Certain isometries related to the bilateral Laplace transform

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Abstract

We study certain isometries between Hilbert spaces, which are generated by the bilateral Laplace transform

$$F_\Phi(z) = \int_{-\infty}^{\infty} e^{zt}\ \Phi(t)dt,\ z \in \mathbb{C}.$$

In particular, we construct an analog of the Bargmann-Fock type reproducing kernel Hilbert space related to this transformation. It is shown that under some integrability conditions on Φ the Laplace transform $F_\Phi(z),\ z = x + iy$ is an entire function belonging to this space. The corresponding isometrical identities, representations of norms, analogs of the Paley-Wiener and Plancherel's theorems are established. As an application this approach drives us to a different type of real inversion formulas for the bilateral Laplace transform in the mean convergence sense.

Keywords: *bilateral Laplace transform, Hilbert space, Sobolev space, real inversion formula, Fourier transform, Hermite polynomials, Bargmann transform, Plancherel theorem*

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1 Introduction

Let us consider the following Hilbert space $\mathcal{H}$ comprising all entire functions $F(z),\ z = x+iy$ with finite norms

$$||F||_{\mathcal{H}} = \left(\frac{1}{2\pi^{3/2}}\int\int_{\mathbb{C}} |F(z)|^2 e^{-x^2} dxdy\right)^{1/2} < \infty. \tag{1.1}$$

We will call this space as the Bargmann-Fock type space (cf. [9, Ch. 28]). In this paper we will show that the image of the space $L_2(\mathbb{R}; e^{t^2}dt)$ of all square integrable functions $\Phi(t)$ with respect to the measure $e^{t^2}dt$ under the bilateral Laplace transform [2], [7]

$$F_\Phi(z) = \int_{-\infty}^{\infty} e^{zt}\Phi(t)dt, \tag{1.2}$$

coincides with the reproducing kernel Hilbert space (1.1) admitting the reproducing kernel $H(z,\bar{u}) = \sqrt{\pi}e^{(z+\bar{u})^2/4}$. Moreover, the bilateral Laplace transform (1.2) is an isometry of the space $L_2(\mathbb{R}; e^{t^2}dt)$ onto the space (1.1). These results will give us an approach to derive a real inversion inversion formula for the transformation (1.2) $F_\Phi(x), x \in \mathbb{R}$. In fact, as far as we aware there is a gap in real inversion theory for the case of the bilateral Laplace transform. Meanwhile, the reproducing kernel approach was extensively used, for instance in [8] to obtain inversion formulas for different kind of integral transformations. Concerning the probabilistic approach to get real inversion formulas see [5].

We will need in the sequel an auxiliary information about the system of Hermite orthogonal polynomials $H_n(x),\ x \in \mathbb{R},\ n = 0, 1, \ldots$ (see [1], [3], [6]). It is defined by the equality

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2},\ n = 0, 1, \ldots \tag{1.3}$$

and has the following integral representation

$$H_n(x) = \frac{2^n(-i)^n e^{x^2}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2+2txi}t^n dt,\ n = 0, 1, \ldots\ . \tag{1.4}$$

This system is orthogonal on $\mathbb{R}$ with respect to the weight e^{-x^2}. The normalized factor is given by

$$\int_{-\infty}^{\infty} e^{-x^2}H_n^2(x)dx = \sqrt{\pi}2^n n!,\ n = 0, 1, \ldots\ . \tag{1.5}$$

For an arbitrary $x \in \mathbb{R}$ when $n \to \infty$ it has the following asymptotic behavior (see [3, Ch. 4])

$$e^{-x^2/2}H_n(x) = \alpha_n\left(\cos\left(\sqrt{2n+1}x - \frac{n\pi}{2}\right) + r_n(x)\right),\ n \to \infty, \tag{1.6}$$

where

$$\alpha_n = \frac{2^n}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right),$$

for an even n and

$$\alpha_n = \frac{2^{n+1}}{\sqrt{\pi(2n+1)}} \Gamma\left(\frac{n}{2}+1\right),$$

when n is odd. Here $\Gamma(z)$ is Euler's Gamma-function [1, Vol. I]. With the use of Stirling's formula for Gamma-function [1] we easily verify that $\alpha_n \equiv \left(2^{n+1} n^n e^{-n}\right)^{1/2}, \ n \to \infty$. Moreover, $r_n(x)$ has the uniform estimate, namely

$$|r_n(x)| \le \text{const.} \frac{|x|^{5/2}}{n^{1/4}},$$

where the constant does not depend on x and n.

From the theory of generalized Fourier series it follows that every $f \in L_2(\mathbb{R}; e^{-t^2} dt)$ can be represented in terms of the series

$$f(x) = \sum_{n=0}^{\infty} c_n H_n(x), \tag{1.7}$$

where

$$c_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} f(t) H_n(t) e^{-t^2} dt, \ n = 0, 1, \dots \ . \tag{1.8}$$

Series (1.7) is convergent with respect to the norm in $L_2(\mathbb{R}; e^{-t^2} dt)$, i.e.

$$\left|\left| f - \sum_{n=0}^{N} c_n H_n(x) \right|\right|_{L_2(\mathbb{R}; e^{-t^2} dt)} = \left(\int_{-\infty}^{\infty} e^{-t^2} \left| f(t) - \sum_{n=0}^{N} c_n H_n(t) \right|^2 dt \right)^{1/2} \to 0, N \to \infty.$$

Moreover, the Parseval equality takes place

$$||f||^2_{L_2(\mathbb{R}; e^{-t^2} dt)} = \sqrt{\pi} \sum_{n=0}^{\infty} 2^n n! |c_n|^2. \tag{1.9}$$

2 Mapping properties

We begin this section establishing the existence and analytic properties for the bilateral Laplace transform (1.2) of $\Phi \in L_2(\mathbb{R}; e^{t^2} dt)$. Indeed, we have

Lemma 1. *Let $\Phi(t) \in L_2(\mathbb{R}; e^{t^2} dt)$. Then the Laplace transform* (1.2) *exists as a Lebesgue integral, which converges absolutely and uniformly on any compact set of $\mathbb{C}$.*

Moreover, it defines an entire function of the second order having the type $\frac{1}{2}$ *and satisfying the following estimate*

$$|F_\Phi(z)| \le \pi^{1/4} e^{|z|^2/2} ||\Phi||_{L_2(\mathbb{R};e^{t^2}dt)}. \tag{2.1}$$

Proof. Indeed, by the straightforward estimation invoking Schwarz's inequality we derive ($z = x + iy$)

$$|F_\Phi(z)| \le \left(\int_{-\infty}^{\infty} e^{-t^2+2xt} dt\right)^{1/2} \left(\int_{-\infty}^{\infty} |\Phi(t)|^2 e^{t^2} dt\right)^{1/2} = \pi^{1/4} e^{x^2/2} ||\Phi||_{L_2(\mathbb{R};e^{t^2}dt)}$$

$$\le \pi^{1/4} e^{|z|^2/2} ||\Phi||_{L_2(\mathbb{R};e^{t^2}dt)}.$$

Moreover, the integrand in (1.2) is analytic in $\mathbb{C}$ with respect to z and as we have seen integral (1.2) is absolutely and uniformly convergent on any compact set of the complex plane. Therefore $F_\Phi(z)$ is an entire function satisfying estimate (2.1). It is not difficult to prove that the order of this entire function is 2 and the type is $\frac{1}{2}$. Lemma 1 is proved.

Lemma 2. *Under conditions of Lemma 1 we have the following isometric identity for the bilateral Laplace transform* (1.2)

$$\frac{1}{2\pi^{3/2}} \int\int_{\mathbb{C}} |F_\Phi(z)|^2 e^{-x^2} dxdy = \int_{-\infty}^{\infty} |\Phi(t)|^2 e^{t^2} dt. \tag{2.2}$$

Proof. We begin by employing the Parseval equality for the Fourier transform [5] to treat the left-hand side of (2.2). Hence we substitute (1.2) into (2.2) and after integration with respect to y we derive

$$\int\int_{\mathbb{C}} |F_\Phi(z)|^2 e^{-x^2} dxdy = 2\pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2+2xt} |\Phi(t)|^2 dxdt.$$

Hence integrating with respect to x we easily arrive at (2.2). Lemma 2 is proved.

As an immediate consequence of this lemma we will show that under the Laplace transform (1.2) the Hilbert space (1.1) forms a reproducing kernel Hilbert space, which admits the reproducing kernel

$$H(z, \bar{u}) = \int_{-\infty}^{\infty} e^{-x^2+(z+\bar{u})x} dx = \sqrt{\pi} e^{(z+\bar{u})^2/4}. \tag{2.3}$$

In fact, the corresponding inner product gives the result

$$\langle F_\Phi(z), H(z,\bar{u})\rangle = \frac{1}{2\pi^{3/2}} \int\int_{\mathbb{C}} F_\Phi(z)\overline{H(z,\bar{u})} e^{-x^2} dxdy$$

$$= \int_{-\infty}^{\infty} e^{t^2}\Phi(t) e^{-t^2+ut} dt = F_\Phi(u),$$

that is the reproducing property is verified. Furthermore, it is straightforward to get that the set of functions $\{e^{zt},\ z \in \mathbb{C}\}$ is complete in $L_2(\mathbb{R}; e^{t^2}dt)$, i.e. the equality $F_\Phi(z) \equiv 0$ yields $\Phi = 0$ almost everywhere. Thus the bilateral Laplace transform is an isometry from $L_2(\mathbb{R}; e^{t^2}dt)$ onto $\mathcal{H}$ (an analog of the Paley-Wiener theorem).

For the real bilateral Laplace transform $F_\Phi(x), x \in \mathbb{R}$ we prove

Lemma 3. *Let $\Phi \in L_2(\mathbb{R}; e^{t^2}dt)$. Then $F_\Phi(x)$ is infinitely smooth, i.e. $F_\Phi \in C^\infty(\mathbb{R})$. Moreover, all derivatives $\frac{d^n}{dx^n}\left(e^{-x^2/2}F_\Phi(x)\right),\ n \in \mathbb{N}_0$ belong to $L_2(\mathbb{R}; dx)$ and satisfy the following inequality*

$$\int_{-\infty}^{\infty}\left|\frac{d^n}{dx^n}\left(e^{-x^2/2}F_\Phi(x)\right)\right|^2 dx \le 2\pi n!||\Phi||^2_{L_2(\mathbb{R};e^{t^2}dt)},\ n = 0, 1, 2, \dots\ . \tag{2.4}$$

Proof. We have

$$e^{-x^2/2}F_\Phi(x) = \int_{-\infty}^{\infty}\Phi(t)e^{t^2/2}e^{-\frac{(x-t)^2}{2}}dt. \tag{2.5}$$

Hence, it is not difficult to verify that on any compact set of $\mathbb{R}$ we can differentiate through with respect to x in the latter integral. As a result we obtain

$$\frac{d^n}{dx^n}\left(e^{-x^2/2}F_\Phi(x)\right) = \int_{-\infty}^{\infty}\Phi(t)e^{t^2/2}\frac{\partial^n}{\partial x^n}e^{-\frac{(x-t)^2}{2}}dt$$
$$= (-1)^n 2^{-n/2}\int_{-\infty}^{\infty}\Phi(t)e^{t^2/2}e^{-\frac{(x-t)^2}{2}}H_n\left(\frac{x-t}{\sqrt{2}}\right)dt, \tag{2.6}$$

where $H_n(y),\ n \in \mathbb{N}_0$ is the system of Hermite polynomials (1.3). Applying the Schwarz inequality, making elementary substitutions and taking into account the value of the normalized factor (1.5) we derive the estimate

$$\left|\frac{d^n}{dx^n}\left(e^{-x^2/2}F_\Phi(x)\right)\right|^2 \le 2^{-n}\int_{-\infty}^{\infty}|\Phi(t)|^2e^{t^2}e^{-\frac{(x-t)^2}{2}}dt$$
$$\times\int_{-\infty}^{\infty}e^{-\frac{(x-t)^2}{2}}H_n^2\left(\frac{x-t}{\sqrt{2}}\right)dt = 2^{-n+\frac{1}{2}}\int_{-\infty}^{\infty}|\Phi(t)|^2e^{t^2}e^{-\frac{(x-t)^2}{2}}dt$$
$$\times\int_{-\infty}^{\infty}e^{-y^2}H_n^2(y)\,dy = \sqrt{2\pi}n!\int_{-\infty}^{\infty}|\Phi(t)|^2e^{t^2}e^{-\frac{(x-t)^2}{2}}dt. \tag{2.7}$$

Hence integrating through with respect to x in (2.7) we change the order of integration via Fubini's theorem and we get the inequality

$$\int_{-\infty}^{\infty}\left|\frac{d^n}{dx^n}\left(e^{-x^2/2}F_\Phi(x)\right)\right|^2 dx \le \sqrt{2\pi}n!\int_{-\infty}^{\infty}|\Phi(t)|^2e^{t^2}\int_{-\infty}^{\infty}e^{-\frac{(x-t)^2}{2}}dxdt$$
$$= 2\pi n!\int_{-\infty}^{\infty}|\Phi(t)|^2e^{t^2}dt,$$

which yields (2.4). Lemma 3 is proved.

3 Plancherel's type theorem

In this section we establish an isometry between two Hilbert spaces, which is realized by the bilateral Laplace transform (1.2) of real variable. The main result will be Plancherel's type theorem for this case of the Laplace transformation.

Let us consider a Sobolev's type space of the infinite order $W_2^{\infty}(\mathbb{R})$ of complex-valued functions on $\mathbb{R}$ that are n times differentiable in a sense of generalized derivatives for all nonnegative integers n. In fact, this is a completion of the corresponding pre-Hilbert space equipped with the inner product

$$(f,g) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} \frac{d^n f}{dx^n} \overline{\frac{d^n g}{dx^n}} dx. \tag{3.1}$$

The norm of this space is given accordingly

$$||f||_{W_2^{\infty}(\mathbb{R})} = \left(\sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} \left| \frac{d^n f}{dx^n} \right|^2 dx \right)^{1/2}. \tag{3.2}$$

Theorem 1. (Plancherel's theorem). *The map* $\Phi(t) \to e^{-x^2/2} F_{\Phi}(x)$, *where* $F_{\Phi}(x)$ *is given by* (1.2) *is a continuous linear map from* $L_2(\mathbb{R}; 2\pi e^{t^2} dt)$ *into* $W_2^{\infty}(\mathbb{R})$ *which is an isometry, i.e. Plancherel's formula holds*

$$\left|\left| e^{-x^2/2} F_{\Phi}(x) \right|\right|_{W_2^{\infty}(\mathbb{R})} = ||\Phi||_{L_2(\mathbb{R}; 2\pi e^{t^2} dt)}. \tag{3.3}$$

Furthermore, if $\Phi, \Psi \in L_2(\mathbb{R}; \sqrt{2\pi} e^{t^2} dt)$ *then Parseval's equality holds*

$$2\pi \int_{\mathbb{R}} e^{t^2} \Phi(t) \overline{\Psi(t)} dt = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} \frac{d^n}{dx^n} e^{-x^2/2} F_{\Phi}(x) \frac{d^n}{dx^n} e^{-x^2/2} \overline{F_{\Psi}(x)} dx, \tag{3.4}$$

where the series in (3.4) *is absolutely convergent.*

Proof. By Lemma 1 $e^{-x^2/2} F_{\Phi}(x)$ is well-defined and by Lemma 3 (see (2.4)) all derivatives are bounded as linear operators $L_2(\mathbb{R}; e^{t^2} dt) \to L_2(\mathbb{R}; dx)$. Further, employing representation (2.6), invoking (1.8) and making elementary substitution we easily see that

$$\frac{d^n}{dx^n} \left(e^{-x^2/2} F_{\Phi}(x) \right) = (-1)^n 2^{-(n-1)/2} \int_{-\infty}^{\infty} \Phi(x - \sqrt{2} y) e^{(x-\sqrt{2}y)^2/2} e^{-y^2} H_n(y)\, dy$$

$$= \sqrt{2\pi} (-1)^n n! 2^{n/2} c_n(x), \tag{3.5}$$

where we denote by $c_n(x)$ Fourier coefficients (1.8) of the function $\Phi(x - \sqrt{2} y) e^{(x-\sqrt{2}y)^2/2}$ for each $x \in \mathbb{R}$, namely

$$c_n(x) = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} \Phi(x - \sqrt{2} y) e^{(x-\sqrt{2}y)^2/2} H_n(y) e^{-y^2} dy, \ n = 0, 1, \dots\ . \tag{3.6}$$

Hence combining with (3.5) the Parseval equality (1.9) yields

$$\int_{-\infty}^{\infty} \left|\Phi(x-\sqrt{2}y)\right|^2 e^{(x-\sqrt{2}y)^2-y^2} dy = \sqrt{\pi}\sum_{n=0}^{\infty} 2^n n! |c_n(x)|^2$$

$$= \frac{1}{2\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{1}{n!} \left| \frac{d^n}{dx^n} \left(e^{-x^2/2} F_\Phi(x) \right) \right|^2 . \tag{3.7}$$

Therefore integrating through in (3.7) with respect to x we use Fubini's theorem and after straightforward calculation of the inner integral we express the left-hand side of the obtained equality as

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \left|\Phi(x-\sqrt{2}y)\right|^2 e^{(x-\sqrt{2}y)^2-y^2} dy = \sqrt{\pi} ||\Phi||^2_{L_2(\mathbb{R};e^{t^2}dt)}.$$

Meanwhile, the order of integration and summation on the corresponding right-hand side can be inverted appealing to Levi's theorem. Thus we easily come up with isometric identity (3.3) and we prove the continuity of the map $e^{-x^2/2}F_\Phi(x)$ from $L_2(\mathbb{R}; 2\pi e^{t^2}dt)$ into $W_2^\infty(\mathbb{R})$. Finally, relation (3.4) follows immediately by using the parallelogram identity. The absolute convergence of the series in (3.4) can be easily established with the Cauchy-Schwarz inequality. Theorem 1 is proved.

Remark 1. We note that in [9, Chap. 28] the reproducing kernel approach has been used as a way of getting isometric identities for the Bargmann transform.

Combining with Lemma 2 we arrive at

Corollary 1. *Any entire function* $F(z)$ *with a finite integral* (1.1) *satisfies the following identity*

$$\frac{1}{\sqrt{\pi}} \int\int_{\mathbb{C}} |F(z)|^2 e^{-x^2} dxdy = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} \left| \frac{d^n}{dx^n} \left(e^{-x^2/2} F(x) \right) \right|^2 dx. \tag{3.8}$$

Corollary 2. *Almost for all* $t \in \mathbb{R}$ *it has the following left inverse operator for the bilateral Laplace transform* (1.2)

$$\Phi(t) = \frac{e^{-t^2}}{2\pi} \frac{d}{dt} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} \frac{d^n}{dx^n} \left(e^{-x^2/2} F_\Phi(x) \right) \frac{\partial^n}{\partial x^n} \left(e^{-x^2/2} \frac{e^{xt}-1}{x} \right) dx. \tag{3.9}$$

Moreover, if $e^{-x^2/2}F_\Phi(x)$ *is such that*

$$\sum_{n=0}^{\infty} \frac{n+1}{n!} \int_{\mathbb{R}} \left| \frac{d^n}{dx^n} e^{-x^2/2} F_\Phi(x) \right|^2 dx < \infty, \tag{3.10}$$

then (3.9) *can be written in the form*

$$\Phi(t) = \frac{e^{-t^2}}{\sqrt{2}\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! 2^n} \int_{\mathbb{R}} \frac{\partial^n}{\partial x^n} \left(e^{-x^2-\sqrt{2}xt} F_{\Phi}\left(\sqrt{2}x + t\right)\right) e^{-x^2} H_n(x) dx. \tag{3.11}$$

Proof. Indeed, the Parseval equality (3.4) gives a key to prove (3.9). Putting

$$\Psi(y) = \begin{cases} 1, & \text{if } y \in [0,t], \\ 0, & \text{if } y \in \mathbb{R}\backslash[0,t], \end{cases}$$

calculating F_Ψ and differentiating through in (3.4) with respect to t we arrive at the inversion (3.9).

In order to establish (3.11) we should motivate the passage of the derivative under signs of the series and the integral in (3.9). In fact, we will appeal to the condition (3.10), the Cauchy-Schwarz and Schwarz inequalities to verify the absolute and uniform convergence with respect to t in the right-hand side of (3.9) after formal differentiation. Precisely, it is not difficult to proceed the following estimates

$$\frac{e^{-t^2}}{2\pi} \sum_{n=0}^{\infty} \frac{1}{n!} \left| \int_{\mathbb{R}} \frac{d^n}{dx^n} \left(e^{-x^2/2} F_{\Phi}(x)\right) \frac{\partial^n}{\partial x^n} e^{xt - x^2/2} dx \right| \le \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$\times \left(\int_{\mathbb{R}} \left| \frac{d^n}{dx^n} \left(e^{-x^2/2} F_{\Phi}(x)\right) \right|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}} \left| \frac{\partial^n}{\partial x^n} e^{-(x-t)^2/2} \right|^2 dx \right)^{1/2} \le \frac{1}{2\pi}$$

$$\left(\sum_{n=0}^{\infty} \frac{n+1}{n!} \int_{\mathbb{R}} \left| \frac{d^n}{dx^n} \left(e^{-x^2/2} F_{\Phi}(x)\right) \right|^2 dx \right)^{1/2} \left(\sum_{n=0}^{\infty} \frac{1}{n! 2^n (n+1)} \right.$$

$$\left. \times \int_{\mathbb{R}} e^{-(x-t)^2} H_n^2 \left(\frac{x-t}{\sqrt{2}} \right) dx \right)^{1/2}. \tag{3.12}$$

The latter integral in (3.12) can be calculated, in turn, by representation (1.4) and the Parseval equality for the Fourier transform. Hence we obtain

$$\int_{\mathbb{R}} e^{-(x-t)^2} H_n^2 \left(\frac{x-t}{\sqrt{2}} \right) dx = \sqrt{2} \int_{-\infty}^{\infty} e^{-2y^2} H_n^2(y)\, dy$$

$$= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(t/2)^2 + tyi} t^n dt \right|^2 dy = \sqrt{2} \int_0^{\infty} e^{-t^2/2} t^{2n} dt = 2^n \Gamma\left(n + \frac{1}{2}\right).$$

Substituting this value into (3.12) and invoking Stirling's asymptotic formula for the Gamma-function (see [1, Vol. I]) we come out with the following uniform estimate

$$\frac{e^{-t^2}}{2\pi} \sum_{n=0}^{\infty} \frac{1}{n!} \left| \int_{\mathbb{R}} \frac{d^n}{dx^n} \left(e^{-x^2/2} F_{\Phi}(x)\right) \frac{\partial^n}{\partial x^n} e^{xt - x^2/2} dx \right| \le \frac{1}{2\pi}$$

$$\left(\sum_{n=0}^{\infty} \frac{n+1}{n!} \int_{\mathbb{R}} \left| \frac{d^n}{dx^n} \left(e^{-x^2/2} F_\Phi(x) \right) \right|^2 dx \right)^{1/2} \left(\sum_{n=0}^{\infty} \frac{1}{n!(n+1)} \Gamma\left(n + \frac{1}{2}\right)\right)^{1/2}$$

$$= \frac{1}{2\pi} \left(\sum_{n=0}^{\infty} \frac{n+1}{n!} \int_{\mathbb{R}} \left| \frac{d^n}{dx^n} \left(e^{-x^2/2} F_\Phi(x) \right) \right|^2 dx \right)^{1/2} \left(O\left(\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(n+1)} \right)\right)^{1/2} < \infty.$$

Therefore making elementary manipulations in the right-hand side of (3.9) we arrive at (3.11). Corollary 2 is proved.

4 A Reproducing kernel approach

As we could see above in Lemma 2 the map $\Phi \to F_\Phi$ is an isometry from $L_2(\mathbb{R}; e^{t^2}dt)$ onto the Bargmann-Fock type space $\mathcal{H}$ with the norm (1.1). As we show next, this map is not just an isometry, but it is, in fact, an invertible isometry from $L_2(\mathbb{R}; e^{t^2}dt)$ onto $\mathcal{H}$. This approach will drive us to an inversion formula for the bilateral Laplace transform (1.2) in the mean convergence sense.

Theorem 2. *Let $F \in \mathcal{H}$. For any non-negative integer we set*

$$F_N(z) = \frac{1}{2\pi} \sum_{n=0}^{N} \frac{1}{n!} \int_{\mathbb{R}} \frac{d^n}{dx^n} \left(e^{-x^2/2} F(x) \right) \frac{\partial^n}{\partial x^n} \left(e^{-x^2/2} H(x, z) \right) dx, \tag{4.1}$$

where the reproducing kernel $H(z, \bar{u})$ is defined by (2.3). *Then $F_N \in \mathcal{H}$ and the sequence $\{F_N\}_{N=0}^{\infty}$ converges to F in $\mathcal{H}$. Moreover, reciprocally*

$$\Phi_N(t) = \frac{e^{-t^2}}{2\pi} \int_{-\infty}^{\infty} e^{xt-x^2} F(x) P_N(x-t) dx \tag{4.2}$$

where the polynomials $P_N(\xi)$ are expressed by

$$P_N(\xi) = \sum_{n=0}^{N} \frac{(-1)^n}{n! 2^n} H_{2n}\left(\frac{\xi}{\sqrt{2}}\right) \tag{4.3}$$

converges to Φ when $N \to \infty$ with respect to the norm in the space $L_2(\mathbb{R}; e^{t^2}dt)$ and $F_{\Phi_N} = F_N$. Finally, the bilateral Laplace transform (1.2) *is an invertible isometry from $L_2(\mathbb{R}; e^{t^2}dt)$ onto $\mathcal{H}$, $F(z) = F_\Phi(z)$ and the Plancherel identity* (2.2) *holds.*

Proof. Recalling (2.3) by straightforward calculations we derive

$$\frac{\partial^n}{\partial x^n} \left(e^{-x^2/2} H(x, z) \right) = \frac{\partial^n}{\partial x^n} \left(e^{-x^2/2} \int_{-\infty}^{\infty} e^{-t^2+(x+z)t} dt \right)$$

$$= \int_{-\infty}^{\infty} e^{-t^2/2+zt} \frac{\partial^n}{\partial x^n} e^{-(x-t)^2/2} dt.$$

Substituting the latter integral into (4.1) and invoking (1.2) we obtain that $F_N(z) = F_{\Phi_N}(z)$, where

$$\Phi_N(t) = \frac{e^{-t^2/2}}{2\pi} \sum_{n=0}^{N} \frac{1}{n!} \int_{-\infty}^{\infty} \frac{d^n}{dx^n} \left(e^{-x^2/2} F(x) \right) \frac{d^n}{dx^n} e^{-(x-t)^2/2} dx. \tag{4.4}$$

Consequently, by Lemma 2 we get $||F_N||^2_{\mathcal{H}} = ||F_{\Phi_N}||^2_{\mathcal{H}} = ||\Phi_N||^2_{L_2(\mathbb{R};e^{t^2}dt)}$.

Meanwhile, invoking (1.1), (1.5), (3.8), the Minkowski, generalized Minkowski inequalities and Cauchy- Schwarz's inequality for the series we find

$$||\Phi_N||_{L_2(\mathbb{R};e^{t^2}dt)} = \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} \left| \sum_{n=0}^{N} \frac{1}{n!} \int_{-\infty}^{\infty} \frac{d^n}{dx^n} \left(e^{-x^2/2} F(x) \right) \frac{\partial^n}{\partial x^n} e^{-(x-t)^2/2} dx \right|^2 dt \right)^{1/2}$$

$$\leq \frac{1}{2\pi} \sum_{n=0}^{N} \frac{1}{n!} \left(\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \frac{d^n}{dx^n} \left(e^{-x^2/2} F(x) \right) \frac{\partial^n}{\partial x^n} e^{-(x-t)^2/2} dx \right|^2 dt \right)^{1/2}$$

$$= \frac{1}{2\pi} \sum_{n=0}^{N} \frac{1}{n!} \left(\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \frac{\partial^n}{\partial t^n} \left(e^{-(x+t)^2/2} F(x+t) \right) \frac{d^n}{dx^n} e^{-x^2/2} dx \right|^2 dt \right)^{1/2}$$

$$\leq \frac{1}{2\pi} \sum_{n=0}^{N} \frac{1}{n!} \int_{-\infty}^{\infty} \left| \frac{d^n}{dx^n} e^{-x^2/2} \right| \left(\int_{-\infty}^{\infty} \left| \frac{\partial^n}{\partial t^n} \left(e^{-(x+t)^2/2} F(x+t) \right) \right|^2 dt \right)^{1/2} dx$$

$$= \frac{1}{2\pi} \sum_{n=0}^{N} \frac{1}{n!} \left(\int_{-\infty}^{\infty} \left| \frac{d^n}{dt^n} \left(e^{-t^2/2} F(t) \right) \right|^2 dt \right)^{1/2} \int_{-\infty}^{\infty} \left| \frac{d^n}{dx^n} e^{-x^2/2} \right| dx$$

$$\leq \frac{1}{2\pi} \left(\sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \left| \frac{d^n}{dt^n} \left(e^{-t^2/2} F(t) \right) \right|^2 dt \right)^{1/2}$$

$$\times \left(\sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_{-\infty}^{\infty} \left| \frac{d^n}{dx^n} e^{-x^2/2} \right| dx \right)^2 \right)^{1/2} = \frac{1}{\sqrt{2\pi}} ||F||_{\mathcal{H}} \left(\sum_{n=N}^{\infty} \frac{1}{n! 2^n} \right.$$

$$\left. \times \left(\int_{-\infty}^{\infty} e^{-x^2} |H_n(x)| \, dx \right)^2 \right)^{1/2} \leq \frac{1}{\sqrt{2}\pi^{1/4}} ||F||_{\mathcal{H}} \left(\sum_{n=0}^{N} \frac{1}{n! 2^n} \int_{-\infty}^{\infty} e^{-x^2} H_n^2(x) dx \right)^{1/2}$$

$$= \sqrt{\frac{N+1}{2}} ||F||_{\mathcal{H}} < \infty. \tag{4.5}$$

Therefore $\Phi_N \in L_2(\mathbb{R}; e^{t^2}dt)$ for each $N \in \mathbb{N}$. Further, we prove that $\{\Phi_N\}_{N=1}^{\infty}$ is a Cauchy sequence. But first denoting by

$$f_n = \left(\frac{1}{n!}\int_{-\infty}^{\infty}\left|\frac{d^n}{dt^n}\left(e^{-t^2/2}F(t)\right)\right|^2 dt\right)^{1/2}, \ n \in \mathbb{N}_0, \tag{4.6}$$

we recall condition (3.10), which turns to be $\sum_{n=0}^{\infty}(n+1)f_n^2 < \infty$. Hence we observe that this is a dense subspace of l_2 since it contains the set of sequences

$$f^{(k)} = (f_1, f_2, \ldots, f_k, 0, 0, \ldots), \ k \in \mathbb{N},$$

which is dense in l_2. Therefore, for each element $f = \{f_n\}_{n=1}^{\infty} \in l_2$ and any $\varepsilon > 0$ there is a sequence $g = \{g_n\}_{n=1}^{\infty}$, $\sum_{n=0}^{\infty}(n+1)|g_n|^2 < \infty$ such that $||f - g||_{l_2} < \varepsilon$. Then at least for sufficiently big N it immediately implies

$$|f_N - g_N| \le \left(\sum_{n=N}^{\infty}|f_n - g_n|^2\right)^{1/2} < \frac{1}{N^{3/2}} < \varepsilon. \tag{4.7}$$

Returning to (4.5), invoking (4.6) and taking $M > N$ we obtain

$$||\Phi_N - \Phi_M||_{L_2(\mathbb{R}; e^{t^2}dt)} \le \frac{1}{2\pi}\sum_{n=N+1}^{M} f_n \frac{1}{\sqrt{n!}}\int_{-\infty}^{\infty}\left|\frac{d^n}{dx^n}e^{-x^2/2}\right| dx = \frac{1}{2\pi}\sum_{n=N+1}^{M}(f_n - g_n)$$

$$\times \frac{1}{\sqrt{n!}}\int_{-\infty}^{\infty}\left|\frac{d^n}{dx^n}e^{-x^2/2}\right| dx + \frac{1}{2\pi}\sum_{n=N+1}^{M} g_n \frac{1}{\sqrt{n!}}\int_{-\infty}^{\infty}\left|\frac{d^n}{dx^n}e^{-x^2/2}\right| dx$$

$$\le \frac{1}{2\pi}\sum_{n=N+1}^{M}\frac{1}{n^{3/2}\sqrt{n!}}\int_{-\infty}^{\infty}\left|\frac{d^n}{dx^n}e^{-x^2/2}\right| dx + \frac{1}{2\pi}\left(\sum_{n=N+1}^{M}(n+1)|g_n|^2\right)^{1/2}$$

$$\times \left(\sum_{n=N+1}^{M}\frac{1}{(n+1)n!}\left(\int_{-\infty}^{\infty}\left|\frac{d^n}{dx^n}e^{-x^2/2}\right| dx\right)^2\right)^{1/2} = \frac{1}{\sqrt{2}\pi}\sum_{n=N+1}^{M}\frac{1}{n^{3/2}\sqrt{n!2^n}}$$

$$\times \int_{-\infty}^{\infty} e^{-x^2}|H_n(x)|\, dx + \frac{1}{\sqrt{2}\pi}\left(\sum_{n=N+1}^{M}(n+1)|g_n|^2\right)^{1/2}$$

$$\times \left(\sum_{n=N+1}^{M}\frac{1}{(n+1)n!2^n}\left(\int_{-\infty}^{\infty} e^{-x^2}|H_n(x)|\, dx\right)^2\right)^{1/2} \le \frac{1}{\sqrt{2}\pi^{3/4}}\sum_{n=N+1}^{M}\frac{1}{n^{3/2}}$$

$$+\frac{1}{\sqrt{2}\pi}\left(\sum_{n=N+1}^{M}(n+1)|g_n|^2\right)^{1/2}\left(\sum_{n=0}^{\infty}\frac{1}{(n+1)n!2^n}\left(\int_{-\infty}^{\infty}e^{-x^2}|H_n(x)|\,dx\right)^2\right)^{1/2}\to 0$$

when $N\to\infty$ if we show that the later series converges

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)n!2^n}\left(\int_{-\infty}^{\infty}e^{-x^2}|H_n(x)|\,dx\right)^2<\infty.$$

In fact, it can be done appealing to the asymptotic formula (1.6) for the Hermite polynomials, the Stirling formula for factorials [1, Vol.I] and the behavior of the following integral (see [3, Ch. 4])

$$\int_1^{\infty}e^{-x^2}x^{-3}H_n^2(x)dx=O\left(\frac{n!2^n}{\sqrt{n}}\right),\ n\to\infty.$$

We have

$$\left(\sum_{n=N}^{\infty}\frac{1}{(n+1)n!2^n}\left(\int_{-\infty}^{\infty}e^{-x^2}|H_n(x)|\,dx\right)^2\right)^{1/2}\le\left(\sum_{n=N}^{\infty}\frac{1}{n!2^n(n+1)}\left(\int_{|x|<1}e^{-x^2}|H_n(x)|\,dx\right)^2\right)^{1/2}$$

$$+\text{const.}\left(\sum_{n=N}^{\infty}\frac{1}{n!2^n(n+1)}\int_{|x|>1}e^{-x^2}|x|^{-3}H_n^2(x)dx\right)^{1/2}=O\left(\left(\sum_{n=N}^{\infty}\frac{1}{\sqrt{n}(n+1)}\right)^{1/2}\right)\to 0,$$

when $N\to\infty$. Thus combining with the above estimates we conclude that $||\Phi_N-\Phi_M||_{L_2(\mathbb{R};e^{t^2}dt)}\to 0,\ N,M\to\infty$, i.e. $\{\Phi_N\}_{N=1}^{\infty}$ is a Cauchy sequence. It has a limit, which we denote by Φ. Thus sequence (4.4) converges to Φ with respect to the norm in $L_2(\mathbb{R};e^{t^2}dt)$. On the other hand, via Lemma 2 we find

$$||F_N-F_M||_{\mathcal{H}}=||F_{\Phi_N}-F_{\Phi_M}||_{\mathcal{H}}=||\Phi_N-\Phi_M||_{L_2(\mathbb{R};e^{t^2}dt)}\to 0,\ N,M\to\infty. \tag{4.8}$$

Therefore the sequence $\{F_N(z)\}_{N=1}^{\infty}$ converges to some function $G(z)\in\mathcal{H}$. We will prove that $G(z)=F(z)=F_\Phi(z)$. Indeed, first we observe that (4.1) is a partial sum of the corresponding series, which is equal to $F(z)$ according to (3.4), Lemma 2 and reproducing property of the kernel (2.3). Furthermore, invoking (3.8), (4.6) with Cauchy-Schwarz's inequalities for the integral and series we deduce for any compact set of $\mathbb{C}$ that

$$|F_N(z)-F_M(z)|\le\frac{1}{2\pi}\sum_{n=N+1}^{M}\frac{1}{n!}\int_{\mathbb{R}}\left|\frac{d^n}{dx^n}\left(e^{-x^2/2}F(x)\right)\frac{\partial^n}{\partial x^n}\left(e^{-x^2/2}H(x,z)\right)\right|dx$$

$$\le\frac{1}{\sqrt{2\pi}}\left(\sum_{n=N+1}^{M}f_n^2\right)^{1/2}||H(\cdot,z)||_{\mathcal{H}}=\frac{1}{\sqrt{2\sqrt{\pi}}}\left(\sum_{n=N+1}^{M}f_n^2\right)^{1/2}e^{(\mathrm{Re}z)^2/2}$$

$$\leq \text{const.} \left(\sum_{n=N+1}^{M} f_n^2 \right)^{1/2} \to 0, N \to \infty.$$

Consequently, the sequence $\{F_N(z)\}_{N=1}^{\infty}$ convergence uniformly on any compact set to $F(z)$. Since it converges to G in $\mathcal{H}$ we have that $F(z) \equiv G(z)$ in $\mathbb{C}$. Hence passing to the limit in (4.8) when $M \to \infty$ we get

$$||F_N - F||_{\mathcal{H}} = ||F_{\Phi_N} - F_\Phi||_{\mathcal{H}} = ||\Phi_N - \Phi||_{L_2(\mathbb{R};e^{t^2}dt)} \to 0, \ N \to \infty.$$

This yields $F(z) = F_\Phi(z)$, the Plancherel identity (2.2) holds and (4.4) is a right inverse operator. In order to establish the representation (4.2) we integrate by parts n times in (4.4) eliminating integrated terms and invoking (1.3) with straightforward calculations.

To complete the proof of the theorem we finally show that F_Φ is an invertible isometry, i.e. (4.4) is left inverse too. Indeed, employing (2.6) we substitute it into (4.4). Making elementary changes of variables we arrive at the equalities

$$\Phi_N(t) = \frac{e^{-t^2/2}}{2\pi} \sum_{n=0}^{N} \frac{(-1)^n}{n! 2^{n/2}} \int_{-\infty}^{\infty} \frac{d^n}{dx^n} e^{-(x-t)^2/2} \int_{-\infty}^{\infty} \Phi(y) e^{y^2/2} e^{-\frac{(x-y)^2}{2}} H_n\left(\frac{x-y}{\sqrt{2}}\right) dydx$$

$$= \frac{e^{-t^2/2}}{2\pi} \sum_{n=0}^{N} \frac{1}{n! 2^n} \int_{-\infty}^{\infty} e^{-(x-t)^2/2} H_n\left(\frac{x-t}{\sqrt{2}}\right) \int_{-\infty}^{\infty} \Phi(y) e^{y^2/2} e^{-\frac{(x-y)^2}{2}} H_n\left(\frac{x-y}{\sqrt{2}}\right) dydx$$

$$= \frac{e^{-t^2/2}}{\sqrt{2}\pi} \sum_{n=0}^{N} \frac{1}{n! 2^n} \int\int_{\mathbb{R}^2} \Phi(y) e^{y^2/2} H_n(u) H_n\left(u + \frac{t-y}{\sqrt{2}}\right) e^{-u^2-(u+(t-y)/\sqrt{2})^2} dudy, \quad (4.9)$$

where the latter double integral exists due to Fubini's theorem and the absolute convergence of the iterated integral in (4.9). The inner integral with respect to u can be calculated invoking (1.4) and Parseval's identity for the Fourier convolution [6, Ch. 2]. As a result we come out with the equality

$$\int_{-\infty}^{\infty} H_n(u) H_n\left(u + \frac{t-y}{\sqrt{2}}\right) e^{-u^2-(u+(t-y)/\sqrt{2})^2} du = \frac{(-1)^n \sqrt{\pi}}{2^{n+1/2}} e^{(y-t)^2/4} H_{2n}\left(\frac{y-t}{2}\right).$$

Substituting this into (4.9) and appealing to the summation formula (4.5.1.5) in [4] we obtain the following representation

$$e^{t^2/2}\Phi_N(t) = \frac{(-1)^N}{\sqrt{\pi} 2^{2N+1} N!} \int_{-\infty}^{\infty} \Phi(y) e^{y^2/2-(y-t)^2/4} H_{2N+1}\left(\frac{y-t}{2}\right) \frac{dy}{y-t}. \quad (4.10)$$

But using again (1.4) and the Fourier transform technique we deduce the formula

$$\int_{-\infty}^{\infty} e^{-x^2} H_{2N+1}(x) \frac{dx}{x} = (-1)^N \sqrt{\pi}\, N! 2^{2N+1}.$$

Hence we have

$$e^{t^2/2}\left[\Phi_N(t)-\Phi(t)\right]=\frac{(-1)^N}{\sqrt{\pi}2^{2N+1}N!}\int_{-\infty}^{\infty}\left[\Phi(y)e^{y^2/2}-\Phi(t)e^{t^2/2}\right]e^{-(y-t)^2/4}$$

$$\times H_{2N+1}\left(\frac{y-t}{2}\right)\frac{dy}{y-t}. \tag{4.11}$$

Calling formula (1.6) of the asymptotic behavior for Hermite's polynomials of the odd order we substitute into (4.11) its right-hand side. Invoking the reduction formula for the Gamma-function [1] we find

$$e^{t^2/2}\left[\Phi_N(t)-\Phi(t)\right]=\frac{2N+1}{\pi\sqrt{4N+3}}\frac{\Gamma\left(N+\frac{1}{2}\right)}{\Gamma(N+1)}\int_{-\infty}^{\infty}\left[\Phi(y)e^{y^2/2}-\Phi(t)e^{t^2/2}\right]e^{-(y-t)^2/8}$$

$$\times\left(\sin\left(\sqrt{N+3/4}(y-t)\right)+r_{2N+1}(y-t)\right)\frac{dy}{y-t}. \tag{4.12}$$

Meanwhile, the asymptotic behavior of the ratio of Gamma-functions [1] yields

$$\frac{2N+1}{\sqrt{4N+3}}\frac{\Gamma\left(N+\frac{1}{2}\right)}{\Gamma(N+1)}=1+O\left(N^{-2}\right),\ N\to\infty.$$

Thus

$$e^{t^2/2}\left[\Phi_N(t)-\Phi(t)\right]=\frac{1}{\pi}\int_{-\infty}^{\infty}\left[\Phi(x+t)e^{(x+t)^2/2}-\Phi(t)e^{t^2/2}\right]e^{-x^2/8}$$

$$\times\left(\sin\left(x\sqrt{N+3/4}\right)+r_{2N+1}(x)\right)\frac{dx}{x}$$

$$+O\left(N^{-2}\right)\int_{-\infty}^{\infty}\left[\Phi(x+t)e^{(x+t)^2/2}-\Phi(t)e^{t^2/2}\right]e^{-x^2/8}$$

$$\times\left(\sin\left(x\sqrt{N+3/4}\right)+r_{2N+1}(x)\right)\frac{dx}{x}=I_N(t)+J_N(t). \tag{4.13}$$

Hence employing the estimate (see (1.6)) $|r_{2N+1}(x)|<\text{const.}|x|^{5/2}/N^{1/4}$, the Schwarz inequality and the elementary inequality $|\sin x|\le|x|$, it is not difficult to verify that $J_N(t)\to 0,\ N\to\infty$ for any $\Phi\in L_2(\mathbb{R};e^{t^2}dt)$ and $t\in\mathbb{R}$. In the same manner we get

$$\int_{-\infty}^{\infty}\left[\Phi(x+t)e^{(x+t)^2/2}-\Phi(t)e^{t^2/2}\right]e^{-x^2/8}r_{2N+1}(x)\frac{dx}{x}\to 0, N\to\infty. \tag{4.14}$$

Further, let us approximate Φ by a sequence of smooth functions $\{\varphi_n\}_{n=1}^{\infty}$ with compact support such that $||\Phi-\varphi_n||_{L_2(\mathbb{R};e^{t^2}dt)}<\varepsilon,\ n\ge n_\varepsilon$ for any $\varepsilon>0$. Moreover denoting by

$f_n(t) = \varphi_n(t)e^{t^2/2}$ we find $f_n(x+t) - f_n(t) = f_n'(t+\theta x)x,\ \theta \in (0,1)$ and the derivative $f_n'(t+\theta x)$ is bounded for all $n \in \mathbb{N}$. Consequently, for each $t \in \mathbb{R}$ the function

$$(f_n(x+t) - f_n(t))\frac{e^{-x^2/8}}{x} \in L_1(\mathbb{R}; dx),$$

and due to the Riemann-Lebesgue lemma [6] we have

$$\frac{1}{\pi}\int_{-\infty}^{\infty} (f_n(x+t) - f_n(t))e^{-x^2/8} \sin\left(x\sqrt{N+3/4}\right)\frac{dx}{x} \to 0,\ N \to \infty. \tag{4.15}$$

Thus combining (4.15) with (4.13), (4.14) we derive that $\lim_{N\to\infty} \varphi_{N,n}(t) = \varphi_n(t)$, where $\varphi_{N,n}$ is defined accordingly (cf. (4.4))

$$\varphi_{N,n}(t) = \frac{e^{-t^2/2}}{2\pi}\sum_{n=0}^{N}\frac{1}{n!}\int_{-\infty}^{\infty}\frac{d^n}{dx^n}\left(e^{-x^2/2}F_{\varphi_n}(x)\right)\frac{d^n}{dx^n}e^{-(x-t)^2/2}dx.$$

Furthermore it is easily follows from the discussions above $\{\Phi_N\}_{N=1}^{\infty}$, $\{\varphi_{N,n}\}_{N=1}^{\infty}$ are Cauchy sequences in $L_2(\mathbb{R}; e^{t^2}dt)$. Hence with Minkowski's inequality we obtain

$$\begin{aligned}||\Phi_N - \Phi||_{L_2(\mathbb{R};e^{t^2}dt)} \le ||\Phi_N - \varphi_{N,n}||_{L_2(\mathbb{R};e^{t^2}dt)} + ||\varphi_{N,n} - \varphi_n||_{L_2(\mathbb{R};e^{t^2}dt)} \\ + ||\varphi_n - \Phi||_{L_2(\mathbb{R};e^{t^2}dt)}.\end{aligned} \tag{4.16}$$

But for $n \ge n_\varepsilon$ the latter norm is less than $\varepsilon/3$. Since we have $||\varphi_{N,n} - \varphi_{M,n}||_{L_2(\mathbb{R};e^{t^2}dt)} < \varepsilon/3$ when $N, M \to \infty$ then the second norm in (4.16) is less than $\varepsilon/3$ via Fatou's lemma. Finally we estimate the norm $||\Phi_N - \varphi_{N,n}||_{L_2(\mathbb{R};e^{t^2}dt)}$. To do this we employ (4.4), (4.13) to write for $n \ge n_\varepsilon$

$$\begin{aligned}e^{t^2/2}\left[\Phi_N(t) - \varphi_{N,n}(t)\right] = \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\Phi(y) - \varphi_n(y)\right]e^{y^2/2-(y-t)^2/8} \\ \times \sin\left((y-t)\sqrt{N+3/4}\right)\frac{dy}{y-t} + \left(\frac{1}{\pi} + O\left(N^{-2}\right)\right)\int_{-\infty}^{\infty}\left[\Phi(y+t) - \varphi_n(y+t)\right] \\ \times e^{(y+t)^2/2-y^2/8}r_{2N+1}(y)\frac{dy}{y} + O\left(N^{-2}\right)\int_{-\infty}^{\infty}\left[\Phi(y+t) - \varphi_n(y+t)\right] \\ \times e^{(y+t)^2/2-y^2/8}\sin\left(y\sqrt{N+3/4}\right)\frac{dy}{y} = I_{1,N}(t) + I_{2,N}(t) + I_{3,N}(t).\end{aligned}$$

Hence

$$||\Phi_N - \varphi_{N,n}||_{L_2(\mathbb{R};e^{t^2}dt)} \le \sum_{i=1}^{3}||I_{i,N}||_{L_2(\mathbb{R};dt)}.$$

Meanwhile with generalized Minkowski's inequality we derive

$$||I_{2,N}||_{L_2(\mathbb{R};dt)} \le \left(\frac{1}{\pi} + O\left(N^{-2}\right)\right) \int_{-\infty}^{\infty} ||\Phi(y+\cdot) - \varphi_n(y+\cdot)||_{L_2(\mathbb{R};e^{(y+t)^2}dt)}$$

$$\times e^{-y^2/8}|r_{2N+1}(y)|\frac{dy}{|y|} < \text{const.}||\Phi - \varphi_n||_{L_2(\mathbb{R};e^{t^2}dt)}$$

$$\times \frac{1}{N^{1/4}} \int_{-\infty}^{\infty} e^{-y^2/8}|y|^{3/2}dy \to 0,\ N \to \infty.$$

Analogously,

$$||I_{3,N}||_{L_2(\mathbb{R};dt)} \le O\left(N^{-3/2}\right) ||\Phi - \varphi_n||_{L_2(\mathbb{R};e^{t^2}dt)} \int_{-\infty}^{\infty} e^{-y^2/8}dy \to 0,\ N \to \infty.$$

Integral $I_{1,N}$ we treat appealing again with the Parseval equality for Fourier's convolution. We have

$$I_{1,N}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\Phi(y) - \varphi_n(y)\right] e^{y^2/2-(y-t)^2/8} \int_{-\sqrt{N+3/4}}^{\sqrt{N+3/4}} e^{iu(y-t)}dudy$$

$$= \frac{1}{2\pi} \int_{-\sqrt{N+3/4}}^{\sqrt{N+3/4}} e^{-iut} \int_{-\infty}^{\infty} \left[\Phi(y) - \varphi_n(y)\right] e^{y^2/2-(y-t)^2/8+iuy}dydu,$$

where the change of the order of integration is due to the absolute and uniform convergence of the latter integral with respect to y. Further, Theorem 64 in [6] and an elementary integral like (2.3) yield immediately the representation

$$\int_{-\infty}^{\infty} \left[\Phi(y) - \varphi_n(y)\right] e^{y^2/2-(y-t)^2/8+iuy}dy = 2\int_{-\infty}^{\infty} \Psi_n^*(\xi)e^{-2(u-\xi)^2+i(u-\xi)t}d\xi,$$

where Ψ^* is the Fourier transform of the function $\Psi_n(y) = (\Phi(y) - \varphi_n(y))e^{y^2/2}$. Therefore by virtue of the uniform convergence of the latter integral with respect to u from the interval $[-\sqrt{N+3/4}, \sqrt{N+3/4}]$ we end up with

$$I_{1,N}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \Psi_n^*(\xi)e^{-i\xi t} \int_{-\sqrt{N+3/4}}^{\sqrt{N+3/4}} e^{-2(u-\xi)^2}dud\xi.$$

Hence by the Plancherel identity for the Fourier transform we get

$$||I_{1,N}||_{L_2(\mathbb{R};dt)} = \sqrt{\frac{2}{\pi}} \left(\int_{-\infty}^{\infty} |\Psi_n^*(\xi)|^2 \left(\int_{-\sqrt{N+3/4}}^{\sqrt{N+3/4}} e^{-2(u-\xi)^2}du\right)^2 d\xi\right)^{1/2}$$

$$\leq \sqrt{\frac{2}{\pi}} \left(\int_{-\infty}^{\infty} |\Psi_n^*(\xi)|^2 \left(\int_{-\infty}^{\infty} e^{-2u^2} du \right)^2 d\xi \right)^{1/2} = ||\Psi^*||_{L_2(\mathbb{R};dt)} = ||\Psi_n||_{L_2(\mathbb{R};dt)}$$

$$= ||\Phi - \varphi_n||_{L_2(\mathbb{R};e^{t^2}dt)} < \frac{\varepsilon}{3}, \ n \geq n_\varepsilon.$$

Combining with (4.16) we verify that $||\Phi_N - \Phi||_{L_2(\mathbb{R};e^{t^2}dt)} < \varepsilon, N \geq N_\varepsilon$ and we conclude the proof of Theorem 2.

References

1. A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions*, Vols. I, II, McGraw-Hill, New York, London and Toronto (1953).

2. I.I. Hirshman and D.V. Widder, *The Convolution Transform*, Princeton Univ. Press, Princeton, NJ (1955).

3. N.N. Lebedev, *Special Functions and Their Applications*, Prentice-Hall, Englewood Cliffs, N.J. (1965).

4. A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, *Integrals and Series*: *Special Functions*, Gordon and Breach, New York and London (1986).

5. J.L. Teugels, Probabilistic proofs of some real inversion formulas, *Math. Nachr.* **146** (1990), 149-157.

6. E.C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* , Chelsea, New York (1986).

7. D.V. Widder, *The Laplace Transform*, Princeton Univ. Press, Princeton, NJ (1941).

8. S.B. Yakubovich, *Index Transforms*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1996.

9. A.I. Zayed, *Function and Generalized Function Transformations*, CRC Press, Boca Raton, FL (1996).

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