and Section 4 to the algebraic strucure called partial $\omega$-monoid. Section 5 discusses the topology of infinite words from a geometric viewpoint, namely Gromov hyperbolic spaces. Sections 6, 7 and 8 are devoted respectively to the fundamental necessary condition of extendability, weak endomorphism extensions and continuous extensions. Conclusions and open problems are summarized in Section 9.

## 2 Preliminaries

Let $A$ denote a finite alphabet. A (finite) *rewriting system* over $A$ is a (finite) subset $R$ of $A^* \times A^*$. Given $u, v \in A^*$, we write $u \longrightarrow_R v$ if

$$u = xry, \quad v = xsy$$

for some $x, y \in A^*$ and $(r, s) \in R$. We denote by $\stackrel{*}{\longrightarrow}$ the reflexive and transitive closure of the relation $\longrightarrow$. The subscript $R$ will be usually omitted. The *congruence* on $A^*$ generated by $R$ will be denoted by $R^\sharp$. Note that $R^\sharp = \stackrel{*}{\longrightarrow}_{R \cup R^{-1}}$. The quotient $M = A^*/R^\sharp$ is said to be the monoid defined by the rewriting system $R$.

A rewriting system $R$ over $A$ is said to be

- *length-reducing* if $|r| > |s|$ for every $(r, s) \in R$;
- *monadic* if $R \subseteq A^+ \times (A \cup \{1\})$;
- *special* if $R \subseteq A^+ \times \{1\}$;
- *noetherian* if there is no infinite chain of reductions $u_1 \to u_2 \to u_3 \to \dots$
- *confluent* if, whenever $u \stackrel{*}{\longrightarrow} v$ and $u \stackrel{*}{\longrightarrow} w$, there exists $z \in A^*$ such that $v \stackrel{*}{\longrightarrow} z$ and $w \stackrel{*}{\longrightarrow} z$:

$$\begin{tikzcd} u \arrow[r, "*"] \arrow[d, "*"] & v \arrow[d, dashed, "*"] \\ w \arrow[r, dashed, "*"] & z \end{tikzcd}$$

- *locally confluent* if, whenever $u \longrightarrow v$ and $u \longrightarrow w$, there exists $z \in A^*$ such that $v \overset{*}{\longrightarrow} z$ and $w \overset{*}{\longrightarrow} z$:

$$\begin{tikzcd} u \arrow[r] \arrow[d] & v \arrow[d, dashed, "*"] \\ w \arrow[r, dashed, "*"] & z \end{tikzcd}$$

**Lemma 2.1** *[3, Theorem 1.1.13] A noetherian rewriting system is confluent if and only if it is locally confluent.*

It is known (see [3, Section 2.2]) that every monoid defined by a finite length-reducing confluent rewriting system can be defined by a finite *normalized* length-reducing confluent rewriting system, i.e., satisfying the two conditions:

- for every $(r, s) \in R$, $|r| > 1$;
- if $(r, s), (arb, s') \in \mathcal{R}$, then $ab = 1$ and $s' = s$.

Therefore, we are entitled to assume whenever convenient that our length-reducing confluent systems are normalized.

Let $R$ be a length-reducing confluent rewriting system over $A$. We say that $w \in A^*$ is *irreducible* (with respect to $R$) if $w \notin \cup_{(r,s)\in R} A^* r A^*$. For every $u \in A^*$, there is exactly one irreducible $v \in A^*$ such that $u \overset{*}{\longrightarrow} v$: existence follows from $R$ being length-reducing, and uniqueness from confluence. We denote this unique irreducible word by $\overline{u}$. It is well known (see [3]) that the equivalence

$$uR^\sharp v \Leftrightarrow \overline{u} = \overline{v}$$

holds for all $u, v \in A^*$, hence $\overline{A^*} = \{\overline{u} \mid u \in A^*\}$ constitutes a set of normal forms for the monoid $M = A^*/R^\sharp$. Moreover,

$$M \cong (\overline{A^*}, \cdot),$$

where $\cdot$ denotes the binary operation on $\overline{A^*}$ defined by $u \cdot v = \overline{uv}$. We denote the monoid $(\overline{A^*}, \cdot)$ by $A_R^*$. We shall often abuse notation and identify $A_R^*$ with $\overline{A^*}$. We write also $A_R^+ = \overline{A^*} \setminus \{1\}$.

We denote by $A^\omega$ the set of all infinite words of the form $a_1 a_2 a_3 \ldots$, with $a_n \in A$ for every $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. Write

$$A^\infty = A^* \cup A^\omega.$$

Given $\alpha \in A^\infty$ and $n \in \mathbb{N}$, we denote by $\alpha^{(n)}$ the $n$-th letter of $\alpha$ (if $\alpha \in A^*$ and $n > |\alpha|$, we set $\alpha^{(n)} = 1$). We write also

$$\alpha^{[n]} = \alpha^{(1)}\alpha^{(2)} \ldots \alpha^{(n)}.$$

An infinite word $\alpha \in A^\omega$ is said to be *irreducible* (with respect to $R$) if $\alpha^{[n]}$ is irreducible for every $n \in \mathbb{N}$. We denote the set of all irreducible infinite words (with respect to $R$) by $A_R^\omega$ and we write

$$A_R^\infty = A_R^* \cup A_R^\omega.$$

For all $\alpha, \beta \in A^\infty$, we define

$$r(\alpha,\beta) = \begin{cases} \min\{n \in \mathbb{N} \mid \alpha^{(n)} \neq \beta^{(n)}\} & \text{if } \alpha \neq \beta \\ \infty & \text{if } \alpha = \beta \end{cases}$$

and we write

$$d(\alpha,\beta) = 2^{-r(\alpha,\beta)},$$

using the convention $2^{-\infty} = 0$. It follows easily from the definition that $d$ is an ultrametric on $A^\infty$, satisfying in particular the axiom

$$d(\alpha,\beta) \leq \max\{d(\alpha,\gamma), d(\gamma,\beta)\}.$$

We shall identify $A^\infty$ with the metric space $(A^\infty, d)$. It is well known that the metric space $A^\infty$ is compact (and therefore complete) [12, Chapter III]. Note that $\lim_{n\to\infty} \alpha_n = \alpha$ if and only if

$$\forall k \in \mathbb{N}\, \exists m \in \mathbb{N}\, \forall n \in \mathbb{N}\, (n \geq m \Rightarrow \alpha_n^{[k]} = \alpha^{[k]}).$$

Note that, since $A^\infty$ is complete, a sequence $u_1, u_2, \ldots \in A^*$ converges if and only if it is a Cauchy sequence, i.e., if the condition

$$\forall k \in \mathbb{N}\, \exists m \in \mathbb{N}\, \forall n, n' \in \mathbb{N}\, (n, n' \geq m \Rightarrow u_n^{[k]} = u_{n'}^{[k]})$$

holds. By transitivity, it follows that $(u_n)_n$ converges if and only if

$$\forall k \in \mathbb{N}\, \exists m \in \mathbb{N}\, \forall n \geq m\ u_n^{[k]} = u_{n+1}^{[k]}. \tag{1}$$

**Proposition 2.2** *If $R$ is a length-reducing confluent rewriting system over $A$, then $A_R^\infty$ is a closed subspace of $(A^\infty, d)$.*

**Proof.** Let $\alpha \in A^\infty \setminus A_R^\infty$. Then $\alpha^{[n]}$ is reducible for some $n \in \mathbb{N}$. Let $B_{2^{-n}}(\alpha)$ denote the open ball with radius $2^{-n}$ and centre $\alpha$. If $\beta \in B_{2^{-n}}(\alpha)$, then $r(\alpha,\beta) > n$ and so $\alpha^{[n]} = \beta^{[n]}$. Hence $\beta \in A^\infty \setminus A_R^\infty$ and so

$$B_{2^{-n}}(\alpha) \subseteq A^\infty \setminus A_R^\infty.$$

Thus $A^\infty \setminus A_R^\infty$ is open and consequently $A_R^\infty$ is closed. $\square$

This immediately yields

**Corollary 2.3** *If $R$ is a length-reducing confluent rewriting system over $A$, then $(A_R^\infty, d)$ is compact (and therefore complete).*

We remark that, since $\alpha = \lim_{n\to\infty} \alpha^{[n]}$ for every $\alpha \in A^\infty$, $(A^\infty, d)$ (respectively $(A_R^\infty, d)$) is the completion of $(A^*, d)$ (respectively $(A_R^*, d)$).

Note also that $d$ induces the discrete topology on $A^*$ since $B_{2^{-(n+1)}}(u) = \{u\}$ for every $u \in A^n$.

If $R$ is finite monadic, rational languages are preserved by reduction:

**Theorem 2.4** *[1] Let $\mathcal{R}$ be a finite monadic confluent rewriting system on $A$ and let $L \subseteq A^*$ be rational. Then $\overline{L}$ is rational and effectively constructible from $L$.*

If we consider the whole reduction class, we are taken into the realm of deterministic context-free languages, as follows from the combined results of Chottin and Sénizergues:

**Theorem 2.5** *[5, 13] Let $\mathcal{R}$ be a finite monadic confluent rewriting system on $A$ and let $L \subseteq A^*$ be rational. Then $D_L = \{u \in A^* \mid \overline{u} \in \overline{L}\}$ is deterministic context-free and effectively constructible from $L$.*

# 3 Convergence of powers

The rewriting systems in this section are all length-reducing confluent. Given a finite length-reducing confluent rewriting system $R = \{(r_1, s_1), \ldots, (r_n, s_n)\}$, we write

$$t_R = \max\{|r_1|, |r_2|, \ldots, |r_n|\}.$$

The next lemma discusses convergence to a finite word.

**Lemma 3.1** *Let $R$ be a length-reducing confluent rewriting system over $A$ and let $(u_n)_n$ be a sequence on $A_R^*$. Then:*

*(i) $\lim_{n\to\infty} u_n \in A_R^*$ if and only if $(u_n)_n$ is stationary;*

*(ii) if $(|u_n|)_n$ is bounded and $(u_n)_n$ converges, then $(u_n)_n$ is stationary.*

**Proof**. (i) Suppose that $\lim_{n\to\infty} u_n = v \in A_R^*$ and take $l = |v| + 1$. Then there exists some $m \in \mathbb{N}$ such that $u_n^{[l]} = v^{[l]} = v$ for every $n \geq m$. It follows that $u_n = v$ for every $n \geq m$ and so $(u_n)_n$ is stationary.

The converse implication is trivial.

(ii) Assume that $|u_n| < K$ for every $n \in \mathbb{N}$. If $\alpha = \lim_{n\to\infty} u_n$, there exists $m \in \mathbb{N}$ such that $u_n^{[K]} = \alpha^{[K]}$ for every $n \geq m$. It follows that $|\alpha^{[K]}| < K$ and so $\alpha \in A_R^*$. By part (i), it follows that $(u_n)_n$ is stationary. $\square$

Now we present necessary and sufficient conditions for a sequence of powers to converge.

Recall that an element of a semigroup is said to be *aperiodic* if it generates an aperiodic semigroup (i.e., with no nontrivial subgroups) and is said to have *finite order* if it generates a finite semigroup. In particular, every element having infinite order must be aperiodic. On the other hand, a finite order element $u$ is aperiodic if and only if $u^n = u^{n+1}$ for some $n \in \mathbb{N}$. If the rewriting system is clear from the context, we shall abuse terminology by saying that $u \in A_R^*$ is aperiodic (respectively has finite order) if $uR^\sharp$ is aperiodic (respectively has finite order) in $M = A^*/R^\sharp$.

**Theorem 3.2** *Let $R$ be a length-reducing confluent rewriting system over $A$ and let $u \in A_R^*$. Then the sequence $(\overline{u^n})_n$ converges in $A_R^\infty$ if and only if $u$ is aperiodic. Moreover, $\lim_{n\to\infty} \overline{u^n} \in A_R^\omega$ if and only if $u$ has infinite order.*

**Proof**. Assume first that $u$ has finite order. Then $(|\overline{u^n}|)_n$ is bounded and it follows from Lemma 3.1 that $(\overline{u^n})_n$ converges if and only if $(\overline{u^n})_n$ is stationary, that is, if and only if $u$ is aperiodic. Moreover, $\lim_{n\to\infty} \overline{u^n} \in A_R^*$ in this case.

Assume now that $u$ has infinite order. In particular, $u$ is aperiodic. Since $u$ has infinite order, there exists some $n \in \mathbb{N}$ such that

$$2|u|t_R < |\overline{u^n}| < |\overline{u^{n+1}}|.$$

Since $R$ is confluent, we have

$$\overline{u^{n+1}} = \overline{u\overline{u^n}} = \overline{\overline{u^n}u}.$$

Since $R$ is length-reducing, the number of steps in $u\,\overline{u^n} \xrightarrow{*} \overline{u^{n+1}}$ is bounded by

$$|u| + |\overline{u^n}| - |\overline{u^{n+1}}| < |u|.$$

Since $\overline{u^n}$ is irreducible, each step uses at most $t_R - 1$ letters of $\overline{u^n}$ and so the prefix of $\overline{u^n}$ involved in the reduction $u\overline{u^n} \xrightarrow{*} \overline{u^{n+1}}$ has length at most $(|u|-1)(t_R-1)$. By duality, and since $|\overline{u^n}| > 2|u|t_R$, one has $\overline{u^n} = fgh$ with $|f|, |h| \leq (|u|-1)(t_R-1)$ and

$$\overline{u^{n+1}} = \overline{uf}gh = fg\overline{hu}. \tag{2}$$

Note that

$$|g| = |\overline{u^n}| - |f| - |h| > 2|\overline{u}|t_R - 2(|u|-1)(t_R-1) = 2|u| + 2t_R - 2 \geq t_R.$$

As $|\overline{u^n}| < |\overline{u^{n+1}}|$, we have $|f| < |\overline{uf}|$ and $|h| < |\overline{hu}|$. By (2), we may write $\overline{uf} = fv$ and $\overline{hu} = wh$ for some $v, w \in A_R^+$. Moreover, we obtain $vg = gw$. By [11, Proposition 1.3.4], there exist $p, q \in A^*$ and $k \geq 0$ such that

$$v = pq, \quad w = qp, \quad g = (pq)^k p.$$

Since $|v| > 0$, $p$ and $q$ are not both empty.

We have $\overline{u^n} = fgh = f(pq)^k ph$ and $\overline{u^{n+1}} = fvgh = f(pq)^{k+1}ph$. We show that

$$\overline{u^{n+m}} = f(pq)^{k+m}ph \tag{3}$$

by induction on $m$. In fact, if $\overline{u^{n+m}} = f(pq)^{k+m}ph$, then

$$\begin{aligned} \overline{u^{n+m+1}} &= \overline{\overline{u^{n+m}}u} = \overline{f(pq)^{k+m}phu} = \overline{f(pq)^{k+m}p\overline{hu}} \\ &= \overline{f(pq)^{k+m}pwh} = \overline{f(pq)^{k+m+1}ph}. \end{aligned}$$

Since $f(pq)^{k+1}ph = \overline{u^{n+1}}$ is irreducible and $|(pq)^k p| = |g| \geq t_R$, it follows easily that $f(pq)^{k+m+1}ph$ is itself irreducible and so $\overline{u^{n+m+1}} = f(pq)^{k+m+1}ph$.

Therefore (3) holds for every $m \in \mathbb{N}$ and it is immediate that $\lim_{n\to\infty} \overline{u^n} = f(pq)^\omega \in A_R^\omega$ as required. $\square$

**Corollary 3.3** *Let $R$ be a length-reducing confluent rewriting system over $A$ and suppose that $u \in A_R^*$ has infinite order. Then there exist $x, y \in A_R^*$, $v \in A_R^+$ and $n_0 \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ such that $\overline{yx} = v^{n_0}$ and*

$$\forall n \geq n_0 \; \overline{u^n} = xv^{n-n_0}y.$$

**Proof**. If we denote $n$ in the proof of Theorem 3.2 by $n_1$, and take $f, g = (pq)^k p, h, v = pq$ to have the same meaning, we have $\overline{u^n} = fv^{n-n_1}gh$ for every $n \geq n_1$ by (3). Writing $x' = f$ and $y' = gh$, we obtain

$$\forall n \geq n_1 \; \overline{u^n} = x'v^{n-n_1}y'. \tag{4}$$

Let $k \geq n_1$ be such that

$$(k - n_1)|v| \geq (t_R - 1)(|y'x'| - n_1|v|). \tag{5}$$

By (4), we have

$$x'v^{2k-n_1}y' = \overline{u^{2k}} = \overline{\overline{u^k}\,\overline{u^k}} = \overline{x'v^{k-n_1}y'x'v^{k-n_1}y'}.$$

Let $m = |y'x'| - n_1|v|$. Since $|x'v^{k-n_1}y'x'v^{k-n_1}y'| - |x'v^{2k-n_1}y'| = m$, and $R$ is length-reducing, we need at most $m$ steps to reduce $x'v^{k-n_1}y'x'v^{k-n_1}y'$. Since $x'v^{k-n_1}y'$ is irreducible and each step involves at most $t_R$ letters, the suffix and the prefix of $x'v^{k-n_1}y'$ involved in the reduction process have length at most $m(t_R - 1) \leq |v^{k-n_1}|$ by (5). Hence $\overline{x'v^{k-n_1}y'x'v^{k-n_1}y'} = x'\overline{v^{k-n_1}y'x'v^{k-n_1}}y'$ and so

$$\overline{v^{k-n_1}y'x'v^{k-n_1}} = v^{2k-n_1}.$$

Let $n_0 = 2k - n_1$. Let $x = x'v^{k-n_1}$ and $y = v^{k-n_1}y'$. We obtain $\overline{yx} = v^{n_0}$. Moreover, for every $n \geq n_0$, we have $n - n_1 \geq n_0 - n_1 = 2(k - n_1) \geq 0$ and so (4) yields

$$\overline{u^n} = x'v^{n-n_1}y' = xv^{n-n_1-2(k-n_1)}y = xv^{n-n_0}y$$

as required. □

We recall that $\alpha \in A^\omega$ is said to be *eventually periodic* if $\alpha = uv^\omega$ for some $u \in A^*$ and $v \in A^+$.

**Corollary 3.4** *Let $R$ be a length-reducing confluent rewriting system over $A$ and suppose that $u \in A_R^*$ has infinite order. Then $\lim_{n\to\infty} \overline{u^n}$ is eventually periodic.*

**Proof**. By Corollary 3.3, there exist $x, y \in A_R^*$, $v \in A_R^+$ and $n_0 \in \mathbb{N}_0$ such that $\overline{yx} = v^{n_0}$ and $\overline{u^n} = xv^{n-n_0}y$ for every $n \geq n_0$. Hence

$$\lim_{n\to\infty} \overline{u^n} = \lim_{n\to\infty} xv^{n-n_0}y = \lim_{n\to\infty} xv^n = xv^\omega$$

is eventually periodic. □

The special case provides further simplification to Corollary 3.3.

**Corollary 3.5** *Let $R$ be a special confluent rewriting system over $A$ and suppose that $u \in A_R^*$ has infinite order. Then there exist $x, y \in A_R^*$, $v \in A_R^+$ and $n_0 \in \mathbb{N}_0$ such that $\overline{yx} = 1$ and*

$$\forall n \geq n_0 \; \overline{u^n} = xv^n y.$$

**Proof**. By Corollary 3.3, there exist $x, y \in A_R^*$, $v \in A_R^+$ and $n_0 \in \mathbb{N}_0$ such that $\overline{yx} = v^{n_0}$ and

$$\forall n \geq n_0 \; \overline{u^n} = xv^{n-n_0}y.$$

If $n_0 = 0$ we are done, hence we assume that $n_0 \geq 1$. Since $\overline{yx} = v^{n_0}$ and $y, x$ are irreducible, we may factor $v = v'v''$ and write

$$y = v^r v' y_0, \quad x = x_0 v'' v^{n_0 - 1 - r},$$

where $\overline{y_0 x_0} = 1$. Let $v_0 = v''v'$. For every $n \geq n_0$, we have

$$\begin{aligned}\overline{u^n} &= x v^{n-n_0} y = x_0 v'' v^{n_0-1-r} v^{n-n_0} v^r v' y_0 \\ &= x_0 v'' v^{n-1} v' y_0 = x_0 v'' (v'v'')^{n-1} v' y_0 = x_0 v_0^n y_0\end{aligned}$$

as required. $\square$

The next example shows that Theorem 3.2 cannot be generalized to noetherian confluent rewriting systems.

**Example 3.6** *There exists a finite noetherian confluent rewriting system over $A = \{a, b\}$ and $u \in A^*$ such that $u$ has infinite order but $(\overline{u^n})_n$ does not converge.*

**Proof**. Let $R = \{(ba, ab), (a^2, b)\}$.

Assume $A$ ordered by $a < b$ and $\mathbb{N}_0 \times \mathbb{N}_0$ (well-)ordered lexicographically. Let $\varphi : A^* \to \mathbb{N}_0 \times \mathbb{N}_0$ be the map defined by

$$u\varphi = (|u|, |\{(i, j) \in \{1, \ldots, |u|\}^2 : (i < j \wedge u^{(i)} > u^{(j)})\}|).$$

It is straightforward to check that

$$u \to v \Rightarrow u\varphi > v\varphi$$

holds for all $u, v \in A^*$. Since $\mathbb{N}_0 \times \mathbb{N}_0$ is well-ordered and therefore contains no infinite descending chain, we conclude that $R$ is noetherian.

To show that $R$ is locally confluent, we only have to complete all possible diagrams of the form

$$\begin{array}{ccc} xyz & \longrightarrow & sz \\ \downarrow & & \\ xs' & & \end{array}$$

where $(xy, s), (yz, s') \in R$ and $y \neq 1$. In particular, $|xyz| \leq 3$ and we only have to consider the cases $xyz = baa$ and $xyz = aaa$. Verifying each one of them, we conclude that R is locally confluent and therefore confluent by Lemma 2.1.

A simple induction shows that

$$\overline{a^{2n}} = b^n, \quad \overline{a^{2n+1}} = ab^n$$

for every $n \in \mathbb{N}_0$. Since the first letter alternates through the whole sequence, $(\overline{a^n})_n$ does not converge. However, $a$ has infinite order, all the reduced forms $\overline{a^n}$ being distinct. $\square$

## 4 Partial $\omega$-monoids

¿From now on, we fix $R = \{(r_1, 1), \ldots, (r_s, 1)\}$ to be a finite special confluent rewriting system over $A$.

We generalize the concept of $\omega$-semigroup [12, Chapter I.4] as follows. A *partial $\omega$-monoid* is a structure of the form $(M_1, M_2, \cdot, \circ, \pi)$, where $\cdot : M_1 \times M_1 \to M_1$ and $\circ : M_1 \times M_2 \to M_2$ are binary operations and $\pi : M_1^\omega = M_1 \times M_1 \times \ldots \to M_1 \cup M_2$ is a surjective partial map, such that:

(w1) $(M_1, \cdot)$ is a monoid;

(w2) if $(u_1, u_2, \ldots)\pi$ is defined and $i_1 < i_2 < \ldots$ is a sequence in $\mathbb{N}$, then $(u_1 \ldots u_{i_1}, u_{i_1+1} \ldots u_{i_2}, u_{i_2+1} \ldots u_{i_3}, \ldots)\pi$ is defined and equal to $(u_1, u_2, \ldots)\pi$;

(w3) if $(u_1, u_2, \ldots)\pi$ is defined and $v \in M_1$, then $(v, u_1, u_2, \ldots)\pi$ is defined and equal to the product of $v$ by $(u_1, u_2, \ldots)\pi$;

(w4) $(1, 1, \ldots)\pi$ is defined and equals 1.

We note that these axioms imply the mixed associative law given by

$$u \circ (v \circ \alpha) = (u \cdot v) \circ \alpha$$

for all $u, v \in M_1$ and $\alpha \in M_2$. In fact, since $\pi$ is onto, we have $\alpha = (u_1, u_2, \ldots)\pi$ for some $(u_n)_n \in M_1^\omega$. By (w2) and (w3), we obtain

$$\begin{aligned} u \circ (v \circ \alpha) &= u \circ (v \circ ((u_1, u_2, \ldots)\pi)) \\ &= u \circ ((v, u_1, u_2, \ldots)\pi) \\ &= (u, v, u_1, u_2, \ldots)\pi \\ &= (u \cdot v, u_1, u_2, \ldots)\pi \\ &= (u \cdot v) \circ ((u_1, u_2, \ldots)\pi) \\ &= (u \cdot v) \circ \alpha. \end{aligned}$$

If $M_1 \cup M_2$ is endowed with a distance $d$ such that:

- the operations $\cdot$ and $\circ$ are continuous (considering the product topology on $M_1 \times (M_1 \cup M_2)$, for instance via the *max* metric on the components);
- $(u_1, u_2, \ldots)\pi$ is defined if and only if $\lim_{n\to\infty} u_1 u_2 \ldots u_n$ exists, in which case they coincide;

then we have a *metric* partial $\omega$-monoid.

It follows easily from (w3) and (w2) that the identity of $M_1$ is a left identity for the mixed product $\circ$.

If $\pi$ is a full map, we have the standard concept of $\omega$-monoid ($\omega$-semigroup if we don't require $(M_1, \cdot)$ to have an identity).

If $u \in M_1$ and $(u, u, u, \ldots)\pi$ is defined, we denote it by $u^\omega$.

An endomorphism of $(M_1, M_2, \cdot, \circ, \pi)$ is a mapping $\varphi : M_1 \cup M_2 \to M_1 \cup M_2$ such that:

(h1) $M_1\varphi \subseteq M_1$;

(h2) for all $u, v \in M_1$, $(u \cdot v)\varphi = (u\varphi) \cdot (v\varphi)$;

(h3) for all $u \in M_1$ and $\alpha \in M_2$,

$$(u \circ \alpha)\varphi = \begin{cases} (u\varphi) \cdot (\alpha\varphi) & \text{if } \alpha\varphi \in M_1 \\ (u\varphi) \circ (\alpha\varphi) & \text{if } \alpha\varphi \in M_2 \end{cases}$$

(h4) if $(u_1, u_2, \ldots)\pi$ is defined, then $(u_1\varphi, u_2\varphi, \ldots)\pi$ is defined and equal to $(u_1, u_2, \ldots)\pi\varphi$.

If we replace axiom (h4) by

(h4') if $u^\omega$ is defined, then $(u\varphi)^\omega$ is defined and equal to $u^\omega\varphi$,

we have a *weak endomorphism*. A (weak) endomorphism is said to be *proper* if $M_2\varphi \subseteq M_2$.

We shall see that $A_R^\infty$ can be viewed naturally as a metric partial $\omega$-monoid, but first we prove some lemmas that help us to understand better the reduction process.

**Lemma 4.1** *Let $u, v, w \in A_R^*$ be such that $|v| \geq |u|(t_R - 1)$ and $vw \in A_R^*$. Then $\overline{uvw} = \overline{uv}w$.*

**Proof**. We use induction on $|u|$. The case $|u| = 0$ being trivial, assume that $|u| > 0$ and the lemma holds for shorter words. We may assume that $uv$ is reducible, otherwise $|v| \geq (t_R - 1)$ and $vw$ irreducible yield $\overline{uvw} = uvw = \overline{uv}w$. Hence we may write $u = u'r'$ and $v = r''v'$ with $(r'r'', 1) \in R$. Note that $r', r'' \neq 1$ since $u, v \in A_R^*$. We have

$$|v'| = |v| - |r''| \geq |u|(t_R - 1) - (t_R - 1) = (|u| - 1)(t_R - 1) \geq |u'|(t_R - 1),$$

hence the induction hypothesis yields $\overline{u'v'w} = \overline{u'v'}w$ and so

$$\overline{uvw} = \overline{u'r'r''v'w} = \overline{u'v'w} = \overline{u'v'}w = \overline{uv}w$$

as required. $\square$

**Lemma 4.2** *For all $u, v \in A_R^*$,*

*(i)* $u = u'u''$ *and* $v = v'v''$ *with* $\overline{uv} = u'v''$, $|u''v'| \leq min\{|u|, |v|\} \cdot t_R$ *and* $|u''| \leq (t_R - 1)|v|$.

*(ii)* $|\overline{uv}| \geq max\{|v| - (t_R - 1)|u|, |u| - (t_R - 1)|v|\}$.

*(iii)* $|u| \leq r(\overline{uv}, u) - 1 + (t_R - 1)|v|$.

**Proof**. (i) Let $u, v \in A_R^*$. Since $R$ is special, we have factorizations $u = u'u''$ and $v = v'v''$ such that $\overline{uv} = u'v''$. We show that $|u''v'| \leq \min\{|u|, |v|\} \cdot t_R$ using induction on $|uv|$. The case $|uv| = 0$ being trivial, assume that $|uv| > 0$ and the lemma holds for smaller lengths. We may assume that $uv$ is reducible, hence we may write $u = u_0u_1$ and $v = v_0v_1$ with $(u_1v_0, 1) \in R$. We have factorizations $u_0 = u'u'_0$ and $v_1 = v'_1v''$ such that $\overline{u_0v_1} = u'v''$. By the induction hypothesis, we have $|u'_0v'_1| \leq \min\{|u_0|, |v_1|\} \cdot t_R$ and $|u'_0| \leq (t_R - 1)|v_1|$. Let $u'' = u'_0u_1$ and $v' = v_0v'_1$. Clearly, there exist factorizations $u = u'u''$ and $v = v'v''$ and

$$\overline{uv} = \overline{u_0u_1v_0v_1} = \overline{u_0v_1} = u'v''.$$

Moreover, since $u_1, v_0 \neq 1$,

$$\begin{aligned}|u''v'| &= |u_0'u_1v_0v_1'| = |u_1v_0| + |u_0'v_1'| \leq t_R + \min\{|u_0|, |v_1|\} \cdot t_R \\ &= (1 + \min\{|u_0|, |v_1|\})t_R \leq \min\{|u|, |v|\} \cdot t_R.\end{aligned}$$

Also

$$\begin{aligned}|u''| &= |u_0'u_1| \leq (t_R - 1)|v_1| + (t_R - 1) \\ &\leq (t_R - 1)|v_0v_1| = (t_R - 1)|v|.\end{aligned}$$

Thus (i) holds.

(ii) It follows from (i) that

$$\begin{aligned}|\overline{uv}| &= |u| + |v| - |u''v'| \geq |u| + |v| - \min\{|u|, |v|\} \cdot t_R \\ &= \max\{|u| + |v| - |u|t_R, |u| + |v| - |v|t_R\} \\ &= \max\{|v| - (t_R - 1)|u|, |u| - (t_R - 1)|v|\}\end{aligned}$$

as claimed.

(iii) Let $u = u'u''$ and $v = v'v''$ with $\overline{uv} = u'v''$ and $\overline{u''v'} = 1$. Then $r(\overline{uv}, u) \geq |u'| + 1$ and $|u''| \leq (t_R - 1)|v|$ by (i), hence the result. □

**Corollary 4.3** *For all $u, v \in A_R^*$, the equation $\overline{ux} = v$ has only finitely many solutions $x \in A_R^*$.*

**Proof**. If $x \in A_R^*$ is a solution of the equation, then $|v| \geq |x| - (t_R - 1)|u|$ by Lemma 4.2(ii) and so $|x| \leq |v| + (t_R - 1)|u|$. Thus there are only finitely many such solutions. □

We define a binary operation

$$\begin{aligned}\circ : A_R^* \times A_R^\omega &\to A_R^\omega \\ (u, \alpha) &\mapsto \overline{u\alpha}\end{aligned}$$

by taking $m = |u|(t_R - 1)$ and

$$\overline{u\alpha} = \overline{u\alpha^{[m]}}\alpha^{(m+1)}\alpha^{(m+2)} \ldots$$

By Lemma 4.1,

$$\overline{u\alpha^{[m+k]}} = \overline{u\alpha^{[m]}}\alpha^{(m+1)}\alpha^{(m+2)} \ldots \alpha^{(m+k)}$$

for every $k \in \mathbb{N}$ and so $\overline{u\alpha} \in A_R^\omega$ is well defined. Alternatively, $\overline{u\alpha} = \lim_{n\to\infty} \overline{u\alpha^{[n]}}$.

The partial operation $\pi : (A_R^*)^\omega \to A_R^\infty$ is defined as follows: for every sequence $(u_n)_n \in (A_R^*)^\omega$, $(u_1, u_2, \ldots)\pi$ is defined if and only if $(\overline{u_1 \ldots u_n})_n$ converges. In such a case, we have

$$(u_1, u_2, \ldots)\pi = \lim_{n\to\infty} \overline{u_1 \ldots u_n}.$$

In particular, $u^\omega = (u, u, \ldots)\pi$ is defined if and only if $u$ is aperiodic, by Theorem 3.2.

**Theorem 4.4** *With the ultrametric $d$, $(A_R^*, A_R^\omega, \cdot, \circ, \pi)$ is a metric partial $\omega$-monoid.*

**Proof**. Since $R$ is confluent, (w1) holds. Axiom (w4) holds trivially.

Suppose that $(u_1, u_2, \ldots)\pi$ is defined and $i_1 < i_2 < \ldots$ is a sequence in $\mathbb{N}$. Write $v_1 = u_1 \ldots u_{i_1}$, $v_2 = u_{i_1+1} \ldots u_{i_2}$, $v_3 = u_{i_2+1} \ldots u_{i_3}$, ... Since $\overline{v_1 \ldots v_n} = \overline{u_1 \ldots u_{i_n}}$ for every $n \in \mathbb{N}$ and $(\overline{u_1 \ldots u_n})_n$ converges, it follows that $\lim_{n\to\infty} \overline{v_1 \ldots v_n} = \lim_{n\to\infty} \overline{u_1 \ldots u_n}$ and (w2) holds.

Next we show that

$$\begin{aligned} A_R^* \times A_R^\infty &\to A_R^\infty \\ (u, \alpha) &\mapsto \overline{u\alpha} \end{aligned}$$

is continuous. If $(u, \alpha) \in A_R^* \times A_R^*$, then for $n = \max\{|u|, |\alpha|\}$ we have $B_{2^{-(n+1)}}(u, \alpha) = \{(u, \alpha)\}$ and we are done.

Assume now that $(u, \alpha) \in A_R^* \times A_R^\omega$ and let $\varepsilon > 0$. Let $m = |u|(t_R - 1)$ and take $n \in \mathbb{N}$ such that $2^{-n} < \varepsilon$. Let $(v, \beta) \in A_R^* \times A_R^\infty$ be such that $d(u, v) < 2^{-(|u|+1)}$ and $d(\alpha, \beta) < 2^{-(n+m)}$. Since $u, v \in A_R^*$ agree up to the $(|u| + 1)$-th letter, we have $u = v$. On the other hand, $\alpha^{[n+m]} = \beta^{[n+m]}$. Since

$$\overline{u\alpha} = \overline{u\alpha^{[m]}}\alpha^{(m+1)}\alpha^{(m+2)} \ldots, \quad \overline{v\beta} = \overline{v\beta^{[m]}}\beta^{(m+1)}\beta^{(m+2)} \ldots,$$

it follows that $\overline{u\alpha}^{[n]} = \overline{v\beta}^{[n]}$ and so $d(\overline{u\alpha}, \overline{v\beta}) < 2^{-n} < \varepsilon$. Thus $A_R^* \times A_R^\infty \to A_R^\infty$ is continuous. In particular, both the product $\cdot : A_R^* \times A_R^* \to A_R^*$ and the mixed product $\circ : A_R^* \times A_R^\omega \to A_R^\omega$ are continuous.

As a consequence, we also have that

$$\alpha = \lim_{n\to\infty} v_n \Rightarrow \overline{u\alpha} = \lim_{n\to\infty} \overline{uv_n} \tag{6}$$

for all $u, v_n \in A_R^*$.

We can now prove (w3). Suppose now that $(u_1, u_2, \ldots)\pi$ is defined and $v \in A_R^*$. It follows that

$$v((u_1, u_2, \ldots)\pi) = \overline{v \lim_{n\to\infty} \overline{u_1 \ldots u_n}} = \lim_{n\to\infty} \overline{vu_1 \ldots u_n} = (v, u_1, u_2, \ldots)\pi.$$

Finally, we observe that the partial map $\pi : (A_R^*)^\omega \to A_R^\infty$ is onto by remarking that $\alpha = \lim_{n\to\infty} \overline{\alpha^{(1)} \ldots \alpha^{(n)}}$ holds for every $\alpha \in A_R^\infty$.

Therefore $(A_R^*, A_R^\omega, \cdot, \circ, \pi)$ is a metric partial $\omega$-monoid. $\square$

¿From now on, when referring to $A_R^\infty$ as a metric partial $\omega$-monoid, we shall be considering the above structure.

The following example shows that the operation $\pi$ is not in general continuous, even when considering the metric $d'$ on $(A_R^*)^\omega$ defined by $d'((u_n)_n, (v_n)_n) = \max\{d(u_n, v_n) \mid n \in \mathbb{N}\}$, which generates a finer topology than the product topology. Note that, since $d(u_n, v_n) \in \{0\} \cup \{2^{-k} \mid k \in \mathbb{N}\}$ for every $n \in \mathbb{N}$, $d'$ is well defined.

**Example 4.5** *Let $A = \{a, b\}$ and $R = \{(ab, 1)\}$. Then $\pi : ((A_R^*)^\omega, d') \to (A_R^\omega, d)$ is not continuous.*

**Proof**. For every $n \in \mathbb{N}$, let $u_n = b^n a^{n+1}$. It follows easily from induction that $\overline{u_1 \ldots u_n} = ba^{n+1}$ for every $n \in \mathbb{N}$. Thus $(u_1, u_2, \ldots)\pi = ba^\omega$. Let $k \in \mathbb{N}$. We show that there exists a sequence $(v_n)_n$ in $A_R^*$ such that $(v_1, v_2, \ldots)\pi$ is defined and

$$d'((u_n)_n, (v_n)_n) < 2^{-k} \wedge d((u_1, u_2, \ldots)\pi, (v_1, v_2, \ldots)\pi) \geq 2^{-2}.$$

Define

$$v_n = \begin{cases} u_n & \text{if } n \leq k \\ b^n & \text{otherwise} \end{cases}$$

Clearly, $r(u_n, v_n) > k$ for every $n \in \mathbb{N}$, hence $d(u_n, v_n) < 2^{-k}$ and $d((u_n)_n, (v_n)_n) < 2^{-k}$. Since $\overline{v_1 \ldots v_{k+2}} = \overline{ba^{k+1}b^{k+1}b^{k+2}} = b^{k+3}$, it follows easily that $(v_1, v_2, \ldots)\pi = b^\omega$ and so

$$d((u_1, u_2, \ldots)\pi, (v_1, v_2, \ldots)\pi) = d(ba^\omega, b^\omega) = 2^{-2}$$

as claimed. Thus $\pi$ is not continuous. □

A natural question to raise is whether or not we can define suitable $\omega$-monoid structures in more general types of rewriting systems. The key feature is of course how the product and the metric are articulated. We show next that (6) does not hold for monadic confluent.

**Example 4.6** *Let $A = \{a, b, c\}$ and $R = \{(ba, c), (ca, b)\}$. Then:*

*(i) $R$ is monadic and confluent;*

*(ii) $(a^n)_n$ converges;*

*(ii) $(ba^n)_n$ does not converge.*

**Proof**. In view of Lemma 2.1, (i) is immediate, as well as (ii). A simple induction shows that

$$\overline{ba^{2n}} = b, \quad \overline{ba^{2n+1}} = c.$$

Thus $(ba^n)_n$ does not converge. □

In the more general setting of length-reducing confluent, we may even have infinitely many adherence values. We recall that $x \in X$ is an *adherence value* of $(u_n)_n$ if:

$$\forall \varepsilon > 0 \, \forall n \in \mathbb{N} \, \exists m \geq n : d(u_m, x) < \varepsilon.$$

This is equivalent to say that there exists some subsequence of $(u_n)_n$ converging to $x$.

**Example 4.7** *Let $A = \{a, b, c\}$ and $R = \{(ba^3, ab), (ca^3, cab), (ac, c), (bc, c), (c^2, c)\}$. Then:*

*(i) $R$ is length-reducing and confluent;*

*(ii) $(a^n)_n$ converges;*

*(iii) the adherence values of $(ca^n)_n$ are all the words in $A_R^\omega$ starting with $c$.*

**Proof**. It follows easily from Lemma 2.1 and direct verification that $R$ is confluent. Thus (i) and (ii) hold.

For every $n \in \mathbb{N}$, let

$$n = \Sigma_{i=0}^k x_i 3^i, \quad (x_i \in \{0, 1, 2\}, \; x_n \neq 0)$$

be the base 3 decomposition of $n$. We show by induction on $n$ that

$$\overline{ca^n} = ca^{x_k} ba^{x_{k-1}} b \ldots ba^{x_1} ba^{x_0}.$$

This is of course trivial for $\overline{ca} = ca$. Assume that it holds for $\overline{ca^n}$. If $x_k = x_{k-1} = \ldots = x_0 = 2$, successive application of the rule $ba^3 \to ab$ followed by $ca^3 \to cab$ yields

$$ca^{x_k}ba^{x_{k-1}}b \ldots ba^{x_1}ba^{x_0}a \xrightarrow{*} ca^3b^k \to cab^{k+1}$$

and so, by the induction hypothesis,

$$\overline{ca^{n+1}} = \overline{ca^{x_k}ba^{x_{k-1}}b \ldots ba^{x_1}ba^{x_0}a} = cab^{k+1}.$$

Since $n+1 = 3^{k+1}$ in this case, the claim follows.

Assume now that $x_j < 2$ and $x_i = 2$ if $0 \leq i < j$. Similarly to the preceding case, we obtain

$$ca^{x_k}ba^{x_{k-1}}b \ldots ba^{x_1}ba^{x_0}a \xrightarrow{*} ca^{x_k}ba^{x_{k-1}}b \ldots ba^{x_j}ab^{j-1}$$

and so

$$\overline{ca^{n+1}} = ca^{x_k}ba^{x_{k-1}}b \ldots ba^{x_j}ab^{j-1}$$

by the induction hypothesis. Since

$$n = \Sigma_{i=j}^{k} x_i 3^i + \Sigma_{i=0}^{j-1} 2 \cdot 3^i$$

yields

$$n+1 = \Sigma_{i=j+1}^{k} x_i 3^i + (x_j+1)3^j$$

in this case, the claim is proved.

It is easy to see that $A_R^\omega$ consists precisely of all the sequences of the form

- $a^\omega$
- $a^{i_1}ba^{i_2}ba^{i_3}b \ldots \qquad (i_1 \in \mathbb{N}_0, i_2, i_3, \ldots \in \{0,1,2\})$
- $ca^{i_1}ba^{i_2}ba^{i_3}b \ldots \qquad (i_1, i_2, \ldots \in \{0,1,2\})$.

If $\alpha = ca^{i_1}ba^{i_2}ba^{i_3}b \ldots \in A_R^\omega$ with $i_1 \neq 0$, then

$$\overline{ca^{p_n}} = ca^{i_1}ba^{i_2}b \ldots ba^{i_n}$$

for $p_n = \Sigma_{j=0}^{n-1} i_{n-j} 3^j$, hence $\alpha = \lim_{n\to\infty} \overline{ca^{p_n}}$ is an adherence value of $(ca^n)_n$ and (iii) holds.

In fact, it is not difficult to check that, for every $\alpha \in A_R^\omega$,

- the sequences $(\overline{a\alpha^{[n]}})_n$ and $(\overline{b\alpha^{[n]}})_n$ converge;
- the sequence $(\overline{c\alpha^{[n]}})_n$ converges if and only if $\alpha \neq a^\omega$.

$\square$

## 5 Hyperbolicity

The definitions introduced throughout this section may be found in [9].

We fix $R$ to be a finite special confluent rewriting system on a finite set $A$. The *Cayley graph* $\Gamma(A,R)$ is a directed $A$-graph defined by

$$V(\Gamma(A,R)) = A_R^*;$$

$$E(\Gamma(A,R)) = \{(u,a,v) \in A_R^* \times A \times A_R^* \mid \overline{ua} = v\}.$$

Note that $\Gamma(A,R)$ is connected (as an undirected graph) since $A_R^*$ is a monoid. We define a metric $s$ on $A_R^*$ by letting $s(u,v)$ be the length of the *shortest undirected path* connecting $u$ and $v$ in $\Gamma(A,R)$. Since $s(u,v) < 1 \Rightarrow u = v$ for all $u,v \in A_R^*$, $s$ induces the discrete topology on $A_R^*$.

A nondirected path $u$ —— $v$ of length $n$ is said to be a *geodesic* if $s(u,v) = n$. It follows from the definition of $s$ that any subpath of a geodesic is itself a geodesic.

We can view $(A_R^*, s)$ as a *geodesic space*. A geodesic space is a metric space $(X,d)$ such that, for all $x,y \in X$, there exists an isometry $f : [0, d(x,y)] \to X$ such that $f(0) = x$ and $f(d(x,y)) = y$, where the real interval $[0, d(x,y)]$ is endowed with the usual metric. The image of $f$ is called a *geodesic* connecting $x$ and $y$. Although $(A_R^*, s)$ is not geodesic in a strict sense, we can embed it in the geometric realization of $\Gamma(A,R)$, where a riemannian metric is defined uniformly for every edge, making it isomorphic to the interval $[0,1]$. The global metric is still the length of the shortest undirected path connecting two given points.

Although there may be more than one geodesic connecting two vertices $x$ and $y$, it is handy to denote it by $[x,y]$. A *geodesic triangle* in $X$ with (not necessarily distinct) vertices $x,y,z$ is the union of three geodesics

$$[x,y] \cup [y,z] \cup [z,x].$$

Let $\delta \geq 0$ be any nonnegative real number. A geodesic space $(X,d)$ is said to be *$\delta$-hyperbolic* if, for every geodesic triangle $[x,y] \cup [y,z] \cup [z,x]$ and every $u \in [x,y]$, we have

$$d(u, [y,z] \cup [z,x]) \leq \delta.$$

We recall that, given $u \in X$ and $Y \subseteq X$ nonempty, we define

$$d(u,Y) = \inf\{d(u,v) \mid v \in Y\}.$$

We say that $(X,d)$ is *hyperbolic* if it is *$\delta$-hyperbolic* for some $\delta \geq 0$.

We intend to show that $(A_R^*, s)$ is indeed hyperbolic. We start with some lemmas. For the whole of this section, we shall write

$$t = t_R, \quad m = \max\{1, t-1\}.$$

**Lemma 5.1** *Let $u,v \in A_R^*$ be adjacent vertices in $A_R^*$. Then one of them is a prefix of the other. Moreover, $|\,|u| - |v|\,| \leq m$.*

**Proof**. Without loss of generality, we may assume that $v = \overline{ua}$. If $ua$ is irreducible, then $u \leq v$ and $|v| = |u| + 1$. Otherwise, since $u$ is irreducible, $v \leq u$ and $|v| \geq |u| - (t-1)$. □

**Corollary 5.2** *For all* $u, v \in A_R^*$, $|\,|u| - |v|\,| \leq ms(u, v)$.

**Proof**. Let $u, v \in A_R^*$. We assume that $|u| \geq |v|$ and

$$u = x_0 \text{ — } x_1 \text{ — } \dots \text{ — } x_n = v$$

is a geodesic. By Lemma 5.1, we have $|x_{i-1}| - |x_i| \leq m$ for $i = 1, \dots, n$. Hence

$$|\,|u| - |v|\,| = \sum_{i=1}^{n} |x_{i-1}| - |x_i| \leq mn = ms(u, v)$$

as required. □

We remark that the combinatorial and the topological definitions of geodesic we introduced so far are perfectly compatible in $A_R^*$. We shall use the most suitable one in each circumstance. In particular, we shall write $p \in [x, y]$ whenever the vertex $p \in A_R^*$ lies in the geodesic $[x, y]$.

**Lemma 5.3** *Let* $[x, y]$ *be a geodesic in* $(A_R^*, s)$. *Then there exists a unique* $p \in [x, y]$ *of minimum length, and* $p$ *is a prefix of every* $u \in [x, y]$.

**Proof**. Let $p$ be the longest common prefix of all words $u \in [x, y]$. It is enough to show that $p \in [x, y]$. Assume that $[x, y]$ is the path

$$x = x_0 \text{ — } x_1 \text{ — } \dots \text{ — } x_n = y.$$

If $p = x$ we are done, so we may assume that $x \in paA^*$ for some $a \in A$. By minimality of $p$, there exists some $i \in \{1, \dots, n\}$ such that $x_{i-1} \in paA^*$ and $x_i \notin paA^*$. By Lemma 5.1, $x_i \leq x_{i-1}$. Since $p \leq x_i$ and $x_i \notin paA^*$, we conclude that $p = x_i \in [x, y]$ as desired. □

We remark that $p$ is not necessarily the longest common prefix of $x$ and $y$. For instance, for $A = \{a\}$ and $R = \{(a^5, 1)\}$, the path $a^4$ — $1$ — $a$ is a geodesic (in fact, the unique geodesic connecting $a^4$ and $a$).

We shall denote $p$ by $\mu[x, y]$. The longest common prefix of $x, y \in A_R^\infty$ will be denoted by $\lambda(x, y)$.

**Lemma 5.4** *Let* $[x, y]$ *be a geodesic in* $(A_R^*, s)$ *and let* $u \in [x, y]$. *Then we may write* $u = qu'$ *where* $|u'| \leq m^2 + m$ *and* $q$ *is a prefix of either* $x$ *or* $y$.

**Proof**. Let $p = \mu[x, y]$. Without loss of generality, we may assume that $u \in [x, p]$. Let $[x, p]$ be the path

$$x = x_0 \text{ — } x_1 \text{ — } \dots \text{ — } x_n = p$$

and assume that $u = x_k$. Let $q = \lambda(u, x)$ and write $u = qu'$. It remains to prove that $|u'| \leq m^2 + m$.

Suppose that $|u'| \geq m^2 + m + 1$. This implies $k \geq 1$. Let $a$ be the first letter of $u'$. By maximality of $q$, $qa \not\leq x$. We show next that

$$\exists i \in \{0, \dots, k-1\} : x_i \in qA^{\leq m}, \tag{7}$$

where $A^{\leq m} = \{w \in A^* : |w| \leq m\}$. Suppose that (7) does not hold. If $q = x$, then $x \in qA^{\leq m}$ and (7) would hold for $i = 0$, hence $q < x$ and we may write $x = qbx'$ with

$b \in A\backslash\{a\}$. Let $j \in \{0, \dots, k\}$ be maximal with respect to $qb \leq x_j$. Since $b \neq a$, we have $qb \not\leq x_k$ and so $j < k$. Write $x_j = qbx'_j$. We have $|bx'_j| \geq m+1$, otherwise (7) would hold for $i = j$. By Lemma 5.1, we have $|\,|x_{j+1}| - |x_j|\,| \leq m$ and one of them is a prefix of the other, hence $qb \leq x_{j+1}$, contradicting the maximality of $j$. Thus (7) holds.

Next we show that

$$\exists j \in \{k+1, \dots, n\} : x_j \in qA^{\leq m}, \tag{8}$$

We have $|u'| \geq m^2 + m + 1 > 1$. Hence $k < n$, otherwise $p = u = q$ and $u' = 1$. Since $x_k = qu'$, it follows from Lemma 5.1 that $q \leq x_{k+1}$. Let $j \in \{k+1, \dots, n\}$ be maximal with respect to $q \leq x_j$. If $j = n$ then $q = p$ and (8) holds trivially. Hence we may assume that $j < n$. By maximality of $j$, we have $q \not\leq x_{j+1}$ and so $x_j \not\leq x_{j+1}$. It follows from Lemma 5.1 that $x_{j+1} \leq x_j$ and $|\,|x_{j+1}| - |x_j|\,| \leq m$. Since $q \leq x_j$, $q \not\leq x_{j+1}$ and $x_{j+1} \leq x_j$, we obtain $x_{j+1} < q$. Thus $|x_j| \leq |x_{j+1}| + m < |q| + m$ and so (8) holds as claimed.

Taking $x_i$ and $x_j$ from (7) and (8), respectively, we get

$$s(x_i, x_j) \leq s(x_i, q) + s(q, x_j) \leq 2m.$$

On the other hand, since $|u'| \geq m^2 + m + 1$, it follows from Corollary 5.2 that

$$s(x_i, x_k) \geq \frac{|x_k| - |x_i|}{m} \geq \frac{|q| + m^2 + m + 1 - |q| - m}{m} = \frac{m^2 + 1}{m}.$$

Similarly,

$$s(x_k, x_j) \geq \frac{m^2 + 1}{m}.$$

Since $x_i$ — $x_k$ — $x_j$ is a geodesic, it follows that

$$2m \geq s(x_i, x_j) = s(x_i, x_k) + s(x_k, x_j) \geq \frac{2(m^2 + 1)}{m}$$

and so $m^2 \geq m^2 + 1$, a contradiction. Therefore $|u'| \leq m^2 + m$ and the lemma holds. □

**Lemma 5.5** *Let $[x, y]$ be a geodesic in $(A_R^*, s)$ and let $u \leq x$ with $|u| \geq |y|$. Then $uA^{<m} \cap [x, y] \neq \emptyset$.*

**Proof**. Let $p = \mu[x, y]$. Since $p \leq x$ by Lemma 5.3 and $|p| \leq |y| \leq |u|$, $u \leq x$, we have $p \leq u$. Suppose that $[x, p] \subseteq [x, y]$ is the path

$$x = x_0 — x_1 — \dots — x_n = p.$$

Let $i \in \{0, \dots, n\}$ be maximal with respect to $u \leq x_i$. If $i = n$, then $u = p \in [x, y]$, hence we may assume that $i < n$. By maximality of $i$, it follows that $u \not\leq x_{i+1}$, and so $x_i \not\leq x_{i+1}$. By Lemma 5.1, we get $x_{i+1} < x_i$ and so $x_{i+1} < u$. Moreover, $|x_i| \leq |x_{i+1}| + m < |u| + m$ and so $x_i \in uA^{<m} \cap [x, y]$. □

**Lemma 5.6** *Let $[x, y]$ be a geodesic in $(A_R^*, s)$. Then*

$$|\lambda(x, y)| - (m^2 - m) \leq |\mu[x, y]| \leq |\lambda(x, y)|.$$

**Proof**. Let $p = \mu[x, y]$ and let $q = \lambda(x, y)$. Since $p$ is itself a common prefix of $x$ and $y$, we have $|p| \leq |q|$.

Since $[x, p] \subseteq [x, y]$ is a geodesic, $q \leq x$ and $|q| \geq |p|$, there exists some $u \in qA^{<m} \cap [x, p]$ by Lemma 5.5. Similarly, there exists some $v \in qA^{<m} \cap [p, y]$. Hence $s(u, v) \leq 2(m-1)$. Since $u$ — $p$ — $v$ is a geodesic, we get $s(u, p) + s(p, v) = s(u, v) \leq 2(m-1)$ and we may assume without loss of generality that $s(u, p) \leq m - 1$. By Corollary 5.2, it follows that

$$|u| - |p| \leq ms(u, p) \leq m(m-1) = m^2 - m,$$

thus

$$|p| \geq |u| - (m^2 - m) \geq |q| - (m^2 - m)$$

as required. □

**Theorem 5.7** *Let $R$ be a finite special confluent rewriting system on a finite set $A$. Then $(A_R^*, s)$ is hyperbolic.*

**Proof**. Let $\delta = 3m^2 + m$. We show that $(A_R^*, s)$ is $\delta$-hyperbolic. Let $[x, y] \cup [y, z] \cup [z, x]$ be a geodesic triangle of $(A_R^*, s)$ and let $u \in [x, y]$. By symmetry, it is enough to show that $s(u, [y, z] \cup [z, x]) \leq \delta$. By Lemma 5.4, we may write $u = qu'$ where $|u'| \leq m^2 + m$ and $q$ is a prefix of either $x$ or $y$. Without loss of generality, we may assume that $q \leq x$. Let $p = \mu[z, x]$.

Suppose that $|q| \geq |p|$. Since $q \leq x$ and $[x, p] \subseteq [x, z]$ is a geodesic, it follows from Lemma 5.5 that there exists some $v \in qA^{<m} \cap [x, p]$. Thus

$$s(u, v) \leq s(u, q) + s(q, v) \leq m^2 + m + m - 1 = m^2 + 2m - 1 \leq 3m^2 + m = \delta$$

and so $s(u, [y, z] \cup [z, x]) \leq \delta$.

We assume now that $|q| < |p|$. Since $q$ and $p$ are both prefixes of $x$, it follows that $q < p$ and so $q < z$. Write $p' = \mu[y, z]$.

Suppose first that $q \not\leq p'$. We may write $[y, z]$ as

$$y = y_0 — y_1 — \ldots — y_n = z.$$

Let $i \in \{0, \ldots, n\}$ be maximal with respect to $q \not\leq y_i$. Since $q \not\leq p' \in [y, z]$, $i$ is well defined. Since $q \leq z$, we have $i < n$ and so $q \leq y_{i+1}$ by maximality of $i$. Hence $y_{i+1} \not\leq y_i$ and so $y_i \leq y_{i+1}$ by Lemma 5.1. Since $q$ and $y_i$ are both prefixes of $y_{i+1}$ and $q \not\leq y_i$, we get $y_i < q$. Now $[y_i, z]$ is a geodesic, $q \leq z$ and $|q| > |y_i|$, thus Lemma 5.5 yields $s(q, [y_i, z]) < m$ and so

$$\begin{aligned} s(u, [y, z] \cup [z, x]) &\leq s(u, [y_i, z]) \leq s(u, q) + s(q, [y_i, z]) \leq m^2 + m + m - 1 \\ &= m^2 + m + m - 1 = m^2 + 2m - 1 < 3m^2 + m = \delta. \end{aligned}$$

Finally, we assume that $q \leq p'$. Let $p'' = \mu[x, y]$. Since $u \in [x, y]$, we have $|u| \geq |p''|$. Thus Lemma 5.6 yields $|u| \geq |\lambda(x, y)| - (m^2 - m)$ and so

$$|q| \geq |u| - (m^2 + m) \geq |\lambda(x, y)| - 2m^2$$

Now $|\lambda(x, y)| \geq \min\{|\lambda(y, z)|, |\lambda(z, x)|\}$ implies that

$$|q| \geq |\lambda(y, z)| - 2m^2 \quad \text{or} \quad |q| \geq |\lambda(z, x)| - 2m^2.$$

In view of Lemma 5.6, this yields

$$|q| \geq |p'| - 2m^2 \quad \text{or} \quad |q| \geq |p| - 2m^2.$$

Since we are assuming that $q \leq p'$ and $q < p$, this implies that $s(q, \{p, p'\}) \leq 2m^2$. Hence

$$\begin{aligned} s(u, [y, z] \cup [z, x]) &\leq s(u, \{p, p'\}) \leq s(u, q) + s(q, \{p, p'\}) \leq m^2 + m + 2m^2 \\ &= 3m^2 + m = \delta. \end{aligned}$$

Therefore $(A_R^*, s)$ is $\delta$-hyperbolic. $\square$

We introduce now the *Gromov product* $g$ (with base point 1) on $A_R^*$ through

$$g(u, v) = \frac{1}{2}(s(u, 1) + s(v, 1) - s(u, v)).$$

Note that $g(u, v) \geq 0$ since $s$ satisfies the triangle inequality. Since $(A_R^*, s)$ is hyperbolic, it follows from [9, Prop. 2.21] that there exists some $\varepsilon \geq 0$ such that

$$\forall x, y, z \in A_R^* \; g(x, z) \geq \min\{g(x, y), g(y, z)\} - \varepsilon.$$

Following [9, Section 7.1], we can view $A_R^\omega$ as the *space of ends* of $A_R^*$: infinite reduced words are viewed as *rays* from the base point 1, every (finite) prefix being a geodesic.

We proceed now to introduce the Gromov topology on the space of ends. A few preparatory lemmas are needed.

**Lemma 5.8** *For all $u, v \in A_R^*$,*

$$\frac{|\lambda(u, v)| - (2m^2 - 2m)}{m} \leq g(u, v) \leq |\lambda(u, v)| + m^2 + m - 2.$$

**Proof**. Let $[u, v]$ be a geodesic. Write $w = \mu[u, v]$ and $p = \lambda(u, v)$. By Lemma 5.6, we have

$$|p| - (m^2 - m) \leq |w| \leq |p|.$$

Since $w \leq p$ by Lemma 5.3, we get $s(w, p) \leq m^2 - m$.

Considering a geodesic $[1, u]$, and since $w \leq u$, there exists some $u' \in wA^{<m} \cap [1, u]$ by Lemma 5.5. Similarly, considering a geodesic $[1, v]$, there exists some $v' \in wA^{<m} \cap [1, v]$. We have

$$s(u', p) \leq s(u', w) + s(w, p) \leq m - 1 + m^2 - m = m^2 - 1.$$

Similarly, $s(v', p) \leq m^2 - 1$. Thus

$$\begin{aligned} s(u, 1) + s(v, 1) - s(u, v) &= s(u, u') + s(u', 1) + s(v, v') + s(v', 1) - s(u, w) - s(w, v) \\ &= (s(u, u') - s(u, w)) + (s(v, v') - s(w, v)) + s(u', 1) + s(v', 1) \\ &\leq s(u', w) + s(v', w) + s(u', 1) + s(v', 1) \\ &\leq 2(m - 1) + s(u', p) + s(v', p) + 2s(p, 1) \\ &\leq 2(m - 1) + 2(m^2 - 1) + 2|p| \\ &= 2(m^2 + m - 2) + 2|p| \end{aligned}$$

and so $g(u, v) \leq |p| + m^2 + m - 2$.

On the other hand, by Corollary 5.2 and our previous remarks,

$$\begin{aligned}
s(u,1)+s(v,1)-s(u,v) &= s(u,u')+s(u',1)+s(v,v')+s(v',1)-s(u,w)-s(w,v)\\
&= (s(u,u')-s(u,w))+(s(v,v')-s(w,v))+s(u',1)+s(v',1)\\
&\geq -s(u',w)-s(v',w)+s(u',1)+s(v',1)\\
&\geq -2(m-1)+\tfrac{|u'|}{m}+\tfrac{|v'|}{m}\\
&\geq -2(m-1)+\tfrac{2|w|}{m}\\
&\geq -2(m-1)+\tfrac{2(|p|-(m^2-m))}{m}\\
&= \tfrac{2(|p|-(2m^2-2m))}{m}
\end{aligned}$$

and so $g(u,v) \geq \frac{|p|-(2m^2-2m)}{m}$ as required. □

Given a mapping $f : \mathbb{N}\times\mathbb{N} \to \mathbb{R}$, we write $\lim_{i,j\to\infty} f(i,j) = \infty$ if

$$\forall M > 0\ \exists k \in \mathbb{N}\ \forall i,j \geq k\ f(i,j) > M.$$

We write also

$$\liminf_{i,j\to\infty} f(i,j) = \lim_{n\to\infty}(\inf\{f(i,j) \mid i,j \geq n\}).$$

**Lemma 5.9** *Let $(u_n)_n$ be a sequence in $A_R^*$. Then $\lim_{i,j\to\infty} g(u_i,u_j) = \infty$ if and only if $(u_n)_n$ converges to some $\alpha \in A_R^\omega$ in $(A_R^\infty, d)$.*

**Proof**. By Lemma 5.8, we have $\lim_{i,j\to\infty} g(u_i,u_j) = \infty$ if and only if

$$\lim_{i,j\to\infty} |\lambda(u_i,u_j)| = \infty. \tag{9}$$

Assume that (9) holds. Since $r(u_i,u_j) = |\lambda(u_i,u_j)|+1$ if $u_i \neq u_j$ and $+\infty$ otherwise, it follows that $\lim_{i,j\to\infty} r(u_i,u_j) = \infty$ and so $\lim_{i,j\to\infty} d(u_i,u_j) = 0$. Hence $(u_n)_n$ is a Cauchy sequence in $(A_R^*, d)$. Since $(A_R^\infty, d)$ is complete by Corollary 2.3, $(u_n)_n$ converges to some $\alpha \in A_R^\infty$. Clearly, (9) implies $\alpha \in A_R^\omega$.

Conversely, assume that $\lim_{n\to\infty} u_n = \alpha \in A_R^\omega$ in $(A_R^\infty, d)$. Let $M > 0$. Then there exists some $k \in \mathbb{N}$ such that $r(u_n,\alpha) > M+1$ for every $n \geq k$. Hence $|\lambda(u_n,\alpha)| > M$ for every $n \geq k$, and so $|\lambda(u_i,u_j)| > M$ for all $i,j \geq k$. Therefore (9) holds as required. □

**Corollary 5.10** *Let $(u_n)_n$ and $(v_n)_n$ be sequences in $A_R^*$. Then $\lim_{i,j\to\infty} g(u_i,v_j) = \infty$ if and only if $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n \in A_R^\omega$ in $(A_R^\infty, d)$.*

**Proof**. By Lemma 5.8, we have $\lim_{i,j\to\infty} g(u_i,v_j) = \infty$ if and only if

$$\lim_{i,j\to\infty} |\lambda(u_i,v_j)| = \infty. \tag{10}$$

Assume that (10) holds. Then

$$\lim_{i,j\to\infty} |\lambda(u_i,u_j)| = \lim_{i,j\to\infty} |\lambda(v_i,v_j)| = \infty$$

and by the proof of Lemma 5.9 we obtain $\lim_{n\to\infty} u_n = \alpha$, $\lim_{n\to\infty} v_n = \beta$ for some $\alpha,\beta \in A_R^\omega$. By (10), we must have $\alpha = \beta$.

Conversely, assume that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = \alpha \in A_R^\omega$ in $(A_R^\infty, d)$. Let $M > 0$. Then there exists some $k \in \mathbb{N}$ such that $r(u_n,\alpha), r(v_n,\alpha) > M+1$ for every $n \geq k$. Hence $|\lambda(u_n,\alpha)|, |\lambda(v_n,\alpha)| > M$ for every $n \geq k$, and so $|\lambda(u_i,v_j)| > M$ for all $i,j \geq k$. Therefore (10) holds as required. □

We can now define the extension of the Gromov product to $A_R^\omega$ as follows: given $\alpha, \beta \in A_R^\omega$, let

$$g(\alpha, \beta) = \sup\{\liminf_{i,j\to\infty} g(u_i, v_j) \mid (u_n)_n, (v_n)_n \text{ are sequences in } A_R^*, \lim_{n\to\infty} u_n = \alpha, \lim_{n\to\infty} v_n = \beta\},$$

where the limits of $(u_n)_n$ and $(v_n)_n$ are taken in $(A_R^\infty, d)$.

Lemma 5.8 admits an immediate extension:

**Corollary 5.11** *For all $\alpha, \beta \in A_R^\omega$ distinct,*

$$\frac{|\lambda(\alpha, \beta)| - (2m^2 - 2m)}{m} \leq g(\alpha, \beta) \leq |\lambda(\alpha, \beta)| + m^2 + m - 2.$$

**Proof**. Let $(u_n)_n, (v_n)_n$ be sequences in $A_R^*$ such that $\lim_{n\to\infty} u_n = \alpha$ and $\lim_{n\to\infty} v_n = \beta$. It suffices to show that

$$\frac{|\lambda(\alpha, \beta)| - (2m^2 - 2m)}{m} \leq \liminf_{i,j\to\infty} g(u_i, v_j) \leq |\lambda(\alpha, \beta)| + m^2 + m - 2.$$

Thus we only need to show that there exists some $n \in \mathbb{N}$ such that

$$\forall i, j \geq n \quad \frac{|\lambda(\alpha, \beta)| - (2m^2 - 2m)}{m} \leq g(u_i, v_j) \leq |\lambda(\alpha, \beta)| + m^2 + m - 2. \tag{11}$$

Let $p = \lambda(\alpha, \beta)$ and take $n \in \mathbb{N}$ such that $|\lambda(u_i, \alpha)|, |\lambda(v_i, \beta)| > |p|$ for every $i \geq n$. Then $p(u_i, v_j) = p$ for all $i, j \geq n$ and (11) follows from Lemma 5.8. □

The Gromov topology $\mathcal{G}$ on $A_R^\omega$ can now be defined as follows. Given $\alpha \in A_R^\omega$ and $\eta > 0$, let

$$V_\eta(\alpha) = \{\beta \in A_R^\omega \mid g(\alpha, \beta) > \eta\}.$$

We take $\{V_\eta(\alpha) \mid \eta > 0\}$ as a fundamental system of neighbourhoods for $\alpha$.

**Theorem 5.12** *The metric $d$ on $A_R^\omega$ induces the Gromov topology $\mathcal{G}$.*

**Proof**. Since the open balls $\{B_\varepsilon(\alpha) \mid \varepsilon > 0\}$ constitute a fundamental system of neighbourhoods for $\alpha$ in $(A_R^\omega, d)$, we only have to compare the two fundamental systems.

Let $\varepsilon > 0$. For every $\beta \in A_R^\omega$, we have

$$\begin{aligned} d(\alpha, \beta) < \varepsilon &\Leftrightarrow 2^{-r(\alpha,\beta)} < 2^{-\log_2(\varepsilon^{-1})} \Leftrightarrow r(\alpha, \beta) > \log_2(\varepsilon^{-1}) \\ &\Leftrightarrow |\lambda(\alpha, \beta)| > \log_2(\varepsilon^{-1}) - 1. \end{aligned}$$

Hence

$$B_\varepsilon(\alpha) = \{\beta \in A_R^\omega \mid |\lambda(\alpha, \beta)| > \log_2(\varepsilon^{-1}) - 1\}.$$

It follows from Corollary 5.11 that

$$g(\alpha, \beta) > \log_2(\varepsilon^{-1}) - 1 + m^2 + m - 2 \Rightarrow |\lambda(\alpha, \beta)| \geq g(\alpha, \beta) - (m^2 + m - 2) > \log_2(\varepsilon^{-1}) - 1,$$

hence $V_\eta(\alpha) \subseteq B_\varepsilon(\alpha)$ for any positive $\eta \geq \log_2(\varepsilon^{-1}) + m^2 + m - 3$.

Conversely, let $\eta > 0$ and take $\varepsilon = 2^{-(m\eta + 2m^2 - 2m + 1)}$. Let $\beta \in B_\varepsilon(\alpha)$. Then $|\lambda(\alpha, \beta)| > \log_2(\varepsilon^{-1}) - 1 = m\eta + 2m^2 - 2m$ and Corollary 5.11 yields

$$g(\alpha, \beta) \geq \frac{|\lambda(\alpha, \beta)| - (2m^2 - 2m)}{m} > \eta.$$

Therefore $B_\varepsilon(\alpha) \subseteq V_\eta(\alpha)$ and so $d$ induces the Gromov topology $\mathcal{G}$. □

It is somewhat surprising that the metric $d$, that can be naturally defined using the *directed* Cayley graph of $A_R^*$, induces a topology defined via the *undirected* Cayley graph of $A_R^*$ (check the definition of $s$). This phenomenon is reminiscent of the geometric behaviour of prefix-automatic monoids in [17].

It is also interesting to note that $A_R^*$ is a word hyperbolic monoid for the definition introduced by Duncan and Gilman [8]. Let $w^r$ denote the *reversal* of a word $w$. A monoid $M$ is called *word hyperbolic* if $M$ has a rational set of representatives $\overline{M} \subseteq A^*$ (where $\overline{x}$ denotes the representative of $x \in M$) such that the language

$$T = \{u\#v\#\overline{uv}^r \mid u, v \in \overline{M}\}$$

is context-free, where # is a new symbol. All the results on context-free languages we shall use can be found in [10].

**Theorem 5.13** *Let $R$ be a finite special confluent rewriting system on a finite set $A$. Then $A_R^*$ is word hyperbolic.*

**Proof**. We may assume that $R$ is normalized.

Since $A_R^*$ is a rational language, it suffices to show that

$$T = \{u\#v\#\overline{uv}^r \mid u, v \in A_R^*\}$$

is context-free. By Lemma 4.2(i), we have

$$T = \{u_1u_2\#v_1v_2\#v_2^ru_1^r \mid u_1u_2, v_1v_2, u_1v_2 \in A_R^*,\ \overline{u_2v_1} = 1\}.$$

Let

$$T_1 = \{u_2\#v_1 \mid u_2, v_1 \in A_R^*,\ \overline{u_2v_1} = 1\}.$$

We show that $T_1$ is context-free. This holds trivially if $R = \emptyset$, hence we may assume that $R \neq \emptyset$. Since $R$ is normalized, $(w, 1) \in R$ implies $|w| \geq 2$. Let $\{(r_i, s_i) \mid i = 1, \ldots, n\}$ denote the set of all possible nontrivial factorizations of relators in $R$ (that is, $(r_is_i, 1) \in R$ and $r_i, s_i \neq 1$). Let

$$T_2 = \{r_{i_1} \ldots r_{i_k}\#s_{i_k} \ldots s_{i_1} \mid k \geq 0,\ i_j \in \{1, \ldots, n\}\}.$$

It is straightforward to check that

$$T_1 = T_2 \ \cap\ A_R^*\#A_R^*.$$

Let $B = \{b_1, \ldots, b_n, b_1^{-1}, \ldots, b_n^{-1}\}$ and let $\varphi : B^* \to A^*$ be the homomorphism defined by

$$b_i\varphi = r_i, \quad b_i^{-1}\varphi = s_i \quad (i = 1, \ldots, n).$$

The language

$$T_3 = \{w\#w^{-1} \mid w \in B^*\}$$

is a well-known context-free language and $T_2 = T_3\varphi$. Since the class of context-free languages is closed for morphisms and intersection with regular languages, it follows that $T_2 = T_3\varphi$ and $T_1 = T_2 \ \cap\ A_R^*\#A_R^*$ are context-free.

Let

$$T_4 = \{u_1\#'v_2\#v_2^ru_1^r \mid u_1, v_2 \in A^*\}$$

and let $\psi : (A \cup \{\#, \#'\})^* \to (A \cup \{\#\})^*$ be the homomorphism defined by

$$\#'\psi = 1, \qquad a\psi = a \quad (a \in A \cup \{\#\}).$$

Since the class of context-free languages is also closed for inverse morphisms and $\{w\#w^r \mid w \in A^*\}$ is context-free, it follows that

$$T_4 = T_1\psi^{-1} \cap A^*\#'A^*\#A^*$$

is context-free.

Finally, let $\theta : (A \cup \{\#, \#'\})^* \to (A \cup \{\#\})^*$ be the context-free substitution defined by

$$\#'\theta = T_1, \qquad a\theta = a \quad (a \in A \cup \{\#\}).$$

Since the class of context-free languages is also closed for context-free substitutions, it follows that

$$T_4\theta = \{u_1u_2\#v_1v_2\#v_2^r u_1^r \mid u_1, v_2 \in A^*,\ u_2, v_1 \in A_R^*,\ \overline{u_2v_1} = 1\}$$

is context-free. Therefore

$$T = T_4\theta \cap A_R^*\#A_R^*\#A_R^*$$

is context-free and $A_R^*$ is word hyperbolic. $\square$

## 6 Extendable homomorphisms

For every $u \in A^*$, taking $n = |u| + 1$, we have $B_{2^{-n}}(u) = \{u\}$. Hence the topology of $(A_R^*, d)$ is discrete and so every endomorphism $\varphi$ of $A_R^* = (\overline{A^*}, \cdot)$ is continuous. Under which conditions can we extend such an endomorphism to an endomorphism of $A_R^\infty$ or to a continuous mapping $\Phi : A_R^\infty \to A_R^\infty$? The next result shows that there is a unique candidate to perform any of these roles.

**Theorem 6.1** *Let $\varphi$ be an endomorphism of $A_R^*$ and let $\Phi : A_R^\infty \to A_R^\infty$ be an extension of $\varphi$. If $\Phi$ is either continuous or an endomorphism of the partial $\omega$-monoid $A_R^\infty$, then $\alpha\Phi = \lim_{n\to\infty} \alpha^{[n]}\varphi$ for every $\alpha \in A_R^\omega$.*

**Proof**. If $\Phi$ is continuous, then it commutes with limits and so

$$\alpha\Phi = (\lim_{n\to\infty} \alpha^{[n]})\Phi = \lim_{n\to\infty} \alpha^{[n]}\Phi = \lim_{n\to\infty} \alpha^{[n]}\varphi$$

holds for every $\alpha \in A_R^\omega$.

Assume now that $\Phi$ is an endomorphism. Let $\alpha \in A_R^\omega$. Since $\alpha = \lim_{n\to\infty} \alpha^{(1)} \ldots \alpha^{(n)}$, then $(\alpha^{(1)}, \alpha^{(2)}, \ldots)\pi$ is defined and equal to $\alpha$. It follows that $(\alpha^{(1)}\Phi, \alpha^{(2)}\Phi, \ldots)\pi$ is defined and equal to $\alpha\Phi$. Thus

$$\alpha\Phi = \lim_{n\to\infty} \overline{(\alpha^{(1)}\Phi) \ldots (\alpha^{(n)}\Phi)} = \lim_{n\to\infty} \alpha^{[n]}\varphi$$

as claimed. $\square$

It follows that the convergence of the sequences $(\alpha^{[n]}\varphi)_n$, for $\alpha \in A_R^\omega$, plays a key role in the extension problems we are about to discuss. We say that an endomorphism $\varphi$ of $A_R^*$ is *extendable* if the sequence $(\alpha^{[n]}\varphi)_n$ converges in $A_R^\infty$ for every $\alpha \in A_R^\omega$. Clearly, if $\Phi$ is a proper endomorphism of $A_R^\infty$, then $\lim_{n\to\infty} \alpha^{[n]}\varphi \in A_R^\omega$ for every $\alpha \in A_R^\omega$. We shall say that $\varphi$ is *properly extendable* if $\lim_{n\to\infty} \alpha^{[n]}\varphi \in A_R^\omega$ for every $\alpha \in A_R^\omega$.

We also introduce the notation

$$h_\varphi = \max\{|a\varphi| : a \in A\}$$

for any endomorphism $\varphi$ of $A_R^*$.

We shall need adequate characterization of idempotency:

**Lemma 6.2** *For every $u \in A_R^*$, the following conditions are equivalent:*

*(i) $u$ is idempotent;*

*(ii) $\overline{vu} = v$ for some $v \in A_R^*$.*

*(iii) $|\overline{u^2}| = |u|$;*

*(iv) $u = xy$ with $\overline{yx} = 1$;*

**Proof**. (i) $\Rightarrow$ (ii). Immediate.

(ii) $\Rightarrow$ (iii). Assume that $\overline{vu} = v$ for some $v \in A_R^*$. We may write $v = v'v''$ and $u = u'u''$ with $\overline{v''u'} = 1$ and $\overline{vu} = v'u''$. It follows that $v'v'' = v = \overline{vu} = v'u''$ and so $v'' = u''$. Therefore $\overline{u^2} = \overline{u'u''u'u''} = \overline{u'u''} = u$. In particular, (iii) holds.

(iii) $\Rightarrow$ (iv). Assume that $|\overline{u^2}| = |u|$. We may write $u = u_1'u_1'' = u_2'u_2''$ with $\overline{u^2} = u_1'u_2''$ and $\overline{u_1''u_2'} = 1$. Since $|\overline{u^2}| = |u|$, we have $|u_2''| = |u_1''|$ and so $u_1'' = u_2''$. Thus $u = u_2'u_2''$ with $\overline{u_2''u_2'} = 1$ and (iv) holds.

(iv) $\Rightarrow$ (i). Immediate. $\square$

The following lemma provides a combinatorial description of extendability.

**Theorem 6.3** *Let $\varphi$ be an endomorphism of $A_R^*$. Then the following conditions are equivalent:*

*(i) $\varphi$ is extendable;*

*(ii) $\forall a \in A\, \forall u \in A_R^*\, (((au)^* \subseteq A_R^* \wedge (auau)\varphi = (au)\varphi) \Rightarrow (aua)\varphi = (au)\varphi)$.*

**Proof**. (i) $\Rightarrow$ (ii). Suppose that there exist $a \in A$ and $u \in A_R^*$ such that $(au)^* \subseteq A_R^*$ and $(auau)\varphi = (au)\varphi$. Since $(au)^* \subseteq A_R^*$, $\alpha = (au)^\omega \in A_R^\omega$. As $\varphi$ is extendable, $(\alpha^{[n]}\varphi)_n$ converges, and therefore its subsequences $((au)^n\varphi)_n$ and $(((au)^n a)\varphi)_n$ converge to the same limit. Actually,

$$(au)^n\varphi = (au)\varphi, \quad ((au)^n a)\varphi = (aua)\varphi,$$

hence both subsequences are constant and $(au)\varphi = (aua)\varphi$.

(ii) $\Rightarrow$ (i). Suppose that $(\alpha^{[n]}\varphi)_n$ does not converge for some $\alpha \in A_R^\infty$. Then $(\alpha^{[n]}\varphi)_n$ fails (1) and so there exists some $k \in \mathbb{N}$ such that

$$\forall m \in \mathbb{N}\, \exists n \geq m : (\alpha^{[n]}\varphi)^{[k]} \neq (\alpha^{[n+1]}\varphi)^{[k]}. \tag{12}$$

By Lemma 4.2(iii),

$$|\alpha^{[n]}\varphi| \leq r(\alpha^{[n]}\varphi, \alpha^{[n+1]}\varphi) - 1 + (t_R - 1)|\alpha^{(n+1)}\varphi| < k + (t_R - 1)h_\varphi.$$

Consequently $|\alpha^{[n+1]}\varphi| \leq k + t_R h_\varphi$. Since there are only finitely many words of length $\leq k + t_R h_\varphi$, it follows from (12) that there exists an infinite sequence $i_1 < i_2 < \ldots$ in $\mathbb{N}$ such that

$$\alpha^{[i_1]}\varphi = \alpha^{[i_2]}\varphi = \ldots = x \neq y = \alpha^{[i_1+1]}\varphi = \alpha^{[i_2+1]}\varphi = \ldots$$

Further refining of the sequence allows us to assume that

$$\alpha^{(i_1+1)} = \alpha^{(i_2+1)} = \ldots = a \in A.$$

Since $A^{t_R}$ is finite, there exist $j, l \in \mathbb{N}$ such that $l > t_R$ and

$$\alpha^{(i_j+2)} \ldots \alpha^{(i_j+1+t_R)} = \alpha^{(i_{j+l}+2)} \ldots \alpha^{(i_{j+l}+1+t_R)}. \tag{13}$$

Write

$$u = \alpha^{(i_j+2)} \ldots \alpha^{(i_{j+l})}$$

and $w = \alpha^{[i_j]}$. Since (13) implies that

$$waua(u^{[t_R]}) = \alpha^{[i_j]}\alpha^{(i_j+1)}\alpha^{(i_j+2)} \ldots \alpha^{(i_{j+l})}\alpha^{(i_{j+l}+1)}\alpha^{(i_{j+l}+2)} \ldots \alpha^{(i_{j+l}+1+t_R)} = \alpha^{[i_{j+l}+1+t_R]}$$

is irreducible and $|u| \geq l - 1 \geq t_R$, we conclude that $(au)^* \subseteq A_R^*$. Moreover,

$$(wa)\varphi = (\alpha^{[i_j]}\alpha^{(i_j+1)})\varphi = \alpha^{[i_j+1]}\varphi = y,$$

$$(wau)\varphi = \alpha^{[i_{j+l}]}\varphi = x = \alpha^{[i_j]}\varphi = w\varphi.$$

Hence

$$(wau)\varphi = w\varphi \neq (wa)\varphi.$$

By Lemma 6.2, $(wau)\varphi = w\varphi$ yields $(auau)\varphi = (au)\varphi$. Suppose that $(aua)\varphi = (au)\varphi$. Then $(wa)\varphi = (waua)\varphi = (wau)\varphi = w\varphi$, a contradiction. Therefore $(aua)\varphi \neq (au)\varphi$ and (ii) does not hold as required. $\square$

We introduce now some notation relative to finite automata (see [2] for details). A finite $A$-*automaton* is described as a quadruple $\mathcal{A} = (S, s_0, T, E)$ where $S$ is a finite set, $s_0 \in S$, $T \subseteq S$ and $E$ is a finite subset of $S \times A \times S$. We denote the language recognized by $\mathcal{A}$ by $L(\mathcal{A})$. Given $s \in S$ and $T' \subseteq S$, we use the notation $L_{s,T'} = L(S, s, T', E)$ whenever the automaton is clear from the context.

Note that, since

$$A_R^* = A^* \setminus (\bigcup_{i=1}^{s} A^* r_i A^*),$$

it is clear that $A_R^*$ is a rational language.

Before proving that extendability is decidable, we prove a few decidability lemmas.

**Lemma 6.4** *Let $L \subseteq A_R^*$ be rational and let*

$$L' = \{u \in A_R^* \mid 1 \in \overline{uL}\}.$$

*Then $L'$ is rational and effectively constructible.*

**Proof**. Let $\{(f_1, g_1), \ldots, (f_k, g_k)\}$ denote all pairs of nonempty words $(f_i, g_i)$ such that $f_i g_i$ is a relator of $R$. Let $B = \{b_1, \ldots, b_k\}$ be a new alphabet, and define two homomorphisms $\varphi, \psi : B^* \to A^*$ by $b_i\varphi = f_i$ and $b_i\psi = g_i$. We show that $L' = (\widetilde{L\psi^{-1}})\varphi \cap A_R^*$ and therefore $L'$ is rational, since the family of rational languages is closed under morphism, inverse morphism (of the free monoid) and reversal.

Assume that $u \in L'$. Then $\overline{uv} = 1$ for some $v \in L$. Since $u$ and $v$ are both irreducible, we may write $u = f_{i_1} \ldots f_{i_n}$, $v = g_{i_n} \ldots g_{i_1}$ for some $i_1, \ldots, i_n \in \{1, \ldots, k\}$. Clearly, $v = (b_{i_n} \ldots b_{i_1})\psi$, hence $b_{i_1} \ldots b_{i_n} \in \widetilde{L\psi^{-1}}$ and so

$$u = f_{i_1} \ldots f_{i_n} = (b_{i_1} \ldots b_{i_n})\varphi \in (\widetilde{L\psi^{-1}})\varphi \cap A_R^*$$

as claimed.

Conversely, assume that $u \in (\widetilde{L\psi^{-1}})\varphi \cap A_R^*$. Then there exists $x = b_{i_n} \ldots b_{i_1} \in B^*$ such that $x\psi \in L$ and $\widetilde{x}\varphi = u$. Let $v = g_{i_n} \ldots g_{i_1} = x\psi$. It follows that

$$u = \widetilde{x}\varphi = (b_{i_1} \ldots b_{i_n})\varphi = f_{i_1} \ldots f_{i_n}$$

and so $\overline{uv} = \overline{f_{i_1} \ldots f_{i_n} g_{i_n} \ldots g_{i_1}} = 1$. Since $u \in A_R^*$, we obtain $u \in L'$ and so $L' = (\widetilde{L\psi^{-1}})\varphi \cap A_R^*$.

Therefore $L'$ is rational and effectively constructible. $\square$

**Lemma 6.5** *Let $L \subseteq A_R^*$ be rational and let*

$$L' = \{u \in A_R^* \mid \exists v \in L : \overline{uv} = u\}.$$

*Then $L'$ is rational and effectively constructible.*

**Proof**. We may assume that $L \neq \emptyset$. Let $\mathcal{A} = (S, s_0, T, E)$ be the minimal automaton of $L$ and let $s \in S$. By Lemma 6.4, the language

$$K_s = L_{s,T} \cap \{w \in A_R^* \mid 1 \in \overline{wL_{s_0,s}}\}$$

is rational and effectively constructible. We show that

$$L' = \bigcup_{s \in S} A_R^* \cap (A_R^* K_s). \tag{14}$$

Assume that $u \in L'$. Then there exists some $v \in L$ such that

$$u = u'u'', \quad v = v'v'', \quad \overline{u''v'} = 1, \quad u'v'' = \overline{uv} = u.$$

In particular, $u'v'' = u = u'u''$ yields $v'' = u''$. Since $v \in L$, we have $v' \in L_{s_0,s}$ and $v'' \in L_{s,T}$ for some $s \in S$. Since $1 = \overline{u''v'} = \overline{v''v'} \in \overline{v''L_{s_0,s}}$, it follows that $v'' \in K_s$. Hence $u = u'u'' = u'v''$ yields $u \in A_R^* \cap (A_R^* K_s)$.

Conversely, assume that $u \in A_R^* \cap (A_R^* K_s)$ for some $s \in S$. Then we may write $u = u'u''$ with $u'' \in K_s$. Then we have $\overline{u''z} = 1$ for some $z \in L_{s_0,s}$. Let $v = zu''$. Since $v \in L_{s_0,s}L_{s,T} \subseteq L(\mathcal{A}) = L$ and $\overline{uv} = \overline{u'u''zu''} = \overline{u'u''} = u$, we obtain $u \in L'$.

Therefore (14) holds and so $L'$ is rational and effectively constructible. $\square$

**Corollary 6.6** *Let $v \in A_R^*$ and*

$$K = \{u \in A_R^* \mid \overline{uv} = u\}.$$

*Then $K$ is rational and effectively constructible.*

**Proof**. It follows immediately from Lemma 6.5, where we consider the singleton set $L = \{v\}$. $\square$

Two more lemmas are needed:

**Lemma 6.7** *It is decidable whether or not a given rational language $L \subseteq A_R^*$ contains some idempotent.*

**Proof**. We may assume that $L \neq \emptyset$. Let $\mathcal{A} = (S, s_0, T, E)$ be the minimal automaton of $L$. We show that $L$ contains some idempotent if and only if $1 \in \cup_{s \in S} \overline{L_{s,T} L_{s_0,s}}$. Since this union is rational and effectively constructible by Theorem 2.4, decidability follows.

Assume that $u \in L$ is idempotent. By Lemma 6.2, we may write $u = xy$ with $\overline{yx} = 1$. Since $xy = u \in L = L(\mathcal{A})$, we have $x \in L_{s_0,s}$ and $y \in L_{s,T}$ for some $s \in S$, hence $1 = \overline{yx} \in \overline{L_{s,T} L_{s_0,s}}$.

Conversely, assume that $1 \in \overline{L_{s,T} L_{s_0,s}}$ for some $s \in S$, say $1 = \overline{yx}$ with $y \in L_{s,T}$ and $x \in L_{s_0,s}$. It follows that $xy \in L_{s_0,s} L_{s,T} \subseteq L(\mathcal{A}) = L$. Since $xy$ is idempotent by Lemma 6.2, $L$ contains an idempotent as required. $\square$

**Lemma 6.8** *Let $L \subseteq A_R^*$ be rational and let $\varphi$ be an endomorphism of $A_R^*$. Then $L\varphi$ is rational.*

**Proof**. Let $\hat{\varphi} : A^* \to A^*$ be the endomorphism defined by $a\hat{\varphi} = a\varphi$ $(a \in A)$. If $u = a_1 \dots a_n \in A_R^*$ $(a_i \in A)$, then

$$\begin{aligned} u\varphi &= \overline{(a_1 \dots a_n)}\varphi = \overline{(a_1\varphi) \dots (a_n\varphi)} = \overline{(a_1\hat{\varphi}) \dots (a_n\hat{\varphi})} \\ &= \overline{(a_1 \dots a_n)\hat{\varphi}} = \overline{u\hat{\varphi}}. \end{aligned}$$

Hence $L\varphi = \overline{L\hat{\varphi}}$. Since rational languages are preserved by free monoid homomorphisms, the lemma follows from Theorem 2.4. $\square$

**Theorem 6.9** *It is decidable whether or not an endomorphism of $A_R^*$ is extendable.*

**Proof**. Let $\varphi$ be an endomorphism of $A_R^*$. By Theorem 6.3, we need to show that condition (ii) in Theorem 6.3 is decidable. Fixing $a \in A$ and considering the negation, we must decide if

$$\exists u \in A_R^* \, ((au)^* \subseteq A_R^* \wedge (auau)\varphi = (au)\varphi \neq (aua)\varphi). \tag{15}$$

Let $\mathcal{A} = (S, s_0, T, E)$ denote the minimal automaton of $A_R^*$ and let $m = |S|$. We define

$$\begin{aligned} \Sigma = \cup_{j=1}^m \{ (s_1, s_2, \dots, s_{2j}) \in S^{2j} \mid\ & s_0, s_2, \dots, s_{2j-2} \text{ are all distinct, } s_{2j} = s_{2i} \\ & \text{for some } i < j \text{ and } a \in \cap_{i=0}^{j-1} L_{s_{2i}, s_{2i+1}} \}. \end{aligned}$$

Clearly, $\Sigma$ is finite and effectively constructible. For every $\sigma = (s_1, s_2, \dots, s_{2j}) \in \Sigma$, define

$$\Lambda(\sigma) = \bigcap_{i=1}^{j} L_{s_{2i-1}, s_{2i}}.$$

We show that (15) holds if and only if

$$\exists \sigma = (s_1, \ldots, s_{2j}) \in \Sigma \; \exists u \in \Lambda(\sigma) : (auau)\varphi = (au)\varphi \neq (aua)\varphi. \tag{16}$$

In fact, if (15) holds for $u \in A_R^*$, then $(au)^m$ labels a path in $\mathcal{A}$ of the form

$$s_0 \xrightarrow{a} s_1 \xrightarrow{u} s_2 \xrightarrow{a} s_3 \xrightarrow{u} \ldots \xrightarrow{u} s_{2m}.$$

Let $j \in \{1, \ldots, m\}$ be such that $s_{2j}$ is the first repetition in the sequence $(s_0, s_2, \ldots, s_{2m})$. Clearly, $\sigma = (s_1, s_2, \ldots, s_{2j}) \in \Sigma$. Moreover, $u \in \Lambda(\sigma)$, hence (16) holds.

Conversely, assume that (16) holds for $\sigma = (s_1, \ldots, s_{2j}) \in \Sigma$ and $u \in \Lambda(\sigma)$. We have a path in $\mathcal{A}$ of the form

$$s_0 \xrightarrow{(au)^i} s_{2l} \circlearrowright (au)^{j-l}$$

where $i \in \{0, \ldots, j-1\}$ is such that $s_{2j} = s_{2i}$. Thus $(au)^n$ labels a path in $\mathcal{A}$ for every $n \in \mathbb{N}$. Therefore $(au)^* \subseteq A_R^*$ and (15) holds.

To show that (16) is decidable, we may fix $\sigma = (s_1, s_2, \ldots, s_{2j}) \in \Sigma$. Writing $v = a\varphi$, we must show that

$$\exists z \in \Lambda(\sigma)\varphi : \overline{vzvz} = \overline{vz} \neq \overline{vzv} \tag{17}$$

is decidable. Since rational languages are closed for Boolean operations, Lemma 6.8 implies that $\Lambda(\sigma)\varphi$ is rational. Define

$$K = A^* \setminus \{w \in A_R^* \mid \overline{wv} = w\}, \quad K' = \overline{v(\Lambda(\sigma)\varphi)}.$$

It follows from Corollary 6.6 and Theorem 2.4 that $K$ and $K'$ are rational and effectively constructible. Moreover, (17) holds if and only if

$$\exists w \in K \cap K' : \overline{ww} = w.$$

By Lemma 6.7, this is decidable and so is the extendability of $\varphi$. $\square$

We consider now the proper case:

**Theorem 6.10** *Let $\varphi$ be an endomorphism of $A_R^*$. Then the following conditions are equivalent:*

*(i) $\varphi$ is properly extendable;*

*(ii) $\varphi$ preserves infinite order;*

*(iii) $\forall u \in A_R^+ \, (u^* \subseteq A_R^* \Rightarrow u^2\varphi \neq u\varphi)$.*

**Proof**. (i) $\Rightarrow$ (ii). Assume that $\varphi$ is properly extendable and suppose that $\varphi$ does not preserve infinite order. Then there exists some $u \in A_R^*$ with infinite order such that $u\varphi$ has finite order. By Corollary 3.5, there exist $x, y \in A_R^*$, $v \in A_R^+$ and $n_0 \in \mathbb{N}_0$ such that $\overline{yx} = 1$ and $\overline{u^n} = xv^ny$ for every $n \geq n_0$. Clearly, $v$ has infinite order. Moreover, $|\overline{u^n}\varphi|$ bounded yields $|v^n\varphi|$ bounded by Lemma 4.2(ii). Let $\alpha = v^\omega = \lim_{n\to\infty} v^n$. Since $\varphi$ is properly extendable, we have $\lim_{n\to\infty} \alpha^{[n]}\varphi \in A_R^\omega$. Thus $\lim_{n\to\infty} v^n\varphi \in A_R^\omega$. However, $|v^n\varphi|$ bounded implies $\lim_{n\to\infty} v^n\varphi \in A_R^*$ by Lemma 3.1, a contradiction.

Therefore $\varphi$ preserves infinite order.

(ii) $\Rightarrow$ (iii). Suppose that there exists some $u \in A_R^+$ such that $u^* \subseteq A_R^*$ and $u^2\varphi = u\varphi$. Since $u^* \subseteq A_R^*$ and $u \neq 1$, $u$ has infinite order. However, $u\varphi$ is an idempotent and so has finite order. Thus $\varphi$ does not preserve infinite order.

(iii) $\Rightarrow$ (i). Suppose that $\varphi$ is not properly extendable. Suppose first that $\varphi$ is not extendable. By Theorem 6.3, there exist $a \in A$ and $v \in A_R^*$ such that $(av)^* \subseteq A_R^*$ and $(avav)\varphi = (av)\varphi \neq (ava)\varphi$. Taking $u = av$, we conclude that (iii) does not hold.

Suppose now that $\varphi$ is extendable. Since $\varphi$ is not properly extendable, there exists some $\alpha \in A_R^\omega$ such that $\lim_{n\to\infty} \alpha^{[n]}\varphi \in A_R^*$. By Lemma 3.1(i), the sequence $(\alpha^{[n]}\varphi)_n$ is stationary and so there exist $v \in A_R^*$ and $m \in \mathbb{N}$ such that $\alpha^{[n]}\varphi = v$ for every $n \geq m$. Let $x$ be a factor of length $t_R$ having two disjoint occurrences in $\alpha^{(m+1)}\alpha^{(m+2)}\alpha^{(m+3)}\ldots$ Since $A^{t_R}$ is finite, such an $x$ exists. Hence we may write $\alpha = \alpha^{[k]}xyx\ldots$ for some $k \geq m$ and $y \in A_R^*$. Let $u = xy \in A_R^+$. Since $xyx$, being a factor of $\alpha$, is irreducible and $|x| = t_R$, we have $u^* = (xy)^* \subseteq A_R^*$. Moreover,

$$v = \alpha^{[k+|u|]}\varphi = (\alpha^{[k]}u)\varphi = \overline{(\alpha^{[k]}\varphi)(u\varphi)} = \overline{v(u\varphi)}$$

and so $u^2\varphi = u\varphi$ by Lemma 6.2. Therefore (iii) does not hold as required. $\square$

**Theorem 6.11** *It is decidable whether or not an endomorphism of $A_R^*$ is properly extendable.*

**Proof**. Let $\varphi$ be an endomorphism of $A_R^*$. By Theorem 6.10, we need to show that

$$\forall u \in A_R^+ \, (u^* \subseteq A_R^* \Rightarrow u^2\varphi \neq u\varphi). \tag{18}$$

is decidable. Let $\mathcal{A} = (S, s_0, T, E)$ be the minimal automaton of $A_R^*$ and write

$$L = \bigcup_{s\in S} (L_{s,s} \setminus \{1\})\varphi.$$

We show that (18) holds if and only if $L$ contains no idempotents.

Suppose first that (18) does not hold. Then there exists some $u \in A_R^+$ such that $u^* \subseteq L(\mathcal{A})$ and $u\varphi$ is idempotent. Since $\mathcal{A}$ is finite, we have $u^k \in L_{s,s}$ for some $k \in \mathbb{N}$ and $s \in S$. Since $u \neq 1$, we have $u^k \in L_{s,s} \setminus \{1\}$. Moreover, $u^k\varphi$ is idempotent since $u\varphi$ is idempotent. Thus $L$ contains the idempotent $u^k\varphi$.

Conversely, suppose that $L$ contains an idempotent, say $u\varphi$ with $u \in L_{s,s} \setminus \{1\}$. Since $u^* \subseteq L_{s,s} \subseteq A_R^*$, it follows that (18) does not hold. Thus (18) holds if and only if $L$ contains no idempotents.

By Lemma 6.8, $L$ is a rational language, and by Lemma 6.7, it is decidable whether or not $L$ contains no idempotents. $\square$

In view of Theorem 6.10, one may wonder if $\varphi$ preserving aperiodicity is equivalent to $\varphi$ being extendable. The answer is negative:

**Proposition 6.12** *Let $\varphi$ be an endomorphism of $A_R^*$. If $\varphi$ is extendable, then $\varphi$ preserves aperiodicity. The converse implication is not always true.*

**Proof**. Assume that $\varphi$ is extendable and $u \in A_R^*$ is aperiodic. Since $\overline{u^{n+1}} = \overline{u^n}$ yields $\overline{u^{n+1}}\varphi = \overline{u^n}\varphi$, we may assume that $u$ has infinite order. By Corollary 3.5, there exist $x, y \in A_R^*$, $v \in A_R^+$ and $n_0 \in \mathbb{N}_0$ such that $\overline{yx} = 1$ and $\overline{u^n} = xv^ny$ for every $n \geq n_0$. If $v\varphi$ has infinite order, then $(|v^n\varphi|)_n$ is unbounded and so is $(|u^n\varphi|)_n$. Hence $u\varphi$ has infinite order and so is aperiodic.

Thus we may assume that $v\varphi$ has finite order. Let $\alpha = v^\omega$. Since $\varphi$ is extendable, $(\alpha^{[n]}\varphi)_n$ converges and so does its subsequence $(v^n\varphi)_n$. Since $v\varphi$ has finite order, then $(|v^n\varphi|)_n$ is bounded and so $(v^n\varphi)_n$ is stationary by Lemma 3.1. Thus $(\overline{u^n}\varphi)_n = ((xv^ny)\varphi)_n$ is stationary and so $u\varphi$ is aperiodic.

To show that the converse implication fails, we take $A = \{a, b, c\}$ and $R = \{(ca, 1)\}$. Let $\varphi$ be the endomorphism of $A_R^*$ defined by

$$a\varphi = a, \quad b\varphi = c\varphi = c.$$

It is clear that the unique finite order elements of $A_R^*$ are those of the form $a^nc^n$ $(n \geq 0)$ and these are idempotents. Hence all elements are aperiodic and so, in particular, $\varphi$ preserves aperiodicity.

However, $\varphi$ is not extendable by Theorem 6.3, since $(ab)^* \subseteq A_R^*$ and

$$(abab)\varphi = \overline{acac} = ac = (ab)\varphi \neq a = \overline{aca} = (aba)\varphi.$$

□

# 7 Weak endomorphism extensions

In the main result of this section, we show that it is decidable whether or not, given an extendable endomorphism $\varphi$ of $A_R^*$, the extension $\Phi : A_R^\infty \to A_R^\infty$ defined by $\alpha\Phi = \lim_{n\to\infty} \alpha^{[n]}\varphi$ is a weak endomorphism. The problem of finding an algorithmic characterization of endomorphism extensions remains open.

Given $u \in A_R^+$, the notation $u^\omega$ may refer to either a word in $A^\omega$ or to $\lim_{n\to\infty} \overline{u^n} \in A_R^\infty$. In dubious cases, we shall use the notation $u^{\hat{\omega}} = \lim_{n\to\infty} \overline{u^n}$.

**Lemma 7.1** *Let $\varphi$ be an extendable endomorphism of $A_R^*$ and let $\Phi : A_R^\infty \to A_R^\infty$ be defined by $\alpha\Phi = \lim_{n\to\infty} \alpha^{[n]}\varphi$. Then $\Phi$ satisfies the endomorphism axioms (h1) – (h3).*

**Proof**. First we note that, since $\varphi$ is extendable, the mapping $\Phi$ is well-defined. Since $\Phi$ extends $\varphi$, axioms (h1) and (h2) are trivially satisfied.

Let $u \in A_R^*$ and $\alpha \in A_R^\omega$. We have

$$\overline{u\alpha} = \overline{u\alpha^{[m]}}\alpha^{(m+1)}\alpha^{(m+2)}\ldots$$

for some $m \in \mathbb{N}$. It follows easily from (6) that

$$\begin{aligned}(\overline{u\alpha})\Phi &= \lim_{n\to\infty}(\overline{u\alpha^{[m]}}\alpha^{(m+1)}\ldots\alpha^{(m+n)})\varphi = \lim_{n\to\infty}\overline{(u\varphi)(\alpha^{[m+n]}\varphi)}\\ &= \lim_{n\to\infty}\overline{(u\varphi)(\alpha^{[n]}\varphi)} = \overline{(u\varphi)\lim_{n\to\infty}\alpha^{[n]}\varphi}\\ &= \overline{(u\varphi)(\alpha\Phi)}.\end{aligned}$$

Therefore (h3) holds as required. □

Therefore the fact of $\Phi$ being an endomorphism (respectively weak endomorphism) of $A_R^\infty$ depends solely of axiom (h4) (respectively (h4')). We need the following lemma:

**Lemma 7.2** *Let $(u_n)_n$, $(v_n)_n$ be sequences in $A_R^*$ such that $(u_n)_n$ converges and $\lim_{n\to\infty} \frac{|u_n|}{|v_n|} = +\infty$. Then $(\overline{u_n v_n})_n$ converges and*

$$\lim_{n\to\infty} \overline{u_n v_n} = \lim_{n\to\infty} u_n.$$

**Proof**. It is enough to show that

$$\forall k \in \mathbb{N}\, \exists m \in \mathbb{N}\, \forall n \geq m\; \overline{u_n v_n}^{[k]} = u_n^{[k]}.$$

We may assume that $v_n \neq 1$ for infinitely many $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{|u_n|}{|v_n|} = +\infty$ and $(u_n)_n$ converges, it follows that $\lim_{n\to\infty} |u_n| = +\infty$.

Let $k \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{|u_n|}{|v_n|} = \lim_{n\to\infty} |u_n| = +\infty$, there exists some $m \in \mathbb{N}$ such that $|u_n| > k + t_R|v_n|$ for every $n \geq m$. Let $n \geq m$. By Lemma 4.2(i), we have factorizations $u_n = u'u''$, $v_n = v'v''$ and $\overline{u_n v_n} = u'v''$ with $|u''v'| < |v_n|t_R$. Thus

$$|u'| = |u_n| - |u''| > k + t_R|v_n| - |v_n|t_R = k$$

and so $\overline{u_n v_n}^{[k]} = u_n^{[k]}$ for every $n \geq m$ as required. $\square$

**Theorem 7.3** *Let $\varphi$ be an extendable endomorphism of $A_R^*$ and let $\Phi : A_R^\infty \to A_R^\infty$ be defined by $\alpha\Phi = \lim_{n\to\infty} \alpha^{[n]}\varphi$. The following conditions are equivalent:*

*(i) $\Phi$ is a weak endomorphism of $A_R^\infty$;*

*(ii)*
$$\forall x, y \in A_R^*\, \forall u \in A_R^+\, ((xu^+y \subseteq A_R^* \wedge \overline{yx} = 1 \wedge u^2\varphi = u\varphi) \Rightarrow y\varphi = 1). \qquad (19)$$

**Proof**. (i) $\Rightarrow$ (ii). Assume that $\Phi$ is a weak endomorphism. Let $x, y \in A_R^*$ and $u \in A_R^+$ be such that $xu^+y \subseteq A_R^*$, $\overline{yx} = 1$ and $u^2\varphi = u\varphi$. Since $\overline{(xuy)^n} = xu^n y$ for every $n \in \mathbb{N}$ and $u \neq 1$, it follows from Lemma 7.2 that

$$(xuy)^{\hat{\omega}} = \lim_{n\to\infty} \overline{(xuy)^n} = \lim_{n\to\infty} xu^n y = \lim_{n\to\infty} xu^n = xu^\omega = xu^{\hat{\omega}}.$$

Since $\Phi$ is a weak endomorphism and $u\varphi = u^2\varphi$, we obtain

$$\begin{aligned}(xuy)\varphi &= \lim_{n\to\infty}((xu^n y)\varphi) = \lim_{n\to\infty} \overline{(xuy)^n}\varphi = \lim_{n\to\infty} \overline{((xuy)\varphi)^n} \\ &= \overline{((xuy)\varphi)^{\hat{\omega}} = (xuy)^{\hat{\omega}}\Phi = (xu^{\hat{\omega}})\Phi = \overline{(x\varphi)(u^{\hat{\omega}}\Phi)}} \\ &= \overline{(x\varphi)(\lim_{n\to\infty} \overline{(u\varphi)^n})} = \overline{(x\varphi)(\lim_{n\to\infty} u^n\varphi)} = \overline{(x\varphi)(u\varphi)} \\ &= (xu)\varphi.\end{aligned}$$

By Lemma 6.2, we obtain $\overline{y^2}\varphi = y\varphi$ and so

$$y\varphi = \overline{y^2 x}\varphi = \overline{yx}\varphi = 1\varphi = 1.$$

Thus (19) holds.

(ii) $\Rightarrow$ (i). Assume that (19) holds. By Lemma 7.1, axioms (h1) – (h3) are satisfied. It remains to check (h4').

Let $u \in A_R^*$ and assume that $u^{\hat{\omega}}$ is defined. We show that $(u\varphi)^{\hat{\omega}}$ is defined and $u^{\hat{\omega}}\Phi = (u\varphi)^{\hat{\omega}}$.

By Theorem 3.2, $u$ is aperiodic. By Proposition 6.12, $u\varphi$ is aperiodic and so $(u\varphi)^{\hat{\omega}}$ is defined. Suppose that $u$ has finite order. Then $\overline{u^m} = \overline{u^{m+1}}$ for some $m \in \mathbb{N}$ and so $\overline{(u\varphi)^m} = \overline{(u\varphi)^{m+1}}$. Therefore

$$u^{\hat{\omega}}\Phi = \lim_{n\to\infty} \overline{u^n}\varphi = \overline{u^m}\varphi = \overline{(u\varphi)^m} = \lim_{n\to\infty} \overline{(u\varphi)^n} = (u\varphi)^{\hat{\omega}}.$$

Assume now that $u$ has infinite order. By Corollary 3.5, there exist $x, y \in A_R^*$, $v \in A_R^+$ and $n_0 \in \mathbb{N}_0$ such that $\overline{yx} = 1$ and $\overline{u^n} = xv^ny$ for every $n \geq n_0$. Hence $v$ has also infinite order. By the proof of Corollary 3.4, we have

$$u^{\hat{\omega}} = \lim_{n\to\infty} \overline{u^n} = xv^{\omega}.$$

Thus $u^{\hat{\omega}}\Phi = \lim_{n\to\infty}(xv^{\omega})^{[n]}\varphi$ and since $(xv^n)_n$ is a subsequence of $((xv^{\omega})^{[n]})_n$, we obtain

$$u^{\omega}\Phi = \lim_{n\to\infty}(xv^n)\varphi. \tag{20}$$

On the other hand,

$$(u\varphi)^{\hat{\omega}} = \lim_{n\to\infty} \overline{u^n}\varphi = \lim_{n\to\infty}(xv^ny)\varphi. \tag{21}$$

If $v\varphi$ has infinite order, all powers $(v\varphi)^n$ are distinct and so $\lim_{n\to\infty}|(v\varphi)^n| = +\infty$. Thus $\lim_{n\to\infty}|(xv^n)\varphi| = +\infty$ by Lemma 4.2(ii). Hence $\lim_{n\to\infty}\frac{|(xv^n)\varphi|}{|y\varphi|} = +\infty$ and Lemma 7.2 yields

$$\lim_{n\to\infty}(xv^n)\varphi = \lim_{n\to\infty}(xv^ny)\varphi.$$

Therefore $u^{\hat{\omega}}\Phi = (u\varphi)^{\hat{\omega}}$ by (20) and (21).

Thus we may assume that $v\varphi$ has finite order. Since $v$ has infinite order, $v\varphi$ is aperiodic by Proposition 6.12. Thus $\overline{(v\varphi)^m} = \overline{(v\varphi)^{m+1}}$ for some $m \in \mathbb{N}$. For $k = \max\{n_0, m\}$, we may write

$$x(v^k)^+y \subseteq A_R^*, \quad \overline{yx} = 1, \quad (v^k)^2\varphi = v^k\varphi.$$

Since $v^k \in A_R^+$, we may apply (19) and obtain $y\varphi = 1$. Together with (20) and (21), this yields

$$u^{\hat{\omega}}\Phi = \lim_{n\to\infty}(xv^n)\varphi = \lim_{n\to\infty}((xv^ny)\varphi) = (u\varphi)^{\hat{\omega}}.$$

Therefore (h4') holds and $\Phi$ is a weak endomorphism as required. $\square$

The properly extendable case is straigthforward:

**Corollary 7.4** *Let $\varphi$ be a properly extendable endomorphism of $A_R^*$ and let $\Phi : A_R^\infty \to A_R^\infty$ be defined by $\alpha\Phi = \lim_{n\to\infty}\alpha^{[n]}\varphi$. Then $\Phi$ is a weak endomorphism of $A_R^\infty$.*

**Proof**. Since $\varphi$ is properly extendable, it preserves infinite order by Theorem 6.10. Hence conditions $xu^+y \subseteq A_R^*$ and $u^2\varphi = u\varphi$ cannot occur simultaneously. Therefore (19) holds trivially and so $\Phi$ is a weak endomorphism by Theorem 7.3. $\square$

The next counterexample shows that the analogue of Corollary 7.4 does not hold for extendable endomorphisms. If we are using an alphabet $A$ containing formal inverses of some of its letters, we say that an endomorphism $\varphi : A_R^* \to A_R^*$ is *matched* if, whenever $a, a^{-1} \in A$, all letters in $a\varphi$ have formal inverses and $a^{-1}\varphi = (a\varphi)^{-1}$ (the formal inverse of the word $a\varphi$).

**Example 7.5** *Let* $A = \{a, b, c, b^{-1}, c^{-1}\}$ *and* $R = \{(bb^{-1}, 1), (cc^{-1}, 1), (c^{-1}c, 1)\}$. *Let* $\varphi : A_R^* \to A_R^*$ *be the matched endomorphism defined by*

$$a\varphi = b^{-1}b, \quad b\varphi = cbc, \quad c\varphi = c^2.$$

*Then* $\varphi$ *is extendable but the mapping* $\Phi : A_R^\infty \to A_R^\infty$ *defined by* $\alpha\Phi = \lim_{n\to\infty} \alpha^{[n]}\varphi$ *is not a weak endomorphism.*

**Proof**. Let $B = \{b, b^{-1}, c, c^{-1}\}$. We may identify $B_R^*$ with the submonoid $\overline{B^*}$ of $A_R^*$. We show that

(A) $\varphi\,|_{\overline{B^*}}$ preserves infinite order.

By Theorem 6.10, we have to show that $u^* \subseteq \overline{B^*}$ implies $u^2\varphi \neq u\varphi$ for an arbitrary $u \in B_R^+$. The case $u \in c^+ \cup (c^{-1})^+$ being trivial, we may assume that

$$u = c^{i_0} b^{-j_1} b^{k_1} c^{i_1} \dots c^{i_{n-1}} b^{-j_n} b^{k_n} c^{i_n}$$

with $n \geq 1$; $i_0, i_n \in \mathbf{Z}$; $i_1, \dots, i_{n-1} \in \mathbf{Z} \setminus \{0\}$ and $j_l, k_l \geq 0$, $j_l + k_l > 0$ for $l = 1, \dots, n$. We have

$$u\varphi = \overline{c^{2i_0}(c^{-1}b^{-1}c^{-1})^{j_1}(cbc)^{k_1}c^{2i_1} \dots c^{2i_{n-1}}(c^{-1}b^{-1}c^{-1})^{j_n}(cbc)^{k_n}c^{2i_n}}. \tag{22}$$

Since $\overline{b^{-1}c^{-1}cb} = b^{-1}b$ and $\overline{bcc^{2i_l}c^{-1}b^{-1}} = bc^{2i_l}b^{-1}$ for $l = 1, \dots, n-1$, it follows easily that no occurrence of either $b$ or $b^{-1}$ is involved in the reduction in (22). Thus

$$u\varphi = \begin{cases} c^{2i_0-1}b^{-1} \dots bc^{2i_n+1} & & \text{if } j_1, k_n > 0 \\ c^{2i_0-1}b^{-1} \dots b^{-1}c^{2i_n-1} & \text{or } c^{2i_0-1}b^{-1}c^{2i_1-1} & \text{if } j_1 > 0 \text{ and } k_n = 0 \\ c^{2i_0+1}b \dots bc^{2i_n+1} & \text{or } c^{2i_0+1}bc^{2i_1+1} & \text{if } j_1 = 0 \text{ and } k_n > 0 \\ c^{2i_0+1}b \dots b^{-1}c^{2i_n-1} & & \text{if } j_1, k_n = 0 \end{cases}$$

If $\overline{(u\varphi)(u\varphi)} = u^2\varphi = u\varphi$, we must have reduction between $b$ and $b^{-1}$ in the product $(u\varphi)(u\varphi)$, hence $j_1, k_n > 0$ and $\overline{c^{2i_n+1}c^{2i_0-1}} = 1$. Thus $i_n = -i_0$. Since $u^2$ is irreducible, $b^{k_n}c^{i_n}c^{i_0}b^{-j_1}$ is irreducible as well. Thus $i_n = i_0 = 0$ and $j_1, k_n > 0$ implies that $bb^{-1}$ is irreducible, a contradiction. Therefore $\varphi\,|_{B_R^*}$ preserves infinite order.

We remark also that

(B) $1(\varphi\,|_{\overline{B^*}})^{-1} = \{1\}$.

It is clear that $c^n\varphi = 1 \Leftrightarrow n = 0$ and the general case follows from neither $b$ or $b^{-1}$ being involved in the reduction in (22).

We show next that

(C) $\overline{B^*}\varphi = \{c^{2n} \mid n \in \mathbf{Z}\} \cup \{c^{2i_0+1}b^{-j_1}b^{k_1}c^{2i_1}b^{-j_2}b^{k_2}c^{2i_2} \dots c^{2i_{n-1}}b^{-j_n}b^{k_n}c^{2i_n+1} \mid n \geq 1;$ $i_0, i_n \in \mathbf{Z}$; $i_1, \dots, i_{n-1} \in \mathbf{Z} \setminus \{0\}$ and $j_l, k_l \geq 0$, $j_l + k_l > 0$ for $l = 1, \dots, n\}$.

Clearly, all elements of $B\varphi$ are contained in the right hand side set, which we denote by $P$. Straightforward checking shows that $\overline{P(B\varphi)} \subseteq P$, hence $\overline{B^*}\varphi \subseteq P$. Conversely, every element of $P$ is clearly a product of words of the form $c^2, c^{-2}, cbc^{-1}$ and $cb^{-1}c^{-1}$. Since $cbc^{-1} = \overline{(cbc)c^{-2}} = (bc^{-1})\varphi$ and $cb^{-1}c^{-1} = \overline{c^2(c^{-1}b^{-1}c^{-1})} = (cb^{-1})\varphi$, it follows that $\overline{B^*}\varphi = P$ as required.

Our next step is to prove that

(D) If $u \in A_R^*$ and $u\varphi \notin \overline{B^*}\varphi$, then $u\varphi$ is idempotent if and only if $u = va^tw$ for some $t \in \mathbb{N}$ and $v, w \in \overline{B^*}$ satisfying $\overline{wv} = 1$.

It suffices to prove the direct inclusion, the opposite inclusion being immediate. Since $u \in A_R^*$ has at least one occurrence of $a$, we may write $u = u_0a^{t_1}u_1 \ldots u_{k-1}a^{t_k}u_k$ with $u_i \in \overline{B^*}$ and $u_1, \ldots, u_{k-1} \neq 1$. Hence

$$u\varphi = \overline{(u_0\varphi)b^{-1}b(u_1\varphi)\ldots(u_{k-1}\varphi)b^{-1}b(u_k\varphi)}.$$

By (B), we have $u_1\varphi, \ldots, u_{k-1}\varphi \neq 1$ and so, by (C), we may in fact write $u\varphi = (u_0\varphi)b^{-1}b(u_1\varphi)\ldots(u_{k-1}\varphi)b^{-1}b(u_k\varphi)$. If $\overline{(u_k\varphi)(u_0\varphi)} \neq 1$, it follows easily from (C) that

$$\overline{b^{-1}b(u_k\varphi)(u_0\varphi)b^{-1}b} = b^{-1}b\overline{(u_k\varphi)(u_0\varphi)}b^{-1}b$$

and so $u\varphi$ is not idempotent. Hence $\overline{b^{-1}b(u_k\varphi)(u_0\varphi)b^{-1}b} = b^{-1}b$ and so

$$\begin{aligned}
&(u_0\varphi)b^{-1}b(u_1\varphi)\ldots(u_{k-1}\varphi)b^{-1}b(u_k\varphi) = u\varphi = \overline{(u\varphi)^2}\\
&= (u_0\varphi)b^{-1}b(u_1\varphi)\ldots(u_{k-1}\varphi)\overline{b^{-1}b(u_k\varphi)(u_0\varphi)b^{-1}b}(u_1\varphi)\ldots(u_{k-1}\varphi)b^{-1}b(u_k\varphi)\\
&= (u_0\varphi)b^{-1}b(u_1\varphi)\ldots(u_{k-1}\varphi)b^{-1}b(u_1\varphi)\ldots(u_{k-1}\varphi)b^{-1}b(u_k\varphi)
\end{aligned}$$

yields $k = 1$ and so $\overline{(u_1\varphi)(u_0\varphi)} = 1$. By (B), we obtain $\overline{u_1u_0} = 1$ and (D) holds.

We prove next that $\varphi$ is extendable. By Theorem 6.3, we must show that

$$((xu)^* \subseteq A_R^* \wedge (xuxu)\varphi = (xu)\varphi) \Rightarrow (xux)\varphi = (xu)\varphi$$

holds for all $x \in A$ and $u \in A_R^*$.

Let $x \in A$ and $u \in A_R^*$ be such that $(xu)^* \subseteq A_R^*$ and $(xuxu)\varphi = (xu)\varphi$. Then $xu$ has infinite order and $(xu)\varphi$ is idempotent, so by (A) $xu$ cannot be in $\overline{B^*}$. We may write $xu = vaw$ with $v \in B^*$, so that $(xu)\varphi = \overline{(v\varphi)b^{-1}b(w\varphi)}$. Since $v\varphi$ does not end in $b$ by (C), we obtain $(xu)\varphi = (v\varphi)b^{-1}\overline{b(w\varphi)}$. It also follows from (C) that $(v\varphi)b^{-1}$ cannot be a prefix of some word in $\overline{B^*}\varphi$ (either $v\varphi = c^{2n}$, but no word in $\overline{B^*}\varphi$ starts with $c^{2n}b^{-1}$, or $v\varphi$ ends in $b^{\pm 1}c^{2i_n-1}$, but no word in $\overline{B^*}\varphi$ contains $b^{\pm 1}c^{2i_n-1}b^{-1}$). Consequently $(xu)\varphi \notin \overline{B^*}\varphi$. By (D), we conclude that $xu = va^tw$ with $t \in \mathbb{N}$, $v, w \in \overline{B^*}$ and $\overline{wv} = 1$. Since $xuxu = va^twva^tw$ is irreducible, it follows that $v = w = 1$ and $xu = a^t$. Then $x = a$ and

$$(xux)\varphi = a^{t+1}\varphi = b^{-1}b = a^t\varphi = (xu)\varphi.$$

Therefore $\varphi$ is extendable.

By Theorem 7.3, $\Phi$ not being a weak endomorphism follows from the existence of $x, y \in A_R^*$ and $u \in A_R^+$ such that $xu^+y \subseteq A_R^*$, $\overline{yx} = 1$, $u^2\varphi = u\varphi$ and $y\varphi \neq 1$. All the conditions are clearly satisfied by $x = b^{-1}$, $u = a$ and $y = b$. $\square$

We can also get decidability for $\Phi$ being a weak endomorphism:

**Theorem 7.6** *Let $\varphi$ be an extendable endomorphism of $A_R^*$ and let $\Phi : A_R^\infty \to A_R^\infty$ be defined by $\alpha\Phi = \lim_{n\to\infty} \alpha^{[n]}\varphi$. Then it is decidable whether or not $\Phi$ is a weak endomorphism of $A_R^\infty$.*

**Proof**. By Theorem 7.3, we only need to show that (19) is decidable. Let $\mathcal{A} = (S, s_0, T, E)$ be the minimal automaton of $A_R^*$ and write $m = |S|$. For every $\sigma = (s_1, s_2, s_3) \in S^3$, define

$$\Lambda(\sigma) = (L_{s_1,s_2} \cap L_{s_2,s_2}) \setminus \{1\}.$$

We show that (19) fails if and only if there exists some $\sigma = (s_1, s_2, s_3) \in S^3$ such that:

(a) $(\Lambda(\sigma))\varphi$ contains an idempotent;

(b) $\exists y \in L_{s_2,s_3}\, (1 \in \overline{yL_{s_0,s_1}} \wedge y\varphi \neq 1)$.

Suppose that (19) fails. Then there exist $x, y \in A_R^*$ and $u \in A_R^+$ such that

$$xu^+y \subseteq A_R^*, \quad \overline{yx} = 1, \quad u^2\varphi = u\varphi, \quad y\varphi \neq 1. \tag{23}$$

In particular, we have a path in $\mathcal{A}$ of the form

$$\begin{tikzcd} s_0 \arrow[r, "x"'] & s_1 \arrow[r, "u^m"'] & s \arrow[loop right, "u^k"] \end{tikzcd}$$

For some $k \in \mathbb{N}$, we may replace $u$ by $u^{mk}$ in (23) and assume that there is a path in $\mathcal{A}$ of the form

$$\begin{tikzcd} s_0 \arrow[r, "x"'] & s_1 \arrow[r, "u"'] & s_2 \arrow[loop above, "u"] \arrow[r, "y"'] & s_3. \end{tikzcd}$$

Since $u \in \Lambda(\sigma)$, (a) holds. Clearly, (b) holds as well.

The converse implication is straightforward, hence we are bound to decide whether or not (a) and (b) hold simultaneously for some $\sigma = (s_1, s_2, s_3) \in S^3$.

Fix $\sigma = (s_1, s_2, s_3) \in S^3$. Since $\Lambda(\sigma)$ is rational, so is $(\Lambda(\sigma))\varphi$ by Lemma 6.8. Thus (a) is decidable by Lemma 6.7. In view of Lemma 6.4, the language

$$K = L_{s_2,s_3} \cap \{y \in A_R^* \mid 1 \in \overline{yL_{s_0,s_1}}\}$$

is rational and effectively constructible. Since (b) holds if and only if there exists some $y \in K$ such that $y\varphi \neq 1$, i.e., if and only if

$$K\varphi \not\subseteq \{1\},$$

decidability follows from Lemma 6.8 and the standard decidability properties of rational languages.

Therefore (19) is decidable as required. $\square$

In the next example, we show that the endomorphism $\varphi$ may have different (proper) weak endomorphism extensions:

**Example 7.7** *Let $A = \{a, b, c\}$ and $R = \{(ac, 1), (bc, 1)\}$. Then the identity mapping of $A_R^*$ admits two proper weak endomorphism extensions in $A_R^\infty$.*

**Proof**. let $\iota : A_R^* \to A_R^*$ denote the identity mapping. Clearly, the identity mapping $\Phi : A_R^\infty \to A_R^\infty$ is a proper weak endomorphism extension of $\iota$. We define a mapping $\Psi : A_R^\infty \to A_R^\infty$ by

$$\alpha\Psi = \begin{cases} \alpha & \text{if } \alpha \text{ is either finite or eventually periodic} \\ c^\omega & \text{otherwise} \end{cases}$$

It is immediate that $\Psi$ satisfies axioms (h1) and (h2). In view of Corollary 3.4, (h4') also holds. Let $u \in A_R^*$ and $\alpha \in A_R^\omega$.

If $\alpha$ is eventually periodic, then so is $\overline{u\alpha}$ and thus

$$(\overline{u\alpha})\Psi = \overline{u\alpha} = \overline{(u\Psi)(\alpha\Psi)}.$$

Assume now that $\alpha$ is not eventually periodic. If $c$ occurs $n$ times in $u$ and $|u| = n + m$, then $u = c^n v$ with $v \in \{a, b\}^*$ and $\overline{uc^m} = \overline{c^n vc^m} = c^n$. Since $\overline{u\alpha}$ is not eventually periodic either, we obtain

$$(\overline{u\alpha})\Psi = c^\omega = \overline{uc^\omega} = \overline{(u\Psi)(\alpha\Psi)}.$$

Therefore (h3) holds and $\Psi$ is a weak homomorphism extending $\iota$.

Since there exist non eventually periodic words in $A_R^\omega$, for instance $aba^2ba^3ba^4b\ldots$, $\Psi$ is not the identity mapping on $A_R^\infty$. $\square$

# 8 Continuous extensions

Clearly, the trivial endomorphism $\varphi : A_R^* \to A_R^*$ defined by $u\varphi = 1$ admits as continuous extension the trivial endomorphism of $A_R^\infty$. Throughout this section, we exclude the trivial case.

We start with one of the simplest situations.

**Lemma 8.1** *If all the elements of $A_R^*$ have finite order, then $|A| = 1$ and $A_R^*$ is a finite cyclic group.*

**Proof**. We assume $R$ to be normalized. Let $a \in A$. Since $\overline{a^*}$ is finite, we have $(a^n, 1) \in R$ for some $n > 1$. Thus we have relations $(a^{n_a}, 1) \in R$ for every $a \in A$. We assume that $n_a$ is minimal for every $a$. Suppose that $(ar, 1) \in R$ with $a \in A$ and $r \in A^+$. Then $a^{n_a}r \to r$ and $a^{n_a}r \to a^{n_a-1}$. Since the reduction process is confluent and $r, a^{n_a-1}$ are both irreducible due to $R$ being normalized, it follows that $r = a^{n_a-1}$ and so all relations in $R$ must be of the form $(a^{n_a}, 1)$.

If $A$ contains some other letter $b \neq a$, $ab \in A_R^*$ would have infinite order, hence $|A| = 1$ and so $R = \{(a^{n_a}, 1)\}$ implies that $A_R^*$ is a finite cyclic group. $\square$

The following lemma provides a simple set-theoretical characterization of uniform continuity, which will play a central role in this section.

**Lemma 8.2** *Let $\varphi$ be a nontrivial endomorphism of $A_R^*$. Then the following conditions are equivalent:*

*(i) $\varphi$ is uniformly continuous;*

*(ii) $w\varphi^{-1}$ is finite for every $w \in A_R^*$.*

**Proof**. The endomorphism $\varphi$ is not uniformly continuous if and only if

$$\exists \varepsilon > 0 \, \forall \delta > 0 \, \exists u, v \in A_R^* \, (d(u,v) < \delta \, \wedge \, d(u\varphi, v\varphi) \geq \varepsilon),$$

that is,

$$\exists k \in \mathbb{N} \, \forall m \in \mathbb{N} \, \exists u_m, v_m \in A_R^* \, (r(u_m, v_m) > m \, \wedge \, r(u_m\varphi, v_m\varphi) \leq k). \tag{24}$$

Let $w_m$ denote the longest common prefix of $u_m$ and $v_m$. We still have $r(u_m, w_m), r(w_m, v_m) > m$. Moreover,

$$\min\{r(u_m\varphi, w_m\varphi), r(w_m\varphi, v_m\varphi)\} \leq r(u_m\varphi, v_m\varphi) \leq k$$

and so we may assume that $u_m$ is a prefix of $v_m$. Then $|u_m| \geq m$. If $v_m = u_m a_1 \ldots a_n$ with $a_1, \ldots, a_n \in A$, then $r(u_m a_1 \ldots a_{i-1}, u_m a_1 \ldots a_i) > m$ for $i = 1, \ldots, n$. Since

$$\min\{r((u_m a_1 \ldots a_{i-1})\varphi, (u_m a_1 \ldots a_i)\varphi) \mid i = 1, \ldots, n\} \leq r(u_m\varphi, v_m\varphi) \leq k,$$

we may assume that $v_m = u_m a_m$ for some $a_m \in A$. Since we may replace $u_m, a_m$ by $u_n, a_n$ if $m < n$, we may assume that the letter $a_m$ is always the same. Thus (24) holds if and only if

$$\exists a \in A \, \exists k \in \mathbb{N} \, \forall m \in \mathbb{N} \, \exists u_m \in A_R^* \, (|u_m| \geq m \wedge r(u_m\varphi, (u_m a)\varphi) \leq k). \tag{25}$$

For every $m \in \mathbb{N}$, by Lemma 4.2(iii), we have

$$|u_m\varphi| \leq r((u_m a)\varphi, u_m\varphi) - 1 + (t_R - 1)|a\varphi| \leq k - 1 + (t_R - 1)h_\varphi.$$

Replacing $u_m$ by some higher index $u_n$ whenever necessary, we may assume that $u_m\varphi$ is constant and so (25) is equivalent to

$$\exists a \in A \, \exists w \in A_R^* \, (\overline{w(a\varphi)} \neq w \, \wedge \, \forall m \in \mathbb{N} \, \exists u_m \in A_R^* \, (|u_m| \geq m \, \wedge \, u_m\varphi = w)),$$

and thus to

$$\exists a \in A \, \exists w \in A_R^* \, (\overline{w(a\varphi)} \neq w \wedge |w\varphi^{-1}| = \infty),$$

that is,

$$\exists w \in A_R^* \, (\overline{w(A_R^*\varphi)} \not\subseteq \{w\} \wedge |w\varphi^{-1}| = \infty). \tag{26}$$

We show that (26) holds if and only if

$$\exists w \in A_R^* : |w\varphi^{-1}| = \infty. \tag{27}$$

To prove this equivalence, we may assume that $R$ is normalized, since both (26) and (27) are independent of the rewriting system.

Assume that $|w\varphi^{-1}| = \infty$ for some $w \in A_R^*$. Suppose that $\overline{w(A_R^*\varphi)} = w$. By Lemma 6.2, $u\varphi$ is idempotent for every $u \in A_R^*$. If $(pq, 1) \in R$, it follows that

$$p\varphi = \overline{p^2 q}\varphi = \overline{pq}\varphi = 1.$$

Considering successively all prefixes of relators, we conclude that $a\varphi = 1$ for every letter $a$ appearing in some relator. Let $A_1$ denote the set of all such letters.

If $\overline{A_1^*}$ is finite, Lemma 8.1 implies that $\overline{A_1^*}$ is a finite cyclic group and 1 its unique idempotent. Since any idempotent of $A_R^*$ belongs to $\overline{A_1^*}$ by Lemma 6.2, it follows that $A_R^*\varphi = 1$ and so $\varphi$ is trivial, a contradiction.

Thus $\overline{A_1^*}$ is infinite, and so $\overline{A_1^*} \subseteq 1\varphi^{-1}$ implies that $1\varphi^{-1}$ is infinite. Since $\varphi$ is not constant by our initial assumption, we have $1(u\varphi) \neq 1$ for some $u \in A_R^*$, thus (26) holds taking $w = 1$.

Since (26) implies (27) trivially, we conclude that they are equivalent.

We have proved that (24) is equivalent to (27). Since they are precisely the negations of conditions (i) and (ii), the lemma holds. □

**Corollary 8.3** *Let $\varphi$ be a nontrivial endomorphism of $A_R^*$. If $\varphi$ is uniformly continuous, then it is properly extendable.*

**Proof**. Assume that $\varphi$ is uniformly continuous. Suppose that $u \in A_R^+$ and $u^* \subseteq A_R^*$. If $u^2\varphi = u\varphi$, then $u\varphi\varphi^{-1}$ contains $u^+$ and is therefore infinite, contradicting Lemma 8.2. Hence $u^2\varphi \neq u\varphi$ and so $\varphi$ is properly extendable by Theorem 6.10. □

In the next theorem, we establish several equivalent conditions to the existence of continuous extensions.

**Theorem 8.4** *Let $\varphi$ be a nontrivial endomorphism of $A_R^*$. Then the following conditions are equivalent:*

*(i) $\varphi$ can be extended to a continuous mapping $\Phi : A_R^\infty \to A_R^\infty$;*

*(ii) $\varphi$ can be extended to a proper uniformly continuous endomorphism of the metric partial $\omega$-monoid $A_R^\infty$;*

*(iii) $\varphi$ is uniformly continuous;*

*(iv) $\varphi$ preserves Cauchy sequences;*

*(v) $w\varphi^{-1}$ is finite for every $w \in A_R^*$.*

*Moreover, if these conditions hold the continuous mapping $\Phi$ is unique and given by $\alpha\Phi = \lim_{n\to\infty} \alpha^{[n]}\varphi$.*

**Proof**. First we note that the given definition of $\Phi$ and its uniqueness follow from Lemma 6.1.

(i) $\Rightarrow$ (ii). If $\Phi : A_R^\infty \to A_R^\infty$ is a continuous extension of $\varphi$, then it is uniformly continuous since $A_R^\infty$ is compact. By Theorem 6.1, $\varphi$ is extendable and so $\Phi$ is proper by

Corollary 8.3. Moreover, the homomorphism axioms (h1) – (h3) hold by Lemma 7.1. It remains to show that (h4) also holds.

Suppose that $(u_1, u_2, \ldots)\pi$ is defined. Then $(\overline{u_1 \ldots u_n})_n$ converges to $(u_1, u_2, \ldots)\pi$. Since $\Phi$ is uniformly continuous, it follows that

$$\lim_{n\to\infty} \overline{(u_1\varphi)\ldots(u_n\varphi)} = \lim_{n\to\infty} \overline{u_1 \ldots u_n}\varphi = \lim_{n\to\infty} \overline{u_1 \ldots u_n}\Phi = (u_1, u_2, \ldots)\pi\Phi,$$

hence $(u_1\varphi, u_2\varphi, \ldots)\pi$ is defined and equals $(u_1, u_2, \ldots)\pi\Phi$. Therefore $\Phi$ is an endomorphism of $A_R^\infty$ and (ii) holds.

(ii) $\Rightarrow$ (i). Trivial.

(i) $\Rightarrow$ (iv). If $\Phi$ is continuous, it preserves convergent sequences (that is, Cauchy sequences, since $A_R^\infty$ is complete). So does its restriction $\varphi$.

(iv) $\Rightarrow$ (iii). Suppose that $\varphi$ is not uniformly continuous. Then there exists some $\varepsilon > 0$ such that

$$\forall n \in \mathbb{N}\, \exists u_n, v_n \in A^* \ (d(u_n, v_n) < \frac{1}{n} \wedge d(u_n\varphi, v_n\varphi) \geq \varepsilon).$$

Since $A_R^\infty$ is compact, the sequence $(u_n)_n$ has an adherence value $\alpha \in A_R^\infty$. Let $(u_{i_n})_n$ be some subsequence converging to $\alpha$. It is straightforward that the sequence $u_{i_1}, v_{i_1}, u_{i_2}, v_{i_2}, \ldots$ is still convergent to $\alpha$, and so is Cauchy. However, $u_{i_1}\varphi, v_{i_1}\varphi, u_{i_2}\varphi, v_{i_2}\varphi, \ldots$ is clearly not Cauchy and so $\varphi$ does not preserve Cauchy sequences.

(iii) $\Rightarrow$ (i). By [7, Corollary XIV.6.2].

(iii) $\Leftrightarrow$ (v). By Lemma 8.2. $\square$

The particular case of groups provides a simple corollary:

**Corollary 8.5** *If $A_R^*$ is a group with no finite nontrivial normal subgroups and $\varphi$ is an endomorphism of $A_R^*$, the following conditions are equivalent:*

*(i) $\varphi$ can be extended to a continuous mapping $\Phi : A_R^\infty \to A_R^\infty$;*

*(ii) $\varphi$ is either trivial or injective.*

*Moreover, if these conditions hold the continuous mapping $\Phi$ is unique and given by $\alpha\Phi = \lim_{n\to\infty} \alpha^{[n]}\varphi$.*

**Proof**. The nontrivial case follows immediately from Theorem 8.4, taking into account that, in a group, $|w\varphi^{-1}| = |\mathrm{Ker}\varphi|$ for every $w \in A_R^*\varphi$ and $\mathrm{Ker}\varphi$ is a normal subgroup. $\square$

This corollary can be of course applied to free groups, but not only:

**Example 8.6** *Let $A = \{a, b, b^{-1}\}$ and $R = \{(a^2, 1), (bb^{-1}, 1), (b^{-1}b, 1)\}$. Then $A_R^*$ is a group (the free product of $\mathbf{Z}$ by $\mathbf{Z}_2$, actually) with no finite nontrivial normal subgroups.*

**Proof**. Let $u \in A_R^+$. If $u$ has finite order, then

$$u \in (b^* \cup (b^{-1})^*)(a(b^+ \cup (b^{-1})^+))^* a(b^* \cup (b^{-1})^*)$$

and so $\{\overline{b^n u b^{-n}} \mid n \in \mathbf{Z}\}$ is infinite. Thus $u$ belongs to no finite normal subgroup. $\square$

**Theorem 8.7** *It is decidable whether or not an endomorphism $\varphi$ of $A_R^*$ can be extended to a continuous mapping $\Phi : A_R^\infty \to A_R^\infty$.*

**Proof**. We may assume that $\varphi$ is nontrivial. By Theorem 8.4, it suffices to show that it is decidable whether or not $|w\varphi^{-1}| = \infty$ for some $w \in A_R^*$.

We show that if $|w\varphi^{-1}| = \infty$ for some $w \in A_R^*$ then

$$\exists w \in A_R^* \,(|w| \leq 2h_\varphi \wedge |w\varphi^{-1}| = \infty). \tag{28}$$

Assume that $|w\varphi^{-1}| = \infty$ for $w = a_1 \dots a_m$ $(a_i \in A)$. Let $(u_n)_n$ be an infinite sequence consisting of distinct elements of $w\varphi^{-1}$. For every $n \in \mathbb{N}$, there exists a finite sequence $0 = i_0 < i_1 < i_2 < \dots < i_{m_n} = m$ such that we may write

$$u_n = g_{n0}b_{n1}g_{n1} \dots b_{nm_n}g_{nm_n}$$

for $b_{nj} \in A$, $g_{nj} \in A_R^*$ satisfying

$$b_{nj}\varphi = p_{nj}a_{i_{j-1}+1} \dots a_{i_j}q_{nj}$$

for $j = 1, \dots, m_n$ and

$$\overline{q_{nj}(g_{nj}\varphi)p_{n,j+1}} = 1$$

for $j = 0, \dots, m_n$, where $q_{n0} = p_{n,m_n+1} = 1$. By the pigeonhole principle, we may refine the sequence $(u_n)_n$ to assume that:

- there exists some $k \in \mathbb{N}$ such that $m_n = k$ for every $n \in \mathbb{N}$;
- the sequence $0 = i_0 < i_1 < i_2 < \dots < i_k = m$ is the same for every $n \in \mathbb{N}$;
- for each $j \in \{1, \dots, k\}$, the letter $b_{nj}$ is the same for every $n \in \mathbb{N}$, say $b_{nj} = b_j$;
- for each $j \in \{0, \dots, k\}$, the words $p_{n,j+1}$ (respectively $q_{nj}$) are the same for every $n \in \mathbb{N}$, say $p_{n,j+1} = p_{j+1}$ (respectively $q_{nj} = q_j$).

Write

$$i_{-1} = -1, \quad , i_{k+1} = m+1, \quad a_0 = a_{m+1} = b_0 = b_{k+1} = p_0 = q_{k+1} = 1.$$

Since the $u_n$ are all distinct, one of the sets $\{g_{nj} \mid n \in \mathbb{N}\}$ is infinite for some $j \in \{0, \dots, k\}$. Let $w' = \overline{p_j a_{i_{j-1}+1} \dots a_{i_{j+1}} q_{j+1}}$. Clearly,

$$|w'| \leq |p_j a_{i_{j-1}+1} \dots a_{i_j}| + |a_{i_j+1} \dots a_{i_{j+1}} q_{j+1}| \leq |b_j\varphi| + |b_{j+1}\varphi| \leq 2h_\varphi.$$

Moreover,

$$\begin{aligned}(b_j g_{nj} b_{j+1})\varphi &= \overline{p_j a_{i_{j-1}+1} \dots a_{i_j} q_j (g_{nj}\varphi) p_{j+1} a_{i_j+1} \dots a_{i_{j+1}} q_{j+1}} \\ &= \overline{p_j a_{i_{j-1}+1} \dots a_{i_j} a_{i_j+1} \dots a_{i_{j+1}} q_{j+1}} = \overline{p_j a_{i_{j-1}+1} \dots a_{i_{j+1}} q_{j+1}} \\ &= w'\end{aligned}$$

for every $n \in \mathbb{N}$. Since $\{g_{nj} \mid n \in \mathbb{N}\}$ is infinite, we obtain $|w'\varphi^{-1}| = \infty$ and so (28) holds.

Since there are only finitely many words of length $\leq 2h_\varphi$, we only need to show that, given a fixed word $w \in A_R^*$, it is decidable whether or not $|w\varphi^{-1}| = \infty$.

Let $\hat{\varphi} : A^* \to A^*$ be the endomorphism defined by $a\hat{\varphi} = a\varphi$ $(a \in A)$. Since $\{w\}$ is a rational language, it follows from Theorem 2.5 that $D_w = \{u \in A^* \mid \overline{u} = w\}$ is (deterministic) context-free. We show that

$$w\varphi^{-1} = D_w\hat{\varphi}^{-1} \cap A_R^*. \tag{29}$$

Let $u \in w\varphi^{-1}$. Since $\overline{u\hat{\varphi}} = u\varphi = w$, we have $u\hat{\varphi} \in D_w$ and so $u \in D_w\hat{\varphi}^{-1}$. Thus $w\varphi^{-1} \subseteq D_w\hat{\varphi}^{-1} \cap A_R^*$.

Conversely, if $u \in D_w\hat{\varphi}^{-1} \cap A_R^*$, then $u\varphi = \overline{u\hat{\varphi}} = w$ and so $u \in w\varphi^{-1}$. Therefore (29) holds.

Since the class of context-free languages is closed for inverse morphisms and intersection with rational languages, and we can test whether or not a context-free language is infinite or not [10], it follows that we can decide whether or not $|w\varphi^{-1}| = \infty$ as required. $\square$

In the following example, we show that a properly extendable endomorphism of $A_R^*$ is not necessarily uniformly continuous, even if it admits a (proper) endomorphism extension.

**Example 8.8** *Let* $A = \{a, b, c, d, d^{-1}, e, e^{-1}, f, f^{-1}, g, g^{-1}\}$ *and* $R = \{(xx^{-1}, 1) \mid x = d, d^{-1}, e, e^{-1}, f, f^{-1}, g, g^{-1}\}$. *Let* $\varphi$ *be the matched endomorphism of* $A_R^*$ *defined by*

$$\begin{array}{lll} a\varphi = fg & d\varphi = d & f\varphi = e^2de^{-2} \\ b\varphi = f^{-1}g^{-1} & e\varphi = ede^{-1} & g\varphi = e^3de^{-3} \\ c\varphi = g^{-1}. & & \end{array}$$

*Then:*

*(i)* $\varphi$ *is properly extendable;*

*(ii)* $\varphi$ *can be extended to an endomorphism of* $A_R^\infty$*;*

*(iii)* $\varphi$ *is not uniformly continuous.*

**Proof**. (i) Let $B = \{a, b, c\}$ and $C = \{d, d^{-1}, e, e^{-1}, f, f^{-1}, g, g^{-1}\}$. We show that

$$1\varphi^{-1} \cap B^* = \{1\}. \tag{30}$$

Since $\overline{C^*}$ is a free group, we can define a group homomorphism $\psi : \overline{C^*} \to \mathbf{Z}^2$ by

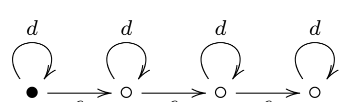
$$d\psi = e\psi = (0,0), \quad f\psi = (1,0), \quad g\psi = (0,1).$$

Let $|u|_x$ denote the number of occurrences of the letter $x$ in $u$. If $u \in B^*$ is such that $u\varphi = 1$, then $(0,0) = u\varphi\psi = (|u|_a - |u|_b, |u|_a - |u|_b - |u|_c)$. This implies $|u|_a = |u|_b$ and $|u|_c = 0$. Then $u \in \{a,b\}^*$ and no cancellation can occur in $u\varphi$, so $u = 1$.

Next we show that

$$1\varphi^{-1} \cap C^* = \{1\}. \tag{31}$$

The submonoid $\overline{C^*}$ of $A_R^*$ is a free group on the set $\{d, e, f, g\}$. If we apply the well-known Reidemeister-Serre-Stallings algorithm [6] to the finite subset $\{d, ede^{-1}, e^2de^{-2}, e^3de^{-3}\}$ of this free group, we obtain the finite automaton



The horizontal edges constitute a maximal subtree and according to the algorithm the four remaining edges yield $\{d, ede^{-1}, e^2de^{-2}, e^3de^{-3}\}$ as a basis of the subgroup of $\overline{C^*}$ generated by them. It follows that $\varphi\mid_{\overline{C^*}}$ is injective and so (31) holds.

¿From (30) and (31) we can deduce that

$$1\varphi^{-1} = \{1\}. \tag{32}$$

Indeed, any $u \in A_R^+$ van be written as $u_1u_2\ldots u_n$, $n \in \mathbb{N}$, with the $u_i$ alternately in $B^+$ and $\overline{C^*}\backslash\{1\}$ (or conversely). Then $u_i\varphi \neq 1$ by (30) and (31), and the $u_i\varphi$ are alternately in $\overline{\{d, d^{-1}, e, e^{-1}\}^*}\backslash\{1\}$ and $\overline{\{f, f^{-1}, g, g^{-1}\}^*}\backslash\{1\}$. Thus $u\varphi = (u_1p)(u_2\varphi)\ldots(u_n\varphi)$ is reduced and so $u\varphi \neq 1$. Therefore (32) holds.

Let $u \in A_R^+$ be such that $u^* \subseteq A_R^*$. Suppose that $u^2\varphi = u\varphi$. Since $u\varphi$ is an idempotent of the free group $\overline{C^*}$, we obtain $u\varphi = 1$, contradicting (32). Thus $u^2\varphi \neq u\varphi$ and so $\varphi$ is properly extendable by Theorem 6.10.

(ii) By Corollary 7.4, the mapping $\Phi : A_R^\infty \to A_R^\infty$ defined by $\alpha\Phi = \lim_{n\to\infty} \alpha^{[n]}\varphi$ is a weak endomorphism. We have to show that $\Phi$ satisfies axiom (h4), that is, if $(u_1, u_2, \ldots)\pi$ is defined, then $(u_1\varphi, u_2\varphi, \ldots)\pi$ is defined and equal to $(u_1, u_2, \ldots)\pi\Phi$. This is equivalent to say that if $\lim_{n\to\infty} \overline{u_1\ldots u_n} = \alpha$, then $(\overline{u_1\ldots u_n}\varphi)_n$ converges and $\lim_{n\to\infty} \overline{u_1\ldots u_n}\varphi = \alpha\Phi = \lim_{n\to\infty} \alpha^{[n]}\varphi$.

Given $u \in A^*$, we define $u\eta$ to be the unique $k \in \mathbb{N}_0$ such that $u \in B^*(C^+B^+)^kC^*$. Equivalently, $u\eta$ is the number of factors of $u$ belonging to $CB$. Since the letters of $B$ cannot be reduced, we have $\overline{uv}\eta \geq u\eta$ for all $u, v \in A_R^*$.

Assume that $\lim_{n\to\infty} \overline{u_1\ldots u_n} = \alpha$.

We suppose first that $(\overline{u_1\ldots u_n}\eta)_n$ is unbounded. Let $k \in \mathbb{N}$. Since $\alpha\Phi = \lim_{n\to\infty} \alpha^{[n]}\varphi$, there exists some $p \geq k$ such that $(\alpha^{[n]}\varphi)^{[k]} = (\alpha\Phi)^{[k]}$ for every $n \geq p$.

Since $(\overline{u_1\ldots u_n}\eta)_n$ is nondecreasing, there exists some $m \in \mathbb{N}$ such that $\overline{u_1\ldots u_n}\eta \geq p$ for every $n \geq m$. Let $\overline{u_1\ldots u_m} = x_0y_1x_1\ldots y_px_pv_m$ with $x_0 \in B^*$, $x_1, \ldots, x_p \in B^+$, $y_1, \ldots, y_p \in C^+$ and $v_m \in A_R^*$. Since the letters of $B$ cannot be reduced, we have

$$\forall n \geq m\, \exists v_n \in A_R^* : \overline{u_1\ldots u_n} = x_0y_1x_1\ldots y_px_pv_n. \tag{33}$$

Once again, it follows from (30) and (31) that

$$\overline{u_1\ldots u_n}\varphi = (x_0\varphi)(y_1\varphi)(x_1\varphi)\ldots(y_p\varphi)(x_pv_n)\varphi$$

for every $n \geq m$. Since $|(x_0\varphi)(y_1\varphi)(x_1\varphi)\ldots(y_p\varphi)| \geq 2p-1 \geq p \geq k$, we obtain $(\overline{u_1\ldots u_n}\varphi)^{[k]}$ $= ((x_0y_1x_1\ldots y_p)\varphi)^{[k]}$ for every $n \geq m$.

On the other hand, (33) implies that $\alpha = x_0y_1x_1\ldots y_px_p\beta$ for some $\beta \in A_R^\infty$ and so (30) and (31) yield

$$\alpha\Phi = (x_0\varphi)(y_1\varphi)(x_1\varphi)\ldots(y_p\varphi)\overline{(x_p\varphi)(\beta\varphi)}.$$

Since $|(x_0\varphi)(y_1\varphi)(x_1\varphi)\ldots(y_p\varphi)| \geq 2p-1 \geq p \geq k$, we obtain

$$(\alpha\Phi)^{[k]} = ((x_0y_1x_1\ldots y_p)\varphi)^{[k]} = (\overline{u_1\ldots u_n}\varphi)^{[k]}$$

for every $n \geq m$. Thus

$$\forall k \in \mathbb{N}\, \exists m \in \mathbb{N}\, (n \geq m \Rightarrow (\overline{u_1\ldots u_n}\varphi)^{[k]} = (\alpha\Phi)^{[k]})$$

and $\lim_{n\to\infty} \overline{u_1 \dots u_n}\varphi = \alpha\Phi$ as required.

We assume now that $(\overline{u_1 \dots u_n}\eta)_n$ is bounded with maximum value $k$. Then there exists some $p \in \mathbb{N}$ such that $\overline{u_1 \dots u_n}\eta = k$ for every $n \geq p$. Since the letters of $B$ cannot be reduced, there exist $x_0 \in B^*$, $x_1, \dots, x_{k-1} \in B^+$ and $y_1, \dots, y_k \in C^+$ such that

$$\forall n \geq p\, \exists v_n \in B^+\, \exists w_n \in C^* : \overline{u_1 \dots u_n} = x_0 y_1 x_1 \dots y_k v_n w_n.$$

If $k = 0$, we may assume $x_0 = 1$ to have $\overline{u_1 \dots u_n} = v_n w_n$.

Suppose first that $(|v_n|)_n$ is bounded. If $|v_n|$ is maximum for $n = r \geq p$, we have $\overline{u_1 \dots u_n} = x_0 y_1 x_1 \dots y_k v_r w_n$ for every $n \geq r$. In particular, $u_n \in C^*$ and $w_n = \overline{w_r u_{r+1} \dots u_n}$ for every $n > r$. Clearly, $\alpha^{[s]} = x_0 y_1 x_1 \dots y_k v_r$ for $s = |x_0 y_1 x_1 \dots y_k v_r|$. It is immediate that $\lim_{n\to\infty} w_n = \alpha^{(s+1)}\alpha^{(s+2)} \dots$ Let $\beta = \alpha^{(s+1)}\alpha^{(s+2)} \dots$

Now we can view $R$ as a rewriting system over $C$. Let $\varphi_C = \varphi\,|_{C_R^*}$ and $\Phi_C = \Phi\,|_{C_R^\infty}$. By (31), Theorem 8.4 and Corollary 8.5, $\Phi_C$ is a continuous endomorphism of the metric partial $\omega$-monoid $C_R^\infty$. In particular,

$$\lim_{n\to\infty} w_n\varphi = \lim_{n\to\infty} w_n\varphi_C = \beta\Phi_C = \beta\Phi.$$

Thus

$$\begin{aligned}\lim_{n\to\infty} \overline{u_1 \dots u_n}\varphi &= \lim_{n\to\infty}(x_0 y_1 x_1 \dots y_k v_r w_n)\varphi = \overline{(x_0 y_1 x_1 \dots y_k v_r)\varphi(\beta\Phi)} \\ &= \overline{x_0 y_1 x_1 \dots y_k v_r \beta}\Phi = (\alpha^{[s]}\beta)\Phi = \alpha\Phi\end{aligned}$$

as required.

Finally, suppose that $(|v_n|)_n$ is unbounded. Since $v_n$ is a prefix of $v_{n+1}$ for every $n \geq p$, we can define $\lim_{n\to\infty} v_n = \beta \in B^\omega$. Clearly,

$$\alpha = \lim_{n\to\infty} \overline{u_1 \dots u_n} = \lim_{n\to\infty} x_0 y_1 x_1 \dots y_k v_n w_n = x_0 y_1 x_1 \dots y_k \beta$$

and so

$$\alpha\Phi = \overline{(x_0 y_1 x_1 \dots y_k)\varphi(\beta\Phi)} \tag{34}$$

by Lemma 7.1. Since $\beta \in B^\omega$ and $\varphi$ is properly extendable, we have $\lim_{n\to\infty} \beta^{[n]}\varphi = \beta\Phi \in A_R^\omega$. Since $(v_p, v_{p+1}, \dots)$ is a subsequence of $(\beta^{[n]})_n$, it follows that $\lim_{n\to\infty}(v_n\varphi) = \beta\Phi$. Moreover, since $(v_n\varphi)(w_n\varphi)$ is irreducible, we have

$$\lim_{n\to\infty}(v_n\varphi)(w_n\varphi) = \lim_{n\to\infty}(v_n\varphi) = \beta\Phi$$

and so (34) yields

$$\begin{aligned}\alpha\Phi &= \overline{(x_0 y_1 x_1 \dots y_k)\varphi(\beta\Phi)} = \overline{(x_0 y_1 x_1 \dots y_k)\varphi(\lim_{n\to\infty}(v_n\varphi)(w_n\varphi))} \\ &= \lim_{n\to\infty} \overline{(x_0 y_1 x_1 \dots y_k)\varphi(v_n\varphi)(w_n\varphi)} = \lim_{n\to\infty}((x_0 y_1 x_1 \dots y_k v_n w_n)\varphi) \\ &= \lim_{n\to\infty}(\overline{u_1 \dots u_n}\varphi).\end{aligned}$$

Therefore $\Phi$ is an endomorphism as claimed.

(iii) To show that $\varphi$ is not uniformly continuous, we show that $g^{-1}\varphi^{-1}$ is infinite. In fact, $(acb)\varphi = c\varphi = g^{-1}$ yields $(a^n cb^n)\varphi = g^{-1}$ for every $n \in \mathbb{N}$. Therefore $g^{-1}\varphi^{-1}$ is infinite and so $\varphi$ is not uniformly continuous by Theorem 8.4. $\square$

## 9 Conclusion

Our results indicate that it is worthwhile studying infinite words on nonstandard algebraic settings. Since homomorphisms constitute the ultimate algebraic concept, our characterization theorems and particularly the positive decidability results provide evidence of that fact.

At this point, the main open questions in this line of research should be:

1. Is it decidable if an endomorphism $\varphi$ of $A_R^*$ admits an endomorphism extension?

2. If an endomorphism $\varphi$ of $A_R^*$ admits a weak endomorphism extension, does $\alpha\Phi = \lim_{n\to\infty} \alpha^{[n]}\varphi$ define a weak endomorphism extension?

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