Convolutions related to the Fourier and Kontorovich-Lebedev transforms revisited

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We continue to investigate boundedness properties in a two-parametric family of Lebesgue spaces for convolutions related to the Fourier and Kontorovich-Lebedev transforms. Norm estimations in the weighted L_p - spaces are obtained and applications to the corresponding class of convolution integral equations are demonstrated. Necessary and sufficient conditions are found for the solvability of these equations in the weighted L_2-spaces.

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1 Introduction

This paper is a continuation of our investigation of convolution operators, given recently in [6] for the Fourier cosine and Kontorovich-Lebedev transformations [3, 4, 5]

$$(F_c f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\cos xt dt, \tag{1}$$

$$K_{ix}[f] = \int_0^\infty K_{ix}(t) f(t) dt, \tag{2}$$

where $K_{ix}(t)$ is the modified Bessel function [1], Vol. 2. We will involve here the Fourier sine transform

$$(F_s f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\sin xt dt, \tag{3}$$

and base on the following formula [2, relation(2.16.48.19)] (see also (13))

$$\int_0^\infty \cos bx K_{ix}(t) dx = \frac{\pi}{2}\exp(-t\cosh b). \tag{4}$$

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Throughout the paper we will deal with a two-parametric family of Lebesgue spaces

$$L_p^{\alpha,\beta} \equiv L_p(\mathbb{R}_+;\ K_0(\beta t)t^\alpha dt),\ \alpha \in \mathbb{R},\ 0 < \beta \le 1, \tag{5}$$

introduced in [6] and normed by

$$||f||_{L_p^{\alpha,\beta}} = \left(\int_0^\infty |f(t)|^p\ K_0(\beta t)\ t^\alpha dt \right)^{1/p} < \infty. \tag{6}$$

It is widely known, that Fourier transforms (1), (3) are well-defined on the space $L_1(\mathbb{R}_+; dt)$. Moreover, if $g(x) = (F_c f)(x) \in L_1(\mathbb{R}_+; dx)$ or $g(x) = (F_s f)(x) \in L_1(\mathbb{R}_+; dx)$ we have reciprocal inversion formulas $f(x) = (F_c g)(x)$, $f(x) = (F_s g)(x)$. In the case of $L_2(\mathbb{R}_+; dt)$- space we should define the cosine and sine Fourier transforms in the mean-square convergence sense, namely

$$(F_{\left\{ {c \atop s} \right\}} f)(x) = \underset{N\to\infty}{\text{l.i.m.}} \sqrt{\frac{2}{\pi}} \int_{1/N}^{N} f(t) \begin{Bmatrix} \cos xt \\ \sin xt \end{Bmatrix}\ dt, \tag{7}$$

and familiar Plancherel's theorem [5], [4], [3] says that $F_c, F_s : L_2(\mathbb{R}_+; dt) \to L_2(\mathbb{R}_+; dt)$ are isometric isomorphisms with reciprocal inversion formulas

$$f(x) = \underset{N\to\infty}{\text{l.i.m.}} \sqrt{\frac{2}{\pi}} \int_{1/N}^{N} (F_{\left\{ {c \atop s} \right\}} f)(t) \begin{Bmatrix} \cos xt \\ \sin xt \end{Bmatrix} dt, \tag{8}$$

and Parseval's equalities

$$||F_{\left\{ {c \atop s} \right\}} f||_{L_2(\mathbb{R}_+;dt)} = ||f||_{L_2(\mathbb{R}_+;dt)}. \tag{9}$$

The Kontorovich-Lebedev operator (2), in turn, is an isometric isomorphism (see [5]) $K_{ix} : L_2(\mathbb{R}_+; tdt) \to L_2(\mathbb{R}_+; x \sinh \pi x dx)$, where integral (2) in general, does not exist in Lebesgue's sense and we understand it in the form

$$K_{ix}[f] = \underset{N\to\infty}{\text{l.i.m.}} \int_{1/N}^{\infty} K_{ix}(t) f(t) dt, \tag{10}$$

where the limit is taken in the mean-square sense with respect to the norm of the space $L_2(\mathbb{R}_+; x \sinh \pi x dx)$. Moreover, the Parseval identity

$$\frac{2}{\pi^2} \int_0^\infty x \sinh \pi x |K_{ix}[f]|^2 dx = \int_0^\infty |f(t)|^2 t dt \tag{11}$$

holds and the inverse operator is defined by the formula

$$f(t) = \underset{N\to\infty}{\text{l.i.m.}} \frac{2}{\pi^2} \int_0^N x \sinh \pi x \frac{K_{ix}(t)}{t} K_{ix}[f] dx, \tag{12}$$

where the convergence is in mean-square with respect to the norm of $L_2(\mathbb{R}_+; tdt)$.

As it is known [3], the modified Bessel function $K_{ix}(t)$ can be represented by the Fourier integral

$$K_{ix}(t) = \int_0^\infty e^{-t\cosh u} \cos xu du, \ t > 0. \tag{13}$$

Hence, when $x \in \mathbb{R}$, it is real-valued and even with respect to the pure imaginary index ix. Furthermore, this integral can be extended to the strip $\delta \in [0, \pi/2)$ in the upper half-plane, i.e.

$$K_{ix}(t) = \frac{1}{2} \int_{i\delta-\infty}^{i\delta+\infty} e^{-t\cosh z + ixz} dz$$

and leads for each $t > 0$ to a uniform estimate

$$|K_{ix}(t)| \le e^{-|x|\arccos\beta} K_0(\beta t), \quad 0 < \beta \le 1, \tag{14}$$

which will be used in the sequel. We note also its asymptotic behaviour [1] at infinity

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}[1 + O(1/z)], \quad z \to \infty, \tag{15}$$

and hear the origin

$$\begin{aligned} z^\nu K_\nu(z) &= 2^{\nu-1}\Gamma(\nu) + o(1), \quad z \to 0, \\ K_0(z) &= -\log z + O(1), \quad z \to 0. \end{aligned} \tag{16}$$

Concerning the properties of spaces $L_p^{\alpha,\beta}$ (5), various embeddings were established in [6]. In particular, for $p = 1, 2$, we have

$$\begin{aligned} ||f||_{L_1^{\alpha,1}} &\le C_\alpha ||f||_{L_2(\mathbb{R}_+;\ dt)}, \\ & \quad C_\alpha = \frac{\pi^{1/4}\,\Gamma^{3/2}(\alpha+1/2)}{2\,\Gamma^{1/2}(\alpha+1)},\ \alpha > -1/2, \end{aligned} \tag{17}$$

$$\begin{aligned} ||f||_{L_1^{\alpha,1}} &\le C_{\alpha,\beta} ||f||_{L_2(\mathbb{R}_+; t^\beta\ dt)}, \\ & \quad C_{\alpha,\beta} = \frac{2^{\alpha-1-\beta/2}\Gamma^2(\alpha + \frac{1-\beta}{2})}{\Gamma^{1/2}(2\alpha-\beta+1)},\ \beta < 2\alpha+1, \end{aligned} \tag{18}$$

$$\begin{aligned} ||f||_{L_1^{\alpha,1}} &\le C_\alpha ||f||_{L_2^{0,1}}, \\ & \quad C_\alpha = 2^{\alpha-1/2}\Gamma(\alpha+1/2),\ \alpha > -1/2, \end{aligned} \tag{19}$$

where $\Gamma(z)$ is Euler's gamma-function [1], Vol. 1. Since $t^\alpha K_0(\beta t)$ is bounded when $\alpha > 0,\ 0 < \beta \le 1$ we arrive at the embedding $L_p(\mathbb{R}_+;\ dt) \subset L_p^{\alpha,\beta},\ 1 \le p < \infty$.

This paper is devoted to commutative convolutions of the form

$$(f * g)_{\left\{{1 \atop 2}\right\}}(x) = \frac{1}{2\pi x} \int_{\mathbb{R}_+^2} f(\tau)g(\theta) \left[e^{-x\cosh(\tau-\theta)} \pm e^{-x\cosh(\tau+\theta)}\right] d\tau d\theta,\ x > 0, \tag{20}$$

and to the noncommutative convolutions

$$(f * g)_{\left\{{3 \atop 4}\right\}}(x) = \frac{1}{2\pi x} \int_{\mathbb{R}_+^2} f(\tau)g(\theta) \left[e^{-\theta\cosh(\tau-x)} \pm e^{-\theta\cosh(\tau+x)}\right] d\tau d\theta,\ x > 0. \tag{21}$$

In particular we will get mapping properties of (20), (21) in spaces (5), and we will establish the following factorization equalities (see (1), (3), (2))

$$K_{ix}[(f*g)_{\{ {1 \atop 2} \}}] = \frac{\pi}{2}\frac{(F_{\{ {c \atop s} \}}f)(x)(F_{\{ {c \atop s} \}}g)(x)}{x\sinh \pi x}, \quad x>0, \tag{22}$$

$$(F_{\{ {c \atop s} \}}(f*g)_{\{ {3 \atop 4} \}})(x) = (F_{\{ {c \atop s} \}}f)(x)K_{ix}[g], \ x>0. \tag{23}$$

Finally, we will apply these results to consider a solvability of the first and second kind convolution integral equations related to operators (20).

2 Mapping properties of convolutions (20)

Denoting by $C_0(\mathbb{R}_+)$ the space of bounded continuous functions vanishing at infinity we start this section considering the Kontorovich-Lebedev transformation (2) $K_{ix} : L_p^{\alpha,\beta} \to C_0(\mathbb{R}_+),\ 1<p<\infty$. The following theorem is proved in [6].

Theorem 1 *The Kontorovich-Lebedev transformation* (2) $K_{ix} : L_p^{\alpha,\beta} \to C_0(\mathbb{R}_+),\ 1<p<\infty,\ \alpha<p-1,\ 0<\beta\le 1$ *is well-defined continuous, linear map with the norm at most* $C_{\alpha,\beta,p}$, *i.e.*

$$||K_{ix}|| \le C_{\alpha,\beta,p} = (2\beta)^{\frac{1-p}{p}}\left(\frac{\beta}{2}\right)^{\frac{\alpha}{p}}\Gamma^{\frac{2(p-1)}{p}}\left(\frac{p-\alpha-1}{2(p-1)}\right). \tag{24}$$

In particular, for the map $K_{ix} : L_p^{0,\beta} \to C_0(\mathbb{R}_+)$ *we have* $||K_{ix}|| \le \left(\frac{2\beta}{\pi}\right)^{\frac{1-p}{p}}$. *Finally, when* $\beta=1$ *it has the exact value* $||K_{ix}|| = \left(\frac{\pi}{2}\right)^{\frac{p-1}{p}}$.

For convolutions (20) we have

Theorem 2 *Let* $f(\tau),\ g(\tau) \in L_1(\mathbb{R}_+; d\tau)$. *Convolutions* (20) *are well-defined for all* $t>0$ *as continuous functions and belong to* $L_p^{\alpha,\beta}$ *with* $\alpha>p-1,\ 0<\beta\le 1,\ 1\le p<\infty$. *Moreover,*

$$||(f*g)_{\{ {1 \atop 2} \}}||_{L_p^{\alpha,\beta}} \le A_{\alpha,p,\beta}||f||_{L_1(\mathbb{R}_+;dt)}||g||_{L_1(\mathbb{R}_+;dt)}, \tag{25}$$

where (see relation (2.16.6.3) in [2])

$$\begin{aligned} A_{\alpha,p,\beta} &= \frac{1}{\pi}\left(\int_0^\infty t^{\alpha-p}e^{-pt}K_0(\beta t)dt\right)^{1/p} \\ &= \pi^{1/(2p)-1}(2p)^{1-\frac{\alpha+1}{p}}\left(\frac{\Gamma^2(\alpha-p+1)}{\Gamma(\alpha-p+3/2)}\right)^{1/p} \\ &\times \left[{}_2F_1\left(\frac{\alpha-p+1}{2},\ \frac{\alpha-p}{2}+1; \alpha-p+3/2; 1-\frac{\beta^2}{p^2}\right)\right]^{1/p}. \end{aligned} \tag{26}$$

In particular, when $\beta=1,\ p=1$ *it has*

$$||(f*g)_{\{ {1 \atop 2} \}}||_{L_1^{\alpha,1}} \le C_\alpha||f||_{L_1(\mathbb{R}_+;dt)}||g||_{L_1(\mathbb{R}_+;dt)}, \tag{27}$$

with

$$C_\alpha = \frac{1}{2^\alpha \sqrt{\pi}} \frac{\Gamma^2(\alpha)}{\Gamma(\alpha + 1/2)}. \tag{28}$$

Besides, generalized Parseval type equalities hold

$$(f * g)_{\{^1_2\}}(t) = \frac{1}{\pi t} \int_0^\infty (F_{\{^c_s\}} f)(x)(F_{\{^c_s\}} g)(x) K_{ix}(t) dx, \quad t > 0, \tag{29}$$

and finally, assuming that $x^{-1}(F_{\{^c_s\}} g)(x) \in L_2((0,1); dx)$*, factorization equalities* (22) *take place, where the Kontorovich-Lebedev operator in its left-hand side is understood by* (10).

Proof. A straightforward estimation of convolutions (20) for all $t > 0$ gives

$$|(f * g)_{\{^1_2\}}(t)| \;\leq\; \frac{e^{-t}}{\pi t} \int_{\mathbb{R}^2_+} |f(\tau)||g(\theta)| d\tau d\theta$$

or

$$|(f * g)_{\{^1_2\}}(t)| \leq \frac{e^{-t}}{\pi t} ||f||_{L_1(\mathbb{R}_+; dt)} ||g||_{L_1(\mathbb{R}_+; dt)}. \tag{30}$$

Hence integral (20) exists for all $t > 0$ and represents a continuous function via its absolute and uniform convergence when $t \geq t_0 > 0$. Furthermore, taking the norm (6) through the latter inequality (30) we immediately obtain (25), where integral (26) is convergent under conditions $\alpha > p - 1,\ 1 \leq p < \infty,\ 0 < \beta \leq 1$ and the result is written in terms of the Gauss hypergeometric function ${}_2F_1(a, b, c; z)$ (see [1], Vol. 1). Letting $p = 1,\ \beta = 1$ it gives (25), (28). Employing again integral representation (4) and appealing to Fubini's theorem we write convolutions (20) in the form

$$\begin{aligned}
(f * g)_{\{^1_2\}}(t) &= \frac{2}{\pi^2 t} \int_{\mathbb{R}^3_+} f(\tau) g(\theta) \begin{Bmatrix} \cos x\tau \cos x\theta \\ \sin x\tau \sin x\theta \end{Bmatrix} K_{ix}(t) d\tau d\theta dx \\
&= \frac{2}{\pi^2 t} \int_0^\infty \left(\int_0^\infty f(\tau) \begin{Bmatrix} \cos x\tau \\ \sin x\tau \end{Bmatrix} d\tau \right) \left(\int_0^\infty g(\theta) \begin{Bmatrix} \cos x\theta \\ \sin x\theta \end{Bmatrix} d\theta \right) K_{ix}(t) dx \\
&= \frac{1}{\pi t} \int_0^\infty (F_{\{^c_s\}} f)(x)(F_{\{^c_s\}} g)(x) K_{ix}(t) dx,
\end{aligned} \tag{31}$$

which proves the generalized Parseval equalities (29). In order to prove (22) we assume first that

$$x^{-1}(F_{\{^c_s\}} g)(x) \in L_2((0,1); dx).$$

Hence it is not difficult to verify that the right-hand side of (22) belongs to $L_2(\mathbb{R}_+; x \sinh \pi x dx)$. Therefore by virtue of the Parseval identity (11) and reciprocities (10), (12) (see also (29)) we get (22), where the integral in the left-hand side is convergent generally in L_2-sense as in (10). Theorem 2 is proved.

Remark 1 *It is not difficult to show by using an elementary inequality* $|\sin x| \leq x,\ x > 0$ *that condition* $x^{-1}(F_s g)(x) \in L_2((0,1); dx)$ *is satisfied when, for instance,* $g(\tau) \in L_1(\mathbb{R}_+; \tau d\tau)$.

Remark 2 *Analogously (see* (30)*) we can obtain the following estimation for the norm of convolutions (20)*

$$\left|\left|(f*g)_{\{^1_2\}}\right|\right|_{L_1(\mathbb{R}_+;t^\alpha dt)} \quad \leq \quad \frac{\Gamma(\alpha)}{\pi}||f||_{L_1(\mathbb{R}_+;dt)}||g||_{L_1(\mathbb{R}_+;dt)}, \ \alpha > 0. \tag{32}$$

In particular,

$$\left|\left|(f*g)_{\{^1_2\}}\right|\right|_{L_1(\mathbb{R}_+;tdt)} \leq \frac{1}{\pi}||f||_{L_1(\mathbb{R}_+;dt)}||g||_{L_1(\mathbb{R}_+;dt)}.$$

Similar to Theorem 2 we prove

Theorem 3 *Let* $f(\tau) \in L_2(\mathbb{R}_+; d\tau)$ *and* $g(\theta) \in L_1(\mathbb{R}_+; d\theta)$ *and* $x^{-1}(F_{\{^c_s\}}g)(x) = O(1), \ x \to 0.$ *Then convolutions* (20) *are well-defined as continuous functions on* $\mathbb{R}_+$. *Moreover,*

$$\left|\left|(f*g)_{\{^1_2\}}\right|\right|_{L_2(\mathbb{R}_+;t^\alpha dt)} \leq C_\alpha||f||_{L_2(\mathbb{R}_+;d\tau)}||g||_{L_1(\mathbb{R}_+;d\tau)}, \tag{33}$$

where

$$C_\alpha = \frac{\Gamma(\alpha-1)}{2^{\frac{\alpha-1}{2}}\pi^{3/4}\Gamma^{1/2}(\alpha-1/2)}, \ \alpha > 1. \tag{34}$$

Finally, Parseval's type equalities (29) *and factorization properties* (22) *are valid.*

Proof. Integration with respect to θ (see (13)) yields

$$\begin{aligned}
\frac{1}{2}\int_0^\infty \left[e^{-x\cosh(\tau-\theta)} \pm e^{-x\cosh(\tau+\theta)}\right] d\theta \quad &< \quad \frac{1}{2}\int_\tau^\infty e^{-x\cosh y}dy + \frac{1}{2}\int_{-\tau}^\infty e^{-x\cosh y}dy \\
&= \quad \frac{1}{2}\int_\tau^\infty e^{-x\cosh y}dy + \frac{1}{2}\int_{-\infty}^\tau e^{-x\cosh y}dy \\
&= \quad \frac{1}{2}\int_{-\infty}^\infty e^{-x\cosh y}dy = \int_0^\infty e^{-x\cosh y}dy = K_0(x).
\end{aligned} \tag{35}$$

Calling Schwarz's inequality for double integrals we deduce

$$\begin{aligned}
\left|(f*g)_{\{^1_2\}}(t)\right| \quad &\leq \quad \frac{1}{2\pi t}\int_{\mathbb{R}^2_+} |f(\tau)||g(\theta)| \left[e^{-x\cosh(\tau-\theta)} \pm e^{-x\cosh(\tau+\theta)}\right] d\tau d\theta \\
&\leq \quad \frac{1}{2\pi t}\left(\int_{\mathbb{R}^2_+} |g(\theta)| \left[e^{-x\cosh(\tau-\theta)} \pm e^{-x\cosh(\tau+\theta)}\right] d\tau d\theta\right)^{1/2} \\
&\times \quad \left(\int_{\mathbb{R}^2_+} |g(\theta)||f(\tau)|^2 \left[e^{-x\cosh(\tau-\theta)} \pm e^{-x\cosh(\tau+\theta)}\right] d\tau d\theta\right)^{1/2} \\
&\leq \quad \frac{1}{\pi t}\left(K_0(t)\int_0^\infty |g(\theta)|d\theta\right)^{1/2} \left(e^{-t}\int_{\mathbb{R}^2_+} |g(\theta)||f(\tau)|^2 d\tau d\theta\right)^{1/2} \\
&= \quad \frac{e^{-t/2}K_0^{1/2}(t)}{\pi t}||f||_{L_2(\mathbb{R}_+;\tau)}||g||_{L_1(\mathbb{R}_+;d\tau)}.
\end{aligned}$$

Thus convolutions (20) exist for all $t > 0$ and are continuous via the absolute and uniform convergence of the corresponding integrals. Estimates (33) follow directly due to the definition of the norm in $L_2(\mathbb{R}_+; t^\alpha dt)$ and relation (2.16.6.4) in [2]. Parseval's type equalities (29) can be proved in the same manner as in Theorem 2.

Since $g \in L_1(\mathbb{R}_+; dt)$ and $x^{-1}(F_{\{{c \atop s}\}}g)(x) = O(1),\ x \to 0$, it means that $\dfrac{|(F_{\{{c \atop s}\}}g)(x)|^2}{x \sinh \pi x}$ is bounded and

$$\frac{(F_{\{{c \atop s}\}}f)(x)(F_{\{{c \atop s}\}}g)(x)}{x \sinh \pi x} \in L_2(\mathbb{R}_+; x \sinh \pi x dx).$$

Consequently, with reciprocities (10), (12) for the Kontorovich- Lebedev transformation we come again to factorization properties (22) and complete the proof of Theorem 3.

As a consequence of (33) and (18) we easily derive

$$\left|\left|(f*g)_{\{{1 \atop 2}\}}\right|\right|_{L_1^{\alpha,1}} \leq C_\alpha ||f||_{L_2(\mathbb{R}_+;d\tau)} ||g||_{L_1(\mathbb{R}_+;d\tau)}, \tag{36}$$

where the constant C_α is given accordingly

$$C_\alpha = \frac{\Gamma^2((\alpha+1)/2)\Gamma(\alpha-1)}{\sqrt{2}\pi^{3/4}\Gamma^{1/2}(\alpha+1)\Gamma^{1/2}(\alpha-1/2)},\ \alpha > 1. \tag{37}$$

Corollary 1 *Under conditions of Theorems 2 or 3 convolutions $(f*g)_{\{{1 \atop 2}\}}(t)$ belong to the space $L_2(\mathbb{R}_+; tdt)$ and the following identities hold*

$$\int_0^\infty \left|(f*g)_{\{{1 \atop 2}\}}(t)\right|^2 tdt = \frac{1}{2}\int_0^\infty \left|(F_{\{{c \atop s}\}}f)(x)(F_{\{{c \atop s}\}}g)(x)\right|^2 \frac{dx}{x \sinh \pi x}.$$

In fact, this is a direct consequence of factorization properties (22) and Parseval's equality (11).

Theorem 4 *Let $f,\ g \in L_2(\mathbb{R}_+; d\tau)$. Then convolutions* (20) *are well-defined as continuous functions on $\mathbb{R}_+$. Moreover,*

$$||(f*g)_{\{{1 \atop 2}\}}||_{L_2(\mathbb{R}_+;t^\alpha dt)} \leq C_\alpha ||f||_{L_2(\mathbb{R}_+;dt)} ||g||_{L_2(\mathbb{R}_+;dt)}, \tag{38}$$

where

$$C_\alpha = \frac{2^{\alpha/2-2}\Gamma^2((\alpha-1)/2)}{\Gamma^{1/2}(\alpha-1)},\ \alpha > 1. \tag{39}$$

Finally, Parseval's type equalities (29) *are valid.*

Proof. Indeed, calling again Schwarz's inequality and using (35) we find

$$\begin{aligned} \left|(f*g)_{\{{1 \atop 2}\}}(t)\right|^2 &\leq \frac{K_0^2(t)}{\pi^2 t^2}\int_0^\infty |f(\tau)|^2 d\tau \int_0^\infty |g(\theta)|^2 d\theta \\ &= \frac{K_0^2(t)}{\pi^2 t^2} ||f||^2_{L_2(\mathbb{R}_+;dt)} ||g||^2_{L_2(\mathbb{R}_+;dt)}. \end{aligned}$$

Hence

$$\left|\left|(f*g)_{\{{1 \atop 2}\}}\right|\right|_{L_2(\mathbb{R}_+;t^\alpha dt)} \leq \frac{1}{\pi}||f||_{L_2(\mathbb{R}_+;dt)} ||g||_{L_2(\mathbb{R}_+;dt)} \left(\int_0^\infty t^{\alpha-2} K_0^2(t) dt\right)^{1/2}.$$

Applying (2.16.33.2) from [2] we obtain the estimation (38) for the norm of convolutions (20). Employing again integral representation (4) and appealing to Fubini's theorem we represent convolutions (20) by the latter equality in (31) and we establish (29) completing the proof of Theorem 4.

3 On the noncommutative convolutions (21)

In this section we will prove similar results of the norm estimations, Parseval type equalities and factorization identities, which are associated with the noncommutative convolutions (21). We start with the boundedness of these convolutions on the space $L_1(\mathbb{R}_+; dt)$.

Theorem 5 *Let $f \in L_1(\mathbb{R}_+; dt)$ and $g \in L_1^{0,\beta},\ 0 < \beta \leq 1$. Then convolutions* (21) *exist for almost all $t > 0$, belong to $L_1(\mathbb{R}_+; dt)$ and*

$$\left|\left|(f*g)_{\left\{ {3 \atop 4} \right\}}\right|\right|_{L_1(\mathbb{R}_+;\ dt)} \leq ||f||_{L_1(\mathbb{R}_+;\ dt)}||g||_{L_1^{0,\beta}}. \tag{40}$$

Moreover, factorization identities (23) *hold true. Further, when $\beta \in (0,1)$, then $(f*g)_{\left\{ {3 \atop 4} \right\}}(t) \in C_0(\mathbb{R}_+)$ and for all $t > 0$ the Parseval type equalities take place*

$$(f*g)_{\left\{ {3 \atop 4} \right\}}(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty (F_{\left\{ {c \atop s} \right\}} f)(x) K_{ix}[g] \begin{Bmatrix} \cos xt \\ \sin xt \end{Bmatrix} dx. \tag{41}$$

Proof. With Fubini's theorem via the estimate (see (35)) we derive

$$\int_0^\infty \left|(f * g)_{\left\{ {3 \atop 4} \right\}}(t)\right| dt \ \leq \int_{\mathbb{R}_+^2} |f(\tau)g(\theta)| K_0(\theta) d\tau d\theta$$

$$\leq \int_{\mathbb{R}_+^2} |f(\tau)g(\theta)| K_0(\beta\theta) d\tau d\theta = ||f||_{L_1(\mathbb{R}_+;\ dt)}||g||_{L_1^{0,\beta}} < \infty. \tag{42}$$

Hence it proves (40) and the existence of convolutions for almost all $t > 0$. Again with Fubini's theorem we prove factorization identities (23), taking the cosine and the sine Fourier transforms (1), (3) of convolutions (21), respectively. Hence we change the order of integration by virtue of (42) and appealing to (13). Supposing now that $\beta \in (0,1)$, we employ representation (4) to substitute in (21), and inequality (14) in order to justify the absolute convergence of the corresponding iterated integrals. Thus via Fubini's theorem we come out with the following chain of equalities

$$\begin{aligned}
(f*g)_{\left\{ {3 \atop 4} \right\}}(t) &= \frac{2}{\pi} \int_{\mathbb{R}_+^3} f(\tau) g(\theta) \begin{Bmatrix} \cos xt \cos x\tau \\ \sin xt \sin x\tau \end{Bmatrix} K_{ix}(\theta)\, d\tau d\theta dx \\
&= \frac{2}{\pi} \int_0^\infty \left(\int_0^\infty f(\tau) \begin{Bmatrix} \cos xt \cos x\tau \\ \sin xt \sin x\tau \end{Bmatrix} d\tau \right) \left(\int_0^\infty g(\theta) K_{ix}(\theta) d\theta \right) \begin{Bmatrix} \cos xt \\ \sin xt \end{Bmatrix} dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty (F_{\left\{ {c \atop s} \right\}} f)(x) K_{ix}[g] \begin{Bmatrix} \cos xt \\ \sin xt \end{Bmatrix} dx,
\end{aligned}$$

which proves (41). Meanwhile, latter integrals with respect to x are absolutely and uniformly convergent and vanish when $t \to \infty$ due to the Riemann-Lebesgue lemma. Consequently, $(f*g)_{\left\{ {3 \atop 4} \right\}}(t) \in C_0(\mathbb{R}_+)$. Theorem 5 is proved.

Extensions of Theorem 5 on spaces $L_p^{0,\beta},\ 1 < p < \infty,\ 0 < \beta \leq 1$ and $L_p(\mathbb{R}_+; dt)$ are given by

Theorem 6 *Let* $f \in L_p(\mathbb{R}_+; dt),\ g \in L_{p'}^{0,\beta},\ 1 < p < \infty,\ p^{-1} + p'^{-1} = 1,\ 0 < \beta \le 1$. *Then convolutions* (21) *exist for all* $t > 0$ *as bounded continuous functions. Moreover,* $(f{*}g)_{\{^3_4\}} \in L_r^{\alpha,\gamma},\ 1 \le r < \infty,\ \alpha > -1,\ 0 < \gamma \le 1$, *where* r *and* p, β *and* γ *have no dependence and*

$$\left|\left|(f{*}g)_{\{^3_4\}}\right|\right|_{L_r^{\alpha,\gamma}} \le C_{\alpha,\gamma}^{1/r} ||f||_{L_p(\mathbb{R}_+;dt)} ||g||_{L_{p'}^{0,\beta}}, \tag{43}$$

where

$$C_{\alpha,\gamma} = (2\gamma)^{-1} \left(\frac{2}{\gamma}\right)^{\alpha} \Gamma^2 \left(\frac{\alpha+1}{2}\right). \tag{44}$$

If we assume $f \in L_1(\mathbb{R}_+; dt) \cap L_p(\mathbb{R}_+; dt),\ 1 < p < \infty$, *then convolutions* (21) *satisfy factorization identities* (23). *Further, when* $\beta \in (0,1)$, *then* $(f{*}g)_{\{^3_4\}}(t) \in C_0(\mathbb{R}_+)$ *and for all* $t > 0$ *Parseval's type equalities* (41) *hold.*

Proof. With the Hölder inequality we easily have

$$\left|(f * g)_{\{^3_4\}}(t)\right| \le \left(\int\limits_{\mathbb{R}_+^2} |f(\tau)|^p e^{-\theta} d\tau d\theta\right)^{1/p} \left(\int\limits_0^\infty |g(\theta)|^{p'} K_0(\theta) d\theta\right)^{1/p'}$$

$$\le \left(\int\limits_0^\infty |f(\tau)|^p d\tau\right)^{1/p} \left(\int\limits_0^\infty |g(\theta)|^{p'} K_0(\beta\theta) d\theta\right)^{1/p'} = ||f||_{L_p(\mathbb{R}_+;dt)} ||g||_{L_{p'}^{0,\beta}}. \tag{45}$$

Therefore double integrals (21) converge absolutely and uniformly and convolutions $(f * g)_{\{^3_4\}}(t)$ are bounded continuous on $\mathbb{R}_+$. Hence (see relation (2.16.2.2) in [2])

$$\begin{aligned}\left|\left|(f{*}g)_{\{^3_4\}}\right|\right|_{L_r^{\alpha,\gamma}} &\le \left(\int\limits_0^\infty x^\alpha K_0(\gamma x) dx\right)^{1/r} ||f||_{L_p(\mathbb{R}_+;dt)} ||g||_{L_{p'}^{0,\beta}} \\ &= (2\gamma)^{-1/r} \left(\frac{\gamma}{2}\right)^{-\alpha/r} \Gamma^{2/r} \left(\frac{\alpha+1}{2}\right) ||f||_{L_p(\mathbb{R}_+;dt)} ||g||_{L_{p'}^{0,\beta}},\ \alpha > -1,\end{aligned}$$

and this yields (43).

If $f \in L_1(\mathbb{R}_+; dt)$ then from the imbedding $L_p^{0,\beta} \subset L_1^{0,\beta}$ with the use of Theorem 5 we easily get $(f{*}g)_{\{^3_4\}}(t) \in L_1(\mathbb{R}; dt)$. Consequently, in the same manner we establish factorization identities (23) and Parseval's type equalities (41), if $\beta \in (0,1)$. Finally, we see that $(f{*}g)_{\{^3_4\}}(t) \in C_0(\mathbb{R}_+)$. Theorem 6 is proved.

Corollary 2 *Under conditions of Theorem 6 convolutions* (21) *exist for all* $t > 0$, *are continuous and belong to* $L_p(\mathbb{R}_+; dt)$. *Besides, the inequality*

$$\left|\left|(f{*}g)_{\{^3_4\}}\right|\right|_{L_p(\mathbb{R}_+;\ dt)} \le \left(\frac{\pi}{2\beta}\right)^{1/p} ||f||_{L_p(\mathbb{R}_+;\ dt)} ||g||_{L_{p'}^{0,\beta}} \tag{46}$$

takes place. In the Hilbert case $p = 2$ *Fourier type Parseval's identities hold true*

$$\int\limits_0^\infty \left|(f{*}g)_{\{^3_4\}}(t)\right|^2 dt = \int\limits_0^\infty \left|(F_{\{^c_s\}} f)(x) K_{ix}[g]\right|^2 dx. \tag{47}$$

Proof. Returning again to estimates (45) and using (35), we find

$$\int_0^\infty |(f*g)_{\{\frac{3}{4}\}}(t)|^p dt \le \int_{\mathbb{R}_+^2} |f(\tau)|^p K_0(\beta\theta) d\tau d\theta$$

$$\times \left(\int_0^\infty |g(\theta)|^{p'} K_0(\beta\theta) d\theta \right)^{p/p'} = \frac{\pi}{2\beta} \left[||f||_{L_p(\mathbb{R}_+;\, dt)} ||g||_{L_{p'}^{0,\beta}} \right]^p.$$

This implies (46). Fourier type Parseval's identities (47) are direct consequences of factorization identities (23) and the Parseval equality for the Fourier transform (9). Corollary 2 is proved.

Remark 3 *One can provide other estimates of convolutions* (21) *in the spaces above. However, in some cases we are unable to guarantee neither factorization identities* (23) *nor Parseval's type equalities* (41)*. Let, for instance, $f \in L_1(\mathbb{R}_+; dt),\ g \in L_2(\mathbb{R}_+; dt) \subset L_2^{\delta,\beta},\ \delta > 0,\ 0 < \beta \le 1$. Then it is not difficult to establish the existence, boundedness and continuity of the convolution $(f * g)_2(t)$ and the following inequality*

$$||(f*g)_{\{\frac{3}{4}\}}||_{L_r^{\alpha,\gamma}} \le C_{\alpha,\gamma}^{1/r} ||f||_{L_1(\mathbb{R}_+;\, d\tau)} ||g||_{L_2(\mathbb{R}_+;\, d\tau)},$$

where the constant $C_{\alpha,\gamma}$ is given by formula (44)*, $\alpha > -1,\ 0 < \gamma \le 1$. But if we suppose $f \in L_1(\mathbb{R}_+; d\tau) \cap L_2(\mathbb{R}_+; d\tau),\ g \in L_2^{0,\beta},\ 0 < \beta < 1$, then we can appeal to previous discussions in order to prove* (23) *and* (41).

Similarly, under conditions $f,\ g \in L_1(\mathbb{R}_+; dt)$ we derive estimates

$$\left|\left|(f*g)_{\{\frac{3}{4}\}}\right|\right|_{L_r^{\alpha,\gamma}} \le C_{\alpha,\gamma}^{1/r} ||f||_{L_1(\mathbb{R}_+;\, d\tau)} ||g||_{L_1(\mathbb{R}_+;\, d\tau)},$$

where the constant is again given by formula (44).

4 Convolution integral equations

This section will be devoted to a class of the first and second kind convolution integral equations related to (20) and (21). We begin to examine a solvability of the following integral equations of the first kind

$$(f*\mu)_{\{\frac{1}{2}\}}(t) = g(t), \quad t \in \mathbb{R}_+, \tag{48}$$

$$(f*\mu)_{\{\frac{3}{4}\}}(t) = g(t), \quad t \in \mathbb{R}_+, \tag{49}$$

$$(\mu*f)_{\{\frac{3}{4}\}}(t) = g(t), \quad t \in \mathbb{R}_+. \tag{50}$$

Functions $g,\ \mu$ are given and f is to be determined. We will establish conditions, which will guarantee the existence and uniqueness of solutions in a closed form for these equations. Similar questions for different convolutions were considered in [4], [5] and [6]. Taking into account symmetric properties of convolution kernels and Parseval equalities (29), (41), equations (48), (49) and (50) can be written, correspondingly, in integral form as

$$\int_0^\infty \mathcal{K}_{\{\frac{1}{2}\}}(t,\tau) f(\tau) d\tau = g(t), \tag{51}$$

where

$$\mathcal{K}_{\{ {1 \atop 2} \}}(t,\tau) = \frac{\sqrt{2}}{\pi\sqrt{\pi t}} \int_0^\infty K_{ix}(t)(F_{\{ {c \atop s} \}}\mu)(x) \left\{ {\cos x\tau \atop \sin x\tau} \right\} dx, \tag{52}$$

$$\int_0^\infty \mathcal{K}_{\{ {3 \atop 4} \}}(t,\tau) f(\tau) d\tau = g(t), \tag{53}$$

where

$$\mathcal{K}_{\{ {3 \atop 4} \}}(t,\tau) = \frac{2}{\pi} \int_0^\infty K_{ix}[\mu] \left\{ {\cos xt \cos x\tau \atop \sin xt \sin x\tau} \right\} dx, \tag{54}$$

$$\int_0^\infty \mathcal{K}_{\{ {5 \atop 6} \}}(t,\tau) f(\tau) d\tau = g(t), \tag{55}$$

where

$$\mathcal{K}_{\{ {5 \atop 6} \}}(t,\tau) = \pi\tau \mathcal{K}_{\{ {1 \atop 2} \}}(\tau, t). \tag{56}$$

Let us give a few examples of the kernels (52), (54), (56). Letting in (54) $\mu(\theta) = \theta^{\alpha-1},\ \alpha > 0$ we get the kernel

$$\mathcal{K}_{\{ {1 \atop 2} \}}(t,\tau) = \frac{\Gamma(\alpha)}{2} \left[\frac{1}{\cosh^\alpha(t-\tau)} \pm \frac{1}{\cosh^\alpha(t+\tau)} \right].$$

When we put $\mu(\theta) = \exp(-\theta\cos\nu),\ 0 < \nu < \pi$ in (54), the same way drives us at the kernel

$$\mathcal{K}_3(t,\tau) = \frac{\cos\nu + \cosh t \cosh\tau}{(\cos\nu + \cosh(t+\tau))(\cos\nu + \cosh(t-\tau))}.$$

Similarly some examples of kernels (54) can be obtained, for instance, taking $\mu(\theta) = \theta^{\alpha-1}e^{-\theta}$, $\mu(\theta) = \frac{\theta^{-1/2}}{\theta+p}e^{-\theta}$, $\mu(\theta) = e^{-\alpha\theta^2}$, $\mu(\theta) = \theta^{-3/2}e^{-\theta-b/\theta}$, $\mu(\theta) = \sinh\theta b$, $\mu(\theta) = \cosh\theta b$.

Particular cases of kernels (52) can be found calculating the Kontorovich-Lebedev integrals. Let $(F_c\mu)(x) = [\cosh\pi x]^{-1}$. Then with relation (2.16.48.12) [2] we obtain

$$\mathcal{K}_1(t,\tau) = \frac{e^{t\cosh\tau}}{\sqrt{2\pi t}} \mathrm{erfc}\left(\sqrt{2t}\cosh\frac{\tau}{2}\right),$$

where $\mathrm{erfc}(z)$ is the error function [1]. The reciprocal function $\mu(\theta)$, which corresponds to this kernel can be calculated by inversion formula (8) for the cosine Fourier transform. Hence with simple manipulations we find the value $\mu(\theta)$ as the integral

$$\mu(\theta) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos x\theta}{\cosh\pi x} dx = \frac{1}{\sqrt{2\pi}\cosh(\theta/2)}.$$

Further, let $(F_c\mu)(x) = \dfrac{x}{\sinh\pi x}$. Then with an elementary integral we get $\mu(\theta) = \dfrac{1}{2\sqrt{2\pi}\cosh^2(\theta/2)}$. We use this μ to calculate $\mathcal{K}_1(t,\tau)$, expressing it in terms of the integral of elementary functions. Indeed, employing the generalized Parseval equality for Fourier transform [3] and relation (2.16.6.4) in [2] we deduce

$$\mathcal{K}_1(t,\tau) = \frac{1}{t\pi\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{x}{\sinh \pi x} K_{ix}(t) e^{ix\tau}\, dx$$

$$= \frac{1}{t\pi\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-t\cosh(\tau-u)} \int\limits_{0}^{\infty} \frac{x\cos xu}{\sinh \pi x}\, dxdu = \frac{1}{2t\pi\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{e^{-t\cosh(\tau-u)}}{1+\cosh u} du.$$

Necessary and sufficient solvability conditions for convolution integral equations (51), (53), (55) are presented by the following theorems.

Theorem 7 *Let* $g \in L_2(\mathbb{R}_+; tdt)$, $\mu \in L_1(\mathbb{R}_+; d\theta)$ *and* $x^{-1}(F_{\{^c_s\}}\mu)(x) = O(1),\ x \to 0$. *Then for the solvability of equations* (51) *in* $L_2(\mathbb{R}_+; d\tau)$ *it is necessary and sufficient that* $\dfrac{x\sinh \pi x K_{ix}[g]}{(F_{\{^c_s\}}\mu)(x)} \in L_2(\mathbb{R}_+; dx)$. *Moreover, the corresponding solution* $f(\tau)$ *is unique and given by the formula*

$$f(\tau) = \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \operatorname*{l.i.m.}_{N\to\infty} \int\limits_{1/N}^{N} \frac{x\sinh \pi x K_{ix}[g]}{(F_{\{^c_s\}}\mu)(x)} \left\{ \begin{matrix} \cos x\tau \\ \sin x\tau \end{matrix} \right\} dx, \tag{57}$$

where the convergence is with respect to the norm in $L_2(\mathbb{R}_+; d\tau)$.

Proof. *Necessity.* Indeed, if we assume that g, μ, f belong to the corresponding L-classes and equations (51) are satisfied, then via Corollary 1 we have the equalities

$$K_{ix}[g] = \frac{\pi}{2} \frac{(F_{\{^c_s\}}f)(x)(F_{\{^c_s\}}\mu)(x)}{x\sinh \pi x}. \tag{58}$$

Further, since $(F_{\{^c_s\}}f)(x) \in L_2(\mathbb{R}_+; dx)$ we get that $\dfrac{x\sinh \pi x K_{ix}[g]}{(F_{\{^c_s\}}\mu)(x)} \in L_2(\mathbb{R}_+; dx)$ and L_2-solutions are given reciprocally by formula (57) (see (8)).

Sufficiency. If conversely, $\dfrac{x\sinh \pi x K_{ix}[g]}{(F_{\{^c_s\}}\mu)(x)} \in L_2(\mathbb{R}_+; dx)$, then $f(\tau)$ is from the space $L_2(\mathbb{R}_+;\ d\tau)$ via formula (57). Furthermore, by virtue of Corollary 1 the left-hand side of (51), which corresponds to convolutions $(f*\mu)_{\{^1_2\}}(t)$ (see (48)), belongs to $L_2(\mathbb{R}_+; tdt)$. In fact, by the same arguments as in Theorem 3 we observe that functions $\dfrac{|(F_{\{^c_s\}}\mu)(x)|^2}{x\sinh \pi x}$ are bounded, and therefore,

$$\int\limits_0^\infty \left|(F_{\{^c_s\}}f)(x)(F_{\{^c_s\}}\mu)(x)\right|^2 \frac{dx}{x\sinh \pi x} < \infty.$$

So, the Kontorovich-Lebedev transform (10) of the left-hand side of (51) is equal to

$$\frac{\pi}{2} \frac{(F_{\{^c_s\}}\mu)(x)(F_{\{^c_s\}}f)(x)}{x\sinh \pi x} = \frac{(F_{\{^c_s\}}\mu)(x)}{x\sinh \pi x} \frac{x\sinh \pi x K_{ix}[g]}{(F_{\{^c_s\}}\mu)(x)} = K_{ix}[g].$$

Hence with the reciprocal inversion (12) and the generalized Parseval equality (29) we find that equations (51) are satisfied, and (57) is a corresponding unique $L_2(\mathbb{R}_+; d\tau)$-solution. Theorem 7 is proved.

In order to examine a solvability of integral equations (53) we will seek possible solutions f in the class $L_1(\mathbb{R}_+; d\tau) \cap L_2(\mathbb{R}_+; d\tau)$. Denoting by $\mathcal{F}_{1,2} \equiv \{h \in L_2(\mathbb{R}_+; dx);\ h = (F_c f)(x)\}$ a set of images of this class under Fourier transforms (7), which is a subspace of $L_2(\mathbb{R}_+; dx)$, we will consider a restriction of this map to $F_{\{^c_s\}} : L_1(\mathbb{R}_+; d\tau) \cap L_2(\mathbb{R}_+; d\tau) \to \mathcal{F}_{1,2},\ \mathcal{F}_{1,2} \subset C_0(\mathbb{R}_+)$.

Theorem 8 *Let $g \in L_2(\mathbb{R}_+; dt)$, $\mu \in L_2^{0,1}$ and $K_{ix}[\mu] = O(1), x \to 0$. Then for the solvability of equations* (53) *in $L_1(\mathbb{R}_+; d\tau)) \cap L_2(\mathbb{R}_+; d\tau)$ it is necessary and sufficient that $\dfrac{(F_{\{^c_s\}} g)(x)}{K_{ix}[\mu]} \in \mathcal{F}_{1,2}$. Moreover, the corresponding solution $f(\tau)$ is unique and given by the formula*

$$f(\tau) = \sqrt{\frac{2}{\pi}} \operatorname*{l.i.m.}_{N\to\infty} \int_{1/N}^{N} \frac{(F_{\{^c_s\}} g)(x)}{K_{ix}[\mu]} \left\{ \begin{matrix} \cos x\tau \\ \sin x\tau \end{matrix} \right\} dx, \tag{59}$$

where the convergence is with respect to the norm in $L_2(\mathbb{R}_+; d\tau)$.

Proof. *Necessity.* Indeed, if under conditions of the theorem equations (53) are satisfied, then convolutions (49) exist and an analog of equalities (58) holds

$$(F_{\{^c_s\}} g)(x) = (F_{\{^c_s\}} f)(x) K_{ix}[\mu]. \tag{60}$$

But $(F_{\{^c_s\}} f)(x) \in \mathcal{F}_{1,2}$. Hence $\dfrac{(F_{\{^c_s\}} g)(x)}{K_{ix}[\mu]} \in \mathcal{F}_{1,2}$ and solutions in $L_2(\mathbb{R}_+; d\tau)$ are given reciprocally by the formula (59).

Sufficiency. Now assuming $\dfrac{(F_{\{^c_s\}} g)(x)}{K_{ix}[\mu]} \in \mathcal{F}_{1,2}$, we get correspondingly $f(\tau) \in L_1(\mathbb{R}_+; d\tau) \cap L_2(\mathbb{R}_+; d\tau)$ via conditions of the theorem. Furthermore, the left-hand side of (53), which is convolutions $(f * \mu)_{\{^3_4\}}(t)$, belongs to $L_2(\mathbb{R}_+; dt)$. This is because $\mu \in L_2^{0,1}$, $K_{ix}[\mu] = O(1), x \to 0$ and therefore, $K_{ix}[\mu]$ is bounded. Then the right-hand side of (47) being written for convolutions $(f * \mu)_{\{^3_4\}}(t)$, is finite. So, Fourier transforms of the left-hand side of (53) are equal to

$$(F_{\{^c_s\}} f)(x) K_{ix}[\mu] = K_{ix}[\mu] \frac{(F_{\{^c_s\}} g)(x)}{K_{ix}[\mu]} = (F_{\{^c_s\}} g)(x).$$

Hence with inversion formulas (8) and generalized Parseval equalities (41) we find that equations (53) are satisfied, and (59) is the corresponding unique solution from $L_1(\mathbb{R}_+; d\tau)) \cap L_2(\mathbb{R}_+; d\tau)$. Theorem 8 is proved.

Equations (55) with noncommutative convolutions will be treated in the class $L_2^{0,1} \cap L_2(\mathbb{R}_+; \tau d\tau) \subset L_1^{0,1} \cap L_2(\mathbb{R}_+; \tau d\tau)$. Denoting by $\mathcal{KL}^2_{ix} \equiv \{h \in L_2(\mathbb{R}_+; x \sinh \pi x\, dx); h = K_{ix}[f]\}$ a set of images of this class under the Kontorovich-Lebedev transform (10) we will consider a restriction of this map to $K_{ix} : L_2^{0,1} \cap L_2(\mathbb{R}_+; \tau d\tau) \to \mathcal{KL}^2_{ix}$.

Theorem 9 *Let $g \in L_2(\mathbb{R}_+; dt)$, $\mu \in L_1(\mathbb{R}_+; dt) \cap L_2(\mathbb{R}_+; dt)$ and $x^{-1}(F_{\{^c_s\}} \mu)(x) = O(1), x \to 0$. Assuming that*

$$\frac{(F_{\{^c_s\}} g)(x)}{(F_{\{^c_s\}} \mu)(x)} \in L_1(\mathbb{R}_+;\ \sinh \pi x dx) \tag{61}$$

and the following conditions are fulfilled

$$\int_0^\infty \sqrt{u}\, e^{u/2} \left| \int_0^\infty \frac{\sinh \pi x (F_{\{^c_s\}} g)(x)}{(F_{\{^c_s\}} \mu)(x)} \sin xu\, dx \right| du < \infty,$$

the solvability of equations (55) *in the class* $L_2^{0,1} \cap L_2(\mathbb{R}_+; \tau d\tau)$ *is guaranteed if and only if*

$$\frac{(F_{\{{c \atop s}\}} g)(x)}{(F_{\{{c \atop s}\}} \mu)(x)} \in \mathcal{K}L^2_{ix}.$$

The corresponding solution is unique and given by the formula

$$f(\tau) = \frac{2}{\pi^2 \tau} \int_0^{\infty} \frac{x \sinh \pi x (F_{\{{c \atop s}\}} g)(x)}{(F_{\{{c \atop s}\}} \mu)(x)} K_{ix}(\tau) dx, \tag{62}$$

where latter integrals exist in the Lebesgue sense.

Proof. *Necessity.* Indeed, if under conditions of the theorem equation (55) takes place, then the convolutions (50) exist by Corollary 2 and satisfy the equalities (see (23))

$$(F_{\{{c \atop s}\}} g)(x) = K_{ix}[f](F_{\{{c \atop s}\}} \mu)(x).$$

However, $K_{ix}[f] \in \mathcal{K}L^2_{ix}$. Hence $\dfrac{(F_{\{{c \atop s}\}} g)(x)}{(F_{\{{c \atop s}\}} \mu)(x)} \in \mathcal{K}L^2_{ix}$ and a solution in $L_2(\mathbb{R}_+; \tau d\tau)$ is given reciprocally by the formula

$$f(\tau) = \underset{N \to \infty}{\text{l.i.m.}} \frac{2}{\pi^2 \tau} \int_0^{N} \frac{x \sinh \pi x (F_{\{{c \atop s}\}} g)(x)}{(F_{\{{c \atop s}\}} \mu)(x)} K_{ix}(\tau) dx. \tag{63}$$

But the latter integral is absolutely convergent due to conditions (61) and the boundedness for all $\tau > 0$ of the product $x K_{ix}(\tau)$, which can be verified by using the integral representation

$$x K_{ix}(\tau) = \tau \int_0^{\infty} e^{-\tau \cosh u} \sinh u \sin xu \, du. \tag{64}$$

We note, that integral (64) can be easily obtained integrating by parts in (13). Consequently, a unique solution (63) of the corresponding equation (55) can be represented by (62). Further, appealing to the generalized Minkowski inequality and relation (2.16.6.2) from [2] we obtain

$$||f||_{L_2^{0,1}} = \left(\int_0^{\infty} K_0(\tau) |f(\tau)|^2 d\tau \right)^{1/2} = \frac{2}{\pi^2} \left(\int_0^{\infty} K_0(\tau) \left| \int_0^{\infty} e^{-\tau \cosh u} \sinh u \right.\right.$$

$$\left.\left. \times \int_0^{\infty} \frac{\sinh \pi x (F_{\{{c \atop s}\}} g)(x)}{(F_{\{{c \atop s}\}} \mu)(x)} \sin xu \, dx \, du \right|^2 d\tau \right)^{1/2} \leq \frac{2}{\pi^2} \int_0^{\infty} \sinh u$$

$$\times \left(\int_0^{\infty} K_0(\tau) e^{-2\tau \cosh u} d\tau \right)^{1/2} \left| \int_0^{\infty} \frac{\sinh \pi x (F_{\{{c \atop s}\}} g)(x)}{(F_{\{{c \atop s}\}} \mu)(x)} \sin xu \, dx \right| du$$

$$< \frac{2}{\pi^2} \int_0^{\infty} \sqrt{u} \, e^{u/2} \left| \int_0^{\infty} \frac{\sinh \pi x (F_{\{{c \atop s}\}} g)(x)}{(F_{\{{c \atop s}\}} \mu)(x)} \sin xu \, dx \right| du < \infty.$$

Therefore $f \in L_2^{0,1} \cap L_2(\mathbb{R}_+; \tau d\tau)$.

Sufficiency. Now assuming $\dfrac{(F_{\{{c \atop s}\}}g)(x)}{(F_{\{{c \atop s}\}}\mu)(x)} \in \mathcal{K}L^2_{ix}$, we get correspondingly $f(\tau) \in L_2^{0,1} \cap L_2\left(\mathbb{R}_+;\ \tau d\tau\right)$ via (63), (62) and conditions of the theorem. Furthermore, the left-hand side of (55), which is convolutions $(\mu{*}f)_{\{{3 \atop 4}\}}(t)$, belongs to $L_2\left(\mathbb{R}_+; dt\right)$. This is because $\mu \in L_1(\mathbb{R}_+; dt) \cap L_2(\mathbb{R}_+; dt)$ and $x^{-1}(F_{\{{c \atop s}\}}\mu)(x) = O(1), x \to 0$ and therefore, $[x \sinh \pi x]^{-1/2}(F_{\{{c \atop s}\}}\mu)(x)$ is bounded. Then since $K_{ix}[f] \in L_2\left(\mathbb{R}_+; x \sinh \pi x\ dx\right)$, we have (see (47))

$$\int\limits_0^\infty \left|(f{*}\mu)_{\{{3 \atop 4}\}}(t)\right|^2 dt = \int\limits_0^\infty x \sinh \pi x \left|K_{ix}[f] \frac{(F_{\{{c \atop s}\}}\mu)(x)}{\sqrt{x \sinh \pi x}}\right|^2 dx < \infty.$$

Hence Fourier transforms (7) of the left-hand side of (55) are equal to

$$\sqrt{\frac{2}{\pi}} K_{ix}[f](F_{\{{c \atop s}\}}\mu)(x) = \frac{(F_{\{{c \atop s}\}}g)(x)}{(F_{\{{c \atop s}\}}\mu)(x)}(F_{\{{c \atop s}\}}\mu)(x) = (F_{\{{c \atop s}\}}g)(x).$$

Therefore equations (55) are satisfied, and (62) is the corresponding unique solution from $L_2^{0,1} \cap L_2(\mathbb{R}_+; \tau d\tau)$. Theorem 9 is proved.

Let us consider briefly other classes of convolution integral equations related to (48), (49), (50).

We start with second kind integral equations involving convolutions (49)

$$f(t) + (f{*}\mu)_{\{{3 \atop 4}\}}(t) = g(t), \quad t \in \mathbb{R}_+. \tag{65}$$

Theorem 10 *Let $g \in L_1(\mathbb{R}_+; dt) \cap L_2(\mathbb{R}_+; dt)$, $\mu \in L_2^{0,1}$ and $K_{ix}[\mu] = O(1), x \to 0$. Moreover, assuming that*

$$1 + K_{ix}[\mu] \neq 0,\ x \in \mathbb{R}_+, \tag{66}$$

Then for the solvability of equations (53) *in $L_1(\mathbb{R}_+; d\tau) \cap L_2(\mathbb{R}_+; d\tau)$ it is necessary and sufficient that $\dfrac{(F_{\{{c \atop s}\}}g)(x)}{1 + K_{ix}[\mu]} \in \mathcal{F}_{1,2}$. Moreover, the corresponding solution $f(\tau)$ is unique and given by the formula*

$$f(\tau) = \sqrt{\frac{2}{\pi}} \operatorname*{l.i.m.}_{N \to \infty} \int\limits_{1/N}^{N} \frac{(F_{\{{c \atop s}\}}g)(x)}{K_{ix}[\mu]} \begin{Bmatrix} \cos x\tau \\ \sin x\tau \end{Bmatrix} dx, \tag{67}$$

where the convergence is with respect to the norm in $L_2(\mathbb{R}_+; d\tau)$.

Proof. *Necessity.* Indeed, if under conditions of the theorem equations (65) are satisfied, then convolutions 21 exist and analogously to (60) it has

$$(F_{\{{c \atop s}\}}g)(x) = (F_{\{{c \atop s}\}}f)(x) + (F_{\{{c \atop s}\}}f)(x)K_{ix}[\mu].$$

But $(F_{\{{c \atop s}\}}f)(x) \in \mathcal{F}_{1,2}$. Hence $\dfrac{(F_{\{{c \atop s}\}}g)(x)}{1 + K_{ix}[\mu]} \in \mathcal{F}_{1,2}$ and solution in $L_2(\mathbb{R}_+; d\tau)$ is given reciprocally by the formula (67).

Sufficiency. Now assuming $\dfrac{(F_{\{{c \atop s}\}}g)(x)}{1 + K_{ix}[\mu]} \in \mathcal{F}_{1,2}$, we get correspondingly $f(\tau) \in L_1(\mathbb{R}_+; d\tau) \cap L_2(\mathbb{R}_+; d\tau)$ via conditions of the theorem. Furthermore, the left-hand side of (65) belongs to $L_2\left(\mathbb{R}_+; dt\right)$. This is because $\mu \in L_2^{0,1}$ and $K_{ix}[\mu] = O(1), x \to 0$ and therefore, $K_{ix}[\mu]$ is bounded. Then the right-hand side of

(47) being written for convolutions $(f*\mu)_{\{^3_4\}}(t)$, is finite. So, the Fourier transform of the left-hand side of (65) is equal to

$$(F_{\{^c_s\}}f)(x) + (F_{\{^c_s\}}f)(x)K_{ix}[\mu] = \frac{(F_{\{^c_s\}}g)(x)}{1+K_{ix}[\mu]} + K_{ix}[\mu]\frac{(F_{\{^c_s\}}g)(x)}{1+K_{ix}[\mu]} = (F_{\{^c_s\}}g)(x).$$

Hence with inversions (8) and generalized Parseval equalities (41) we find that equations (65) are satisfied, and (67) is correspondingly a unique solution from $L_1(\mathbb{R}_+;d\tau)) \cap L_2(\mathbb{R}_+;d\tau)$. Theorem 10 is proved.

Next, we will consider integral equation with two kernels, which corresponds to convolution $(\mu*f)_3(t)$

$$(\mu*f)_3(t) \ + \int_0^\infty e^{-u\cosh t} f(u)du = g(t), \quad t \in \mathbb{R}_+. \tag{68}$$

Taking into account factorization properties (23) we prove the following result.

Theorem 11 *Let $g \in L_2(\mathbb{R}_+;dt)$, $\mu \in L_1(\mathbb{R}_+;dt) \cap L_2(\mathbb{R}_+;dt)$ and $x^{-1}(F_c\mu)(x) = O(1), x \to 0$. If*

$$(F_c\mu)(x) + \sqrt{\frac{2}{\pi}} \neq 0, \quad x \in \mathbb{R}_+, \tag{69}$$

$$(F_c g)(x) \in L_1((1,\infty);\sinh \pi x dx) \cap L_2((1,\infty); x\sinh \pi x dx) \tag{70}$$

and the following condition holds

$$\int_0^\infty \sqrt{u}\, e^{u/2} \left| \int_0^\infty \frac{\sinh \pi x (F_c g)(x)}{(F_c\mu)(x) + \sqrt{\frac{2}{\pi}}} \sin xu \, dx \right| du < \infty, \tag{71}$$

then there exists a unique solution of equation (68) *in the class $L_2^{0,1} \cap L_2(\mathbb{R}_+;\tau d\tau)$ given by the formula*

$$f(\tau) = \frac{2}{\pi^2 \tau} \int_0^\infty \frac{x\sinh \pi x (F_c g)(x)}{(F_c\mu)(x) + \sqrt{\frac{2}{\pi}}} K_{ix}(\tau)dx, \tag{72}$$

where the latter integral exists in the Lebesgue sense.

Proof. In fact, from conditions of the theorem it follows that convolution $(\mu*f)_3(t) \in L_2(\mathbb{R}_+;dt)$. Moreover, by virtue of the generalized Minkowski and Schwarz inequalities and after calculation of the inner integral, we find (see (15), (16))

$$\left(\int_0^\infty \left| \int_0^\infty e^{-u\cosh t} f(u)du \right|^2 dt \right)^{1/2} \leq \int_0^\infty |f(u)| K_0^{1/2}(2u)du$$

$$\leq \left(\int_0^\infty K_0(u)|f(u)|^2 du \right)^{1/2} \left(\int_0^\infty \frac{K_0(2u)}{K_0(u)} du \right)^{1/2} < \infty.$$

Therefore one can consider the left-hand side of equation (68) from $L_2(\mathbb{R}_+;dt)$. Taking the cosine Fourier transform (7) from its both sides, accounting (13) and the Fubini theorem we come out with the equality

$$K_{ix}[f]\left[(F_c\mu)(x)+\sqrt{\frac{2}{\pi}}\right]=(F_cg)(x).$$

Hence via (69), (70) and since $(F_cg)(x)\in L_2(\mathbb{R}_+;dx)$, we obtain

$$K_{ix}[f]=\frac{(F_cg)(x)}{(F_c\mu)(x)+\sqrt{\frac{2}{\pi}}}\in L_2(\mathbb{R}_+;x\sinh\pi x dx)\cap L_1(\mathbb{R}_+;\sinh\pi x dx)$$

and

$$f(\tau)=\underset{N\to\infty}{\text{l.i.m.}}\frac{2}{\pi^2\tau}\int_0^N\frac{x\sinh\pi x(F_cg)(x)}{(F_c\mu)(x)+\sqrt{\frac{2}{\pi}}}K_{ix}(\tau)dx.$$

Now in the same manner as in Theorem 9 we show using (70) and (71) that the latter solution belongs to $L_2^{0,1}\cap L_2(\mathbb{R}_+;tdt)$ and can be written in the form (72). Theorem 11 is proved.

Our final result is about a solvability of the following systems of convolution integral equations

$$\begin{cases} h(t)+(f*\mu_1)_{\left\{\substack{1\\2}\right\}}(t)=g_1(t),\\ f(t)+(\mu_2*h)_{\left\{\substack{3\\4}\right\}}(t)=g_2(t),\end{cases}\tag{73}$$

where $\mu_i,\ g_i,\ i=1,2$ are given functions and a pair of solutions (f,h) is seeking in $(L_2(\mathbb{R}_+;dt),\ L_2(\mathbb{R}_+;tdt))$. This means that $f\in L_2(\mathbb{R}_+;dt)$ and $h\in L_2(\mathbb{R}_+;tdt)$, respectively.

Theorem 12 *Let $g_1\in L_2(\mathbb{R}_+;tdt)$, $\mu_1\in L_1(\mathbb{R}_+;dt)$, $g_2\in L_2(\mathbb{R}_+;dt)$, $\mu_2\in L_1(\mathbb{R}_+;dt)\cap L_2(\mathbb{R}_+;dt)$ and $x^{-1}K_{ix}[\mu_i]=O(1), x\to 0,\ i=1,2$. If*

$$\Delta(x)=1-\frac{\pi}{2}\frac{(F_{\left\{\substack{c\\s}\right\}}\mu_1)(x)(F_{\left\{\substack{c\\s}\right\}}\mu_2)(x)}{x\sinh\pi x}\neq 0,\quad x\in\mathbb{R}_+,\tag{74}$$

$$K_{ix}[g_1]\in L_2((0,1);dx),\tag{75}$$

$$(F_{\left\{\substack{c\\s}\right\}}g_2)(x)\in L_2((1,\infty);x\sinh\pi x\ dx),\tag{76}$$

then there exists a unique solution (f,h) of systems (73) *in the class $(L_2(\mathbb{R}_+;dt)$, $L_2(\mathbb{R}_+;tdt))$ given by formulas*

$$f(t)=\underset{N\to\infty}{\text{l.i.m.}}\sqrt{\frac{2}{\pi}}\int_0^N\left\{\begin{matrix}\cos x\tau\\ \sin x\tau\end{matrix}\right\}\begin{vmatrix} K_{ix}[g_1] & \frac{\pi}{2}\frac{(F_{\left\{\substack{c\\s}\right\}}\mu_1)(x)}{x\sinh\pi x}\\ (F_{\left\{\substack{c\\s}\right\}}g_2)(x) & 1\end{vmatrix}\frac{dx}{\Delta(x)},\tag{77}$$

$$h(t)=\underset{N\to\infty}{\text{l.i.m.}}\frac{2}{\pi^2}\int_0^N x\sinh\pi x\ K_{ix}(t)\begin{vmatrix} 1 & K_{ix}[g_1]\\ (F_{\left\{\substack{c\\s}\right\}}\mu_2)(x) & (F_{\left\{\substack{c\\s}\right\}}g_2)(x)\end{vmatrix}\frac{dx}{\Delta(x)}.\tag{78}$$

Proof. Indeed, by using Corollaries 1, 2 systems (73) can be written in equivalent forms as linear algebraic systems in terms of the Fourier and Kontorovich-Lebedev transforms with respect to unknown pair $((F_{\{^c_s\}}f)(x), K_{ix}[h])$. Its nonzero determinant $\Delta(x)$ is given by (74). Precisely, in a matrix form it becomes

$$\begin{pmatrix} 1 & \frac{\pi}{2}\frac{(F_{\{^c_s\}}\mu_1)(x)}{x\sinh \pi x} \\ (F_{\{^c_s\}}\mu_2)(x) & 1 \end{pmatrix} \cdot \begin{pmatrix} K_{ix}[h] \\ (F_{\{^c_s\}}f)(x) \end{pmatrix} = \begin{pmatrix} K_{ix}[g_1] \\ (F_{\{^c_s\}}g_2)(x) \end{pmatrix}.$$

Therefore Cramer's rule and inversion formulas (8), (12) lead us to the unique pair of solutions (f, h) represented by (77), (78), where

$$(F_{\{^c_s\}}f)(x) = \frac{1}{\Delta(x)} \begin{vmatrix} K_{ix}[g_1] & \frac{\pi}{2}\frac{(F_{\{^c_s\}}\mu_1)(x)}{x\sinh \pi x} \\ (F_{\{^c_s\}}g_2)(x) & 1 \end{vmatrix},$$

$$K_{ix}[h] = \frac{1}{\Delta(x)} \begin{vmatrix} 1 & K_{ix}[g_1] \\ (F_{\{^c_s\}}\mu_2)(x) & (F_{\{^c_s\}}g_2)(x) \end{vmatrix}$$

belong to $L_2(\mathbb{R}_+; dx),\ L_2(\mathbb{R}_+; x\sinh \pi x dx)$, respectively. The latter fact is guaranteed by conditions (74), (75), (76) and Corollaries 1, 2. Theorem 12 is proved.

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