A note on the C^0-centralizer of an open class of bidimensional Anosov diffeomorphisms

Jorge Rocha

June 15, 2005

Abstract

In this note we prove that the C^0-centralizer of a bidimensional Anosov diffeomorphism having only a fixed point, f, is generated by f and a square root of the Identity, if f reverses the orientation, or by a square root of f and a square root of the Identity, otherwise.

1 Introduction

Let f be a C^r diffeomorphism of a compact manifold M, $r \in \mathbb{N} \cup \{\infty, \omega\}$. For $s \leq r$, the C^s-centralizer of f is the set

$$Z^s(f) = \{g \in Diff^s(f) : \; g \circ f = f \circ g\}.$$

The C^s-centralizer of f is said to be trivial if it reduces to the powers of f. There are examples of diffeomorphisms f whose centralizer is not trivial: for example in the case where f is the time one map associated to a flow or when f is a linear contraction on $\mathbb{R}$ (in this case $Z^0(f)$ is not even abelian).

In [S] Smale conjectured that there is an open and dense subset of C^r diffeomorphisms on M whose elements have trivial centralizer. This conjecture was proved by Kopell, [K], when M is a circle and $r \geq 2$. For any compact manifold Palis and Yoccoz showed that generically, in the open subset of C^∞ stable diffeomorphisms, the C^∞ centralizer is trivial ([PY1]), and that there exists an open and dense subset of the Anosov diffeomorphisms on a torus whose elements have trivial C^∞ centralizer ([PY2]). The author transposed part of these results to the context of real analytic stable diffeomorphisms ([R2] and [R3]), and to the C^1 centralizer for stable bidimensional C^∞ diffeomorphisms ([R1]). In the non stable context, Burslem showed that there exists a C^1 open and dense subset of the partially hyperbolic diffeomorphisms whose elements have discrete centralizer ([B]).

In the opposite direction Plykin ([P]) showed that, with some conditions, the centralizer of an Anosov automorphism on a n dimensional torus is isomorphic to $\mathbb{Z}^l \oplus F$, for an appropriate l, where F is a finite commutative group. In this note, using in a simple way ideas introduced in [PY1] and developed in [R1] and [R2], we calculate explicitly the C^0 centralizer of an open class of bidimensinal Anosov diffeomorphisms. More precisely we prove the following result.

Theorem 1. *If f is an Anosov diffeomorphism of the two torus having only a fixed point then either*

i) $Z^0(f)$ *is the abelian group generated by* f *and* h*, where* h *is a homeomorphism satisfying* $h^2 \equiv Id$*, if* f *reverses the orientation,*

or

ii) $Z^0(f)$ *is the abelian group generated by* g *and* h*, where* g *and* h *are homeomorphisms satisfying* $g^2 \equiv f$ *and* $h^2 \equiv Id$*, if* f *preserves the orientation.*

It is interesting to observe that this result implies some kind of (semi) rigidity on conjugations. More precisely if f and g are two Anosov diffeomorphisms, having only a fixed point and orientation reversing, and h_1, h_2 are two homeomorphisms that conjugate f and g, that is

$$g = h_1 \circ f \circ {h_1}^{-1} = h_2 \circ f \circ {h_2}^{-1},$$

then ${h_2}^{-1} \circ h_1 \in Z^0(f)$, therefore there are $s \in \{0,1\}$ and $k \in \mathbb{Z}$ such that $h_2 = h_1 \circ \left(h^s \circ f^k\right)$, where $h \in Z^0(f)$ and $h^2 = Id$. The same holds in the orientation preserving case replacing f by a *square root of* f.

2 Proof of the Theorem

We consider the torus $\mathbb{T}^2$ as the quotient space obtained by the following equivalence relation on $\mathbb{R}^2$:

$$(x, y) \sim (z, t) \ \ if \ \ (x - z) \in \mathbb{Z} \ and \ (y - t) \in \mathbb{Z}.$$

We denote by Π the projection from $\mathbb{R}^2$ to $\mathbb{T}^2$ and write $[(x, y)] = \Pi((x, y))$.

An automorphism of $\mathbb{T}^2$, f_A, is the map induced on the torus by a linear map of $\mathbb{R}^2$ whose matrix, with respect to the canonic basis, is of the form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, b, c, d \in \mathbb{Z}$ and $|det(A)| = |ad - dc| = 1$. Moreover the automorphism f_A is said to be *hyperbolic* if the linear map A has two real eigenvalues with modulus not equal to one.

It is well known that if f is an Anosov diffeomorphism of $\mathbb{T}^2$ then f is conjugated to some hyperbolic automorphism f_A, (see [M]), that is there exists a homeomorphism of $\mathbb{T}^2$, g, such that $g \circ f = f_A \circ g$. If f has only a fixed point then the same holds for f_A, that is $[(0, 0)]$ is the only fixed point of f_A, and it is easy to see that the map $\Gamma : Z^0(f) \to Z^0(f_A)$, defined by $\Gamma(h) = g \circ h \circ g^{-1}$, is a group automorphism. Therefore we just have to prove the Theorem for hyperbolic automorphisms of $\mathbb{T}^2$ with exactly one fixed point.

Now, if $h \in Z^0(f_A)$ and f_A as only a fixed point then $h([(0, 0)]) = [(0, 0)]$; moreover if f_A is hyperbolic then h is an automorphism of $\mathbb{T}^2$ (see [PY2]). That is to determine the centralizer of a hyperbolic automorphism f_A having only a fixed point we just need to solve the equation $f_B \circ f_A = f_A \circ f_B$, where B is a linear map.

Let f_A be a hyperbolic automorphism; a straightforward calculation shows that the condition of existence of unique fixed point is equivalent to

- either $|tr(A)| = |a + d| = 1$, if $det(A) = -1$, or

- $tr(A) = 3$, if $det(A) = 1$.

First observe that if $det(A) = 1$ and $tr(A) = 3$ then $det(A - Id) = -1$, $tr(A - Id) = 1$, $f_{(A-Id)}$ is a hyperbolic automorphism and $f^2_{(A-Id)} = f_A$. As $f_{(A-Id)} = f_A \circ f_{(-Id)}$ and $f_{(-Id)} \in Z^0(f_B)$, for all B, it follows that $f_B \circ f_A = f_A \circ f_B$ implies that $f_B \circ f_{(A-Id)} = f_{(A-Id)} \circ f_B$, that is $Z^0(f_A) = Z^0(f_{(A-Id)})$.

Second observe that if $det(A) = -1$ and $tr(A) = -1$ then $det(-A) = -1$, $tr(A) = 1$ and $Z^0(f_A) = Z^0(f_{(-A)})$.

From these two observations it follows that to prove the Theorem we just need to prove that the C^0-centralizer of f_A is generated by f_A and $f_{(-Id)}$, where f_A is a hyperbolic automorphism with $det(A) = -1$ and $tr(A) = 1$.

Let us fix such a f_A. The linear map A has two eigenvalues $\lambda^u = \frac{1+\sqrt{5}}{2}$ and $\lambda^s = \frac{1-\sqrt{5}}{2}$ whose eigenspaces E^u and E^s are generated by vectors $v^u = (1, \frac{\lambda^u - a}{b})$ and $v^s = (1, \frac{\lambda^s - a}{b})$, respectively. These two lines have irrational slope therefore $\Pi(E^s)$ and $\Pi(E^u)$ are both dense on $\mathbb{T}^2$ and they intersect transversally with a constant angle. Let $\mathbb{H} = \Pi(E^s) \cap \Pi(E^u)$ denote the *set of homoclinic points* of f_A. Each point of $\mathbb{H}$ has an E^s-coordinate and an E^u-coordinate which can be determinated in the following way. Let $[(z, w)]$ be a point of $\mathbb{H}$, then there are real numbers x and y such that

$$[(z, w)] = \Pi(xv^s) = \Pi(yv^u),$$

which is equivalent to

$$y\left(1, \frac{\lambda^u - a}{b}\right) = x\left(1, \frac{\lambda^s - a}{b}\right) + (m, n),$$

for some $m, n \in \mathbb{Z}$. From this it follows that the points of $\mathbb{H}$ correspond to those points of the product space $E^s \times E^u$ whose (x, y)-coordinates are of the form

$$x = \frac{m(a - \lambda^u) + nb}{(\lambda^u - \lambda^s)}, \qquad y = \frac{m(a - \lambda^s) + nb}{(\lambda^u - \lambda^s)}, \tag{1}$$

where $m, n \in \mathbb{Z}$. Let us denote this subset of $E^s \times E^u$ by $\mathbb{H}_0$. As $\Pi(E^s)$ is dense in $\mathbb{T}^2$, from the previous expression of the $x-coordinate$ of a homoclinic point it follows that $\mathbb{H}$ is a dense subset of $\mathbb{T}^2$ and, as $\Pi(E^s)$ and $\Pi(E^u)$ intersect with a constant angle, we get that $\mathbb{H}_0$ is a closed and discrete subset of $E^s \times E^u$.

If $f_B \in Z^0(f_A)$ then $B(E^s) = E^s$ and $B(E^u) = E^u$ thus $f_B(\Pi(E^s)) = \Pi(E^s)$, $f_B(\Pi(E^u)) = \Pi(E^u)$, $f_B(\mathbb{H}) = \mathbb{H}$ and $(B^s, B^u)(\mathbb{H}_0) = \mathbb{H}_0$, where $B^s = B_{|E^s}$ and $B^u = B_{|E^u}$. The maps B^s and B^u are one dimensional linear maps so let us denote by $\mu^s(B)$ and $\mu^u(B)$ their eigenvalues, respectively. To each map $f_B \in Z^0(f_A)$ we can associate an element $(\sigma^s(B), \sigma^u(B), \alpha^s(B), \alpha^u(B))$ of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{R} \times \mathbb{R}$ in the following way:

- $\sigma^s(B) = 1$ if $\mu^s(B) > 0$, otherwise $\sigma^s(B) = -1$,

- $\sigma^u(B) = 1$ if $\mu^u(B) > 0$, otherwise $\sigma^u(B) = -1$,

- $\alpha^s(B) = \frac{\log(|\mu^s(B)|)}{\log(|\lambda^s|)}$,

- $\alpha^u(B) = \frac{\log(|\mu^u(B)|)}{\log(|\lambda^u|)}$, thus defining a map Θ from $Z^0(f_A)$ to the additive group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{R} \times \mathbb{R}$.

The next two lemmas contain the main properties of the map Θ and are crucial for the proof of the Theorem.

Lemma 2.1. *One has*

i) $\Theta(f_A) = (-1, 1, 1, 1)$, $\Theta(f_{Id}) = (1, 1, 0, 0)$, $\Theta(f_{(-Id)}) = (-1, -1, 0, 0)$,

ii) if $f_B, f_C \in Z^0(f_A)$ *then,* $\forall\, n, m \in \mathbb{Z}$,

$$\Theta(f_B^n \circ f_C^m) = (\sigma^s(B)^n \times \sigma^s(C)^m, \sigma^u(B)^n \times \sigma^u(C)^m, n\alpha^s(B) + m\alpha^s(C), n\alpha^u(B) + m\alpha^u(C))\,;$$

in particular

$$\Theta(f_B^n \circ f_A^m) = (\sigma^s(B)^n \times (-1)^m, \sigma^u(B)^n, n\alpha^s(B) + m, n\alpha^u(B) + m),$$

iii) if f_B, $f_C \in Z^0(f_A)$, $\sigma^s(B) = \sigma^s(C)$ *and* $\alpha^s(B) = \alpha^s(C)$ *(or* $\sigma^u(B) = \sigma^u(C)$ *and* $\alpha^u(B) = \alpha^u(C)$*), then* $f_B = f_C$,

iv) if f_B, $f_C \in Z^0(f_A)$, $\sigma^s(B) \neq \sigma^s(C)$ *and* $\alpha^s(B) = \alpha^s(C)$ *(or* $\sigma^u(B) \neq \sigma^u(C)$ *and* $\alpha^u(B) = \alpha^u(C)$*), then* $B = -C$,

v) if $f_B \in Z^0(f_A)$ *then* $f_B^n = f_A$ *iff* $\sigma^s(B)^n = -1$ *and* $\alpha^s(B) = \frac{1}{n}$, *in particular* n *is an odd number.*

Proof. Items i) and ii) follow directly from the definition of Θ.

To prove iii) observe that the hypothesis imply that $B^s = C^s$ (or $B^u = C^u$), that is $(f_B)_{|\Pi(E^s)} = (f_C)_{|\Pi(E^s)}$ (or $(f_B)_{|\Pi(E^u)} = (f_C)_{|\Pi(E^u)}$). As $\Pi(E^s)$ and $\Pi(E^u)$ are dense in $\mathbb{T}^2$, one gets $f_B = f_C$.

If f_B and f_C satisfy the hypothesis iv) then, by ii), $\Theta(f_B \circ f_C^{-1}) = (-1, \star, 0, \star)$, which, in view of iii) and i), implies that $f_B \circ f_C^{-1} = f_{(-Id)}$, that is $B = -C$.

Finally v) is a direct consequence of i), ii) and iii).

□

Lemma 2.2. *One has*

i) for every $f_B \in Z^0(f_A)$, $\alpha^s(B) = \alpha^u(B) \in \mathbb{Q}$,

ii) there exists $q_0 \in \mathbb{N}$ *such that* $\alpha^s(B) \geq \frac{1}{q_0}$, *for every* $f_B \in Z^0(f_A)$ *such that* $\alpha^s(B) > 0$,

iii) $Z^0(f_A)$ *is the (abelian) group generated by* $f_{(-Id)}$ *and* f_B, *where* f_B *is an automorphism that satisfies* $f_B^k = f_A$, *for some* $k \in \mathbb{N}$.

Proof. To prove i) assume that there exists $f_B \in Z^0(f_A)$ such that $\alpha^s(B) \in \mathbb{R} \setminus \mathbb{Q}$. We can choose monotone sequences n_k and m_k, $k \in \mathbb{N}$, such that $\{(n_k\alpha^s(B) + m_k)\}_k$ converges to 0 and $n_k < 0$, for all $k \in \mathbb{N}$, if $\alpha^s(B) \leq \alpha^u(B)$, otherwise $n_k > 0$, for all $k \in \mathbb{N}$. As $f_B(\mathbb{H}) = \mathbb{H}$ it follows that $(B^s, B^u)(\mathbb{H}_0) = \mathbb{H}_0$. Let us choose a point $P \in \mathbb{H}_0 \setminus \{(0,0)\}$, then the sequence (or a proper subsequence) of points $P_k \in \mathbb{H}_0$, $P_k = (B^s, B^u)^{n_k} \circ (A^s, A^u)^{m_k}(P)$ is injective and converges to a point Q that, since $\mathbb{H}_0$ is closed, belongs to $\mathbb{H}_0$, which contradicts the fact that this set is discrete.

Now fix $f_B \in Z^0(f_A)$ such that $\alpha^s(B) = \frac{p}{q}$, where $(p, q) = 1$ and $q > 0$. Let $m, n \in \mathbb{Z}$ be such that $np + mq = 1$; a direct computation gives $\alpha^s((B^n \circ A^m))^q = 1$, therefore, by i) and iii) of

Lemma 2.1, we conclude that $f_{(B^n \circ A^m))^q} = f_A$ or $f_{(B^n \circ A^m))^q} = f_{(-A)}$. In both cases we get that $q(n\alpha^u(B) + m) = 1$, that is $\alpha^u(B) = \frac{p}{q}$.

If ii) does not hold then there exists a sequence of automorphisms f_{B_n}, $n \in \mathbb{N}$, such that $\alpha^s(B_n) = \frac{1}{q_n}$ is an injective sequence converging to 0. Fix a point $P \in \mathbb{H}_0 \backslash \{(0,0)\}$; as $(B_n^s, B_n^u)(\mathbb{H}_0) = (\mathbb{H}_0)$ it follows that, taking a subsequence if necessary, $P_n = (B_n^s, B_n^u)(P)$ is an injective sequence of points of $\mathbb{H}_0$ converging to a point of $\mathbb{H}_0$, which contradicts the fact that this set is discrete.

To prove iii) let k be the smallest natural number that satisfies ii) and $f_B \in Z^0(f_A)$ be an automorphism such that $\alpha^s(B) = \frac{1}{k}$. If $f_C \in Z^0(f_A)$ and $C \notin \{Id, -Id\}$ then $\alpha^s(C) = \frac{1}{p}$, where $|p| \leq k$. Replacing C by C^{-1}, if necessary, we can assume that p is positive.

Write $k = rs_1$ and $p = rs_2$, where $r = (k,p)$. Now choose integers n and l such that $ns_1 + ls_2 = 1$ and consider the map $f_{B^l \circ C^n} \in Z^0(f_A)$. A direct calculation shows that $\alpha^s(B^l \circ C^n) = \frac{1}{ks_2}$, which, by the choice of k, implies that $s_2 = 1$, that is p divides k. Finally, from iii) of Lemma 2.1 we conclude that $f_C = f_{B^{s_1}}$ or $f_C = f_{(-Id)} \circ f_{B^{s_1}}$.

□

Remark 2.3. *Item (iii) of previous lemma shows that $Z^0(f_A)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$ thus obtaining, in this context and using a different argument, the same result of Plykin ([P]).*

To end the proof of the Theorem it is enough to show that the k we got at item iii) of Lemma 2.2 is equal to 1. Assume that there exists $f_B \in Z^0(f_A)$ such that $\alpha^u(B) = \frac{1}{k}$, where $k > 1$ is odd, and $\sigma^u(B) = 1$ (if $\sigma^u(B) = -1$ we take $f_{(-B)}$ instead of f_B). Fix the point $P \in \mathbb{H}_0$ whose coordinates, in the space $E^s \times E^u$, are $(\frac{b}{\sqrt{5}}, \frac{b}{\sqrt{5}})$, obtained by taking $n = 1$ and $m = 0$ in equation (1) and observing that $\lambda^u - \lambda^s = \sqrt{5}$.

Now, the point $(B^s, B^u)(P)$ belongs to $\mathbb{H}_0$ and therefore, according to (1), its E^u coordinate satisfies the equation

$$(\lambda^u)^{\frac{1}{k}} \frac{b}{\sqrt{5}} = \frac{m(a - \lambda^s) + nb}{(\lambda^u - \lambda^s)}, \tag{2}$$

for some $m, n \in \mathbb{Z}$. Defining $r = 2ma - m + 2nb$ and $s = m$ and recalling that $\lambda^u = \frac{1+\sqrt{5}}{2}$ and that $\lambda^s = \frac{1-\sqrt{5}}{2}$, then the previous equation implies that the following equation is satisfied for some $r, s \in \mathbb{Z}$

$$\left(\frac{1+\sqrt{5}}{2}\right) b^k = \left(\frac{r + s\sqrt{5}}{2}\right)^k. \tag{3}$$

As $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$ is a unique factorization domain and $\frac{1+\sqrt{5}}{2}$ is a fundamental unity it follows that equation (3) has solutions if and only if $k = 1$ (in this case $r = s = b$, that is $m = 1$ and $n = 1 - a$), which is a contradiction, thus ending the proof of the Theorem.

References

[B] L. Burslem, *Centralizers of partially hyperbolic diffeomorphisms*, Erg. Theory and Dyn. Syst., 24, 1, 2004, 55 − 87

[IR] K. Ireland, M. Rosen, *A Classical Introduction to Modern Number Theory*, Graduate Texts in Math., Springer-Verlag, Second Edition, 1990

[K] N. Kopell, *Commuting Diffeomorphisms*, Gobal Analysis (Proc. Symp. Pure Math. XIV, 1968), AMS, 1970, 165 – 184

[M] A. Manning, *There are no new Anosov Diffeomorphisms on Tori*, Am. J. of Math., 96, 1974, 422 – 429

[PY1] J. Palis and J.C. Yoccoz, *Rigidity of centralizers of diffeomorphisms*, Ann. Scient. Ec Norm. Sup., 22, 1, 1989, 81 – 98

[PY2] J. Palis and J.C. Yoccoz, *Centralizers of Anosov diffeomorphisms on Tori*, Ann. Scient. Ec Norm. Sup., 22, 1, 1989, 99 – 108

[P] R. V. Plykin, *The structure of centralizers of Anosov diffeomorphisms of a torus*, Russ. Math. Surv.,, 53 6, 1998, 1363 – 1364

[R1] J. Rocha, *Rigidity of the C^1 centralizer of bibimensional diffeomorphisms*, Dyn. Syst.-Santiago do Chile, Pitman Research Notes in Math. Series, 285, 1993, 211 – 229

[R2] J. Rocha, *Rigidity of centralizers of real analytic diffeomorphisms*, Erg. Theory and Dyn. Syst., 13, 1, 1993, 175 – 197

[R3] J. Rocha, *Centralizers and Entropy*, Bol. Soc. Bras. Mat., 25, 2, 1994, 213 – 222

[S] S. Smale, *Mathematical problems for the next century*, Mathematics: Frontiers and Perspectives, Am. Math. Soc, 2000, 271 – 294

Jorge Rocha (jrocha@fc.up.pt)
Departamento de Matemática Pura, Univ. Porto
Rua do Campo Alegre 687
4169-007 Porto
Portugal