

# Weak convergence of a bootstrap geometric-type estimator with applications to risk theory

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## Abstract

Based on least square considerations, Brito and Moreira Freitas (2003) proposed a geometric-type estimator for estimating an exponential tail coefficient. We consider here the tail bootstrap method introduced by Bacro and Brito (1998) and show that this procedure works for this estimator. Moreover, we extend the application given in Brito and Moreira Freitas (2003), by showing that the results obtained may be applied to the related problem of estimating the adjustment coefficient in the Sparre Andersen model, under the standard conditions.

*Key words and phrases:* Adjustment coefficient; Bootstrap; Parameter estimation; Random walk; Sparre Andersen model.

*JEL Classification:* C13, C14

*AMS 2000 Classification:* 62G05, 62P05

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# 1 Introduction

Bootstrap methods have been shown to be very useful in a vast range of situations. In this paper, we shall be concerned first with the estimation of a general exponential tail coefficient. Secondly, we shall discuss an important application in risk theory, namely the estimation of the adjustment coefficient.

Let  $Z_1, Z_2, \dots$  be independent, nonnegative random variables (r.v.'s) with common distribution function (d.f.)  $F$  satisfying

$$1 - F(z) = P(Z_1 > z) = r(z)e^{-Rz}, \quad z > 0, \quad (1)$$

where  $r$  is a regularly varying function at infinity and  $R$  is a positive constant. Denoting by  $F^{-1}$  the left continuous inverse of  $F$ , i.e.,  $F^{-1}(s) := \inf\{x: F(x) \geq s\}$ , (1) is equivalent to

$$F^{-1}(1 - s) = -\frac{1}{R} \log s + \log \tilde{L}(s), \quad 0 < s < 1, \quad (2)$$

where  $\tilde{L}$  is a slowly varying function at zero (see e.g. Schultze and Steinebach (1996) and the references therein).

The problem of estimating the tail coefficient  $R$  has received considerable attention and several estimators have been proposed in the literature. Recently, Brito and Moreira (2001) have introduced an estimator of  $R$ , called *geometric-type* estimator,  $\hat{R}(k_n)$ , and defined as follows. Let  $Z_{1,n} \leq Z_{2,n} \leq \dots \leq Z_{n,n}$  denote the order statistics of the sample  $Z_1, Z_2, \dots, Z_n$  and assume that  $(k_n)$  is a sequence of positive integers satisfying

$$1 \leq k_n < n, \quad \lim_{n \rightarrow \infty} k_n = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} k_n/n = 0. \quad (3)$$

The estimator  $\hat{R}(k_n)$  is given by

$$\hat{R}(k_n) = \sqrt{\frac{\sum_{i=1}^{k_n} \log^2(n/i) - \frac{1}{k_n} \left( \sum_{i=1}^{k_n} \log(n/i) \right)^2}{\sum_{i=1}^{k_n} Z_{n-i+1,n}^2 - \frac{1}{k_n} \left( \sum_{i=1}^{k_n} Z_{n-i+1,n} \right)^2}}.$$

This estimator is related to the least square estimators proposed by Schultze and Steinebach (1996) and denoted by  $\hat{R}_1 \equiv \hat{R}_1(k_n)$  and  $\hat{R}_3 \equiv \hat{R}_3(k_n)$ . Indeed,  $\hat{R}(k_n)$  arises in a natural way from a geometrical adaptation of the procedure used by Schultze and Steinebach and is given by the geometric mean of  $\hat{R}_1$  and  $\hat{R}_3$ . The asymptotic properties of  $\hat{R}(k_n)$  were investigated in Brito and Moreira Freitas (2003). In particular, these authors established the consistency of the estimator and proved that, under general

regularity conditions, the distribution of  $k_n^{1/2} \left( \widehat{R}(k_n) - R \right)$  is asymptotically normal.

We recall also that the above estimation problem is equivalent to the estimation of the tail index of a Pareto type distribution. Setting  $X_i = e^{Z_i}$  with  $Z_i, i = 1, 2, \dots$  as above, we have

$$1 - G(x) = P(X_1 > x) = x^{-1/\alpha} L(x), \quad x > 1, \quad (4)$$

where  $\alpha = 1/R$  and  $L(x) = r(\log x)$  is slowly varying at infinity. In this context, several estimators have been proposed. One of the most commonly used estimators for  $\alpha$  is the Hill estimator (1975), defined by

$$H_n(k_n) = \frac{1}{k_n} \sum_{i=1}^{k_n} \log X_{n-i+1,n} - \log X_{n-k_n,n},$$

where  $X_{1,n} \leq X_{2,n} \leq \dots \leq X_{n,n}$  denote the order statistics of the sample  $X_1, X_2, \dots, X_n$  (for the asymptotic properties of  $H_n(k_n)$  see e.g. Deheuvels et al. (1988) and Haeusler and Teugels (1985)).

In a wide range of problems, bootstrap methods provide accurate approximations to the distribution of pivotal quantities such as the studentized mean (see e.g. Efron and Tibshirani (1993)). Over the recent years several modifications of the bootstrap method introduced by Efron (1979) have been suggested in the literature. We shall consider here the *tail bootstrap* procedure, proposed by Bacro and Brito (1998) for estimating tail coefficients. Let  $l_n$  be a nondecreasing sequence of positive integers and denote by  $W_1, W_2, \dots, W_{l_n}$  the  $l_n$  exceedances of the random level  $Z_{n-l_n,n}$ , that is

$$W_i := Z_{n-l_n+i,n} - Z_{n-l_n,n}, \quad 1 \leq i \leq l_n. \quad (5)$$

The tail bootstrap is based on resampling, with replacement, the sample  $W_1, W_2, \dots, W_{l_n}$  instead of the initial sample  $Z_1, Z_2, \dots, Z_n$ .

If we take,  $l_n = k_n, n \in \mathbb{N}$ , then we may write

$$\widehat{R}(k_n) = \frac{\sqrt{i_n(k_n)}}{\sqrt{\frac{1}{k_n} \sum_{i=1}^{k_n} W_i^2 - \left( \frac{1}{k_n} \sum_{i=1}^{k_n} W_i \right)^2}}, \quad (6)$$

where

$$i_n(k_n) := \frac{1}{k_n} \sum_{i=1}^{k_n} \log^2(n/i) - \left( \frac{1}{k_n} \sum_{i=1}^{k_n} \log(n/i) \right)^2. \quad (7)$$

By the above representation for  $\widehat{R}(k_n)$ , we can derive the corresponding bootstrap version in an obvious way. Let  $\mathbf{W}^* = (W_1^*, \dots, W_{k_n}^*)$  be a sample

drawn with replacement from  $W_1, W_2, \dots, W_{k_n}$ . The tail bootstrap version of the estimator is then given by

$$\widehat{R}^*(k_n) = \frac{\sqrt{i_n(k_n)}}{\sqrt{\frac{1}{k_n} \sum_{i=1}^{k_n} W_i^{*2} - \left(\frac{1}{k_n} \sum_{i=1}^{k_n} W_i^*\right)^2}}. \quad (8)$$

The main emphasis in this paper is on the theoretical validation of this bootstrap estimator. We follow a standard approach to establish the bootstrap consistency. We show namely that, conditioned on the sample tail  $Z_{n-k_n, n}, \dots, Z_{n, n}$ ,  $\frac{1}{\sqrt{2\widehat{R}(k_n)}} k_n^{1/2} \left(\widehat{R}^*(k_n) - \widehat{R}(k_n)\right)$  converges weakly, in probability, to the same limit as  $\frac{1}{\sqrt{2R}} k_n^{1/2} \left(\widehat{R}(k_n) - R\right)$ .

As a consequence of this result, we obtain that is possible to construct tail bootstrap confidence intervals for the exponential tail coefficient  $R$ , based on the estimator  $\widehat{R}(k_n)$ , with asymptotically correct coverage rates.

We present our main results in Section 2. The corresponding proofs are given in Section 3. The application in risk theory is considered in Section 4. In this section, we also discuss a general question concerning the risk model assumptions. In particular, we observe that, the consistency properties of the Hill-type estimator,  $\widehat{R}_1$ ,  $\widehat{R}_3$ , and  $\widehat{R}$  for the problem of estimating the exponential tail coefficient, also hold for the estimation of the adjustment coefficient in the Sparre Andersen model under the standard conditions. Moreover, the interesting property of the universal asymptotic normality of the last three estimators still holds in this context. Finally, in Section 5 we present a small-scale simulation study in order to illustrate the finite sample behaviour of the tail bootstrap approximation.

## 2 Main results

Our results on the bootstrap consistency are motivated by the asymptotic properties of the estimator  $\widehat{R}(k_n)$ , derived by Brito and Moreira Freitas (2003). These results are summarized below. In the sequel,  $\xrightarrow{D}$  and  $\stackrel{D}{=}$  stand, respectively, for convergence and equality in distribution.

**Theorem 1** (*Brilo and Moreira Freitas (2003), Theorem 6*) *Assume that  $F$  satisfies (1) and  $k_n$  is a sequence of positive integers such that (3) holds. If we suppose that, as  $n \rightarrow \infty$ ,*

$$k_n^{1/2} \sup_{1/k_n \leq y \leq 1} \left| \log \left( \frac{\widetilde{L}(yt^{k_n/n})}{\widetilde{L}(k_n/n)} \right) \right| \rightarrow 0 \quad (9)$$

*uniformly in  $t$  on compact sets of  $(0, \infty)$ , then*

$$\frac{1}{\sqrt{2R}} k_n^{1/2} \left(\widehat{R}(k_n) - R\right) \xrightarrow{D} N(0, 1).$$

In order to get more explicit conditions, it is necessary to specify the asymptotic behaviour of the slowly varying function  $L(x) = r(\log x)$  introduced in (4). Consider the following asymptotic relation

$$\text{(SR1)} \quad L(tx)/L(x) - 1 = O(g(x)) \text{ as } x \rightarrow \infty \text{ for each } t > 0,$$

where  $g$  is a positive function satisfying  $g(x) \rightarrow 0$  as  $x \rightarrow \infty$ . As shown in Bacro and Brito (1998), under the above assumptions, condition (9) may be much simplified, as stated in the following corollary.

**Corollary 1** (*Brito and Moreira Freitas (2003), Corollary 1*) *Assume that the slowly varying function  $L$  in (4) satisfies (SR1) with  $g$  regularly varying at infinity with index  $\gamma < 0$ . Then, if*

$$k_n^{1/2} g(\exp(F^{-1}(1 - k_n/n))) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

we have

$$\frac{1}{\sqrt{2R}} k_n^{1/2} \left( \widehat{R}(k_n) - R \right) \xrightarrow{D} N(0, 1).$$

Our main result is given in the following theorem.

**Theorem 2** *Assume that  $F$  satisfies (1) and  $k_n$  is a sequence of positive integers such that (3) holds. If we suppose that, as  $n \rightarrow \infty$ ,*

$$k_n^{1/2} \sup_{1/k_n \leq y \leq 1} \left| \log \left( \frac{\widetilde{L}(yt \frac{k_n}{n})}{\widetilde{L}(\frac{k_n}{n})} \right) \right| \rightarrow 0$$

uniformly in  $t$  on compact sets of  $(0, \infty)$ , then, in probability, for all  $x$ ,

$$P \left[ \frac{1}{\sqrt{2\widehat{R}(k_n)}} k_n^{1/2} \left( \widehat{R}^*(k_n) - \widehat{R}(k_n) \right) \leq x \mid (Z_{n-k_n, n}, \dots, Z_{n, n}) \right] \rightarrow \Phi(x)$$

as  $n \rightarrow \infty$ .

We now state the bootstrap analogue to Corollary 1.

**Corollary 2** *Assume that the slowly varying function  $L$  in (4) satisfies (SR1) with  $g$  regularly varying at infinity with index  $\gamma < 0$  and such that*

$$k_n^{1/2} g(\exp(F^{-1}(1 - k_n/n))) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Then in probability, for all  $x$ ,

$$P \left[ \frac{1}{\sqrt{2\widehat{R}(k_n)}} k_n^{1/2} \left( \widehat{R}^*(k_n) - \widehat{R}(k_n) \right) \leq x \mid (Z_{n-k_n, n}, \dots, Z_{n, n}) \right] \rightarrow \Phi(x)$$

as  $n \rightarrow \infty$ .

We are now in a position to define tail bootstrap confidence intervals for  $R$ , based on  $\widehat{R}(k_n)$ . Keeping in mind the application in risk theory (see Section 4), we consider the lower bounds with coverage probability  $p$ ,  $0 < p < 1$ ,

$$BI(k_n, p) = \left\{ R : \frac{1}{\sqrt{2R}} k_n^{1/2} \left( \widehat{R}(k_n) - R \right) \leq \widehat{H}^{-1}(p) \right\},$$

where  $\widehat{H}$  denotes the tail bootstrap distribution function

$$\widehat{H}(x) = P \left[ \frac{1}{\sqrt{2\widehat{R}(k_n)}} k_n^{1/2} \left( \widehat{R}^*(k_n) - \widehat{R}(k_n) \right) \leq x \mid (Z_{n-k_n, n}, \dots, Z_{n, n}) \right].$$

One of the important applications of Theorems 1 and 2 is the construction of confidence intervals for  $R$ . The following corollaries show that the tail bootstrap confidence intervals give asymptotically correct coverage rates. These results are easily derived using standard techniques (see e.g. Beran and Ducharme (1981), Proposition 1.3).

**Corollary 3** *Assume that  $F$  satisfies (1) and  $k_n$  is a sequence of positive integers such that (3) holds. If we suppose that, as  $n \rightarrow \infty$ ,*

$$k_n^{1/2} \sup_{1/k_n \leq y \leq 1} \left| \log \left( \frac{\widetilde{L}(yt^{k_n/n})}{\widetilde{L}(k_n/n)} \right) \right| \rightarrow 0$$

*uniformly in  $t$  on compact sets of  $(0, \infty)$ , then*

$$P(R \in BI(k_n, p)) \rightarrow p \text{ as } n \rightarrow \infty.$$

**Corollary 4** *Assume that the slowly varying function  $L$  in (4) satisfies (SR1) with  $g$  regularly varying at infinity with index  $\gamma < 0$  and such that*

$$k_n^{1/2} g(\exp(F^{-1}(1 - k_n/n))) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

*Then, as  $n \rightarrow \infty$ ,*

$$P[R \in BI(k_n, p)] \rightarrow p.$$

### 3 Proofs

Throughout this section we shall assume that (1) holds. We assume also that  $U_1, U_2, \dots$  is a sequence of independent uniform  $U(0, 1)$  random variables. The order statistics of the sample  $(U_1, U_2, \dots, U_n)$  are denoted by  $U_{1,n} \leq U_{2,n} \leq \dots \leq U_{n,n}$ .

In order to prove Theorem 2, we begin by establishing the following proposition.

**Proposition 1** Assume that  $F$  satisfies (1) and  $k_n$  is a sequence of positive integers such that (3) holds. If we suppose that, as  $n \rightarrow \infty$ ,

$$k_n^{1/2} \sup_{1/k_n \leq y \leq 1} \left| \log \left( \frac{\tilde{L}(yt^{k_n/n})}{\tilde{L}(k_n/n)} \right) \right| \rightarrow 0 \quad (10)$$

uniformly in  $t$  on compact sets of  $(0, \infty)$ , then, in probability, for all  $x$ ,

$$P \left[ \frac{\widehat{R}(k_n)}{\sqrt{8}} k_n^{1/2} \left( \frac{1}{\widehat{R}^*(k_n)} - \frac{1}{\widehat{R}(k_n)} \right) \leq x \mid (Z_{n-k_n, n}, \dots, Z_{n, n}) \right] \rightarrow \Phi(x)$$

as  $n \rightarrow \infty$ .

For proving this result we derive an adequate representation of the pivotal quantity in terms of an asymptotically r.v. plus some error terms. The major tool used for studying the main term is the following result obtained by Bickel and Freedman (1981), which establishes the validity of the bootstrap for means.

**Theorem 3** (Bickel and Freedman (1981), Theorem 2.1) Suppose  $X_1, X_2, \dots$  are independent and identically distributed random variables and have finite positive variance  $\sigma^2$ . Then, as  $n$  and  $m$  tend to  $\infty$

(a)  $P \left[ m^{1/2} \left( (1/m) \sum_{i=1}^m X_i^* - (1/n) \sum_{i=1}^n X_i \right) \leq x \mid (X_1, \dots, X_n) \right] \rightarrow \Phi(x/\sigma)$ , almost surely;

(b) for  $\epsilon$  positive,

$$P \left[ \left| (1/m) \sum_{i=1}^m (X_i^* - (1/m) \sum_{i=1}^m X_i^*)^2 - \sigma \right| > \epsilon \mid (X_1, \dots, X_n) \right] \rightarrow 0,$$

almost surely.

We also use the following auxiliary lemma.

**Lemma 1** (Brito and Moreira Freitas (2003), Lemma 2) Let  $k_n$  be a sequence of positive integers such that  $1 \leq k_n \leq n$ . For the sequence  $i_n(k_n)$  defined in (7) we have

$$i_n(k_n) = 1 + O \left( \frac{\log^2 k_n}{k_n} \right).$$

**Proof of Proposition 1.** Consider the sequence  $(W_i)_{1 \leq i \leq k_n}$  defined by (5).

Since  $Z_i \stackrel{D}{=} F^{-1}(U_i)$ ,  $i \geq 1$ , we write, without loss of generality,

$$W_i = F^{-1}(U_{n-k_n+i,n}) - F^{-1}(U_{n-k_n,n}).$$

As in Bacro and Brito (1998) (cf. Theorem 1) we shall make use of the following equivalent representation for  $W_i$ ,  $i = 1, \dots, k_n$ ,

$$W_i = -\frac{1}{R} \log Y_i + \log \frac{\tilde{L}(Y_i(1 - U_{n-k_n,n}))}{\tilde{L}(1 - U_{n-k_n,n})}, \quad (11)$$

where

$$Y_i = \frac{1 - U_{n-k_n+i,n}}{1 - U_{n-k_n,n}}, \quad \text{for } i = 1, \dots, k_n.$$

We recall that  $(Y_i)_{1 \leq i \leq k_n}$  is distributed as the vector of the order statistics of an i.i.d.  $k_n$ -sample from an uniform  $U(0, 1)$  distribution.

In order to simplify the presentation of the proof we will introduce some notation. Let  $T_1, T_2, \dots$  and  $T'_1, T'_2, \dots$  be two sequences of random variables. We denote by  $S_n^2(\mathbf{T})$  the sample variance of  $\mathbf{T} = (T_1, \dots, T_n)$  and by  $S_n(\mathbf{T}, \mathbf{T}')$  the sample covariance between  $\mathbf{T}$  and  $\mathbf{T}' = (T'_1, \dots, T'_n)$ , that is,

$$S_n^2(\mathbf{T}) = \frac{1}{n} \sum_{i=1}^n (T_i - \frac{1}{n} \sum_{i=1}^n T_i)^2$$

and

$$S_n(\mathbf{T}, \mathbf{T}') = \frac{1}{n} \sum_{i=1}^n \left( (T_i - \frac{1}{n} \sum_{i=1}^n T_i) (T'_i - \frac{1}{n} \sum_{i=1}^n T'_i) \right).$$

We define also, for  $i = 1, \dots, k_n$ ,

$$E_i := -\frac{1}{R} \log Y_i, \quad F_i := \log \frac{\tilde{L}(Y_i(1 - U_{n-k_n,n}))}{\tilde{L}(1 - U_{n-k_n,n})},$$

and write  $\mathbf{E} = (E_1, \dots, E_{k_n})$  and  $\mathbf{F} = (F_1, \dots, F_{k_n})$ . The corresponding bootstrap versions are then given by

$$E_i^* := -\frac{1}{R} \log Y_i^*, \quad F_i^* := \log \frac{\tilde{L}(Y_i^*(1 - U_{n-k_n,n}))}{\tilde{L}(1 - U_{n-k_n,n})}, \quad i = 1, \dots, k_n,$$

where  $(Y_1^*, \dots, Y_{k_n}^*)$  is a sample of i.i.d. r.v.'s drawn with replacement from the sample  $(Y_1, \dots, Y_{k_n})$ .

With these notations and using the representation (11) in the definition of the estimators  $\widehat{R}(k_n)$  and  $\widehat{R}^*(k_n)$  (cf. (6), (8)), we may write

$$\begin{aligned} & \frac{i_n(k_n)}{\widehat{R}^{2*}(k_n)} - \frac{i_n(k_n)}{\widehat{R}^2(k_n)} \\ &= S_{k_n}^2(\mathbf{E}^*) - S_{k_n}^2(\mathbf{E}) + S_{k_n}^2(\mathbf{F}^*) - S_{k_n}^2(\mathbf{F}) + 2S_{k_n}(\mathbf{E}^*, \mathbf{F}^*) - 2S_{k_n}(\mathbf{E}, \mathbf{F}). \end{aligned}$$

The proof will be done in two steps. Denote by  $P^*$  the conditional probability, given  $U_{n-k_n, n}, Y_1, \dots, Y_{k_n}$ . We will show that

$$P \left[ \frac{R^2}{\sqrt{8}} k_n^{1/2} (S_{k_n}^2(\mathbf{E}^*) - S_{k_n}^2(\mathbf{E})) \leq x \mid (U_{n-k_n, n}, Y_1, \dots, Y_{k_n}) \right] \rightarrow \Phi(x) \text{ a. s.} \quad (12)$$

and that, in probability

$$\frac{R^2}{\sqrt{8}} k_n^{1/2} |S_{k_n}^2(\mathbf{F}^*) - S_{k_n}^2(\mathbf{F}) + 2S_{k_n}(\mathbf{E}^*, \mathbf{F}^*) - 2S_{k_n}(\mathbf{E}, \mathbf{F})| = o_{P^*}(1). \quad (13)$$

We start by establishing (12). As  $(Y_1, \dots, Y_{k_n})$  is independent of  $U_{n-k_n, n}$ , it suffices to prove that

$$P \left[ \frac{R^2}{\sqrt{8}} k_n^{1/2} (S_{k_n}^2(\mathbf{E}^*) - S_{k_n}^2(\mathbf{E})) \leq x \mid (Y_1, \dots, Y_{k_n}) \right] \rightarrow \Phi(x) \text{ almost surely.}$$

Recall that  $S_{k_n}^2(\mathbf{E}) = \frac{1}{k_n} \sum_{i=1}^{k_n} (E_i - \frac{1}{k_n} \sum_{i=1}^{k_n} E_i)^2$  is the sample variance of an unit exponential i.i.d.  $k_n$ -sample.

We will use the following decomposition

$$\begin{aligned} & k_n^{1/2} (S_{k_n}^2(\mathbf{E}^*) - S_{k_n}^2(\mathbf{E})) \\ &= k_n^{1/2} \left( S_{k_n}^2(\mathbf{E}^*) - \frac{1}{k_n} \sum_{i=1}^{k_n} (E_i - 1/R)^2 + \frac{1}{k_n} \sum_{i=1}^{k_n} (E_i^* - 1/R)^2 \right) \\ & \quad + k_n^{1/2} \left( -S_{k_n}^2(\mathbf{E}) + \frac{1}{k_n} \sum_{i=1}^{k_n} (E_i - 1/R)^2 - \frac{1}{k_n} \sum_{i=1}^{k_n} (E_i^* - 1/R)^2 \right). \end{aligned}$$

Applying Theorem 3 to the i.i.d. r.v.'s  $(E_1 - 1/R)^2, (E_2 - 1/R)^2, \dots$ , we obtain that, almost surely,

$$P \left[ k_n^{1/2} \left( \frac{1}{k_n} \sum_{i=1}^{k_n} (E_i^* - 1/R)^2 - \frac{1}{k_n} \sum_{i=1}^{k_n} (E_i - 1/R)^2 \right) \leq x \right] \rightarrow \Phi(xR^2/\sqrt{8}).$$

Now observe that

$$k_n^{1/2} \left( -S_{k_n}^2(\mathbf{E}) + \frac{1}{k_n} \sum_{i=1}^{k_n} (E_i - 1/R)^2 \right) = \left( k_n^{1/4} \left( \frac{1}{k_n} \sum_{i=1}^{k_n} E_i - 1/R \right) \right)^2,$$

which converges to zero, almost surely. Moreover,

$$\begin{aligned} & k_n^{1/2} \left( S_{k_n}^2(\mathbf{E}^*) - \frac{1}{k_n} \sum_{i=1}^{k_n} (E_i^* - 1/R)^2 \right) \\ &= - \left( k_n^{1/4} \left( \frac{1}{k_n} \sum_{i=1}^{k_n} E_i^* - \frac{1}{k_n} \sum_{i=1}^{k_n} E_i \right) + k_n^{1/4} \left( \frac{1}{k_n} \sum_{i=1}^{k_n} E_i - 1/R \right) \right)^2, \end{aligned}$$

which is an  $o_{P^*}(1)$  almost surely, by Theorem 3 a) and the above observation.

For proving (13) we begin by noting that

$$\begin{aligned} & k_n^{1/2} |S_{k_n}^2(\mathbf{F}^*) - S_{k_n}^2(\mathbf{F}) + 2S_{k_n}(\mathbf{E}^*, \mathbf{F}^*) - 2S_{k_n}(\mathbf{E}, \mathbf{F})| \\ & \leq k_n^{1/2} |S_{k_n}^2(\mathbf{F}^*) + 2S_{k_n}(\mathbf{E}^*, \mathbf{F}^*)| + k_n^{1/2} |S_{k_n}^2(\mathbf{F}) + 2S_{k_n}(\mathbf{E}, \mathbf{F})| \\ & =: C_n^* + C_n. \end{aligned}$$

The well-known inequality

$$(S_{k_n}(\mathbf{E}, \mathbf{F}))^2 \leq S_{k_n}^2(\mathbf{E})S_{k_n}^2(\mathbf{F}),$$

implies that

$$k_n^{1/2} |S_{k_n}(\mathbf{E}, \mathbf{F})| = O_P(1)(k_n S_{k_n}^2(\mathbf{F}))^{1/2}. \quad (14)$$

By a routine calculation, it is easily verified that

$$k_n^{1/2} S_{k_n}^2(\mathbf{F}) \leq k_n^{1/2} \sup_{Y_{k_n} \leq y \leq 1} \left( \log \frac{\tilde{L}(y(1 - U_{n-k_n, n}))}{\tilde{L}(k_n/n)} \right)^2. \quad (15)$$

To proceed the proof we need the following auxiliary result.

**Lemma 2** *If (10) holds, then*

$$k_n^{1/2} |S_{k_n}^2(\mathbf{F}) + 2S_{k_n}(\mathbf{E}, \mathbf{F})| \xrightarrow{P} 0.$$

**Proof.** Choose any  $(\lambda_1, \lambda_2) \in ]1, +\infty[ \times ]1, +\infty[$  and consider the event

$$A_n(\lambda_1, \lambda_2) = \left\{ \lambda_1^{-1} \leq \frac{n}{k_n} (1 - U_{n-k_n, n}) \leq \lambda_1, \lambda_2^{-1} \leq k_n Y_{k_n} \leq \lambda_2 \right\}.$$

We have on  $A_n(\lambda_1, \lambda_2)$

$$k_n^{1/2} \sup_{Y_{k_n} \leq y \leq 1} \left| \log \frac{\tilde{L}(y(1 - U_{n-k_n, n}))}{\tilde{L}(k_n/n)} \right| \leq k_n^{1/2} \sup_{\frac{1}{k_n} \leq y \leq 1} \sup_{\frac{1}{\lambda_1 \lambda_2} \leq t \leq \lambda_1} \left| \log \frac{\tilde{L}(y t k_n/n)}{\tilde{L}(k_n/n)} \right|,$$

which converges to zero as  $n \rightarrow \infty$ , by condition (10).

Since  $P\{A_n(\lambda_1, \lambda_2)\} \rightarrow 1$  as  $n \rightarrow \infty$ , then it follows from (14) and (15) that  $k_n^{1/2} |S_{k_n}^2(\mathbf{F}) + 2S_{k_n}(\mathbf{E}, \mathbf{F})| \xrightarrow{P} 0$ .  $\square$

By Lemma 1,  $C_n$  converges to 0, in probability. Now, we will study the term  $C_n^*$ . Applying Theorem 3 b) to the r.v.'s  $E_1, E_2, \dots$ , we obtain

$$S_{k_n}^2(\mathbf{E}^*) = \frac{1}{R^2} + o_{P^*}(1) \quad \text{almost surely,}$$

and then

$$(S_{k_n}(\mathbf{E}^*, \mathbf{F}^*))^2 \leq \left( \frac{1}{R^2} + o_{P^*}(1) \right) S_{k_n}^2(\mathbf{F}^*).$$

Hence,

$$C_n^* = k_n^{1/2} S_{k_n}^2(\mathbf{F}^*) + O_{P^*}(k_n S_{k_n}^2(\mathbf{F}^*))^{1/2}.$$

Since  $Y_{1,k_n}^* \geq Y_{k_n}$ , it is easily verified that

$$k_n^{1/2} S_{k_n}^2(\mathbf{F}^*) \leq k_n^{1/2} \left( \sup_{Y_{k_n} \leq y \leq 1} \left| \log \frac{\tilde{L}(y(1 - U_{n-k_n, n}))}{\tilde{L}(k_n/n)} \right| \right)^2,$$

which implies that, by the proof of Lemma 2,  $C_n^* = o_{P^*}(1)$ , in probability. We conclude that

$$C_n^* + C_n = o_{P^*}(1),$$

in probability, and hence (13) is proved.

Now, to complete the proof of the proposition it is enough to note that

$$\begin{aligned} & \frac{\widehat{R}^2(k_n)}{\sqrt{8}} k_n^{1/2} \left( \frac{1}{\widehat{R}^{2^*}(k_n)} - \frac{1}{\widehat{R}^2(k_n)} \right) \\ &= \frac{1}{i_n(k_n)} \frac{\widehat{R}^2(k_n)}{R^2} \left( \frac{R^2}{\sqrt{8}} k_n^{1/2} \left( \frac{i_n(k_n)}{\widehat{R}^{2^*}(k_n)} - \frac{i_n(k_n)}{\widehat{R}^2(k_n)} \right) \right) \end{aligned}$$

and to recall that  $\widehat{R}^2(k_n)$  is a consistent estimator of  $R^2$  and that  $i_n(k_n)$  converges to one, by Lemma 1.  $\square$

**Proof of Theorem 2.** First note that we have

$$\widehat{R}^*(k_n) - \widehat{R}(k_n) = \left( \frac{1}{\widehat{R}^{2^*}(k_n)} - \frac{1}{\widehat{R}^2(k_n)} \right) \left( -\frac{1}{2\sqrt{\xi_{k_n}^3}} \right),$$

with  $\min\left(\frac{1}{\widehat{R}^{2*}(k_n)}, \frac{1}{\widehat{R}^2(k_n)}\right) < \xi_{k_n} < \max\left(\frac{1}{\widehat{R}^{2*}(k_n)}, \frac{1}{\widehat{R}^2(k_n)}\right)$ .

By Proposition 1, conditioned on the sample tail  $Z_{n-k_n, n}, \dots, Z_{n, n}$ ,

$$k_n^{1/2} \left( \frac{1}{\widehat{R}^{2*}(k_n)} - \frac{1}{\widehat{R}^2(k_n)} \right)$$

converges weakly, in probability, to  $N(0, 8/R^4)$ , which implies that

$$\frac{1}{\widehat{R}^{2*}(k_n)} - \frac{1}{\widehat{R}^2(k_n)} = o_{P^*}(1),$$

in probability. Since  $1/\widehat{R}^2(k_n)$  is a consistent estimator of  $1/R^2$ , we have that  $\frac{1}{\widehat{R}^{2*}(k_n)} - \frac{1}{R^2} = o_{P^*}(1)$  and, consequently,  $\xi_{k_n} - \frac{1}{R^2} = o_{P^*}(1)$ , in probability. Hence, conditioned on  $Z_{n-k_n, n}, \dots, Z_{n, n}$ ,

$$\frac{1}{\sqrt{2}R} k_n^{1/2} \left( \widehat{R}^*(k_n) - \widehat{R}(k_n) \right)$$

converges weakly, in probability, to  $N(0, 1)$  which implies the result.  $\square$

**Proof of Corollaries 2 and 4.** See the proof of Corollary 1 in Bacro and Brito (1998).

**Proof of Corollary 3.** See the proof of Proposition 1.3 in Beran and Ducharme (1981).

## 4 Estimating the adjustment coefficient in risk theory

The problem of estimating the exponential tail coefficient  $R$  is motivated by an important problem in risk theory. Consider the Sparre Andersen model for claims arriving at an insurance company, and assume that the sequence  $C_1, C_2, \dots$  of claims occur at times  $T_1, T_1 + T_2, \dots$ , where  $\{C_i\}$  and  $\{T_i\}$  are independent sequences of i.i.d. r.v.'s with finite means. Denoting by  $C(t)$  the total sum of claims up to time  $t$ , we may write,

$$C(t) = \sum_{i=1}^{N(t)} C_i,$$

where

$$N(t) = \max\{n \geq 0 : \sum_{i=1}^n T_i \leq t\}, \quad t \geq 0,$$

is the number of claims observed up to time  $t$ . Starting with an initial capital  $x$  and compensating the claim process  $\{C(t)\}$  by incoming premiums with constant rate  $\gamma > 0$  per unit time, the risk reserve of the company is defined by

$$S(t) = x + \gamma t - C(t).$$

The probability of ruin is then given by

$$\begin{aligned} U(x) &= P\left(\inf_{t>0} S(t) < 0\right) \\ &= P\left(\sup_{n \geq 1} \sum_{i=1}^n (C_i - \gamma T_i) > x\right). \end{aligned}$$

Define i.i.d. r.v.'s by  $D_i := C_i - \gamma T_i$  for  $i = 1, 2, \dots$  and consider the associated random walk  $S_0 = 0$ ,  $S_n = D_1 + \dots + D_n$ ,  $n = 1, 2, \dots$ . We assume that  $E(D_1) < 0$  so that, as is well-known,  $U(x) = P\{\sup_{n \geq 1} S_n > x\} < 1$ .

We consider also the following standard conditions

**(H1)** There exists  $R > 0$  such that  $E(e^{RD_1}) = 1$ .

**(H2)**  $E(|D_1| | e^{RD_1}) < \infty$ .

The solution  $R$  is called the *adjustment coefficient*. Its importance is a consequence of the well-known Lundberg inequality, which, under **(H1)** gives an exponential upper bound for the ruin probability

$$U(x) \leq e^{-Rx}, \text{ for all } x > 0,$$

and of the Cramér-Lundberg approximation, that, under both conditions **(H1)** and **(H2)**, gives the asymptotic relationship

$$U(x) \sim ae^{-Rx}, \text{ as } x \rightarrow \infty,$$

where  $a$  is a positive constant (see e.g. Rolski et al. (1999), Sections 6.5.2 e 6.5.3). Different approaches have been used for estimating the adjustment coefficient  $R$  (see e.g. Embrechts and Mikosch (1991), Pitts et al. (1996) and references therein). In particular, Csörgő and Steinebach (1991) suggested to estimate  $R$  by means of a sequence of auxiliary r.v.'s  $\{Z_k\}$ , recursively defined as follows.

$$\begin{aligned} M_0 &= 0, \quad M_n = \max\{M_{n-1} + D_n, 0\} \quad \text{for } n = 1, 2, \dots, \\ \nu_0 &= 0, \quad \nu_k = \min\{n \geq \nu_{k-1} + 1 : M_n = 0\} \quad \text{for } k = 1, 2, \dots, \\ Z_k &= \max_{\nu_{k-1} < j \leq \nu_k} M_j \quad \text{for } k = 1, 2, \dots \end{aligned}$$

$Z_1, Z_2, \dots$  defines a sequence of i.i.d. r.v.'s. In the context of queueing models the r.v.  $Z_k$  may be interpreted as the maximum waiting time in the  $k$ -th

busy cycle of a GI/G/1 queueing system. In this context, Cohen (1969) computed the exact distribution of  $Z_1$  in the cases where

**(H3)**  $\{C(t)\}$  is a compound Poisson process or the claims  $C_i$  are exponentially distributed.

As a consequence, Csörgő and Steinebach observed that, in both cases,

$$P(Z_1 > z) = ce^{-Rz}(1 + O(e^{-Az})) \quad \text{as } z \rightarrow \infty, \quad (16)$$

with positive constants  $c$  and  $A$ . Hence, under one of the assumptions stated in **(H3)** condition (1) holds. Based on this fact, Schultze and Steinebach (1996) used their least square estimators, proposed for estimating the exponential tail coefficient in the family (1), to estimate the adjustment coefficient under **(H3)**. In Brito and Moreira Freitas (2003) the same approach was used to propose  $\hat{R}(k_n)$  as a consistent estimator of the adjustment coefficient. But, as is observed in Richter et al. (1993), **(H3)** is a too strong condition to obtain the consistency of the estimators. In particular, **(H3)** implies **(SR1)**, which is a very convenient assumption in order to obtain rates of consistency for these type of estimators, as seen in Corollaries 2 and 4.

By the following result of Iglehart (1972), concerning the tail behaviour of random walks, the standard conditions for the Sparre Andersen model stated in **(H1)** and **(H2)** are in fact sufficient to ensure that  $Z_1$  has the exponential tail behaviour (1).

**Theorem 4** (Iglehart (1972), Theorem 1) *Let  $X_1, X_2, \dots$  be a sequence of i.i.d. r.v.'s,  $S_0 = 0$ ,  $S_n = X_1 + \dots + X_n$ ,  $n = 1, 2, \dots$  and  $\eta = \min\{n \geq 1 : S_n \leq 0\}$ .*

*If  $E(X_1) < 0$  and there exists a number  $\lambda > 0$  such that  $E(e^{\lambda X_1}) = 1$  and  $E(|X_1| e^{\lambda X_1}) < \infty$ , then*

$$P(\max\{0, \dots, S_{\eta-1}\} > z) \sim ce^{-\lambda z},$$

*where  $c$  is a positive constant.*

Since in the Sparre Andersen model,  $Z_1 = \max\{0, \dots, S_{\nu_1-1}\}$ , the application of Theorem 4 to the r.v.'s  $D_i$  yields

**Corollary 5** *If **(H1)** and **(H2)** hold, then*

$$P(Z_1 > z) = ce^{-Rx}(1 + o(1)) \quad \text{as } z \rightarrow \infty, \quad (17)$$

*where  $c$  is a positive constant.*

This implies, in particular, the consistency of the Schultze and Steinebach (1996) estimators,  $\widehat{R}(k_n)$  and also the Hill-type estimator in the Sparre Andersen model under the standard conditions and gives an answer to a question raised in Deheuvels and Steinebach (1990) concerning the tail behaviour of the auxiliary r.v.'s.

Consider now the particular cases **(H3)**. Recall that Corollaries 2 and 4 may be applied to the family given by (16) (cf. Brito and Moreira Freitas (2003), Corollary 1). For this family,  $L(x) = c\{1 + O(x^{-A})\}$ . The relation **(SR1)** is then satisfied with the regularly varying function  $g(x) = x^{-A}$ . In this case,

$$k_n^{1/2}(\exp(F^{-1}(1 - k_n/n)))^{-A} \sim c^{-A/R} k_n^{1/2} (k_n/n)^{A/R} \text{ as } n \rightarrow \infty.$$

Then, if  $k_n \rightarrow \infty$  such that  $k_n = o(n^{2A/(2A+R)})$ , the results of Corollaries 2 and 4 hold.

For the case of a compound Poisson claim process  $\{C(t)\}$  with exponentially distributed claims the exact d.f. of the r.v.'s  $Z_1, Z_2, \dots$  is given by

$$F(z) = \frac{1 - ae^{-(1-a)z/\beta}}{1 - a^2e^{-(1-a)z/\beta}}, \quad z > 0, \quad (18)$$

where  $a := \beta/\alpha$  and  $\alpha := E(\gamma T_1) > E(C_1) =: \beta$  (cf. Cohen (1969)). Moreover,

$$R = \frac{\alpha - \beta}{\alpha\beta}.$$

In this case,  $1 - F(z) = a(1 - a)e^{-Rz}\{1 + O(e^{-Rz})\}$  as  $z \rightarrow \infty$ , and so the condition of Corollaries 2 and 4 is satisfied for  $k_n = o(n^{2/3})$ .

## 5 Simulation results

In this section we perform a small simulation study, in order to examine the finite sample behavior of the tail bootstrap approximation.

For the sake of comparison with previous simulation studies, we consider the family defined by (18) with  $(\alpha, \beta) = (24000, 10000)$ .

We successively take  $n = 5000$  with  $k_n = 600$  and  $k_n = 800$ , and  $n = 10000$  with  $k_n = 1000$  and  $k_n = 2000$ . The number of tail bootstrap replications was equal to 20000.

The resulting histograms are given in Figs. 1-4. They look quite normal as predicted by the theory.

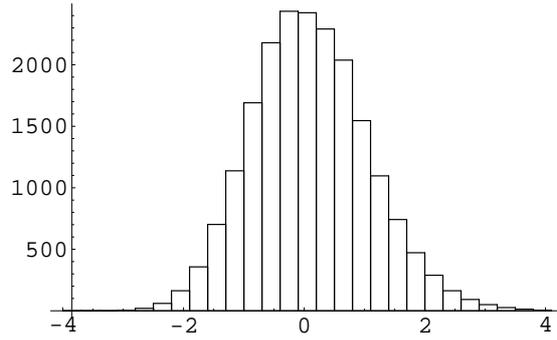


Figure 1: Histogram of 20000 bootstrap replications for the estimator  $\widehat{R}^*(k_n)$  with  $n = 5000$  and  $k_n = 600$ . Mean = 0.0986; SD = 0.9669.

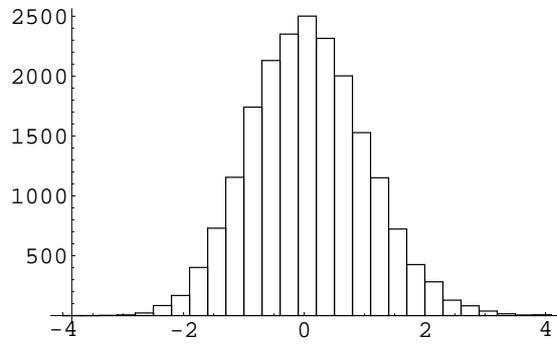


Figure 2: Histogram of 20000 bootstrap replications for the estimator  $\widehat{R}^*(k_n)$  with  $n = 5000$  and  $k_n = 800$ . Mean = 0.0735; SD = 0.9607.

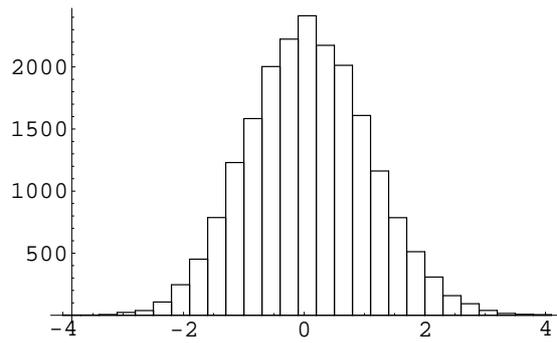


Figure 3: Histogram of 20000 bootstrap replications for the estimator  $\widehat{R}^*(k_n)$  with  $n = 10000$  and  $k_n = 1000$ . Mean = 0.0786; SD = 1.0089.

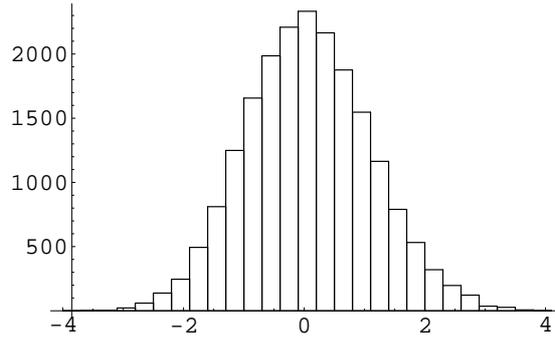


Figure 4: Histogram of 20000 bootstrap replications for the estimator  $\hat{R}^*(k_n)$  with  $n = 10000$  and  $k_n = 2000$ . Mean = 0.0591; SD = 1.0151.

## Acknowledgements

Both authors acknowledge support from CMUP, financed by FCT (Portugal) through the programs POCTI and POSI, with national and European Community structural funds.

## References

- Bacro, J.N. and Brito, M. (1998). A tail bootstrap procedure for estimating the tail Pareto index. *Journal of Statistical Planning and Inference* 71, 245-260.
- Beran, R. and Ducharme, G.R. (1991). Asymptotic theory for bootstrap methods in statistics. *Publication C.R.M. Universit es de Montreal*.
- Bickel, P.J. and Freedman, D.A. (1981). Some asymptotic theory for the bootstrap. *The Annals of Statistics* 9, 1196-1217.
- Brito, M. and Moreira, A.C. (2001). Estimac o do Coeficiente de Cauda Exponencial. In: *“Um Olhar sobre a Estatstica”* (P. Oliveira and E. Athayde, eds), Ed. SPE, 223-237.
- Brito, M. and Moreira Freitas, A.C. (2003). Limiting behaviour of a geometric-type estimator for tail indices. *Insurance: Mathematics and Economics* 33, 211-226.
- Cohen, J. W. (1969). *The single server queue*. North-Holland, Amsterdam.
- Cs rg , M. and Steinebach, J. (1991). On the estimation of the adjustment coefficient in risk theory via intermediate order statistics. *Insurance: Mathematics and Economics* 10, 37-50.
- Deheuvels, P., Haeusler, E. and Mason, D.M. (1988). Almost sure convergence of the Hill estimator. *Mathematical Proceedings of the Cambridge Philosophical Society* 104, 371-381.

- Deheuvels, P. and Steinebach, J. (1990). On some alternative estimates of the adjustment coefficient in risk theory. *Scandinavian Actuarial Journal* 135-159.
- Efron, B. (1979). Bootstrap methods: another look at the Jackknife. *The Annals of Statistics* 7, 1-26.
- Efron, B. and Tibshirani, R.J. (1993). *An Introduction to the Bootstrap*. Chapman & Hall, New York.
- Embrechts, P. and Mikosch, T. (1991). A bootstrap procedure for estimating the adjustment coefficient. *Insurance: Mathematics and Economics* 10, 181-190.
- Haeusler, E. and Teugels, J.L. (1985). On asymptotic normality of Hill's estimator for the exponent of regular variation. *The Annals of Statistics* 13, 743-756.
- Hill, B.M. (1975). A simple approach to inference about the tail of a distribution. *The Annals of Statistics* 3, 1163-1174.
- Iglehart, D.L. (1972). Extreme values in the GI/G/1 queue. *Ann. Math. Statist.* 43, 627-635.
- Pitts, S.M., Grübel, R. and Embrechts, P. (1996). Confidence bounds for the adjustment coefficient. *Advances in Applied Probability* 28, 802-827.
- Richter, W.-D., Steinebach, J. and Taube, S. (1993). On a class of estimators for an exponential tail coefficient with applications in risk theory. *Statistics & Decisions*, Supplement Issue 3, 145-173.
- Rolski, T., Schmidli, H., Schmidt, V. and Teugels, J. (1999). *Stochastic processes for insurance and finance*. Wiley, New York.
- Schultze, J. and Steinebach, J. (1996). On least squares estimates of an exponential tail coefficient. *Statistics & Decisions* 14, 353-372.