DOUBLE STANDARD MAPS

MICHAŁ MISIUREWICZ AND ANA RODRIGUES

ABSTRACT. We investigate the family of double standard maps of the circle onto itself, given by $f_{a,b}(x) = 2x + a + (b/\pi)\sin(2\pi x) \pmod 1$, where the parameters a, b are real and $0 \le b \le 1$. Similarly to the well known family of (Arnold) standard maps of the circle, $A_{a,b}(x) = x + a + (b/(2\pi))\sin(2\pi x) \pmod 1$, any such map has at most one attracting periodic orbit and the set of parameters (a, b) for which such orbit exists is divided into tongues. However, unlike the classical Arnold tongues, that begin at the level $b = 0$, for double standard maps the tongues begin at higher levels, depending on the tongue. Moreover, the order of the tongues is different. For the standard maps it is governed by the continued fraction expansions of rational numbers; for the double standard maps it is governed by their binary expansions. We investigate closer two families of tongues with different behavior.

1. INTRODUCTION

It is a usual procedure that in order to understand the behavior of a system in higher dimension one investigates first a one-dimensional system that is somewhat similar. The classical example is the Hénon map and similar systems, where a serious progress occurred only after unimodal interval maps have been thoroughly understood.

Another example of this type was investigation by V. Arnold of the family of *standard maps* of the circle, given by the formula

$$A_{a,b}(x) = x + a + \frac{b}{2\pi}\sin(2\pi x) \pmod 1 \tag{1.1}$$

(when we write "mod 1," we mean that both the arguments and the values are taken modulo 1). Those maps, called also *Arnold maps*, should not be confused with *Taylor-Chirikov* maps, which are defined by similar formulas in the annulus, and are also called *standard maps*. The family (1.1) appeared in [1] and its study was useful in the creation of the KAM Theory. This family has been investigated by various authors since then, see for instance [12, 6] and other papers cited there.

Recently, very interesting families of branched covering maps in the plane have been studied, [3, 4, 5]. It motivates finding similar (in some sense) one-dimensional maps and studying them. If we consider a branched covering map of the plane that has only one branching point and degree 2, a good choice is to study degree 2 circle maps. To begin with, one should concentrate on some specific family of such maps. Perhaps the most natural choice is the family similar to standard maps, but with

Date: May 25, 2006.

2000 *Mathematics Subject Classification.* 37E10.

Key words and phrases. Standard map, Arnold tongues.

The first author was partially supported by NSF grant DMS 0456526. The second author was supported by FCT Grant SFRH/BD/18631/2004. The authors are grateful to Janina Kotus for the help in the proof of Theorem 3.5.

the sinusoid added not to the identity but to the doubling map (we also rescale the parameter b in order to keep its critical value at 1). In such a way we get the following family of *double standard maps*

$$f_{a,b}(x) = 2x + a + \frac{b}{\pi}\sin(2\pi x) \pmod 1. \tag{1.2}$$

There are also other reasons for studying this family. It is a hybrid between the family of standard maps and the family of expanding circle maps (see, e.g., [11]). Both families are of special interest, so it is an important problem to investigate what the result of the cross-breeding may be. Moreover, the circle maps with cubic critical points (this is what we get when we put $b = 1$ in (1.2)) already proved to be interesting (see, e.g., [8]).

A widely accepted method of investigating new dynamical system or their families consists of initial numerical investigation, formulating questions and conjectures based on it, and subsequent attempts to answer the questions and prove the conjectures. We will follow this scheme. In this paper we study this family for the values of b from 0 to 1. In this range, the maps are local homeomorphisms, while for $b > 1$ they are bimodal circle maps – a class with quite different features, similar to unimodal interval maps.

In Section 2 we realize the first, easiest part of the plan. Namely, we make computer experiments, look at the pictures and try to understand what we see. In Section 3 we develop some tools that will be useful in the next sections. In Section 4 we explain the order in which tongues appear as the parameter a increases. In Section 5 we look closer at the only tongue for which the direct computations are reasonably simple, namely at the period 1 tongue. We describe its shape and produce explicit estimates for which values of b this is the only tongue. In Sections 6 and 7 we investigate, for $b = 1$, two families of attracting periodic orbits with opposite behaviors. One of them consists of orbits which would be very strongly repelling if it did not happen that one point of such orbit is very close to the critical point. The other one consists of intermittent orbits, for which repelling properties are extremely weak. For both families we estimate the size of the windows in the parameter space and in the phase space. Finally, in Section 8 we estimate the size of the tongues in the direction of the parameter b for those two families.

2. Numerical results

In this section we will present several computer generated pictures for the family of double standard maps and describe the apparent features of this family, based on the pictures.

The usual pictures produced for the standard (Arnold) family of maps present the situation in the (a, b)-plane and show the parameter values for which there is an attracting periodic orbit (phase locking regions). Those parameter values are grouped in regions called *Arnold tongues* (see Figure 1). Note that (mod 1) we have $A_{-a,b}(-x) = -A_{a,b}(x)$, so $A_{-a,b}$ is conjugate to $A_{a,b}$ via the map $x \mapsto -x$ (mod 1). Therefore the picture is symmetric with respect to the line $a = 1/2$ and therefore we only need to show it for $1/2 \le a \le 1$. The same applies to the maps $f_{a,b}$ replacing $A_{a,b}$.

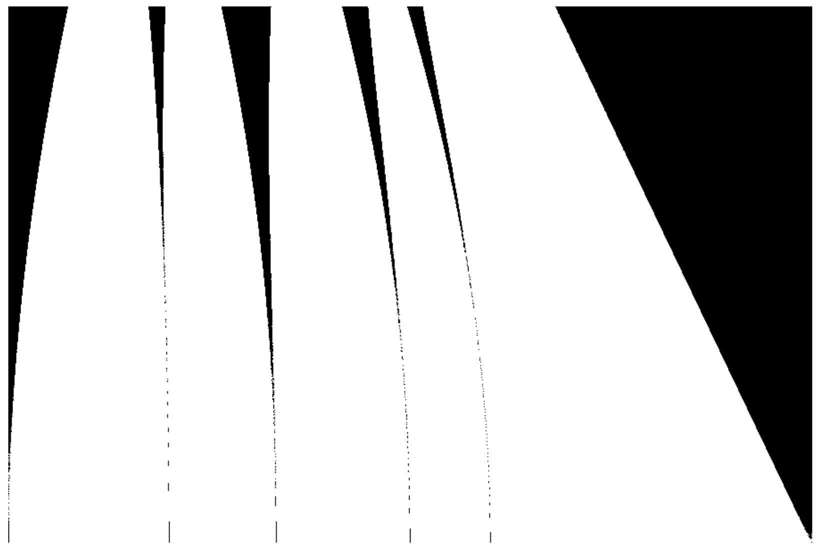

FIGURE 1. Arnold tongues for the family of standard maps

FIGURE 2. Arnold tongues for the family of double standard maps

Let us describe Figure 1 precisely. The vertical axis is b, from 0 to 1. The horizontal axis is a, from 1/2 to 1. The tongues shown are all tongues of period 5 or less, and their order from left to right is 2, 5, 3, 4, 5, 1. They correspond to the rotation numbers $1/2 < 3/5 < 2/3 < 3/4 < 4/5 < 1/1$.

Let us compare this picture to the analogous one for the double standard maps (see Figure 2). Here the vertical axis is b, from 1/2 to 1. The horizontal axis is a, from 1/2 to 1. The tongues shown are all tongues of period 5 or less (in fact, almost all, because the last one is so small that it does not show on the picture), and their order from left to right is 1, 5, 5, 4, 5, 5, 4, 3, 5, 5, 4, 5, 5, 4, 3, 5, 5, 2, 5, 5,

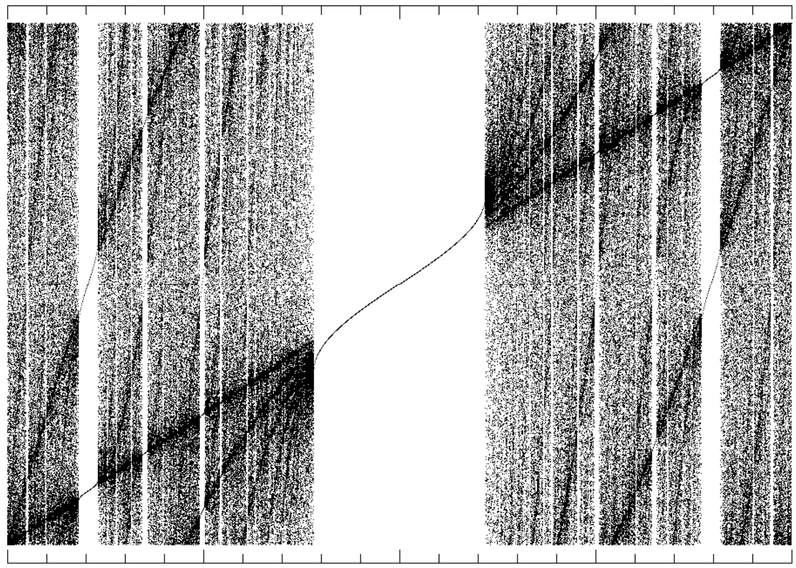

FIGURE 3. The (a, x)-plot, with a and x from 0 to 1 and $b = 1$

4, 5, 3, 5, 4, 5. As we will explain later, they correspond to the rational numbers $0/1 < 1/31 < 2/31 < 1/15 < 3/31 < 4/31 < 2/15 < 1/7 < 5/31 < 6/31 < 3/15 < 7/31 < 8/31 < 4/15 < 2/7 < 9/31 < 10/31 < 1/3 < 11/31 < 12/31 < 6/15 < 13/31 < 3/7 < 14/31 < 7/15 < 15/31$ (the denominators are of the form $2^n - 1$, where n is the period). This order is completely different than for the standard maps. Another big difference is that here the tongues begin not at the level $b = 0$, like for standard maps, but at much higher levels. The lowest tongue tip is at $b = 1/2$, for the period 1 tongue. There cannot be anything lower, because if $0 < b < 1/2$ then the map is expanding.

Thus, the natural conjecture is that for the double standard family of maps the phase locking regions come in tongues, whose shapes are similar to the classical Arnold tongues. We have to explain their order. It seems that for a given value of $b \in [0, 1)$ there are only finitely many of them (however, we will see in Section 8 that this is not true). In particular, only the period 1 tongue begins as low as 1/2.

We would like to know the size of the tongues in both a and b directions. The a-size should be measured at the level $b = 1$. Then, since we fix the value of b, it makes sense to look at the picture in the (a, x)-plane (like the classical pictures for the family of the logistic or real quadratic families of maps). Figure 3 presents the global picture, with both a and x varying from 0 to 1.

Since 1/2 is the unique critical point of $f_{a,1}$ and the map has negative Schwarzian derivative, for every a there is at most one attracting periodic orbit (see, e.g., [9]). If such an orbit exists, one of its points must be close to 1/2. Since $f_{a,1}(1/2) = a$, there must be a point of such orbit close to a. Figure 3 suggests that in order to see well how the attracting periodic orbit varies with a, it is better to look close to the diagonal $x = a$, rather than close to the line $x = 1/2$, where the line is very steep unless the period is very small. And indeed, blow-ups at many regions close to the diagonal show a graph of a periodic point as a function of a that is not so steep in its

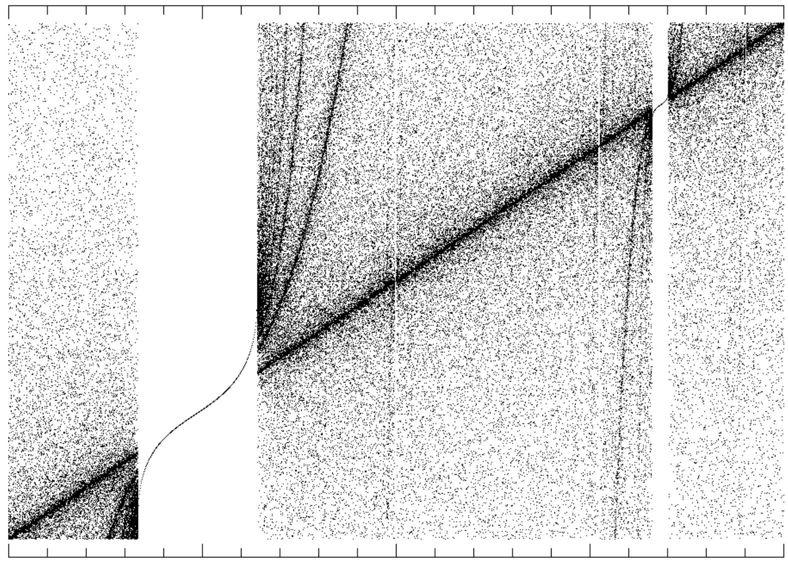

FIGURE 4. The (a, x)-plot, with a and x from 0.69053 to 0.69055 and $b = 1$

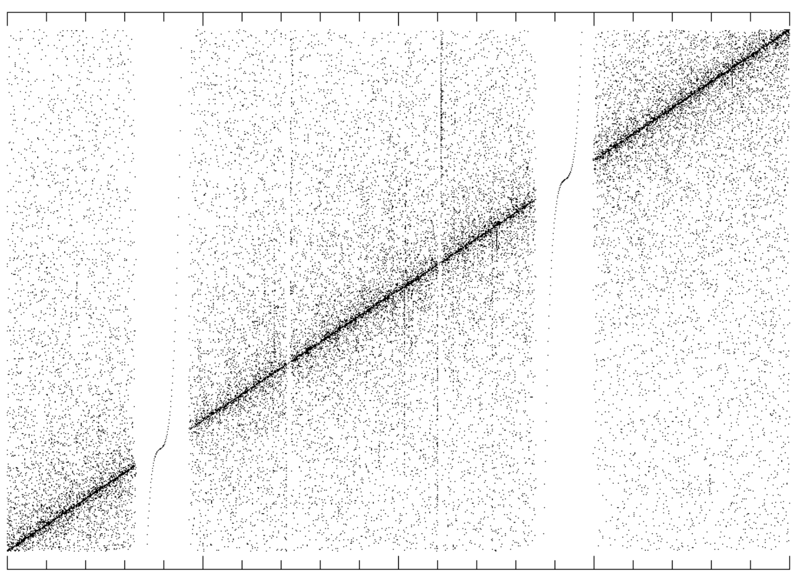

FIGURE 5. The (a, x)-plot, with a and x from 0.61087 to 0.61093 and $b = 1$

middle part (although of course it has to be vertical at the boundary of the window). This is illustrated on Figure 4, where a and x vary from 0.69053 to 0.69055. However, there are periodic orbits close to the boundaries of tongues of small period, that we can call *resonant* or *intermittent*, for which this graph is much steeper. Figure 5 shows what happens near the boundary of the period 1 tongue. There a and x vary from 0.61087 to 0.61093.

3. Tools

In this section we prove some preliminary results, that will serve as tools for more detailed investigation of the family of double standard maps. In most of the paper, $f_{a,b}$ will denote the standard map given by equation (1.2) and $F_{a,b}$ its lifting to the real line, that is, the map given by the same formula, but not considered modulo 1. However, in this section (except the very end), we prove some properties of those maps, which do not depend on the precise formula. Therefore at the moment we will only assume that $F_{a,b}$ are maps from the real line to itself, satisfying the following properties:

(1) Each $F_{a,b}$ is continuous increasing (as a function of x),
(2) $F_{a,b}(x+k) = F_{a,b}(x) + 2k$ for every integer k,
(3) $F_{a,b}(x)$ is increasing as a function of a and continuous jointly in x, a, b.

While the fact that local homeomorphisms of the circle of degree 2 are semiconjugate to the doubling map is well known, we need additionally monotonicity properties of the semiconjugacy as the function of a. Therefore we include a simple proof which also gives us this monotonicity.

The first lemma establishes semiconjugacy as a certain limit.

Lemma 3.1. *Under the assumptions (1) and (2), the limit*

$$\Phi_{a,b}(x) = \lim_{n\to\infty} \frac{F^n_{a,b}(x)}{2^n} \tag{3.1}$$

exists uniformly in x. The limit $\Phi_{a,b}(x)$ is a continuous increasing function of x. Moreover, $\Phi_{a,b}(x+k) = \Phi_{a,b}(x) + k$ for every integer k and $\Phi_{a,b}(F_{a,b}(x)) = 2\Phi_{a,b}(x)$ for every x, so $\Phi_{a,b}$ semiconjugates $F_{a,b}$ with multiplication by 2.

Proof. Since $F_{a,b}$ is continuous and satisfies (2), it has a fixed point $x_{a,b}$. For this fixed point we know that $F_{a,b}(x_{a,b}) = x_{a,b}$, and therefore by (2), $F_{a,b}(x_{a,b}+k) = x_{a,b}+2k$. From this by induction we get $F^n_{a,b}(x_{a,b}+k) = x_{a,b}+2^n k$ for any integer k and $n \geq 0$. Now, if we take any x and large m, we know that there exists an integer k such that $x_{a,b}+k \leq F^m_{a,b}(x) \leq x_{a,b}+k+1$. Then for any $n \geq 0$ we get $x_{a,b}+2^n k \leq F^{m+n}_{a,b}(x) \leq x_{a,b}+2^n(k+1)$, and therefore

$$\frac{x_{a,b}}{2^{m+n}} + \frac{k}{2^m} \leq \frac{F^{m+n}_{a,b}(x)}{2^{m+n}} \leq \frac{x_{a,b}}{2^{m+n}} + \frac{k+1}{2^m}.$$

This implies that for every $r, s \geq n$ and every x we have

$$\left| \frac{F^r_{a,b}(x)}{2^r} - \frac{F^s_{a,b}(x)}{2^s} \right| \leq \frac{1}{2^m}. \tag{3.2}$$

Therefore the sequence $\left(\frac{F^r_{a,b}(x)}{2^r}\right)_{r=1}^{\infty}$ satisfies the uniform Cauchy's condition, so it converges uniformly.

Since the limit in (3.1) is uniform and the functions $\frac{F^n_{a,b}(x)}{2^n}$ are continuous and increasing, the function $\Phi_{a,b}$ is also continuous and increasing.

Since

$$\begin{aligned}\Phi_{a,b}(x+k) &= \lim_{n\to\infty}\frac{F^n_{a,b}(x+k)}{2^n} = \lim_{n\to\infty}\frac{F^n_{a,b}(x)+2^nk}{2^n}\\ &= \lim_{n\to\infty}\frac{F^n_{a,b}(x+k)}{2^n}+k=\Phi_{a,b}(x)+k\end{aligned}$$

and

$$\Phi_a(F_{a,b}(x)) = \lim_{n\to\infty}\frac{F^n_{a,b}(F_a(x))}{2^n} = 2\lim_{n\to\infty}\frac{F^{n+1}_{a,b}(x)}{2^{n+1}} = 2\Phi_{a,b}(x),$$

the last two properties of $\Phi_{a,b}$ also hold. □

Lemma 3.2. *Under the assumptions of Lemma 3.1, the map $\Phi_{a,b}$ is a lifting of a monotone degree one map $\varphi_{a,b}$ of the circle to itself, which semiconjugates $f_{a,b}$ with the doubling map $D : x \mapsto 2x \pmod 1$. Moreover, if p is a periodic point of $f_{a,b}$ of period n then $\varphi_{a,b}(p)$ is a periodic point of D of period n.*

Proof. The first statement follows directly from Lemma 3.1. Assume that p is a periodic point of $f_{a,b}$ of period n. Since $f^n_{a,b}(p) = p$, we get $D^n(\varphi_{a,b}(p)) = \varphi_{a,b}(p)$. Suppose that the period of $\varphi_{a,b}(p)$ for D is not n. Then it has to be a factor of n and there must be points $x \neq y$ on the orbit of p which are mapped to the same point under $\varphi_{a,b}$. Since $\varphi_{a,b}$ is monotone, a whole arc A joining x with y has to be mapped by $\varphi_{a,b}$ to one point. We may assume that this arc goes from x to y in the anticlockwise direction. For every point q of the orbit of p there is k such that $f^k_{a,b}(x) = q$. Then $f^k_{a,b}(A)$ contains an arc joining q with its anticlockwise neighbor from the orbit of p, and it is mapped by $\varphi_{a,b}$ to one point. This proves that the whole circle is mapped by $\varphi_{a,b}$ to one point, a contradiction. Hence, the period of $\varphi_{a,b}(p)$ for D must be n. □

The next lemma adds monotonicity with respect to a.

Lemma 3.3. *Under an additional assumption (3), $\Phi_{a,b}(x)$ is increasing as a function of a and continuous as a function of x, a, b (jointly).*

Proof. The inequality (3.2) is uniform jointly in x, a, b, so by (3), the limit (3.1) is continuous jointly in x, a, b. Since $F_{a,b}(x)$ is increasing in a and x, the iterates $F^n_a(x)$ are increasing in a, and therefore the limit (3.1) is increasing in a. □

The fourth lemma is of a different nature. It deals with a map close to a saddle-node (in the intermittency regime). While other, in a sense stronger, tools can be used for this purpose (see [10]), this one has a simple proof and gives explicit estimates unavailable otherwise.

Lemma 3.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 orientation preserving diffeomorphism. Choose $x_0 \in \mathbb{R}$ and set $x_i = f^i(x_0)$ for $i \in \mathbb{Z}$. Assume that $x_{-1} < x_0 < \dots < x_n < x_{n+1}$. Then:*

(1) *If f' is increasing on (x_{-1}, x_{n-1}) then*

$$(f^n)'(x_0) \geq \frac{x_n - x_{n-1}}{x_0 - x_{-1}}.$$

(2) *If f' is decreasing on (x_0, x_n) then*

$$(f^n)'(x_0) \geq \frac{x_{n+1} - x_n}{x_1 - x_0}.$$

(3) *If f' is increasing on (x_0, x_n) then*

$$(f^n)'(x_0) \leq \frac{x_{n+1} - x_n}{x_1 - x_0}.$$

(4) *If f' is decreasing on (x_{-1}, x_{n-1}) then*

$$(f^n)'(x_0) \leq \frac{x_n - x_{n-1}}{x_0 - x_{-1}}.$$

Proof. We have

$$(f^n)(x_0) = \prod_{i=0}^{n-1} f'(x_i).$$

By the Mean Value Theorem, for each i there is $\xi_i \in (x_{i-1}, x_i)$ such that $(x_{i+1} - x_i)/(x_i - x_{i-1}) = f'(\xi_i)$. If f' is increasing, we have $f'(\xi_i) \leq f'(x_i)$, so

$$(f^n)'(x_0) \geq \prod_{i=0}^{n-1} \frac{x_{i+1} - x_i}{x_i - x_{i-1}} = \frac{x_n - x_{n-1}}{x_0 - x_{-1}}.$$

If f' is decreasing, we have $f'(\xi_i) \leq f'(x_{i-1})$, so

$$(f^n)'(x_0) \geq \prod_{i=0}^{n-1} \frac{x_{i+2} - x_{i+1}}{x_{i+1} - x_i} = \frac{x_{n+1} - x_n}{x_1 - x_0}.$$

The last two inequalities are proved in the same way. □

Let us now return to double standard maps. The way we think of the circle on which the maps $f_{a,b}$ act, this is the circle $\mathbb{R}/\mathbb{Z}$.

Theorem 3.5. *If $0 \leq b \leq 1$ then the double standard map $f_{a,b}$, given by* (1.2)*, has at most one attracting or neutral periodic orbit.*

Proof. We can complexify $f_{a,b}$ by conjugating it via $e^{2\pi i x}$. Then we get the map

$$g_{a,b}(z) = e^{2\pi i a} z^2 e^{b\left(z - \frac{1}{z}\right)}, \tag{3.3}$$

of the unit circle to itself. This map is the restriction of the map of $\mathbb{C} \setminus \{0\}$ to itself given by the same formula. By the results in the theory of iterations of complex maps (see [2], Theorem 7), it follows that for a map (3.3) any attracting periodic orbit of $g_{a,b}$ has to attract a critical point. A neutral periodic orbit on the unit circle is parabolic, so it is on a boundary of a periodic Leau domain. Therefore this result applies also to such orbit. If $b < 1$ then here is only one pair of critical points, symmetric (in the complex sense) with respect to the unit circle, and the map preserves this symmetry. If $b = 1$, there is just one critical point, -1. Therefore, there can be at most one attracting or neutral periodic orbit. □

4. Order of tongues

Suppose that a double standard map $f_{a,b}$ has an attracting periodic orbit P of period n. By Theorem 3.5, the trajectories under $g_{a,b}^n$ of both critical points of $g_{a,b}$ (or of the critical point -1 if $b = 1$) converge to $e^{2\pi ip}$ for some $p \in P$. Let $\varphi_{a,b}$ be the semiconjugacy from Lemma 3.2. Then by that lemma, $\varphi_{a,b}(p)$ is a periodic point of period n of the doubling map D. We will denote this point by $T(P)$ and call the *type of the orbit* P.

For a periodic point T of D we define the *tongue of type* T as the set of parameter values (a, b) (where we think of a as taken modulo 1 and b is from $[0, 1]$) for which there exists a periodic orbit of type T. If the period of T is n, we will say that the tongue of type T has period n.

Since $g_{a,b}$ and $\varphi_{a,b}$ depend continuously of (a, b), each tongue is open.

We first investigate some properties of double standard maps with an attracting or neutral periodic orbit. We will use them later in this section in the case $b = 1$, but we state and prove them in a more general case.

Lemma 4.1. *Assume that* p *is an attracting or neutral periodic point of* $f_{a,b}$ *of period* n*. Let* J *be the set of all points* x *for which* $\varphi_{a,b}(x) = \varphi_{a,b}(p)$*. Then* J *is either a closed interval (modulo 1) or a singleton and* $f_{a,b}^n|_J$ *is an orientation preserving homeomorphism of* J *onto itself. The endpoints of* J *are fixed points of* $f_{a,b}^n$*, and one of the following four possibilities holds (see Figure 6). In the first three cases* J *is an interval.*

1. *The left endpoint of* J *is neutral, topologically attracting from the right and topologically repelling from the left; the right endpoint of* J *is repelling; there are no no other fixed points of* $f_{a,b}^n$ *in* J*.*
2. *The right endpoint of* J *is neutral, topologically attracting from the left and topologically repelling from the right; the left endpoint of* J *is repelling; there are no no other fixed points of* $f_{a,b}^n$ *in* J*.*
3. *Both endpoints of* J *are repelling; there is an attracting fixed point of* $f_{a,b}^n$ *in the interior of* J*; there are no no other fixed points of* $f_{a,b}^n$ *in* J*.*
4. *The set* J *consists of one neutral fixed point of* $f_{a,b}^n$*, repelling from both sides.*

Proof. By Lemma 3.1, $\varphi_{a,b}$ is an increasing continuous function. Therefore J is a closed interval or a singleton. Since $\varphi_{a,b}(J)$ is a set consisting of a fixed point of D^n and $f_{a,b}$ is an orientation preserving local homeomorphism, we see that $f_{a,b}^n|_J$ is an orientation preserving homeomorphism of J onto itself. It follows that the endpoints of J are fixed points of $f_{a,b}^n$. None of them can be attracting from the "outside", because the whole immediate basin of attraction would be contained in J.

Since $f_{a,b}^n$ is analytic, it has finitely many fixed points in J and if $x < y$ are consecutive fixed points, either x is attracting from the right and y is repelling from the left, or x is repelling from the right and y is attracting from the left (here by attracting and repelling we mean topologically attracting and repelling; such a point can be neutral). By Theorem 3.5, there can be at most one fixed point topologically attracting from one or both sides. All this restricts the possibilities to the four ones listed in the statement of the lemma. □

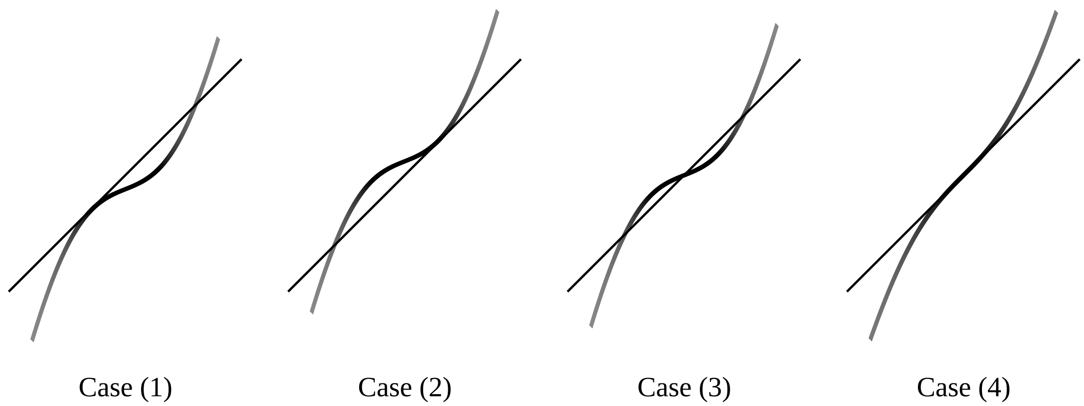

FIGURE 6. Four cases

Since $f^n_{a,b}$ and its derivative depend continuously on (a, b) and a change of (a, b) that strictly increases (respectively decreases) $f_{a,b}$ also strictly increases (respectively decreases) $f^n_{a,b}$, we get immediately (look at Figure 6 and trace possible changes of the graph) the following lemma.

Lemma 4.2. *A small change in (a, b) that strictly increases $f_{a,b}$, applied to case (1) of Lemma 4.1, or a small change in (a, b) that strictly decreases $f_{a,b}$, applied to case (2), results in case (3), with the periodic point p depending continuously on (a, b). A small change in (a, b) that strictly decreases $f_{a,b}$, applied to case (1) of Lemma 4.1, or a small change in (a, b) that strictly increases $f_{a,b}$, applied to case (2), results in disappearing of an attracting or neutral periodic point of period n. A small change in (a, b) applied to case (3), results in case (3), with the periodic point p depending continuously on (a, b).*

We are interested in the order of the tongues as we vary a. While Lemma 3.3 gives us monotonicity of $\varphi_{a,b}$ with respect to a, we cannot be sure where the point p from the definition of $T(P)$ is located. Fortunately, if $b = 1$, we know where on the circle the critical point of $f_{a,b}$ is located. Elementary computations show that this point is at $1/2$ and that $f_{a,1}$ has negative Schwarzian derivative. Therefore the whole interval joining p with $1/2$ is attracted to p under the iterates of $f^n_{a,1}$, where n is the period of P.

To simplify notation, we will write f_a for $f_{a,1}$ and φ_a for $\varphi_{a,1}$.

Lemma 4.3. *If f_a has an attracting periodic orbit P then $T(P) = \varphi_a(1/2)$.*

Proof. Let p be the point of P from the definition of $T(P)$ and let n be the period of P. As we observed, the whole interval joining p with $1/2$ is attracted to p under the iterates of $f^n_{a,1}$. Then from the definition of $\Phi_{a,b}$ it follows that $\varphi_a(p) = \varphi_a(1/2)$. Thus, $T(P) = \varphi_a(1/2)$. □

The next result is a kind of converse to Lemma 4.3. It describes the situation when $\varphi_a(1/2)$ is a periodic point of D.

Proposition 4.4. *Let q be a periodic point of D of period n. Then the set of values of a for which $\varphi_a(1/2) = q$ is a closed interval I (modulo 1). If $a \in I$ then f_a has an attracting or neutral periodic point $p(a)$ of period n. The set $J(a) = \varphi_a^{-1}(1/2)$ is a closed interval (modulo 1). Its interior (together with $p(a)$ if $p(a)$ is an endpoint of $J(a)$) is the immediate basin of attraction of $p(a)$ and contains $1/2$. If a is the left (respectively right) endpoint of I then the left (respectively right) endpoint of $J(a)$ is $p(a)$ and it is neutral; the other endpoint of $J(a)$ is a repelling periodic point of period n and there are no periodic points in $J(a)$ other than these two. If a is in the interior of I, then $p(a)$ is attracting; both endpoints are repelling periodic points of period n and there are no periodic points in $J(a)$ other than these three.*

Proof. By Lemma 3.3, $a \mapsto \varphi_a(1/2)$ is an increasing continuous function. Therefore I is a closed interval or a singleton. Assume that $a \in I$. By Lemma 4.1, $J(a)$ is a closed interval or a singleton, $f_a^n|_{J(a)}$ is an orientation preserving homeomorphism of $J(a)$ onto itself, the endpoints of $J(a)$ are fixed points of f_a^n, and one of the cases (1)-(4) of that lemma holds. By the definition, $1/2 \in J(a)$. If $J(a)$ is a singleton, then $1/2$ is periodic for f_a, and since $f_a'(1/2) = 0$, it is superattracting. On the other hand, by Lemma 4.1, it is neutral, a contradiction. This proves that $J(a)$ is an interval and leaves only cases (1)-(3).

In all three cases there is an attracting or neutral fixed point $p(a)$ of f_a^n and its immediate basin of attraction is the interior of $J(a)$ (together with $p(a)$ if $p(a)$ is an endpoint of $J(a)$). Since $1/2$ is in the interior of $J(a)$, it belongs to the immediate basin of attraction of $p(a)$. The sets $f_a^k(J(a))$ are disjoint from $J(a)$ for $k = 1, 2, \dots, n-1$, so the fixed points of $f_a^n|_{J(a)}$ are periodic points of period n of f_a.

It remains to prove that I is an interval and that the cases (1), (2) and (3) correspond to a being the left endpoint, right endpoint and an interior point of I, respectively. However, this is a straightforward consequence of Lemma 4.2. □

Now, Lemma 4.3, Proposition 4.4 and the fact that the function $a \mapsto \varphi_a(1/2)$ is increasing and continuous, imply the main theorem of this section.

Theorem 4.5. *As a increases, the types of the tongues of f_a vary in the order of rational numbers.*

In particular, this theorem explains the order mentioned in Section 2. As a varies from $1/2$ to 1, the periodic points of D of period 5 or less are $0/1 < 1/31 < 2/31 < 1/15 < 3/31 < 4/31 < 2/15 < 1/7 < 5/31 < 6/31 < 3/15 < 7/31 < 8/31 < 4/15 < 2/7 < 9/31 < 10/31 < 1/3 < 11/31 < 12/31 < 6/15 < 13/31 < 3/7 < 14/31 < 7/15 < 15/31$, and they have periods 1, 5, 5, 4, 5, 5, 4, 3, 5, 5, 4, 5, 5, 4, 3, 5, 5, 2, 5, 5, 4, 5, 3, 5, 4, 5, respectively.

If we want to apply the same methods to the tongues at the level $b = b_0$ with $b_0 < 1$, the problems arise already when we want to prove an analogue of Lemma 4.3.

Observe that the derivative of $f_{a,b}$ attains its minimum at $x = 1/2$. It is a natural conjecture that if there is an attracting periodic orbit P then $1/2$ is in its immediate basin of attraction, which would prove the desired lemma. While numerical experiments seem to support this conjecture, it is nevertheless false.

Let (a_0, b_0) be the coordinates of a tip of a period 2 tongue in the parameter plane. To be more precise, b_0 is the infimum of the values of b for which $f_{a,b}$ has an attracting periodic orbit of period 2 and there is a sequence $(a_n, b_n)_{n=1}^{\infty}$ convergent to (a_0, b_0)

such that f_{a_n,b_n} has an attracting periodic point x_n of period 2. We may assume that $x_n \to x_0$ as $n \to \infty$. Then x_0 is a periodic point of period 2 of f_{a_0,b_0}. By Lemma 4.2, for (a_0, b_0) case (4) of Lemma 4.1 has to occur. In particular, $(f^2_{a_0,b_0})''(x_0) = 0$.

Small change of (a, b) cannot produce a large basin of attraction of a periodic point of period 2, so the length of the immediate basin of attraction of x_n shrinks to 0. Therefore, if $1/2$ is always in this basin of attraction, we must have $x_0 = 1/2$. We will show that this is impossible.

Let $f = f_{a_0,b_0}$ and assume that $x_0 = 1/2$. Since $(f^2)'' = (f'' \circ f)(f')^2 + (f' \circ f)f''$ and $(f^2)''(1/2) = 0$, we get $f''(f(1/2))(f'(1/2))^2 + f'(f(1/2))f''(1/2) = 0$. However, $f''(1/2) = 0$, so $f''(f(1/2))(f'(1/2))^2 = 0$. Since $f'(1/2) \neq 0$, we get $f''(f(1/2)) = 0$. The only points at which f'' vanishes are 0 and $1/2$ (modulo 1), so either $f(1/2) = 1/2$ or $f(1/2) = 0$ (modulo 1). In the first case, $1/2$ is a fixed point of f, and since $b_0 > 1/2$, this point is attracting, a contradiction. In the second case, since $f(0) = a_0$ and $f(1/2) = 1+a_0$ (modulo 1), we get $1/2 = a_0 = 0$ (modulo 1), also a contradiction.

This proves that in a period 2 tongue, close to its tip, there must be values of a, b such that $1/2$ is not in the immediate basin of attraction of the attracting periodic orbit of period 2.

We finish this section with a corollary to Lemma 4.2.

Proposition 4.6. *Whenever a piece of the boundary of a tongue consists of points for which the case (1) or (2) of Lemma 4.1 holds, it has slope with the absolute value at least* π*.*

Proof. Observe that the partial derivative of $f_{a,b}(x)$ with respect to a is 1, while the partial derivative with respect to b is $\sin(2\pi x)/\pi$, which has modulus at most $1/\pi$. Therefore any change in (a, b) in the direction of a vector $(1, y)$, where $|y| < \pi$, strictly increases $f_{a,b}$. Similarly, any change in (a, b) in the direction of a vector $(-1, y)$, where $|y| < \pi$, strictly decreases $f_{a,b}$. The statement of the lemma follows from this and Lemma 4.2. □

The problem with the application of this proposition is that we have to exclude the possibility of pieces of the boundaries of tongues consisting of points for which case (4) holds. This would require the solution to three analytic equations: $f^n_{a,b}(x) = x$, $(f^n_{a,b})'(x) = 1$ and $(f^n_{a,b})''(x) = 0$ in the (a, b, x)-space to contain a curve. Generically, this is not a case. However, we do not know how generic the family of double standard maps is.

5. Period 1 tongue

Let us investigate closer the tongue corresponding to period 1. Elementary computations show that its boundary is given by the curves

$$a = \frac{1}{2} \pm \frac{\sqrt{4b^2 - 1} - \arctan\sqrt{4b^2 - 1}}{2\pi} \tag{5.1}$$

and the corresponding fixed point is then

$$x = -\frac{1}{2} \pm \frac{\arctan\sqrt{4b^2 - 1}}{2\pi}.$$

Set $b = 1/2 + t$. Then (5.1) becomes

$$a = \frac{1}{2} \pm \frac{2\sqrt{t+t^2} - \arctan(2\sqrt{t+t^2})}{2\pi} \tag{5.2}$$

and the derivative of the right-hand side of (5.2) is

$$\frac{2\sqrt{t+t^2}}{\pi(1+2t)}.$$

At $t = 0$ this is of order $t^{1/2}$, so the tangency of the two lines bounding period 1 tongue is of order $t^{3/2}$.

This tongue begins at the level $b = 1/2$. We will show that all other tongues begin substantially higher.

For the double standard maps we have

$$f'_{a,b}(x) = 2 + 2b\cos(2\pi x). \tag{5.3}$$

Therefore, $f'_{a,b}$ has one minimum, at $x = 1/2 \pmod 1$, one maximum, at $x = 0 \pmod 1$, is decreasing on $(0, 1/2)$ and increasing on $(1/2, 1)$. This allows us to apply Lemma 3.4 to $f_{a,b}$, or rather, since we use inequalities, to $F_{a,b}$. Clearly, the formula for $F'_{a,b}$ is the same as for $f'_{a,b}$.

Lemma 5.1. *Assume that* $x \in (0,1)$, $k \geq 1$, $F^{k-1}_{a,b}(x) \leq 1$ *and* $F_{a,b}(t) > t$ *for* $t \in (x, F^k_{a,b}(x))$. *Then*

$$(F^k_{a,b})'(x) \geq \frac{F^k_{a,b}(x) - F^{k-1}_{a,b}(x)}{F_{a,b}(x) - x} \cdot (F_{a,b})'(1/2). \tag{5.4}$$

Proof. Assume first that $x < 1/2$ and $F^k_{a,b}(x) > 1/2$. Then the orbit of x is $x_0 < \cdots < x_n < y_0 < \cdots < y_m < \dots$, where $1/2$ is between x_n and y_0 and $n + m + 1 = k$ (so $F^k_{a,b}(x) = y_m$. Then by Lemma 3.4,

$$(F^k_{a,b})'(x) \geq \frac{y_0 - x_n}{x_1 - x_0} \cdot (F_{a,b})'(x_n) \cdot \frac{y_m - y_{m-1}}{y_0 - x_n} \geq \frac{y_m - y_{m-1}}{x_1 - x_0} \cdot (F_{a,b})'(1/2).$$

If $F^k_{a,b}(x) \leq 1/2$ (that is, there are no y_i's), then the estimate that we get from Lemma 3.4

$$(F^k_{a,b})'(x) \geq \frac{F_{a,b}(x_n) - x_n}{x_1 - x_0},$$

so by the Mean Value theorem we get

$$(F^k_{a,b})'(x) \geq \frac{F_{a,b}(x_n) - x_n}{x_n - x_{n-1}} \cdot \frac{x_n - x_{n-1}}{x_1 - x_0} \geq (F_{a,b})'(1/2) \cdot \frac{x_n - x_{n-1}}{x_1 - x_0}.$$

Similarly, if $x \geq 1/2$ (that is, there are no x_i's), then we get

$$(F^k_{a,b})'(x) \geq \frac{y_m - y_{m-1}}{y_0 - (F_{a,b})^{-1}(y_0)} \geq (F_{a,b})'(1/2) \cdot \frac{y_m - y_{m-1}}{y_1 - y_0}.$$

∎

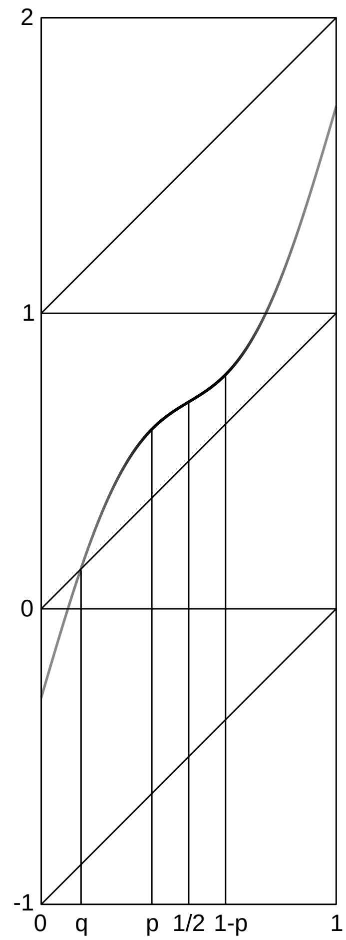

FIGURE 7. The graph of $F_{a,b}$ for $a = -0.3$, $b = 0.7$

By our analysis of the derivative $F'_{a,b}$, if $1/2 < b < 1$ then on $[0, 1]$ there are 2 points where it is equal to 1. The first of them is in $(0, 1/2)$:

$$p = \frac{1}{2\pi} \arccos\left(\frac{-1}{2b}\right), \tag{5.5}$$

and the other one is $1 - p$ (note that the graph of $F_{a,b}$ is centrally symmetric about the point $(1/2, F_{a,b}(1/2))$, see Figure 1. If $F_{a,b}$ has no attracting or neutral fixed point then it has a unique fixed point q. If $-1/2 < a \leq 0$ then $F_{a,b}(0) = a \leq 0$ and $F_{a,b}(1/2) = 1 + a > 1/2$, so $0 \leq q < 1/2$ (see Figure 1).

Lemma 5.2. *Assume that $-1/2 < a \leq 0$, $1/2 < b < 1$, $F_{a,b}(1 - p) \leq 1$, $f_{a,b}$ has no attracting or neutral fixed point, and*

$$\big(1 - F_{a,b}^{-1}(1)\big) F'_{a,b}(1/2) > F_{a,b}(p) - p. \tag{5.6}$$

Then every periodic orbit of $f_{a,b}$ is repelling.

Proof. Any periodic orbit of $f_{a,b}$ of period larger than 1 can be divided into blocks as follows. Any point of the orbit that is in $[0, q)$ forms a block of length 1. The rest of the points of the orbits are divided in a natural way into maximal blocks of the form $(x, f_{a,b}(x), \dots, f_{a,b}^{k-1})$ satisfying the assumptions of Lemma 5.1 (on those blocks we can replace $f_{a,b}$ by $F_{a,b}$). In order to prove the lemma it is enough to show that the derivative of $F_{a,b}$ along any block is larger than 1. This is true for the blocks of length 1 with the point in $[0, q)$, because $F'_{a,b}$ is decreasing in $[0, q]$ and is larger than 1 at q. It is also true for the other blocks if the right-hand side of (5.4) is larger than 1.

Look at such a block. If $x > 1 - p$ then $F'_{a,b}$ is larger than 1 at all points of the block, so the product of those derivatives is also larger than 1. Assume now that $x \le 1 - p$. Then

$$F_{a,b}(x) - x \le F_{a,b}(p) - p.$$

Moreover, since $F_{a,b}(1 - p) \le 1$ and $F'_{a,b}$ is increasing in $[1 - p, 1]$, we get

$$F_{a,b}^k(x) - F_{a,b}^{k-1}(x) \ge 1 - F_{a,b}^{-1}(1).$$

Therefore, the right-hand side of (5.4) is larger than 1 by the inequality (5.6). □

We get a similar result also in another situation, not involving intermittency.

Lemma 5.3. *Assume that $-1/2 < a \le 0$, $1/2 < b < 1$, $f_{a,b}$ has no attracting or neutral fixed point, and*

$$F'_{a,b}(1/2) \cdot F'_{a,b}(f_{a,b}(p)) > 1 \quad \text{and} \quad F'_{a,b}(1/2) \cdot F'_{a,b}(f_{a,b}(1 - p)) > 1. \tag{5.7}$$

Then every periodic orbit of $f_{a,b}$ is repelling.

Proof. From (5.7) it follows that $F_{a,b}(p) > 1 - p$ and $F_{a,b}(1 - p) < p + 1$. Therefore $F_{a,b}([p, 1 - p]) \subset [1 - p, p + 1]$. If $x \in [p, 1 - p]$ then $F'_{a,b}(x) \ge F'_{a,b}(1/2)$ and

$$F'_{a,b}(F_{a,b}(x)) \ge \min\big(F'_{a,b}(p), F'_{a,b}(1 - p)\big).$$

By (5.7) we get $F'_{a,b}(x) \cdot F'_{a,b}(F_{a,b}(x)) > 1$. Therefore, we can divide any periodic orbit of $f_{a,b}$ into blocks of length 1 or 2 (if the point x on the orbit is in $[p, 1 - p]$ then it is the first point of a block of length 2), and the derivative of $f_{a,b}$ along any block is larger than 1. This completes the proof. □

Now we can prove the main theorem of this section.

Theorem 5.4. *If $0 \le b < 0.5$ then all periodic orbits of $f_{a,b}$ are repelling. Set $b_0 = 0.578$. If $0.5 \le b \le b_0$ then all periodic orbits of $f_{a,b}$, except perhaps one fixed point, are repelling.*

Proof. If $0 \le b < 0.5$ then $f'_{a,b}$ is larger than 1 everywhere, so all periodic orbits are repelling. Similarly, if $b = 0.5$ then there cannot be attracting periodic orbits, and the only neutral periodic orbit can have period 1. Assume that $0.5 < b \le b_0$. If there is an attracting or neutral fixed point, then by Theorem 3.5 that there are no other attracting or neutral periodic orbits. Therefore it remains to consider the case when there is no attracting or neutral fixed point.

Further reduction can be made in the range of the parameter a. Since we consider our map modulo 1, we may assume that $a \in [-1, 0]$. Moreover, the map $x \mapsto 1 - x$ conjugates $f_{a,b}$ with $f_{-1-a,b}$. Therefore we may assume that $a \in [-1/2, 0]$. If $a = -1/2$ then the point $1/2$ is attracting, so our final assumption on a is that $-1/2 < a \le 0$, the same as in Lemmas 5.2 and 5.3.

Set $a_0 = -0.285$. If $a \le a_0$, we would like to apply Lemma 5.2. To do this, we have to check that $f_{a,b}(1-p) \le 1$ and that (5.6) is satisfied.

For a fixed b, the largest value of $F_{a,b}(1-p)$ is attained when $a = a_0$. Thus, we have to check that $F_{a_0,b}(1-p) \le 1$. Since

$$\arccos \frac{-1}{2b} = \sqrt{1 - \frac{1}{4b^2}},$$

this inequality is equivalent to

$$2\pi(1+a_0) \le 2\arccos\frac{-1}{2b} + \sqrt{4b^2-1}. \tag{5.8}$$

The derivative of the right-hand side of (5.8) is

$$\frac{1}{\sqrt{4b^2-1}}\left(4b - \frac{2}{b}\right),$$

so the right-hand side of (5.8) attains its minimum at $b = \sqrt{2}/2$. The value of this minimum is $(3/2)\pi + 1 > 5.7$, while the value of the left-hand side of (5.8) is less than 4.5. This proves that $F_{a,b}(1-p) \le 1$.

Fix b and consider both sides of (5.6) as functions of a. Since p is independent of a, the derivative of the right-hand side with respect to a is 1. On the left-hand side the factor $F'_{a,b}(1/2)$ is independent of a and smaller than 1. Since $F_{a,b}(1-p) \le 1$, we get that $F^{-1}_{a,b}(1) \ge 1-p$, so $F'_{a,b}(F^{-1}_{a,b}(1)) \ge 1$. Therefore, since the derivative of $F_{a,b}(x)$ for any fixed x is 1, and by the Implicit Function Theorem, the absolute value of the derivative of $F^{-1}_{a,b}(1)$ with respect to a is smaller than or equal to 1. Therefore the derivative of the left-hand side with respect to a is smaller than 1. This means that we have to check (5.6) only for $a = a_0$. Now, for the fixed value of a, as b increases, on $(1/2, 1)$ $f_{a,b}$ decreases, so $1 - F^{-1}_{a,b}(1)$ decreases, $F'_{a,b}(1/2)$ decreases, and on $(0, 1/2)$ $x \mapsto F_{a,b}(x) - x$ increases (as a function of b), so $F_{a,b}(p) - p$ increases. Therefore we have to check (5.6) only for $b = b_0$. For those values of a and b the left-hand side of (5.6) is larger than 0.224, while the right-hand side is smaller than 0.224. This shows that (5.6) holds, so by Lemma 5.2, if $a \le a_0$ (plus the assumptions of the theorem) then all periodic orbits of $f_{a,b}$, except perhaps one fixed point, are repelling.

Assume now that $a > a_0$. Then we would like to use Lemma 5.3, so we have to verify that (5.7) holds. Consider the first inequality of (5.7). Since $p < 1/2$ and $b > 1/2$, we have $F_{a,b}(p) > F_{a,1/2}(p)$. Therefore it is enough to show that

$$F'_{a,b}(1/2) \cdot F'_{a,b}(F_{a,1/2}(p)) > 1.$$

The value of p decreases as b increases. Therefore

$$1/2 > p \ge \frac{1}{2\pi}\arccos\left(\frac{-1}{2b_0}\right) > 0.4163,$$

so

$$1 = f_{0,1/2}(1/2) > (f_{a,1/2}(p)) > f_{a_0,1/2}(0.4163) > 0.627.$$

This shows that $F_{a,1/2}(p)$ is in the region where $F'_{a,b}$ increases. Hence,

$$F'_{a,b}(f_{a,1/2}(p)) > 2 + 2 \cdot 0.578 \cdot \cos(2\pi \cdot 0.627) > 1.19,$$

so

$$F'_{a,b}(1/2) \cdot f'_{a,b}(f_{a,1/2}(p)) > 0.844 \cdot 1.19 > 1.004.$$

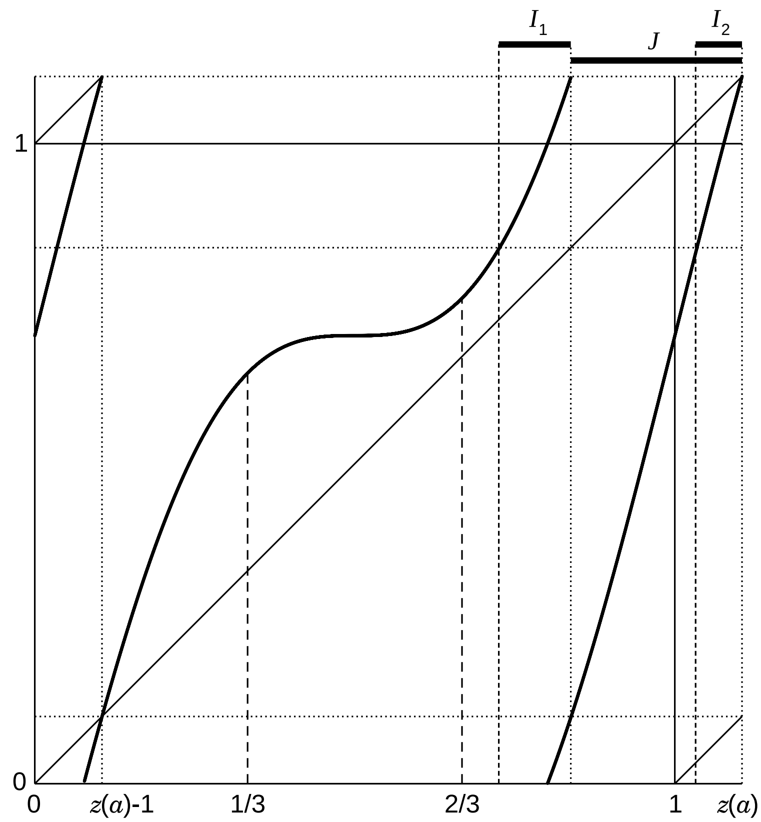

FIGURE 8. The map f_a for $a \in (a_l, a_r)$

Consider now the second inequality of (5.7). Since $p > 0.4$, we have $1 - p < 0.6$, so $F_{a,1/2}(1-p) < 1.2$. Moreover, $1 - p > p$. The derivative of $F_{a,b}$ is the same at $1 + t$ and at $1 - t$, so we get

$$F'_{a,b}(F_{a,1/2}(1-p)) > \min(F'_{a,b}(0.627), F'_{a,b}(0.8)) = F'_{a,b}(0.627) > 1.19,$$

and then we get the same estimate as for the first inequality. This completes the proof. $\square$

6. Mostly repelling attracting periodic orbits

In this section we consider again the case $b = 1$, and we use the same notation as in Section 4.

We will consider here a special class $\mathcal{P}$ of attracting periodic orbits. They are attracting periodic orbits for f_a of type $0.\overline{0001 * 1 * 1 \cdots * 1}$ (the line over a finite sequence means that it is repeated periodically), where each $*$ can be 0 or 1. There are values of a, $a_l \approx -0.32221099$ and $a_r \approx -0.28609229$ for which $\Phi_{a_l}(1/2) = 1/16$ and $\Phi_{a_r}(1/2) = 1/8$. We have $1/16 = 0.0001\overline{0}$ and $1/8 = 0.000\overline{1}$. The numbers of the form $0.\overline{0001 * 1 * 1 \cdots * 1}$ are between those two, so any a for which f_a has a periodic orbit of such type is in (a_l, a_r).

Let $z(a)$ be the fixed point of $F_a - 1$ (see Figure 8). Then $\Phi_a(z(a)) = 1$. If $P \in \mathcal{P}$ then $z(a)$ is not in the basin of attraction of P, so the sets $\Phi_a^{-1}(1)$, as well as $\Phi_a^{-1}(1/2^j)$

for $j = 1, 2, \dots,$ consist of one point each. We have $F_{a_r}(1/2) = a_r + 1 > 2/3$, so $F^2_{a_r}(2/3) < z(a_r)$. As a decreases, $F^2_a(2/3)$ decreases, while $z(a)$ increases. Therefore $2/3 < F_a^{-2}(z(a))$ for all $a \in (a_l, a_r)$. The point $F_a^{-2}(z(a))$ is the unique point whose image under Φ_a is $1/4$, so the binary expansion of $\Phi_a(2/3)$ starts with 0.00. On the other hand, $z(a_l) < 4/3$, so $z(a) < 4/3$ for all $a \in (a_l, a_r)$. Since the only points where $F'_a \le 1$ are in $[1/3, 2/3]$ and the integer shifts of this interval (note that F'_a does not depend on a), we see that there exists a constant $\lambda > 1$ such that whenever $a \in (a_l, a_r)$ and the binary expansion of $\Phi_a(x)$ does not start with 0.00, we have $F'_a(x) > \lambda$.

Assume that $P \in \mathcal{P}$ is an orbit of period n and let $p \in P$ be the point for which $1/2$ is in the immediate basin of attraction of p for f_a^n (we need p not modulo 1, so we choose $p \in [0, 1)$). Since $F_{a_l}(1/3) = 2/3 + a_l + \sqrt{3}/(2\pi) > 1/2$, there exists a constant $c > 0$ such that whenever $a \in (a_l, a_r)$ and $x \in [1/3, 2/3]$ then $f'_a(f_a(x)) > c$. Thus, if $p \ge 1/3$ then $f'_a(f_a(p)) > c$. We cannot easily exclude the case $p < 1/3$. However, then $[1/3, 1/2]$ is contained in the basin of immediate attraction of p for f_a^n. Then there is $\varepsilon > 0$ such that the length of the basin of immediate attraction of $f_a(p)$ for f_a^n is larger than ε, independently of a. We have to have $\varepsilon\lambda^{n-2} < 1$, so there is N such that if $n \ge N$, this is impossible. From now on, we exclude from $\mathcal{P}$ the orbits of period less than N.

Moreover, we have $p \le 2/3$, since otherwise our periodic orbit would not contain a point with derivative less than 1, so it would not be attracting.

To summarize, we get the following structure of an orbit $P \in \mathcal{P}$. There is a point $p \in [1/3, 2/3] \cap P$, such that $1/2$ is in the basin of immediate attraction of p for f_a^n. The derivative of f_a at $f_a(p)$ is larger than c and at the points of P other than p and $f_a(p)$ is larger than λ.

To describe the situation in more geometrical terms, let us look at Figure 8. The interval J consists of points whose image under Φ_a has binary expansion starting with 0.1. For the intervals I_1 and I_2 this is respectively 0.01 and 0.11. The point p is in $[1/3, 2/3]$. Its image under f_a^2 is in I_1, and further images under the iterates of f_a^2 are in I_1 and I_2. Since $f_a(I_1) = f_a(I_2) = J$ and $f_a(J)$ is the whole circle, we can get periodic orbits that under the iterates of f_a^2 go through I_1 and I_2 in the prescribed order. Moreover, the derivative of f_a on $I_1 \cup I_2 \cup J$ is larger than λ.

We are interested in the sizes in the directions of a and p of the region where our orbit $P \in \mathcal{P}$ is attracting (we will refer to them as the *P-windows* in the directions of a and p). Denote those windows by $[a_1, a_2]$ and $[p_1, p_2]$ respectively. Since p depends on a, we will write $p(a)$. Thus, we have $p_i = p(a_i)$ for $i = 1, 2$. We will express those sizes in terms of the exponent of $P \setminus \{p(a)\}$

$$\alpha(a) = (f_a^{n-1})'(f_a(p(a)). \tag{6.1}$$

We have to choose some specific value of a, and the most natural such value is a_0 for which $p(a_0) = 1/2$.

Theorem 6.1. *There exist positive constants K_1, K_2, K_3, K_4 such that if a periodic orbit P belongs to $\mathcal{P}$ then for the P-windows $[p_1, p_2]$ in the direction of p and $[a_1, a_2]$ in the direction of a we have*

$$K_1(\alpha(a_0))^{-1/2} \le p_2 - p_1 \le K_2(\alpha(a_0))^{-1/2} \tag{6.2}$$

and

$$K_3(\alpha(a_0))^{-3/2} \leq a_2 - a_1 \leq K_4(\alpha(a_0))^{-3/2}. \tag{6.3}$$

In particular, the size of the P-window in the direction of a is of order of the cube of the size of the P-window in the direction of p.

Proof. Let us compute the partial derivatives of the iterates of f_a with respect to a. We have to treat f as a function of 2 variables. Use notation $f(a,x) = f_a(x)$. Then:

$$\frac{\partial f_a^{n+1}(x)}{\partial a} = \frac{\partial f(a, f_a^n(x))}{\partial a} = \frac{\partial f}{\partial a}(a, f_a^n(x)) + \frac{\partial f}{\partial x}(a, f_a^n(x)) \cdot \frac{\partial f_a^n(x)}{\partial a}.$$

Since in our case $\partial f/\partial a = 1$, we obtain

$$\frac{\partial f_a^{n+1}(x)}{\partial a} = 1 + f_a'(f_a^n(x)) \cdot \frac{\partial f_a^n(x)}{\partial a}.$$

Therefore by induction we get

$$\frac{\partial f_a^n(x)}{\partial a} = \sum_{i=0}^{n-1} (f_a^i)'(f_a^{n-i}(x)). \tag{6.4}$$

We have $f_a^n(p(a)) = p(a)$. Differentiate both sides of this equality with respect to a:

$$\frac{\partial f_a^n}{\partial a}(p(a)) + (f_a^n)'(p(a)) \cdot p'(a) = p'(a).$$

Therefore we get, substituting the formula (6.4),

$$p'(a) = \frac{\sum_{i=0}^{n-1} (f_a^i)'(f_a^{n-i}(p(a)))}{1 - (f_a^n)'(p(a))}. \tag{6.5}$$

Using notation (6.1), we get

$$(f_a^n)'(p(a)) = f_a'(p(a)) \cdot \alpha(a).$$

We have

$$\frac{\sum_{i=0}^{n-1} (f_a^i)'(f_a^{n-i}(p(a)))}{\alpha(a)} = \sum_{i=0}^{n-1} \frac{1}{(f_a^{n-i})'(p(a))} = \sum_{i=0}^{n-1} \frac{1}{(f_a^i)'(p(a))}.$$

The term of the last sum above corresponding to $i = 0$ is equal to 1, so the whole sum is larger than or equal to 1. On the other hand, if $i > 0$ then we have $(f_a^i)'(p(a)) \geq c\lambda^{i-1}$, so

$$\sum_{i=0}^{n-1} \frac{1}{(f_a^i)'(p(a))} \leq 1 + \sum_{i=1}^{\infty} \frac{1}{c\lambda^{i-1}} = 1 + \frac{\lambda}{c(\lambda - 1)}.$$

Thus, there is a constant $C > 0$, independent of $P \in \mathcal{P}$, such that

$$C \leq \frac{\alpha(a)}{\sum_{i=0}^{n-1} (f_a^i)'(f_a^{n-i}(p(a)))} \leq 1 \tag{6.6}$$

Now we can rewrite the differential equation (6.5) as

$$\frac{da}{dp} = \frac{\alpha(a)}{\sum_{i=0}^{n-1} (f_a^i)'(f_a^{n-i}(p))} \left(\frac{1}{\alpha(a)} - f_a'(p) \right). \tag{6.7}$$

The right-hand side of (6.5) is positive if $a \in (a_1, a_2)$ and infinite if $a = a_i$, $i = 1, 2$. Therefore the right-hand side of (6.7) is positive if $a \in (a_1, a_2)$ and zero if $a = a_i$, $i = 1, 2$. In particular, $1/\alpha(a_i) = f'_{a_i}(p_i)$.

Let us estimate the distortion of α as a varies from a_1 to a_2. Then $p(a)$ increases and a increases, so $f^i_a(p(a))$ increases for every $i \geq 0$. If additionally $i \leq n$, then

$$f^n_{a_1}(p_1) = f^{n-i}_{a_1}(f^i_{a_1}(p_1)) \leq f^{n-i}_{a_1}(f^i_{a_2}(p_2)) \leq f^{n-i}_{a_2}(f^i_{a_2}(p_2)) = f^n_{a_2}(p_2).$$

Since $f^n_{a_1}(p_1) = p_1 + k$ and $f^n_{a_2}(p_2) = p_2 + k$ for the same integer k, we get

$$f^{n-i}_{a_1}(f^i_{a_2}(p_2)) - f^{n-i}_{a_1}(f^i_{a_1}(p_1)) \leq p_2 - p_1 < 1.$$

For $j = 2, 3, \dots, n-1$ the interval $[f^i_{a_1}(p_1), f^i_{a_2}(p_2)]$, and therefore also its subinterval $[f^i_{a_1}(p_1), f^i_{a_1}(p_2)]$, is in the region where the derivative of f_a (which is independent of a) is larger than λ; if $j = 1$, we should replace λ by c. Therefore if $2 \leq i \leq n$, we get

$$f^i_{a_2}(p_2) - f^i_{a_1}(p_1) < \frac{1}{\lambda^{n-i}},$$

and

$$f_{a_2}(p_2) - f_{a_1}(p_1) < \frac{1}{c\lambda^{n-1}}.$$

In those regions the logarithm of the derivative of f_a is Lipschitz continuous with some constant L, so by the chain rule we get for any $a, b \in [a_1, a_2]$

$$|\log \alpha(a) - \log \alpha(b)| \leq L\left(\frac{1}{c\lambda^{n-1}} + \sum_{i=2}^{n-1} \frac{1}{\lambda^{n-i}}\right) \leq L\left(\frac{1}{c} + \sum_{j=1}^{\infty} \frac{1}{\lambda^j}\right).$$

The right-hand side of this inequality is a constant independent of the orbit $P \in \mathcal{P}$. Therefore there exists a constant $D > 1$, independent of the orbit $P \in \mathcal{P}$, such that

$$\frac{1}{D} \leq \frac{\alpha(a)}{\alpha(b)} \leq D \tag{6.8}$$

for every $a, b \in [a_1, a_2]$.

We have

$$f'_a\left(\frac{1}{2} + t\right) = 2(1 - \cos(2\pi t)),$$

so there exist positive constants E_1, E_2 such that if $1/3 \leq 1/2 + t \leq 2/3$ then

$$E_1 t^2 \leq f'_a\left(\frac{1}{2} + t\right) \leq E_2 t^2. \tag{6.9}$$

For $i = 1, 2$, since $1/\alpha(a_i) = f'_{a_i}(p_i)$, we get

$$E_1\left(\frac{1}{2} - p_i\right)^2 \leq \frac{1}{\alpha(a_i)} \leq E_2\left(\frac{1}{2} - p_i\right)^2. \tag{6.10}$$

From inequalities (6.8) and (6.10) we get

$$\sqrt{\frac{1}{DE_2} \cdot \frac{1}{\alpha(a_0)}} \leq \left|\frac{1}{2} - p_i\right| \leq \sqrt{\frac{D}{E_1} \cdot \frac{1}{\alpha(a_0)}}. \tag{6.11}$$

Therefore (6.2) holds for some positive constants K_1, K_2 independent of $P \in \mathcal{P}$.

By (6.8), we have $1/\alpha(a) - f_a'(p) \le D/\alpha(a_0)$, so from (6.7) and (6.6) we get

$$a_2 - a_1 \le \frac{D}{\alpha(a_0)}(p_2 - p_1). \tag{6.12}$$

On the other hand, the right-hand side of (6.7) is non-negative, so in view of (6.6) and since (by (6.8)) $1/\alpha(a) \ge (1/D)(1/\alpha(a_0))$,

$$a_2 - a_1 \ge C \int_{1/2-s}^{1/2+s} \left(\frac{1}{D} \cdot \frac{1}{\alpha(a_0)} - f_a'(p) \right) dp, \tag{6.13}$$

where

$$s = \sqrt{\frac{1}{DE_2} \cdot \frac{1}{\alpha(a_0)}}$$

(note that by (6.11) we have $[1/2 - s, 1/2 + s] \subset [p_1, p_2]$). By (6.9) we have

$$\int_{1/2-s}^{1/2+s} f_a'(p)\, dp \le E_2 \int_{1/2-s}^{1/2+s} t^2\, dt = \frac{2E_2}{3} s^3 = \frac{2}{3} \cdot \frac{1}{D\sqrt{DE_2}} (\alpha(a_0))^{-3/2}.$$

Therefore, from (6.13) we get

$$a_2 - a_1 \ge C \left(\frac{2s}{D} \cdot \frac{1}{\alpha(a_0)} - \frac{2}{3} \cdot \frac{1}{D\sqrt{DE_2}} (\alpha(a_0))^{-3/2} \right) = C \left(\frac{4}{3} \cdot \frac{1}{D\sqrt{DE_2}} (\alpha(a_0))^{-3/2} \right).$$

From this, (6.12) and (6.2) we get (6.3) for some positive constants K_3, K_4 independent of $P \in \mathcal{P}$. □

Let us make several comments about Theorem 6.1. The first one is that if instead of looking at the point p of the orbit P for which $1/2$ is in its immediate basin of attraction, we look at the next point along the orbit, $q = f_a(p)$ (the one that has a in its basin of attraction), then the scaling of the P-window in the direction of q will be the same as the scaling of the P-window in the direction of a. Indeed, if this window is $[q_1, q_2]$ then

$$q_2 - q_1 = F_{a_2}(p_2) - F_{a_1}(p_1) = (F_{a_1}(p_2) - F_{a_1}(p_1)) + (a_2 - a_1)$$

and since the map f_{a_1} in $[1/3, 2/3]$ is cubic up to a multiplicative constant (and in view of (6.3)), we get

$$K_5(\alpha(a_0))^{-3/2} \le q_2 - q_1 \le K_6(\alpha(a_0))^{-3/2} \tag{6.14}$$

for some positive constants K_5, K_6 independent of $P \in \mathcal{P}$. Therefore we get the following corollary to Theorem 6.1. It is consistent with Figures 3 and 4 (remember that when we consider f_a, we take a modulo 1).

Corollary 6.2. *There exist positive constants K_7, K_8 such that if a periodic orbit P belongs to $\mathcal{P}$ then for the P-windows $[q_1, q_2]$ in the direction of q and $[a_1, a_2]$ in the direction of a,*

$$K_7 \le \frac{q_2 - q_1}{a_2 - a_1} \le K_8.$$

The second comment is that since we expressed the sizes of the P-windows in terms of $\alpha(a_0)$, we have some information how those sizes behave as the period of $P \in \mathcal{P}$ goes to infinity. Then $\alpha(a_0)$ grows exponentially with the period n, in the sense that

$$c_1\lambda^{n-2} \leq \alpha(a_0) \leq c_2\Lambda^{n-2} \tag{6.15}$$

and $c_1, c_2 > 0$, $\Lambda \geq \lambda > 1$ (this follows immediately from the definition of α and our earlier estimates). However, whether $(1/n)\log\alpha(a_0)$ is closer to $\log\lambda$ or $\log\Lambda$, depends on a concrete orbit P.

The third comment is that although the orbits from $\mathcal{P}$ are kind of special, there are infinitely many of them. Moreover, the only properties of $\mathcal{P}$ that we used were that the growth of the derivatives along the pieces of the orbit $P \in \mathcal{P}$ not passing through p is exponential in the length of the piece, uniformly in $\mathcal{P}$. Thus, there are many other families similar to $\mathcal{P}$ for which the same properties can be proved.

7. Intermittent periodic orbits

Now we consider periodic orbits with the behavior in a sense opposite to the behavior of the orbits considered in Section 6. Again the case is $b = 1$, so we work with the maps f_a and their liftings F_a. Set

$$a_I = \frac{\sqrt{3}}{2\pi} - \frac{2}{3} \approx -0.3910022190.$$

We have $F_{a_I}(2/3) = 2/3$ and $F'_{a_I}(2/3) = 1$. Thus, $2/3$ is a neutral fixed point and if a is slightly larger than a_I then we observe intermittency for f_a. The trajectories of points in a rather large interval containing $1/2$ are increasing and spend a lot of time very close to $2/3$.

We denote by $\mathcal{R}$ the class of attracting periodic orbits for f_a such that if $p \in P \in \mathcal{R}$ and $1/2$ is in the immediate basin of attraction of p and n is the period of P then $p < F_a(P) < F_a^2(p) < \cdots < F_a^{n-1}(p)$ and $p = F_a^n(p) - 1$. It follows that $\Phi_a(p) = D^n(\Phi_a(p)) - 1$, so $\Phi_a(p) = 1/(2^n - 1)$. Therefore the type of such an orbit is $1/(2^n - 1)$. Since we encounter all those types for the values of a slightly larger than a_I, they cannot appear anywhere else, and they are our intermittent ones.

The general philosophy for intermittency is that as the period of the attracting periodic orbits increases, we have the same behavior (even quantitatively) in the directions of the variables x and b, while in the direction of a we have scaling depending on the order of tangency of the graph of F_{a_I} to the diagonal. This we will see in Theorems 7.2 and 8.2. We can also observe the repetition of the same behavior on Figures 9 and 10. Note that we see there wide windows coming in pairs. Such a pair consists of orbits of types $1/(2^n - 1)$ and $2/(2^{n+1} - 1)$. Clearly, our considerations can be applied also to the latter types, as well as to every intermittent family.

As in the preceding section, we want to estimate sizes of the P-windows for $P \in \mathcal{R}$. This time we will do it in terms of the period of P.

We will be using a result of Jonker [7]. Although stated formally for circle homeomorphisms, it is local and applies to any intermittent behavior in one dimension. Let us restate it for our family of maps. Lemma 2.5 of [7] (we skip the dependence of some constants on other constants) gives us the following lemma.

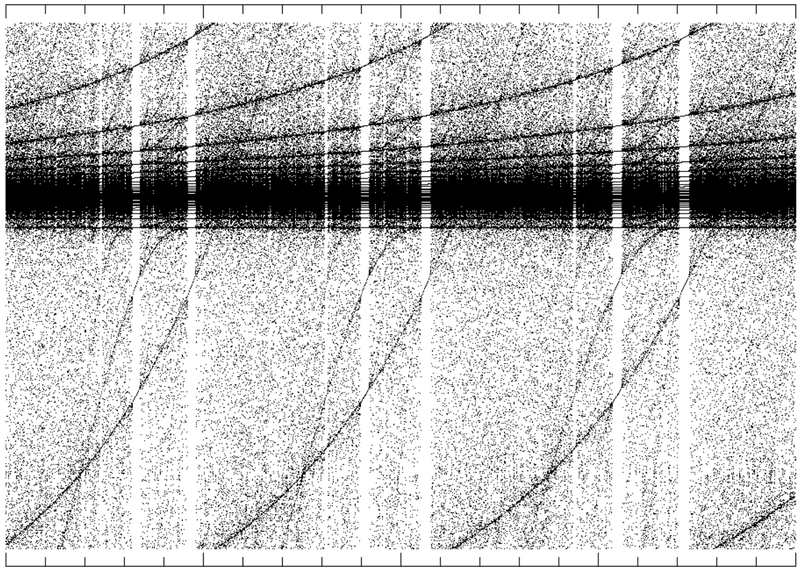

FIGURE 9. The (a, x)-plot, with a from 0.6107 to 0.6111, x from 0 to 1 and $b = 1$

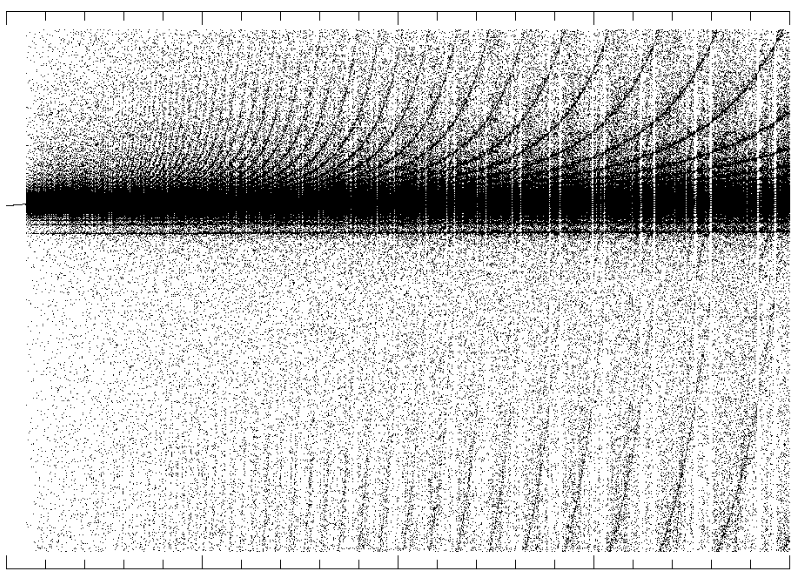

FIGURE 10. The (a, x)-plot, with a from 0.6089 to 0.6129, x from 0 to 1 and $b = 1$

Lemma 7.1 ([7]). *There are $\varepsilon, \tau > 0$ such that if $m \geq 1$ then there exist constants $K_1, K_2 > 0$ such that*

$$K_1(a - a_I)^{-3/2} < \frac{\partial f_a^q(x)}{\partial a} < K_2(a - a_I)^{-3/2}$$

whenever $a_I < a_0 < a_I + \varepsilon$, $x \in [2/3 - \tau, F_a^m(2/3 - \tau))$ and $F_a^q(x) \in [F_a^{-m}(2/3 + \tau), 2/3 + \tau)$.

The main result of this section is the following theorem.

Theorem 7.2. *There exist positive constants M_1, M_2, M_3, M_4 such that if P_n is the periodic orbit of $\mathcal{R}$ of period n then for the P-windows $[p_1, p_2]$ in the direction of p and $[a_1, a_2]$ in the direction of a, then*

$$M_1 \le p_2 - p_1 \le M_2 \tag{7.1}$$

and

$$M_3 n^{-3} \le a_2 - a_1 \le M_4 n^{-3}. \tag{7.2}$$

Moreover, there exist positive constants M_5, M_6 such that if c_n is the value of the parameter a for which $1/2 \in P_n$, then

$$M_5 n^{-2} \le c_n - a_{_I} \le M_6 n^{-2}. \tag{7.3}$$

Proof. We need estimates of the partial derivative with respect to a along our periodic orbit. If we start and end close to $1/2$ then we can split a trajectory piece of length n into 3 pieces of lengths k, $n-2k$ and k, so that Lemma 7.1 applies to the middle piece. As the parameter a approaches $a_{_I}$, the maps f_a converge to $f_{a_{_I}}$, so we can find k that will work for all sufficiently large periods.

Computations similar as in Section 6 give us the following formulas:

$$\frac{\partial f_a^{n-k}}{\partial a}(f_a^k(p)) = \frac{\partial f_a^k}{\partial a}(f_a^{n-k}(p)) + (f_a^k)'(f_a^{n-k}(p))\frac{\partial f_a^{n-2k}}{\partial a}(f_a^k(p)) \tag{7.4}$$

and

$$\frac{\partial f_a^n}{\partial a}(p) = \frac{\partial f_a^{n-k}}{\partial a}(f_a^k(p)) + (f_a^{n-k})'(f_a^k(p))\frac{\partial f_a^k}{\partial a}(p). \tag{7.5}$$

By substituting (7.4) to (7.5) we get

$$\frac{\partial f_a^n}{\partial a}(p) = \frac{\partial f_a^k}{\partial a}(f_a^{n-k}(p)) + (f_a^k)'(f_a^{n-k}(p))\frac{\partial f_a^{n-2k}}{\partial a}(f_a^k(p)) + (f_a^{n-k})'(f_a^k(p))\frac{\partial f_a^k}{\partial a}(p). \tag{7.6}$$

Let us now estimate the derivative with respect to x. The point p has $1/2$ in its immediate basin of attraction, while other points of the orbit of p do not. Therefore $F_a(p) > 1/2$ and $p < F_a(1/2)$. If n is the period of p, then $F_a(F_a^{n-1}(p)) = p + 1 < F_a(1/2) + 1 = F_a(1)$, so $F_a^{n-1}(p) < 1$. This proves that for $i = 1, 2, \dots, n$ the points $F_a^i(p)$ belong to the interval $(1/2, 1)$ on which F_a' is increasing. Therefore by Lemma 3.4 we get

$$(F_a^{n-2})'(F_a(p)) \le \frac{F_a^n(p) - F_a^{n-1}(p)}{F_a^2(p) - F_a(p)} \le (F_a^{n-2})'(F_a^2(p)). \tag{7.7}$$

To get estimates from both sides of $(F_a^{n-1})'(F_a(p))$ we need additionally the upper estimate of $F_a'(F_a^{n-1}(p))$ and the lower estimate of $F_a'(F_a(p))$. The first one is simple, because the maximal value of the derivative of F_a is 4. The second one requires the proof that $F_a(p)$ cannot be too close to $1/2$.

In the same way as (7.7), we get

$$(F_a^{n-3})'(F_a(x)) \le \frac{F_a^{n-1}(x) - F_a^{n-2}(x)}{F_a^2(x) - F_a(x)}$$

for every $x \in [1/2, F_a(p)]$. Since, as we noticed, $F_a' \leq 4$, we get

$$(F_a^{n-1})'(F_a(x)) \leq 16 \frac{F_a^{n-1}(x) - F_a^{n-2}(x)}{F_a^2(x) - F_a(x)}. \tag{7.8}$$

Since $[1/2, F_a(p)] \subset [1/2, F_a(1/2)]$, for a sufficiently close to a_I the values of of $F_a^2(x) - F_a(x)$ are uniformly (in a and x) bounded away from 0. Clearly, $F_a^{n-1}(x) - F_a^{n-2}(x)$ are uniformly bounded from above, so together with (7.8) we get that $(F_a^{n-1})'(F_a(x))$ is uniformly bounded from above. Therefore there is $\delta_1 > 0$ such that if a is sufficiently close to a_I and $F_a(p) < 1/2 + \delta_1$ then $(F_a^n)'(F_a(x)) < 1$ for all $x \in [1/2, F_a(p)]$. This means that $1/2$ is in the immediate basin of attraction of $F_a(p)$, which is impossible. Therefore we must have $F_a(p) \geq 1/2 + \delta_1$. Consequently, there is $\delta_2 > 0$ such that if a is sufficiently close to a_I then $F_a'(F_a(p)) > \delta_2$.

This estimate together with $F_a'(F_a^{n-1}(p)) \leq 4$ and (7.7) gives us

$$\delta_2 \frac{F_a^n(p) - F_a^{n-1}(p)}{F_a^2(p) - F_a(p)} \leq (F_a^{n-1})'(F_a(p)) \leq 4 \frac{F_a^n(p) - F_a^{n-1}(p)}{F_a^2(p) - F_a(p)} \tag{7.9}$$

The same type of estimates as in the preceding paragraph show that if a is sufficiently close to a_I then $F_a^n(p) - F_a^{n-1}(p)$ is uniformly bounded from above and $F_a^2(p) - F_a(p)$ is uniformly bounded away from 0. By this and by (7.9) we conclude that there are constants $K_3, K_4 > 0$ such that

$$K_3 \leq (F_a^{n-1})'(F_a(p)) \leq K_4. \tag{7.10}$$

This estimate holds for all a sufficiently close to a_I. This leaves out finitely many periods, and for each of them clearly an estimate of this type holds. Therefore, by changing constants K_3 and K_4, we get (7.10) for all orbits from $\mathcal{R}$.

Let us return to (7.6). As we said, Lemma 7.1 applies to the middle piece, so there are constants $K_1, K_2 > 0$ such that

$$K_1 (a - a_I)^{-3/2} < \frac{\partial f_a^{n-2k}}{\partial a}(f_a^k(p)) < K_2 (a - a_I)^{-3/2}.$$

As n goes to infinity, a approaches a_I, so $\partial f_a^k / \partial a (f_a^{n-k}(p))$, $(f_a^k)'(f_a^{n-k}(p))$ and $\partial f_a^k / \partial a(p)$ converge to continuous positive functions of p. Thus, they are bounded from above and bounded away from 0 by constants independent of n and p. Clearly, the considerations that led to (7.10) give the same results for $(f_a^{n-k})'(f_a^k(p))$. Taking all this into account, we get from (7.6)

$$K_5 + K_6 (a - a_I)^{-3/2} \leq \frac{\partial f_a^n}{\partial a}(p) \leq K_7 + K_8 (a - a_I)^{-3/2},$$

where K_5, K_6, K_7, K_8 are positive constants. Since $(a - a_I)^{-3/2}$ goes to infinity as $a \to a_I$, we conclude that there are constants $K_9, K_{10} > 0$ such that

$$K_9 (a - a_I)^{-3/2} \leq \frac{\partial f_a^n}{\partial a}(p) \leq K_{10} (a - a_I)^{-3/2}. \tag{7.11}$$

Although this does not appear explicitly in (7.10) and (7.11), the point p depends on a, so we will write $p(a)$ instead of p now. As for the case of mostly repelling attracting orbits, we have

$$p'(a) = \frac{\frac{\partial f_a^n}{\partial a}(p(a))}{1 - (f_a^n)'(p(a))} \tag{7.12}$$

(this is (6.5) without (6.4) plugged in).

As in Section 6, denote $\alpha(a) = (f_a^{n-1})'(f_a(p(a)))$ and for $P \in \mathcal{R}$ of period n consider P-windows $[a_1, a_2]$ and $[p_1, p_2]$ in the directions of a and p respectively. In view of (7.10), (6.8) holds in the present case, too, so in the same way as in Section6, we get (6.2). Applying (7.10) again, we conclude that there are constants $M_1, M_2 > 0$, independent of n, such that (7.1) holds.

In order to estimate the size of the P-window in the direction of a, we have to solve approximately (up to a multiplicative constant) (7.12). Observe that in the region where we want to solve it both the numerator and denominator of the right-hand side of (7.12) are positive. Therefore p is a strictly increasing function of a, and thus we can write a as a function of p, $a = a(p)$. We have $p(a_i) = a_i$ for $i = 1, 2$ and thus

$$\int_{p_1}^{p_2} (1 - (f_{a(p)}^n)'(p))\, dp = \int_{a_1}^{a_2} \frac{\partial f_a^n}{\partial a}(p(a))\, da. \tag{7.13}$$

Clearly, $1 - (f_{a(p)}^n)'(p) \leq 1$, so by (7.1) the left-hand side of (7.13) is bounded from above by M_2. On the other hand, $(f_{a(p)}^n)'(p) = \alpha(a(p)) f'_{a(p)}(p)$. By (7.10), $\alpha(a(p))$ is bounded from below by K_3 and from above by K_4. Moreover, $f'_{a(p)}(p) = 2(1 - \cos(2\pi(1/2 - p)))$. There is $s > 0$, independent of n, such that $|p - 1/2| < s$ then

$$1 - \cos(2\pi(1/2 - p)) < \frac{1}{2K_4}.$$

Then

$$(f_{a(p)}^n)'(p) = \alpha(a(p)) \cdot 2(1 - \cos(2\pi(1/2 - p))) < K_4 \cdot \frac{2}{2K_4} = 1,$$

so $[1/2 - s, 1/2 + s] \subset [p_1, p_2]$. Therefore

$$\int_{p_1}^{p_2} (1 - (f_{a(p)}^n)'(p))\, dp \geq \int_{1/2-s}^{1/2+s} K_3 \cdot 2(1 - \cos(2\pi(1/2 - p)))\, dp.$$

The right-hand side of the above equation is a positive constant, call it K_{11}, independent of n. Thus, we have

$$K_{11} \leq \int_{p_1}^{p_2} (1 - (f_{a(p)}^n)'(p))\, dp \leq M_2. \tag{7.14}$$

Let us consider the right-hand side of (7.13). By (7.11), we have

$$\begin{aligned}
2K_9((a_1 - a_I)^{-1/2} - (a_2 - a_I)^{-1/2}) &= \int_{a_1}^{a_2} K_9(a - a_I)^{-3/2}\, da \\
&\leq \int_{a_1}^{a_2} \frac{\partial f_a^n}{\partial a}(p(a))\, da \\
&\leq \int_{a_1}^{a_2} K_{10}(a - a_I)^{-3/2}\, da \\
&= 2K_{10}((a_1 - a_I)^{-1/2} - (a_2 - a_I)^{-1/2}).
\end{aligned}$$

Together with (7.13) and (7.14) this gives us the existence of constants $K_{12}, K_{13} > 0$, independent of n, such that

$$K_{12} \leq \frac{1}{\sqrt{a_1 - a_I}} - \frac{1}{\sqrt{a_2 - a_I}} \leq K_{13}. \tag{7.15}$$

Now we have to investigate the dependence between $a-a_I$ and n. Recall that P_n is the periodic orbit from $\mathcal{R}$ of period n and c_n is the value of the parameter a for which $1/2 \in P_n$. As a moves from c_{n+1} to c_n then F_a^n moves from $F_{c_{n+1}}^n(1/2) = F_{c_{n+1}}^n(p(c_{n+1}))$ to $F_{c_n}^n(1/2) = 1/2$, so the distance it covers is between $1/2$ and 1. The estimates that resulted in (7.11) hold also in this situation (maybe with slightly worse constants), so we get

$$K_{14}(a-a_I)^{-3/2} \le \frac{\partial f_a^n}{\partial a}(1/2) \le K_{15}(a-a_I)^{-3/2}$$

for all $a \in [c_{n+1}, c_n]$, where the constants $K_{14}, K_{15} > 0$ are independent of n. Therefore

$$1 \ge \int_{c_{n+1}}^{c_n} K_{14}(a-a_I)^{-3/2}\, da = 2K_{14}\left(\frac{1}{\sqrt{c_{n+1}-a_I}} - \frac{1}{\sqrt{c_n-a_I}}\right)$$

and

$$\frac{1}{2} \le \int_{c_{n+1}}^{c_n} K_{15}(a-a_I)^{-3/2}\, da = 2K_{15}\left(\frac{1}{\sqrt{c_{n+1}-a_I}} - \frac{1}{\sqrt{c_n-a_I}}\right).$$

Thus,

$$\frac{1}{4K_{15}} \le \frac{1}{\sqrt{c_{n+1}-a_I}} - \frac{1}{\sqrt{c_n-a_I}} \le \frac{1}{2K_{14}}. \tag{7.16}$$

Summing it from $n=1$ to $m-1$ we get

$$\frac{m-1}{4K_{15}} + \frac{1}{\sqrt{c_1-a_I}} \le \frac{1}{\sqrt{c_m-a_I}} \le \frac{m-1}{2K_{14}} + \frac{1}{\sqrt{c_1-a_I}}.$$

Therefore there exist constants $K_{16}, K_{17} > 0$, independent of n, such that

$$K_{16}n \le \frac{1}{\sqrt{c_n-a_I}} \le K_{17}n.$$

This gives us (7.3) with $M_5 = 1/K_{17}^2$ and $M_6 = 1/K_{16}^2$.

Now, by (7.15), (7.3), the identity

$$x-y = (x\sqrt{y}+y\sqrt{x})\left(\frac{1}{\sqrt{y}} - \frac{1}{\sqrt{x}}\right),$$

and since $c_{n+1} < a_1 < a_2 < c_{n-1}$, we conclude that there exist constants $M_3, M_4 > 0$, independent of n, such that (7.2) holds. $\square$

The scaling we obtained for orbits from $\mathcal{R}$ is completely different than the scaling for the orbits from $\mathcal{P}$. In particular, switching from the point p to its image q will not change this scaling.

8. Length of tongues

Despite a clear picture emerging from the numerical experiments, we do not know much about the shape of the tongues, except the tongue of period 1. We do not even know whether the tongues are connected. Therefore, since the bulk of our knowledge concerns the level $b=1$, it makes sense to define *proper tongues* as those components of the tongues that have non-empty intersection with the line $b=1$. By Propositions 4.3 and 4.4, the intersection of any tongue with the line $b=1$ is connected and nonempty, and therefore there is exactly one proper tongue of each type.

For the types considered in Section 6, we have enough information to estimate the length of the proper tongues. We measure the length of a tongue in the direction of b. The first result seems to confirm the conjecture that at a given level $b < 1$ there are only finitely many tongues.

Theorem 8.1. *Let s, t be periodic points of D with $1/16 < s < t < 1/8$. Then there exist constants $\lambda > 1$, $N > 0$ and $K_5 > 0$ such that any proper tongue of a type between s and t, period $n \geq N$, and such that the orbit of this type for some f_a belongs to $\mathcal{P}$, has length smaller than $K_5\lambda^{-n}$.*

Proof. There are a_s, a_t such that f_{a_s} has a periodic attracting orbit of type s and f_{a_t} has a periodic attracting orbit of type t. If ε is sufficiently small, then for any $b \in [1-\varepsilon, 1]$ the map $f_{a_s,b}$ has a periodic attracting orbit of type s and $f_{a_t,b}$ has a periodic attracting orbit of type t. Since tongues are pairwise disjoint, the proper tongue of any type $r \in (s,t)$ intersected with the set $[0,1) \times [1-\varepsilon, 1]$ is contained in $(a_s, a_t) \times [1-\varepsilon, 1]$.

If ε is sufficiently small and $b \in [1-\varepsilon, 1]$, the maps $f_{a,b}$ are uniformly close to the maps f_a. Therefore for the orbits of the types described in the statement of the theorem, the same estimates for the derivatives (with respect to x) as in Section 6 hold, perhaps with slightly smaller λ and c. Therefore, using the notation of that section, we get

$$(f_{a,b}^{n-1})'(f_{a,b}(p)) \geq c\lambda^{n-1}.$$

On the other hand, the minimum of $f'_{a,b}$ occurs at $1/2$ and is equal to $2-2b$. If our orbit is attracting, we get $(2-2b)c\lambda^{n-1} < 1$, so $1-b < (\lambda/(2c))\lambda^{-n}$. Thus, if $K_5\lambda^{-n} < \varepsilon$, where $K_5 = \lambda/(2c)$, we see that the length of the proper tongue that we consider is smaller than $K_5\lambda^{-n}$. To complete the proof, we note that there exists N such that if $n \geq N$ then $K_5\lambda^{-n} < \varepsilon$. □

Let us now consider periodic orbits of the types considered in Section 7. Here we will see that if $b < 1$ is sufficiently close to 1 then there are infinitely many tongues at that level. Once we know where they are situated, we can produce a picture showing them (see Figure 11). Let us remark that a straightforward method used to detect attracting periodic orbits does not work well here, since a point that moves only slightly due to intermittency may be mistaken for a fixed point.

Theorem 8.2. *There exists a constant $L > 0$ such that any proper tongue such that the orbit of this type for some f_a belongs to $\mathcal{R}$, has length larger than L.*

Proof. As we noticed in Section 7, the type of an orbit from the statement of the theorem is $1/(2^n-1)$ if its period is n. For each value of b and each n there exists a unique value, $a(b,n)$ of a, such that for $f_{a(b,n),b}$ the point $1/2$ is periodic and $\Phi_{a(b,n),b}(1/2) = 1/(2^n-1)$. Clearly, if n is fixed, then $a(b,n)$ depends continuously on b.

If b is sufficiently close to 1, then the inequality (7.7) with p replaced by $1/2$ and F_a replaced by $F_{a(b,n),b}$ can be proved in exactly the same way as in Section 7. The upper estimate of the derivative of $F_{a(b,n),b}$ by 4 still holds. Therefore we get an analogue of the estimate from (7.10),

$$(F_{a(b,n),b}^{n-1})'(F_{a(b,n),b}(1/2)) \leq K_4$$

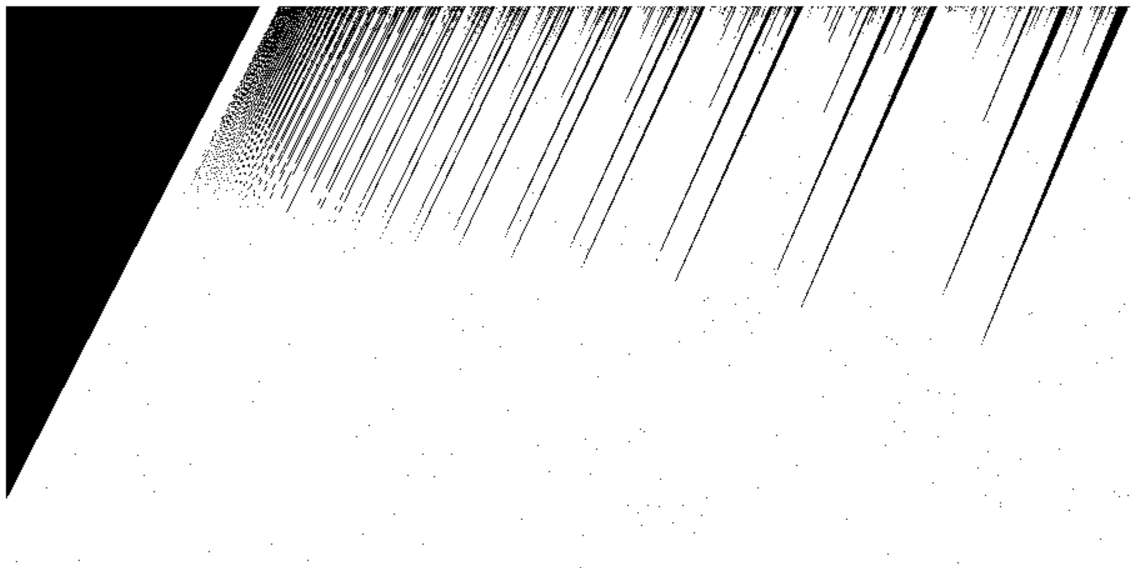

FIGURE 11. Tongues of period 50 or less in the intermittent region, $0.6 \leq a \leq 0.64$, $0.96 \leq b \leq 1$

with the value of K_4 possibly changed (but independent of n and of b provided b is sufficiently close to 1). Now, if b is sufficiently close to 1 then $F'_{a(b,n),b}(1/2) < 1/K_4$, so

$$(F^n_{a(b,n),b})'(F_{a(b,n),b}(1/2)) \leq 1.$$

This proves that $(a(b,n),b)$ belongs to the tongue of type $1/(2^n-1)$. The estimates on how close b should be to 1 are independent of n and since $a(b,n)$ depends continuously on b, this is the proper tongue. This completes the proof. □

References

[1] V. I. Arnold, *Small denominators, I. Mappings of the circumference onto itself*, Amer. Math. Soc. Translations **46** (1965), 213–284.

[2] W. Bergweiler, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc. **29** (1993), 151–188.

[3] A. Blokh, C. Cleveland and M. Misiurewicz, *Expanding polymodials*, in "Modern Dynamical Systems and Applications", eds. M. Brin, B. Hasselblatt and Ya. Pesin, Cambridge University Press, Cambridge, 2004, pp. 253–270.

[4] A. Blokh, C. Cleveland and M. Misiurewicz, *Julia sets of expanding polymodials*, Ergodic Theory Dynam. Systems **25** (2005), 1691–1718.

[5] A. Blokh and M. Misiurewicz, *Branched derivatives*, Nonlinearity **18** (2005), 703–715.

[6] A. Epstein, L. Keen and C. Tresser, *The set of maps $F_{a,b} : x \mapsto x + a + \frac{b}{2\pi}\sin(2\pi x)$ with any given rotation interval is contractible*, Comm. Math. Phys. **173** (1995), 313–333.

[7] L. B. Jonker, *The scaling of Arnold's tongues for differentiable homeomorphisms of the circle*, Commun. Math. Phys. **129** (1990), 1–25.

[8] G. Levin and G. Świątek, *Universality of critical circle covers*, Comm. Math. Phys. **228** (2002), 371–399.

[9] W. de Melo and S. van Strien, "One-dimensional dynamics", Springer-Verlag, Berlin, 1993

[10] S. E. Newhouse, J. Palis and F. Takens, *Bifurcations and stability of families of diffeomorphisms*, Publ. Math. IHES **57** (1983), 5–71.

[11] M. Shub and D. Sullivan, *Expanding endomorphisms of the circle revisited*, Ergodic Theory Dynam. Systems **5** (1985), 285–289.

[12] W. Wenzel, O. Biham and C. Jayaprakash, *Periodic orbits in the dissipative standard map*, Phys. Rev. A **43** (1991), 6550–6557.

Department of Mathematical Sciences, IUPUI, 402 N. Blackford Street, Indianapolis, IN 46202-3216
E-mail address: `mmisiure@math.iupui.edu`

Departamento de Matemática Pura, Centro de Matemática[1], Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal
E-mail address: `ana.rodrigues@fc.up.pt`

[1]CMUP is supported by FCT through POCTI and POSI of Quadro Comunitário de apoio III (2000-2006) with FEDER and national funding