Dynamics of generic multidimensional linear differential systems

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Abstract

We prove that there exists a residual set $\mathcal{R}$ (with respect to the C^0 topology) of all d-dimensional linear differential systems based in a μ-invariant flow and with transition matrix evolving in $GL(d,\mathbb{R})$ such that if $A \in \mathcal{R}$, then, for μ-a.e. point, the Oseledets splitting along the orbit is dominated (uniform projective hyperbolicity) or else the Lyapunov spectrum is trivial. Moreover, in the conservative setting, we obtain the dichotomy: dominated splitting *versus* zero Lyapunov exponents.

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1 Introduction

Let $\varphi^t : X \to X$ be a continuous flow defined in a compact Hausdorff space X and $A : X \to \mathfrak{sl}(d,\mathbb{R})$ be a continuous map, where $\mathfrak{sl}(d,\mathbb{R})$ is the Lie algebra of all $d \times d$ matrices with trace equal to zero. Given any $p \in X$, the solution $\Phi^t_A(p)$ of the non-autonomous linear differential equation $u(t)' = A(\varphi^t(\cdot)) \cdot u(t)$, with initial condition $\Phi^0_A(p) = \mathrm{Id}$, is a linear flow which lies in the special linear group $SL(d,\mathbb{R})$. As a typical example we have the linear Poincaré flow (see [9] B.3) of a divergence-free (zero divergence) vector field $F : X \to TX$ such that $\|F(p)\| = 1$ for all regular point $p \in X$ and $\dim(X) = d+1$. However, the linear Poincaré flow of any divergence-free

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vector field F in general does not evolve in $SL(d,\mathbb{R})$. Nevertheless, it still has a restriction which is invariant as we will see in Section 2.1 when we define the linear differential systems, which mimic the volume-preserving flows.

Moreover, beyond the conservative setting, we will consider general linear differential systems with solutions in the linear group $GL(d,\mathbb{R})$, which might be induced by a flow in manifolds with dimension $d+1$.

Given a linear differential system A, the *Lyapunov exponents* measure the asymptotic exponential growth rate of $\|\Phi^t_A(p)\cdot v\|$ for $v\in\mathbb{R}^d_p$. These real numbers play a central role in ergodic theory and the absence of zero exponents yield valuable description of the dynamics of A. Therefore, it is very important to detect zero Lyapunov exponents. If $A(t)=A$ is constant (e.g. the flow is over a fixed point), then the Lyapunov exponents are exactly the real parts of the eigenvalues of A. In general, the eigenvalues of the matrix $A(t)$ are meaningless if one aims to study the asymptotic solutions. If we assume that φ^t leaves invariant a Borel regular probability measure μ, then, due to the multiplicative ergodic theorem (see for example [16]), we have that the Lyapunov exponents are well defined for almost every orbit.

Throughout this work we use a weak form of hyperbolicity, called *dominated splitting* which, broadly speaking, means that we have an invariant splitting along the orbit into two subspaces such that one is most expanding (or less contracted) than the other, by *uniform* rates.

The main target of this paper is to understand in detail the Lyapunov exponents for typical continuous-time general families of linear differential systems in any dimension. By using a much more careful and elaborate technique, we obtain, in particular, the higher dimensional generalization of the theorems in [4]. Let us start with our main result.

Theorem 1.1 *There exists a C^0-residual subset $\mathcal{R}$ of d-dimensional linear differential systems with fundamental matrix evolving in $GL(d,\mathbb{R})$ such that if $A\in\mathcal{R}$, then for μ-a.e. point $x\in X$ we have dominated splitting or else the Lyapunov exponents are all equal.*

Since we are going to make conservative perturbations and, in conservative setting, having equal Lyapunov exponents is tantamount to all exponents being zero, we derive the following Corollary.

Corollary 1.1 *There exists a C^0-residual subset $\mathcal{R}$ of d-dimensional conservative linear differential systems such that if $A\in\mathcal{R}$, then for μ-a.e. point $x\in X$ we have dominated splitting or else the Lyapunov exponents are zero.*

We point out that the first manifestation of these kind of dichotomies appeared in the breakthrough approach of Mañé (see [20]). In [21] Mañé gave an outline of what could be the proof for area-preserving diffeomorphisms in surfaces. Followingly, Bochi presented the complete proof (see [7]). In the groundbreaking paper of Bochi and Viana (see [8]), it is proved the generalization of the dichotomy for any dimension and also a version for simplectomorphisms and also for discrete cocycles. Afterwards we start the approach of *Mañé-Bochi-Viana's* results for the continuous-time systems by proving the 2-dimensional linear differential systems version (see [4]) and also by proving the 3-dimensional volume-preserving flows case (see [5]).

Let us mention now some results in the opposite direction to ours: in [6] we prove that, in the setting of *dynamical* linear differential systems, ergodicity and dominated splitting assure that zero Lyapunov can be removed by small C^1-perturbations, at least when the central direction is 1-dimensional. We also mention the early work of Millionshchikov (see [22] and [23]) where it is proved abundance (dense and open set with respect to C^0-topology) of simple spectrum (all Lyapunov exponents are different) for a class of linear differential systems. Fabbri, in [12], proved the C^0-genericity (open and dense) of hyperbolicity on the torus for two-dimensional linear differential systems (see also Fabbri-Johnson [14]). For determining the positivity of Lyapunov exponents we mention that Knill (see [18]) proved that for a C^0-dense set of two-dimensional bounded and measurable conservative discrete cocycles we have positive exponents. We remark that Fabbri's result is the continuous-time counterpart of Knill's theorem on tori. Subsequently, in [3], Arnold and Cong used a different strategy and generalized Knill's result to $GL(d,\mathbb{R})$ valued discrete cocycles. A very interesting recent result of Cong (see [10]) says that a generic bounded cocycle has simple spectrum, moreover the Oseledets splitting is dominated. As a consequence of this result, in the conservative 2-dimensional discrete and bounded case we have abundance of uniform hyperbolicity. Going back to linear differential systems we also recall the results of Fabbri [13], Fabbri-Johnson [15] and the early paper of Kotani [17]. Furthermore, Nerurkar (see [24]) proved the positivity of Lyapunov exponents for a dense set in a class of conservative linear differential systems.

It is possible to prove the dichotomy of Theorem 1.1 for systems with solutions evolving in more general subgroups of $GL(d,\mathbb{R})$. Actually, in order to obtain similar results these systems must satisfy the *accessibility condition* (see [4] Definition 5.1) which guarantees that we can mix directions and so the strategy of the proof still works. Note that in [24] Nerurkar also used a definition of accessibility. Notwithstanding the fact that most common subgroups are accessible, we must prove it or construct the perturbations in order to interchange directions, like for example, linear differential systems with transition matrix evolving in the symplectic group.

2 Preliminaries

2.1 Linear Differential Systems

We consider a non-atomic probability space (X,μ) where X is a compact and Hausdorff space and μ is a Borel measure. Let $\varphi^t : X \to X$ be a flow continuous in the space parameter and C^1 in the time parameter and assume that μ is φ^t-invariant. For $d \in \mathbb{N}$ let $A : X \to GL(d,\mathbb{R})$ be a continuous map. For each $p \in X$ we consider the non-autonomous linear differential equation:

$$\frac{d}{dt}u(s)|_{s=t} = A(\varphi^t(p)) \cdot u(t), \tag{2.1}$$

called *linear variational equation.* The solution of (2.1) we call the *fundamental matrix* of the system A. This solution is a linear flow $\Phi_A^t(p) : \mathbb{R}_p^d \to \mathbb{R}_{\varphi^t(p)}^d$ which may be seen as the skew-product flow,

$$\begin{array}{rccc} \Phi^t: & X \times \mathbb{R}_p^d & \longrightarrow & X \times \mathbb{R}_{\varphi^t(p)}^d \\ & (p, v) & \longrightarrow & (\varphi^t(p), \Phi_A^t(p) \cdot v). \end{array}$$

Since for all $p \in X$ we have $A(p) = \frac{d}{dt}\Phi_A^t(p)|_{t=0}$, Φ_A^t is also called the infinitesimal generator of A. We recall a basic relation which says that for all $p \in X$ and $t \in \mathbb{R}$ we have $\Phi_A^{t+s}(p) = \Phi_A^s(\varphi^t(p)) \circ \Phi_A^t(p)$.

We will be interested in two of the most common systems; the *traceless* ones, where $\mathrm{Tr}(A) = 0$ and the systems where $\Phi_A^t \in GL(d, \mathbb{R})$, say where Φ_A^t evolves in the linear group of matrices with non-zero determinant. We denote the last-mentioned systems by $\mathcal{G}$ and the traceless systems by $\mathcal{T}$. Another kind of systems with deserve a special interest are the *modified volume-preserving* systems which simulate the volume-preserving vector fields in manifolds with dimension $d+1$. To define formally these systems we consider a continuous nonnegative subexponential function $a : X \to \mathbb{R}$ such that $a(X \setminus \mathrm{Fix}(\varphi^t)) \neq 0$ where $\mathrm{Fix}(\varphi^t)$ denotes the set of fixed points of φ^t. We say that A is modified volume-preserving, denoting by $\mathcal{T}_a$ if:

$$\det\Phi_A^t(p) = \begin{cases} 1, & \text{if } p \in \mathrm{Fix}(\varphi^t) \\ \frac{a(p)}{a(\varphi^t(p))}, & \text{if } p \notin \mathrm{Fix}(\varphi^t). \end{cases}$$

We note that the function $a(\cdot)$ plays the part of the norm of the vector field.

Let $A \in \mathcal{T}$, then any conservative perturbation of A, say $A + H$, must satisfy $\Phi_{A+H}^t \in SL(d, \mathbb{R})$, otherwise we jump out of $\mathcal{T}$ which we not interested. It is immediate to see that if $H \in \mathcal{T}$, then $A + H \in \mathcal{T}$. Using the Liouville formula,

$$\det\Phi_A^t(p) = \exp\left(\int_0^t \mathrm{Tr}A(\varphi^s(p))\right) ds,$$

it is straightforward to see that if $A \in \mathcal{T}_a$, respectively $A \in \mathcal{G}$, and $H \in \mathcal{T}$, then $A + H \in \mathcal{T}_a$, respectively $A + H \in \mathcal{G}$.

To compute the distance between linear differential systems we will basically work with two norms; the uniform convergence norm (or the C^0-norm) which is given by

$$\|A - B\| = \max_{p \in X} \|A(p) - B(p)\|$$

and the L^∞-norm, which is given by

$$\|A - B\|_\infty = \mathrm{esssup}\|A(p) - B(p)\|.$$

2.2 Multiplicative ergodic theorem

In our setting the Oseledets theorem [25] guarantees that for μ-a.e. point $p \in X$, there exists an Φ_A^t-invariant splitting called *Oseledets's splitting* of the fiber $\mathbb{R}_p^d = E_p^1 \oplus ... E_p^{k(p)}$ and real numbers called *Lyapunov exponents* $\hat{\lambda}_1(p) > ... > \hat{\lambda}_{k(p)}(p)$, with $k(p) \leq d$, such that:

$$\lim_{t \to \pm\infty} \frac{1}{t} \log \|\Phi_A^t(p) \cdot v^i\| = \hat{\lambda}_i(p),$$

for any $v^i \in E_p^i \setminus \{\vec{0}\}$ and $i = 1, ..., k(p)$. If we do not count the multiplicities, then we have $\lambda_1(p) \geq \lambda_2(p) \geq ... \geq \lambda_d(p)$. Moreover, given any of these subspaces E^i and E^j, the angle between them along the orbit is subexponential, say

$$\lim_{t \to \pm\infty} \frac{1}{t} \log \sin(\measuredangle(E^i_{\varphi^t(p)}, E^j_{\varphi^t(p)})) = 0. \tag{2.2}$$

If the flow φ^t is ergodic, then the Lyapunov exponents and the dimensions of the associated subbundles are μ-a.e. constant. For a simplified proof of this theorem for linear differential systems see [16]. We denote by $\mathcal{O}(A)$ the μ-generic points given by the multiplicative ergodic theorem.

2.3 Multilinear Operators algebra

Let $\mathcal{H}$ be a Hilbert space and $n \in \mathbb{N}$. The n^{th} exterior product of $\mathcal{H}$, denoted by $\wedge^n(\mathcal{H})$, is also a vector space. If $\dim(\mathcal{H}) = d$, then $\dim(\wedge^n(\mathcal{H})) = \binom{d}{n}$. Given an orthonormal basis of $\mathcal{H}$, $\{e_j\}_{j \in J}$, then the family of exterior products $e_{j_1} \wedge e_{j_2} \wedge ... \wedge e_{j_n}$ for $j_1 < ... < j_n$ with $j_\alpha \in J$ constitutes an orthonormal basis of $\wedge^n(\mathcal{H})$. Given two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and a linear operator $L : \mathcal{H}_1 \to \mathcal{H}_1$ we define the operator $\wedge^n(L)$ by

$$\begin{array}{rccc} \wedge^n(L): & \wedge^n(\mathcal{H}_1) & \longrightarrow & \wedge^n(\mathcal{H}_2) \\ & \psi_1 \wedge ... \wedge \psi_n & \longrightarrow & L(\psi_1) \wedge ... \wedge L(\psi_n). \end{array}$$

Note that given a linear differential system A over φ^t, and since for the operators $L_t : \mathcal{H}_p \to \mathcal{H}_{\varphi^t(p)}$ and $L_s : \mathcal{H}_{\varphi^t(p)} \to \mathcal{H}_{\varphi^{t+s}(p)}$ we have $\wedge^n(L_s L_t) = \wedge^n(L_s) \wedge^n (L_t)$, we obtain that $\wedge^n(\Phi_A^t)$ is also a linear differential system over φ^t which we also denote by $\wedge^n(A)$. For details on multilinear algebra of operators in Hilbert spaces see [26] chapter V.

This operator, in the particular case of $\dim(\mathcal{H}) = d$, will be very useful to prove our results since we can recover the spectrum and splitting information of the dynamics of $\wedge^n(\Phi_A^t)$ from the one obtained by applying Oseledets's theorem to Φ_A^t. This information will be over the same full measure set and with this approach we deduce our results. Next we present the multiplicative ergodic theorem for exterior power (for a proof see [2] Theorem 5.3.1).

Lemma 2.1 *The Lyapunov exponents $\lambda_i^{\wedge n}(p)$ for $i \in \{1, ..., \binom{d}{n}\}$ (repeated with multiplicity) of the n^{th} exterior product operator $\wedge^n(A)$ at p are the numbers:*

$$\sum_{j=1}^{n} \lambda_{i_j}(p), \text{ where } 1 \leq i_1 < ... < i_n \leq d.$$

This nondecreasing sequence starts with $\lambda_1^{\wedge n}(p) = \lambda_1(p) + \lambda_2(p) + ... + \lambda_n(p)$ and ends with $\lambda_{q(n)}^{\wedge n}(p) = \lambda_{d+1-n}(p) + \lambda_{d+2-n}(p) + ... + \lambda_d(p)$. Moreover the splitting of $\wedge^n(\mathbb{R}_p^d(i))$ for $0 \leq i \leq q(n)$ (of $\wedge^n(A)$) associated to $\lambda_i^{\wedge n}(p)$ can be obtained from the splitting $\mathbb{R}_p^d(i)$ (of A) as follows; take an Oseledets's basis $\{e_1(p), ..., e_d(p)\}$ of $\mathbb{R}_p^d$ such that $e_i(p) \in E_p^k$ for $\dim(E_p^1) + ... + \dim(E_p^{k-1}) < i \leq \dim(E_p^1) + ... + \dim(E_p^k)$. Then the Oseledets space is generated by the n-vectors:

$$e_{i_1} \wedge ... \wedge e_{i_n} \text{ such that } 1 \leq i_1 < ... < i_n \leq d \text{ and } \sum_{j=1}^{n} \lambda_{i_j}(p) = \lambda_i^{\wedge n}(p).$$

2.4 Dominated splitting

Let $\varphi^t : X \to X$ be a flow and $\Lambda \subseteq X$ a compact and φ^t-invariant set. Let A be a linear differential system. Given any linear map L we denote by $\mathfrak{m}(L)$ the *co-norm* which is defined by $\|L^{-1}\|^{-1} = \inf_{v \neq \vec{0}} \|L \cdot v\|$. We say that $\mathbb{R}_\Lambda^d = U \oplus S$ is an *m-dominated splitting* for A if $\Phi_A^t(p) \cdot U_p = U_{\varphi^t(p)}$ and $\Phi_A^t(p) \cdot S_p = S_{\varphi^t(p)}$ for $p \in \Lambda$, the dimensions of U_p and S_p are constant on Λ and there exists $\alpha \in (0, 1)$ such that for every $x \in \Lambda$ the following inequality holds:

$$\frac{\|\Phi_A^m(p)|_{S_p}\|}{\mathfrak{m}(\Phi_A^m(p)|_{U_p})} \leq \alpha. \tag{2.3}$$

From now on we fix $\alpha = 1/2$. For details on dominated structures see [9]. We note that dominated splitting is a weak form of *uniform hyperbolicity* (or *exponential dichotomy*, see [11]). The dimension of U is called *index* of the splitting.

Given A as above, $n \in \{1, ..., d-1\}$ and $m \in \mathbb{N}$ we denote by $\Lambda_n(A, m) \subseteq X$ the set of points p such that exists a splitting $\mathbb{R}_{\varphi^t(p)}^d = U_{\varphi^t(p)} \oplus S_{\varphi^t(p)}$ for $t \in \mathbb{R}$ such that:

- $\Phi_A^r(p) \cdot U_{\varphi^t(p)} = U_{\varphi^{r+t}(p)}$ and $\Phi_A^r(p) \cdot S_{\varphi^t(p)} = S_{\varphi^{r+t}(p)}$ for all $r, t \in \mathbb{R}$;
- $\dim(U_{\varphi^t(p)}) = n$ for all $t \in \mathbb{R}$;
- $\frac{\|\Phi_A^m(\varphi^t(p))|_{S_{\varphi^t(p)}}\|}{\mathfrak{m}(\Phi_A^m(\varphi^t(p))|_{U_{\varphi^t(p)}})} \leq \frac{1}{2}$ for all $t \in \mathbb{R}$.

In other words the points in $\Lambda_n(A, m)$ are the points with m-dominated splitting of index n. In the d-dimensional case, a dominated splitting induces an uniform hyperbolicity in the projective space $\mathbb{R}P^{d-1}$. We define also the open set $\Gamma_n(A, m) := X \setminus \Lambda_n(A, m)$. Note that all the points in this set have iterates such that (2.3) is false. Since we will be interested in perturbing systems with index n and with lack of hyperbolicity (in order to prove that for the perturbed system B its Lyapunov exponents are equal, say $\lambda_n(B,p) = \lambda_{n+1}(B,p)$), it will be useful to define the "bad set" $\Gamma_n^\sharp(A,m)$ of points $p \in \Gamma_n(A,m) \cap \mathcal{O}(A)$ such that $\lambda_n(A,p) > \lambda_{n+1}(A,p)$. Let $\Gamma_n^*(A,m)$ be the set of nonperiodic points of $\Gamma_n^\sharp(A,m)$. Finally, let $\Gamma_n(A,\infty) = \bigcap_{m\in\mathbb{N}} \Gamma_n(A,m)$ and $\Gamma_n^\sharp(A,\infty) = \bigcap_{m\in\mathbb{N}} \Gamma_n^\sharp(A,m)$.

In [8] Lemma 4.1 is proved that for every system A and $n \in \mathbb{N}$, the set $\Gamma_n^\sharp(A,\infty)$ contains no periodic points. Hence for any small given $\delta > 0$, we can increase $m \in \mathbb{N}$ in order to obtain that $\mu(\Gamma_n^\sharp(A,m) \setminus \Gamma_n^*(A,m)) < \delta$. Since our perturbations will be performed along large segments of an orbit, it follows that the presence of periodic points, and consequently overlapping, may difficult our aim. Nevertheless, previous remark says that we can avoid periodic points because they are negligible from the measure μ point of view.

2.5 Entropy functions

Let us consider the following function where $\mathcal{L}$ is one of the sets $\mathcal{T}$, $\mathcal{T}_a$ or $\mathcal{G}$:

$$\begin{array}{rccc} \mathcal{E}_n: & \mathcal{L} & \longrightarrow & [0,+\infty) \\ & A & \longmapsto & \int_X \lambda_1(\wedge^n(A),p)d\mu(p). \end{array} \tag{2.4}$$

With this function we compute the integrated *largest* Lyapunov exponent of the n^{th} exterior power operator. We consider also the function $\mathcal{E}_n(A,\Gamma)$ where $\Gamma \subseteq X$ is a φ^t-invariant set defined by:

$$\mathcal{E}_n(A,\Gamma) = \int_\Gamma \lambda_1(\wedge^n(A),p)d\mu(p).$$

Let us denote $\Sigma_n(A,p) := \lambda_1(A,p)+...+\lambda_n(A,p)$. By using Lemma 2.1 we conclude that for $n = 1,...,d-1$ we have $\Sigma_n(A,p) = \lambda_1(\wedge^n(A),p)$ and therefore we obtain $\mathcal{E}_n(A,\Gamma) = \mathcal{E}_1(\wedge^n(A),\Gamma)$. By using Proposition 2.2 of [8] we get that:

$$\mathcal{E}_n(A,\Gamma) = \inf_{k\in\mathbb{N}} \frac{1}{k} \int_\Gamma \log \| \wedge^n (\Phi_A^k(p)) \| d\mu(p), \tag{2.5}$$

and so the entropy function (2.4) is upper semi-continuous for all $n \in \{1,...,d-1\}$.

Let us consider now the function:

$$\begin{array}{rccc} \mathcal{E}: & \mathcal{L} & \longrightarrow & \mathbb{R}^{d-1} \\ & A & \longmapsto & (\mathcal{E}_1(A),\mathcal{E}_2(A),...,\mathcal{E}_{d-1}(A)). \end{array}$$

In section 5 we derive Theorem 1.1 from the following proposition.

Proposition 2.1 *If A is a continuity point of $\mathcal{E}$, then for μ-a.e. point $p \in X$, the Oseledets splitting of A at p is either dominated or trivial.*

3 Perturbations of linear differential systems

We begin by proving a basic perturbation lemma which will be the main tool for proving our results.

Lemma 3.1 *Given $A \in \mathcal{T}$ ($\mathcal{T}_a$ or $\mathcal{G}$) and $\epsilon > 0$, there exists an angle $\xi > 0$, such that for all $p \in X$ (non-periodic or with period larger than 1) and a 2-dimensional vector space $V_p \subset \mathbb{R}^d_p$, there exists a measurable traceless system H such that:*

1. $\|H\| < \epsilon$;
2. *H is supported in $\varphi^t(p)$ for $t \in [0, 1]$;*
3. *$\Phi^t_{A+H}(p) = \Phi^t_A(p)$ in W_p (the orthogonal complement of V_p in $\mathbb{R}^d_p$);*
4. *$\Phi^1_{A+H}(p) \cdot v = \Phi^1_A(p) \circ R_\xi \cdot v$, $\forall v \in V_p$, where R_ξ is the rotation of angle ξ on V_p;*

Proof. Take $K := \max_{p \in X} \|\Phi^{\pm t}_A(p)\|$ for $t \in [0, 1]$. We claim that it is sufficient to take the angle $\xi > 0$ such that:

$$\xi \leq \frac{\epsilon}{2K^2}.$$

Let $\eta : \mathbb{R} \to [0, 1]$ be any C^∞ function such that $\eta(t) = 0$ for $t \leq 0$, $\eta(t) = 1$ for $t \geq 1$, and $0 \leq \eta'(t) \leq 2$, for all t. We define the 1-parameter family of linear maps $\Psi^t(p) : \mathbb{R}^d_p \to \mathbb{R}^d_p$ for $t \in [0, 1]$ as follows; we fix two orthonormal basis $\{u_1, u_2\}$ of V_p and $\{u_3, u_4, ..., u_d\}$ of W_p. For $\theta \in [0, 2\pi]$, we consider the rotation of angle θ whose matrix relative to the basis $\{u_1, u_2\}$ is

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Since $V_p \oplus W_p = \mathbb{R}^d_p$, given any $u \in \mathbb{R}^d_p$ we decompose $u = u_V + u_W$, where $u_V \in V_p$ and $u_W \in W_p$. For $t \in \mathbb{R}$ we define

$$\mathcal{R}^t \cdot u = R_{\eta(t)\xi}(u_V) + u_W.$$

Now we consider the 1-parameter family of linear maps $\Psi^t(p) : \mathbb{R}^d_p \to \mathbb{R}^d_{\varphi^t(p)}$ where $\Psi^t(p) := \Phi^t_A(p) \circ \mathcal{R}^t$. We take time derivatives and we obtain:

$$\begin{aligned} (\Psi^t(p))' &= (\Phi^t_A(p))'\mathcal{R}^t + \Phi^t_A(p)(\mathcal{R}^t)' = \\ &= A(\varphi^t(p))\Phi^t_A(p)\mathcal{R}^t + \Phi^t_A(p)(\mathcal{R}^t)' = \\ &= A(\varphi^t(p))\Psi^t(p) + \Phi^t_A(p)(\mathcal{R}^t)'(\Psi^t(p))^{-1}\Psi^t(p) = \\ &= \left[A(\varphi^t(p)) + H(\varphi^t(p))\right] \cdot \Psi^t(p). \end{aligned}$$

Hence we define the perturbation by,

$$H(\varphi^t(p)) = \Phi^t_A(p)(\mathfrak{R}^t)'(\mathfrak{R}^t)^{-1}(\Phi^t_A(p))^{-1},$$

where $(\mathfrak{R}^t)'$ and $(\mathfrak{R}^t)^{-1}$ are respectively $(\mathcal{R}^t)'$ and $(\mathcal{R}^t)^{-1}$ but written in the canonical base of $\mathbb{R}^d_p$ instead. Since

$$(\mathcal{R}^t)' \cdot u = \eta'(t)\xi \begin{pmatrix} -\sin(\eta(t)\xi) & -\cos(\eta(t)\xi) \\ \cos(\eta(t)\xi) & -\sin(\eta(t)\xi) \end{pmatrix} \cdot u_V$$

and also

$$(\mathcal{R}^t)^{-1} \cdot u_V = R_{-\eta(t)\xi}(u_V) = \begin{pmatrix} \cos(\eta(t)\xi) & \sin(\eta(t)\xi) \\ -\sin(\eta(t)\xi) & \cos(\eta(t)\xi) \end{pmatrix} \cdot u_V,$$

we obtain that if $u_V = (\psi_1, \psi_2, \overbrace{0, 0, ..., 0}^{(d-2)\times})$ (in the coordinate system $\{u_1, ..., u_d\}$) then,

$$(\mathcal{R}^t)'(\mathcal{R}^t)^{-1} \cdot u = \xi\eta'(t)(-\psi_2, \psi_1, \overbrace{0, 0, ..., 0}^{(d-2)\times}).$$

Clearly we have $\mathrm{Tr}((\mathcal{R}^t)'(\mathcal{R}^t)^{-1}) = 0$ and since the *Trace* is invariant by any change of coordinates we obtain $\mathrm{Tr}((\mathfrak{R}^t)'(\mathfrak{R}^t)^{-1}) = 0$ and consequently

$$\mathrm{Tr}(\Phi^t_A(p)(\mathfrak{R}^t)'(\mathfrak{R}^t)^{-1}(\Phi^t_A(p))^{-1}) = 0.$$

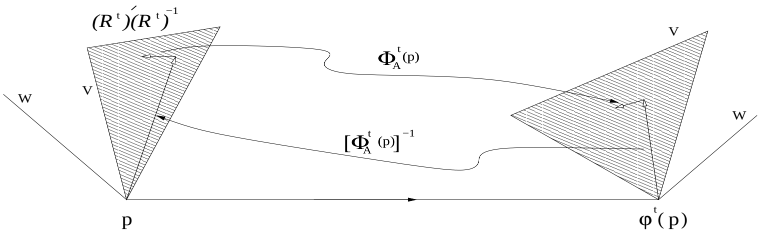

Figure 1: The action of the perturbation $H(\varphi^t(p))$, $t \in [0, 1]$.

Therefore we define the linear differential system $B = A + H$ which is measurable and clearly conservative. In fact, as we mention previously, if $A \in \mathcal{T}_a$ (respectively $A \in \mathcal{G}$) and $Tr(H) = 0$, then $A + H \in \mathcal{T}_a$ (respectively $A \in \mathcal{G}$). In Figure 1 we give the geometric idea of how H acts. Now to prove 1. we compute the norm of H:

$$\begin{aligned} \|H(\varphi^t(p))\| &= \|\Phi^t_A(p)(\mathfrak{R}^t)'(\mathfrak{R}^t)^{-1}(\Phi^t_A(p))^{-1}\| \leq \\ &\leq K^2\|(\mathcal{R}^t)'(\mathcal{R}^t)^{-1}\| \leq 2K^2\xi \leq \epsilon. \end{aligned}$$

Moreover, by choice of η, we have that $\mathrm{Supp}(H)$ is $\varphi^t(p)$ for $t \in [0, 1]$ and 2. is proved. Note that the perturbed system B generates the linear flow $\Phi^t_{A+H}(p)$ which is the same as Ψ^t, hence given $u \in W_p$ we have:

$$\Phi^t_B(p) \cdot u = \Psi^t(p) \cdot u = \Phi^t_A(p)[R_{\eta(t)\xi}(u_V) + u_W] = \Phi^t_A(p) \cdot u_W = \Phi^t_A(p) \cdot u,$$

and 3. follows. Finally to prove 4. taking $u \in V_p$ we obtain,

$$\begin{aligned} \Phi^1_B(p)\cdot u &= \Psi^1(p)\cdot u = \Phi^1_A(p)\circ \mathcal{R}^1\cdot u = \Phi^1_A(p)[R_{\eta(1)\xi}(u_V)+u_W] = \\ &= \Phi^1_A(p)R_\xi(u_V) = \Phi^1_A(p)\circ R_\xi \cdot u, \end{aligned}$$

and Lemma 3.1 is proved.

Lemma 3.2 *Given $A \in \mathcal{T}$ ($\mathcal{T}_a$ or $\mathcal{G}$) and $\epsilon > 0$, there exists an angle $\xi > 0$, such that for all $p \in X$ (non-periodic or with period larger than 1) and a 2-dimensional vector space $V_{\varphi^1(p)} \subset \mathbb{R}^d_{\varphi^1(p)}$, there exists a measurable traceless system H such that:*

1. *$\|H\| < \epsilon$;*
2. *H is supported in $\varphi^t(p)$ for $t \in [0,1]$;*
3. *$\Phi^t_{A+H}(p) = \Phi^t_A(p)$ in W_p (the orthogonal complement of V_p in $\mathbb{R}^d_p$);*
4. *$\Phi^1_{A+H}(p)\cdot v = \tilde{R}_\xi \circ \Phi^1_A(p)\cdot v$, $\forall v \in V_p$, where $\tilde{R}_\xi$ is the elliptical rotation of angle ξ on $V_{\varphi^1(p)}$;*

Proof. We keep the same notation of Lemma 3.1 and we define the following 1-parameter linear map acting on $\mathbb{R}^d_{\varphi^t(p)}$:

$$\tilde{\mathcal{R}}^t := \Phi^t_A(p)\cdot \mathcal{R}^t \cdot [\Phi^t_A(\varphi^t(p))]^{-1}.$$

We denote $\Psi^t = \tilde{\mathcal{R}}^t \cdot \Phi^t_A$ and we take derivatives (in order to t) and obtain,

$$\begin{aligned} (\Psi^t)' &= (\tilde{\mathcal{R}}^t\cdot\Phi^t_A)' = (\Phi^t_A\cdot\mathcal{R}^t)' = \\ &= [A(\varphi^t(p)+H(\varphi^t(p))]\cdot(\Phi^t_A\cdot\mathcal{R}^t) = \\ &= [A(\varphi^t(p)+H(\varphi^t(p))]\cdot(\Phi^t_A\cdot\mathcal{R}^t\cdot(\Phi^t_A)^{-1}\cdot\Phi^t_A) = \\ &= [A(\varphi^t(p)+H(\varphi^t(p))]\cdot(\tilde{\mathcal{R}}^t\cdot\Phi^t_A) = \\ &= [A(\varphi^t(p)+H(\varphi^t(p))]\cdot\Psi^t, \end{aligned}$$

now it is analogous to the proof of Lemma 3.1. The lemma is proved.

In the next lemma we use the almost conformal property to produce a small perturbation which allows us to perform rotations of large angle.

Lemma 3.3 *Given $A \in \mathcal{T}$ ($\mathcal{T}_a$ or $\mathcal{G}$) and $\epsilon, c, \xi > 0$. There exists $m \in \mathbb{N}$ such that: Given any non-periodic point $p \in X$, a 2-dimensional vector space $V_p \subset \mathbb{R}^d_p$ and S_p,$U_p \subseteq V_p$ with $S_p \neq U_p$. Suppose that for all $t, r \in [0,m]$ with $0 \le t+r \le m$ we have:*

(i) $\measuredangle(S_{\varphi^t(p)}, U_{\varphi^t(p)}) > \xi$;

(ii) $\frac{\|\Phi^r_A(\varphi^t(p))|_{S_t}\|}{\mathfrak{m}(\Phi^r_A(\varphi^t(p))|_{U_t})} \le c$.

(iii) $\frac{\|\Phi^m_A(p)|_{S_p}\|}{\mathfrak{m}(\Phi^m_A(p)|_{U_p})} \ge \frac{1}{2}$.

Then, there exists a measurable traceless system H, such that, for all $\alpha \in [0, 2\pi]$ we have:

1. $\|H\| < \epsilon$;
2. H *is supported in* $\varphi^t(p)$ *for* $t \in [0, m]$;
3. $\Phi^t_{A+H}(p) = \Phi^t_A(p)$ *in* W_p *(the orthogonal complement of* V_p *in* $\mathbb{R}^d_p$*);*
4. $\Phi^m_{A+H}(p) \cdot v = \Phi^m_A(p) \circ R_\alpha \cdot v$, $\forall v \in V_p$.

Proof. Once again we use the ideas of the proof of Lemma 3.1. We take:

$$\theta < \frac{\epsilon \sin^6(\xi)}{16c}.$$

We claim that taking $m = \alpha/\theta$ (and assume this number is an integer) will allows us to prove the lemma.

This time we take a C^∞ function $\eta : \mathbb{R} \to [0, m]$ such that $\eta(t) = 0$ for $t \leq 0$, $\eta(t) = m$ for $t \geq m$, and $0 \leq \eta'(t) \leq 2$, for all t. We define $R_{\eta(t)\theta}$ like in Lemma 3.1. The main difficulty is to control the size of the perturbation,

$$H(\varphi^t(p)) = \Phi^t_A(p)(\mathcal{R}^t)'(\mathcal{R}^t)^{-1}(\Phi^t_A(p))^{-1} \text{ for } t \in [0, m].$$

Let $\Phi^t_A(p)/W_p : \mathbb{R}^d_p/W_p \to \mathbb{R}^d_{\varphi^t(p)}/\Phi^t_A(p)(W_p)$ be the induced linear map from the quotient space $\mathbb{R}^d_p/W_p$ into the quotient space $\mathbb{R}^d_{\varphi^t(p)}/\Phi^t_A(p)(W_p)$. It follows directly from Lemma 3.8 of [8] that *(i)*, *(ii)* and *(iii)* implies the following inequality for all $t \in [0, m]$,

$$\frac{\|\Phi^t_A(p)/W_p\|}{\mathfrak{m}(\Phi^t_A(p)/W_p)} \leq \frac{8c}{\sin^6(\xi)}.$$

Since for $v \in W_p$ and $t \in \mathbb{R}$ we have $H(\varphi^t(p)) \cdot v = \vec{0}$ we obtain that,

$$\begin{aligned}
\|H(\varphi^t(p))\| &= \|\Phi^t_A(p)(\mathcal{R}^t)'(\mathcal{R}^t)^{-1}(\Phi^t_A(p)/W_p)^{-1}\| \\
&\leq 2\theta\|\Phi^t_A(p)/W_p\|\|(\Phi^t_A(p)/W_p)^{-1}\| = \\
&= 2\theta\frac{\|\Phi^t_A(p)/W_p\|}{\mathfrak{m}(\Phi^t_A(p)/W_p)} \leq \\
&\leq 2\theta\frac{8c}{\sin^6(\xi)} < \epsilon.
\end{aligned}$$

Therefore 1. follows. The conclusions 2. and 3. are immediate. Finally to prove 4. we note that in time-m we rotate $\eta(m)\theta = m\theta = \alpha$ and the lemma is proved.

The following lemma will be crucial in the sequel.

Lemma 3.4 *Given a system $A \in \mathcal{T}$ ($\mathcal{T}_a$ or $\mathcal{G}$) and $\epsilon > 0$, there exists $m \in \mathbb{N}$ with the following property: For all non-periodic point p with a splitting $\mathbb{R}^d_p = U_p \oplus S_p$ satisfying*

$$\frac{\|\Phi^m_A(p)|_{S_p}\|}{\mathfrak{m}(\Phi^m_A(p)|_{U_p})} \geq \frac{1}{2}, \tag{3.6}$$

there exists a measurable traceless system H supported in $\varphi^{[0,m]}(p)$ and such that there exist vectors $\mathfrak{u} \in U_p \setminus \{\vec{0}\}$ and $\mathfrak{s} \in \Phi_A^m(S_p) \setminus \{\vec{0}\}$ satisfying $\Phi_{A+H}^m(p)(\mathfrak{u}) = \mathfrak{s}$.

Proof. Let $\xi > 0$ be given by Lemma 3.1 and Lemma 3.2 depending on $\epsilon > 0$. Let also $c > 0$ be such that:

$$c > \frac{1}{\sin^{-2}\xi} \text{ and } c \geq \max_{p \in X} \left\{ \frac{\|\Phi_A^1(p)\|}{\mathfrak{m}(\Phi_A^1(p))} \right\}. \tag{3.7}$$

Take $m \in \mathbb{N}$ given by Lemma 3.3 and depending on ϵ, c and ξ.
Small angle: Let us denote $S_t = \Phi_A^t(S_p)$ and $U_t = \Phi_A^t(U_p)$ for $t \in [0, m]$. First we assume that

$$\exists t \in [0, m] \text{ such that } \measuredangle(S_t, U_t) \leq \xi. \tag{3.8}$$

Then we take unit vectors $s_t \in S_t$ and $u_t \in U_t$ with $\measuredangle(s_t, u_t) < \xi$. If $t \in [0, m-1]$, then we use Lemma 3.1 with $V_{\varphi^t(p)} = \langle s_t, u_t \rangle$ (where $\langle e_1, e_2 \rangle$ denotes the vector space spanned by e_1 and e_2) and define $H(\varphi^r(p))$ for $r \in [0, 1]$ and zero otherwise. On the other hand, if $t \in (m-1, m]$, then we use Lemma 3.2 and define $H(\varphi^r(p))$ for $r \in [t-1, t]$ and zero otherwise. In both cases we obtain vectors $\mathfrak{u} \in U_p \setminus \{\vec{0}\}$ and $\mathfrak{s} \in \Phi^m(S_p) \setminus \{\vec{0}\}$ such that $\Phi_{A+H}^m(p)(\mathfrak{u}) = \mathfrak{s}$.

Now we assume that exist $r, t \in \mathbb{R}$ with $0 \leq r + t \leq m$ such that:

$$\frac{\|\Phi_A^r(\varphi^t(p))|_{S_t}\|}{\mathfrak{m}(\Phi_A^r(\varphi^t(p))|_{U_t})} \geq c. \tag{3.9}$$

We choose unit vectors $s_t \in S_t$ and $u_t \in U_t$ which realizes both norms, say $\|\Phi_A^r(\varphi^t(p)) \cdot s_t\| = \|\Phi_A^r(\varphi^t(p))|_{S_t}\|$ and $\|\Phi_A^r(\varphi^t(p)) \cdot u_t\| = \mathfrak{m}(\Phi_A^r(\varphi^t(p))|_{U_t})$. We define also the unit vectors,

$$u_{t+r} = \frac{\Phi_A^r(\varphi^t(p)) \cdot u_t}{\|\Phi_A^r(\varphi^t(p)) \cdot u_t\|} \in U_{t+r} \text{ and } s_{t+r} = \frac{\Phi_A^r(\varphi^t(p)) \cdot s_t}{\|\Phi_A^r(\varphi^t(p)) \cdot s_t\|} \in S_{t+r}.$$

The vector $\hat{u}_t := u_t + \sin(\xi) s_t$ satisfy $\measuredangle(\hat{u}_t, u_t) < \xi$ so a ϵ-small perturbation B_1 given by Lemma 3.1 with $V_{\varphi^t(p)} = \langle s_t, u_t \rangle$ will send u_t into $\Phi_A^1(\varphi^t(p)) \cdot (\mathbb{R}\hat{u}_t)$.

Let $\gamma = \|\Phi_A^r(\varphi^t(p)) \cdot u_t\| (\sin\xi \|\Phi_A^r(\varphi^t(p)) \cdot s_t\|)^{-1}$ we define a vector in $\mathbb{R}_{\varphi^{t+r}(p)}$ by $\hat{s}_{t+r} := \gamma u_{t+r} + s_{t+r}$. We have that,

$$\begin{aligned}
\Phi_A^r(\varphi^t(p)) \cdot \hat{u}_t &= \Phi_A^r(\varphi^t(p)) \cdot u_t + \sin(\xi) \Phi_A^r(\varphi^t(p)) \cdot s_t = \\
&= \Phi_A^r(\varphi^t(p)) \cdot u_t + \frac{\|\Phi_A^r(\varphi^t(p)) \cdot u_t\|}{\gamma \|\Phi_A^r(\varphi^t(p)) \cdot s_t\|} \Phi_A^r(\varphi^t(p)) \cdot s_t = \\
&= \gamma^{-1} \|\Phi_A^r(\varphi^t(p)) \cdot u_t\| . (\gamma u_{t+r} + s_{t+r}) = \\
&= \gamma^{-1} \|\Phi_A^r(\varphi^t(p)) \cdot u_t\| . \hat{s}_{t+r}.
\end{aligned}$$

Hence the vectors $\Phi_A^r(\varphi^t(p)) \cdot \hat{u}_t$ and $\hat{s}_{t+r}$ are co-linear. Moreover, by (3.7), (3.9) and definition of γ, u_t and s_t, we have

$$\gamma = \frac{\mathfrak{m}(\Phi_A^r(\varphi^t(p))|_{U_t})}{\|\Phi_A^r(\varphi^t(p))|_{S_t}\|} \sin^{-1}\xi \leq (c \sin\xi)^{-1} < \sin\xi.$$

Therefore we obtain that $\measuredangle(s_{t+r}, \hat{s}_{t+r}) < \xi$ and using Lemma 3.2 we are able to produce a time-1 ϵ-perturbation B_2 based at $\varphi^{t+r-1}(p)$ such that $\Phi^1_{B_2}(\varphi^{t+r-1}(p)) = R_\xi \circ \Phi^1_A(\varphi^{t+r-1}(p))$, where R_ξ acts in $V_{\varphi^{t+r}(p)} = \langle s_{t+r}, \hat{s}_{t+r}\rangle$ and sends $\hat{s}_{t+r}$ into s_{t+r}. We note that choosing $c > 0$ sufficiently large, see (3.7), guarantees disjoint perturbations. We concatenate as follows:

$$\mathbb{R}u_0 \overset{\Phi^t_A(p)}{\longrightarrow} \mathbb{R}u_t \overset{\Phi^1_{B_1}(\varphi^t(p))}{\longrightarrow} \mathbb{R}[\Phi^1_A(\varphi^t(p)) \cdot \hat{u}_t] \overset{\Phi^{r-1}_A(\varphi^{t+1}(p))}{\longrightarrow} \mathbb{R}[\Phi^r_A(\varphi^t(p)) \cdot \hat{u}_t].$$

We go back by time-1 and then we perform our second perturbation B_2:

$$\mathbb{R}[\Phi^{-1}_A(\Phi^r_A(\varphi^t(p)) \cdot \hat{u}_t)] \overset{\Phi^1_{B_2}(\varphi^{t+r-1}(p))}{\longrightarrow} \mathbb{R}s_{t+r} \overset{\Phi^{m-t-r}_A(\varphi^{t+r}(p))}{\longrightarrow} \mathbb{R}s_m.$$

Large angle: Finally we treat the case when we do not have (3.8) and also (3.9). Then, the condition

$$\forall r, t \in \mathbb{R} : 0 \leq t + r \leq m \text{ we have } \frac{\|\Phi^m_A(\varphi^t(p))|_{S_t}\|}{\mathfrak{m}(\Phi^m_A(\varphi^t(p))|_{U_t})} \leq c,$$

the angles bounded away from ξ and the hypothesis (3.6) will allows us to obtain almost conformality. Hence we can use Lemma 3.3 and rotate by a large angle keeping the size of the perturbation controlled.
Now for $\alpha = \measuredangle(s_0, u_0)$ we are able to use Lemma 3.3 and obtain a system $B = A+H$ with H supported in $\varphi^t(p)$ for $t \in [0, m]$, such that there exists nonzero vectors $\mathfrak{u} \in \mathbb{R}$ and $\mathfrak{s} \in \mathbb{R}\Phi^m_A(s_0)$ such that $\Phi^m_{A+H}(p)(\mathfrak{u}) = \mathfrak{s}$ and Lemma 3.4 is proved.

4 On the decay of the entropy function

Next lemma gives us a local strategy to use the nondominance and different Lyapunov exponents in order to cause a decay of the largest Lyapunov exponent of the n^{th} exterior power system. We follow Proposition 4.2 of [8] adapting it to the flow setting. We only give the main steps of the proof, for all the details see [8].

Lemma 4.1 *Let $A \in \mathcal{T}$ ($\mathcal{T}_a$ or $\mathcal{G}$), $\epsilon, \delta > 0$ and $n \in \{1, ..., d-1\}$. There exist $m \in \mathbb{N}$ and a measurable function $T : \Gamma^*_n(A, m) \to \mathbb{R}$ such that for μ-a.e. point $q \in \Gamma^*_n(A, m)$ and every $t > T(q)$ there exists $H \in \mathcal{T}$ supported on the segment $\varphi^t(q)$ for $t \in [0, m]$ such that*

1. *$\|H\| < \epsilon$;*
2. *$\frac{1}{t} \log \| \wedge^n (\Phi^t_{A+H}(q))\| < \frac{\Sigma_{n-1}(A,q)+\Sigma_{n+1}(A,q)}{2} + \delta$.*

Proof. Fix A and $n \in \{1, ..., d-1\}$ we have,

$$\frac{1}{2}[\Sigma_{n-1}(A, q) + \Sigma_{n+1}(A, q)] = \lambda_1(q) + ... + \lambda_{n-1}(q) + \frac{\lambda_n(q) + \lambda_{n+1}(q)}{2}. \quad (4.10)$$

Let $\mathbb{R}^d_{\Gamma_n^*(A,m)} = U \oplus S$, where U correspond to the vector space spanned by the Lyapunov exponents $\lambda_1(q)$,...,$\lambda_n(q)$ and S correspond to the vector space spanned by $\lambda_{n+1}(q)$,...,$\lambda_d(q)$. We note that by definition of $\Gamma_n^*(A,m)$ we have $\lambda_n(q) > \lambda_{n+1}(q)$. By a recurrence result (see Lemma 3.12 of [7]) for μ-generic point $q \in \Gamma_n^*(A,m)$, there exists $T(q)$ such that $p = \varphi^s(q) \in \Delta_n(A,m)$, where $s \approx T(q)/2$.

Let E_q be the vector space associated to $\lambda_1^{\wedge n}(q)$ (the largest Lyapunov exponent of the n^{th} exterior power system) and F_q be the vector space associated to the other Lyapunov exponents. We obtain a splitting $\wedge^n(\mathbb{R}^d) = E \oplus F$. Since both $\lambda_i(q)'s$ and $\lambda_j^{\wedge n}(q)'s$ are written in nonincreasing *order* it follows that,

$$\lambda_1^{\wedge n}(q) = \sum_{i=1}^{n} \lambda_i(q) \text{ and } \lambda_2^{\wedge n}(q) = \sum_{i=1}^{n-1} \lambda_i(q) + \lambda_{n+1}(q). \tag{4.11}$$

Since $\lambda_n(q) > \lambda_{n+1}(q)$ we get $\lambda_1^{\wedge n}(q) > \lambda_2^{\wedge n}(q)$ and also that $\dim(E_q) = 1$. By using Lemma 4.4 of [8] and Lemma 3.4 we get that, $\wedge^n(\Phi^m_{A+H}(p)) : \wedge^n(\mathbb{R}^d_p) \to \wedge^n(\mathbb{R}^d_{\varphi^m(p)})$ satisfies the property,

$$\wedge^n(\Phi^m_{A+H}(p))(E_p) \subset F_{\varphi^m(p)}. \tag{4.12}$$

We decompose the action of the map $\wedge^n(\Phi^m_{A+H}(p))$ in three steps (see Figure 2); the first (between q and p) and the third (between $\varphi^m(p)$ and $\varphi^t(q)$), with matrix in the basis given by Oseledets's directions denoted respectively by:

$$A_1 = \begin{pmatrix} A_1^{uu} & 0 \\ 0 & A_1^{ss} \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} A_2^{uu} & 0 \\ 0 & A_2^{ss} \end{pmatrix}.$$

The second step (between p and $\varphi^m(p)$), with matrix in the basis given by the Oseledets directions is denoted by:

$$B = \begin{pmatrix} B^{uu} & B^{us} \\ B^{su} & B^{ss} \end{pmatrix}.$$

The outcome of the inclusion (4.12) is that $B^{uu} = 0$. Therefore we obtain:

$$\wedge^n(\Phi^t_{A+H}(q)) = \begin{pmatrix} 0 & A_2^{uu} B^{us} A_1^{ss} \\ A_2^{ss} B^{su} A_1^{uu} & A_2^{ss} B^{ss} A_1^{ss} \end{pmatrix}. \tag{4.13}$$

Since $t >> m$ the entries of B are small compared with t. As $p = \varphi^s(q)$ with $s \approx T(q)/2$, we obtain that $A_i^{ss} < \exp(t(\lambda_2^{\wedge n}(q)/2) + \delta/4))$ and $A_i^{uu} < \exp(t(\lambda_1^{\wedge n}(q)/2 + \delta/4))$ for $i = 1, 2$, because once again t(depending on δ) is very large and Oseledets's theorem guarantees these estimates.

Therefore we obtain that all entries in (4.13) are bounded say,

$$\log(\| \wedge^n (\Phi^t_{A+H}(q))\|) < t\left(\frac{\lambda_1^{\wedge n}(q) + \lambda_2^{\wedge n}(q)}{2} + \frac{\delta}{2}\right).$$

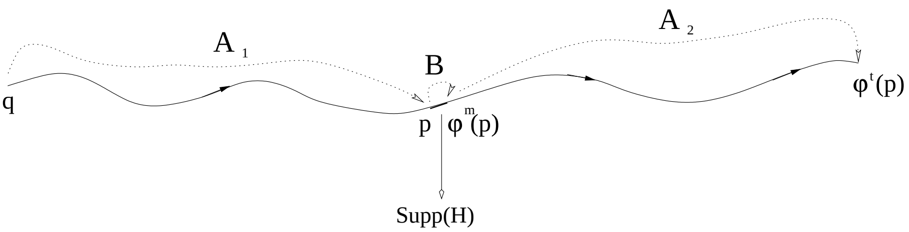

Figure 2: The $A_i's$ for $i = 1, 2$ goes approximately $t/2$ where $t >> m$.

Now using (4.10) and (4.11) we obtain,

$$\begin{aligned}
\frac{1}{t}\log(\| \wedge^n (\Phi^t_{A+H}(q))\|) &< \frac{1}{2}\left(\sum_{i=1}^{n}\lambda_i(q) + \sum_{i=1}^{n-1}\lambda_i(q) + \lambda_{n+1}(q) + \delta\right) = \\
&= \lambda_1(q) + ... + \lambda_{n-1}(q) + \frac{\lambda_n(q) + \lambda_{n+1}(q)}{2} + \delta = \\
&= \frac{1}{2}(\Sigma_{n-1}(A, q) + \Sigma_{n+1}(A, q)) + \delta,
\end{aligned}$$

and the lemma is proved.

In the next lemma we make the previous lemma global.

Lemma 4.2 *Let $A \in \mathcal{T}$ ($\mathcal{T}_a$ or $\mathcal{G}$), $\epsilon, \delta > 0$ and $n \in \{1, ..., d-1\}$. There exist $m \in \mathbb{N}$ and a continuous system $B = A + H$ (with $Tr(A) = Tr(B)$) such that,*

1. *$H(\cdot) = [0]$ outside the open set $\Gamma_n(A, m)$;*
2. *$\|H\|_\infty < \epsilon$;*
3. *$\int_{\Gamma_n(A,m)} \Sigma_n(B, q)d\mu(q) < \delta + \int_{\Gamma_n(A,m)} \frac{\Sigma_{n-1}(A,q)+\Sigma_{n+1}(A,q)}{2} d\mu(q)$.*

Proof. First by Lemma 4.1 and by Ambrose-Kakutani Theorem (see [1]) we can developed a Kakutani tower argument (completely described in Proposition 4.2 and Lemma 7.4 of [8]) to construct the measurable system $\tilde{B}$ such that $\|A-\tilde{B}\|_\infty < \epsilon/2$. Then, since Luzin's theorem asserts that measurable functions are almost continuous we produce the continuous system. For complete details see Proposition 7.3 of [8].

Now we define the discontinuity "jump" of the function $\mathcal{E}_n$ defined in section 2.5 by:

$$J_n(A) := \int_{\Gamma_n(A,\infty)} \lambda_n(A, p) - \lambda_{n+1}(A, p)d\mu(p).$$

In the next lemma we follow [8] (Proposition 4.17):

Lemma 4.3 *Given $A \in \mathcal{T}$ ($\mathcal{T}_a$ or $\mathcal{G}$), $\epsilon, \delta > 0$ and $n \in \{1, ..., d-1\}$, there exists $B \in \mathcal{T}$ (respectively in $\mathcal{T}_a$ or $\mathcal{G}$) ϵ-close to A such that*

$$\int_X \Sigma_n(B, \cdot)d\mu < \int_X \Sigma_n(A, \cdot)d\mu - 2J_n(A) + \delta.$$

Proof. By Lemma 4.2 we obtain B such that $A = B$ outside $\Gamma_n(A, m)$ such that,

$$\int_{\Gamma_n(A,m)} \Sigma_n(B, \cdot)d\mu < \delta + \int_{\Gamma_n(A,m)} \frac{\Sigma_{n-1}(A, \cdot) + \Sigma_{n+1}(A, \cdot)}{2} d\mu.$$

Clearly $X = \Gamma_n(A, m) \sqcup (X \setminus \Gamma_n(A, m))$ so we split the integral:

$$\int_X \Sigma_n(B, \cdot)d\mu < \delta + \int_{\Gamma_n(A,m)} \frac{\Sigma_{n-1}(A, \cdot) + \Sigma_{n+1}(A, \cdot)}{2} d\mu + \int_{X \setminus \Gamma_n(A,m)} \Sigma_n(A, \cdot)d\mu.$$

Now since $\Sigma_n(A, \cdot) := \lambda_1(A, \cdot) + ... + \lambda_n(A, \cdot)$ we note that,

$$2J_n(A) = \int_{\Gamma_n(A,\infty)} \left(\Sigma_n(A, \cdot) - \frac{\Sigma_{n-1}(A, \cdot) + \Sigma_{n+1}(A, \cdot)}{2} \right) d\mu.$$

Moreover, since $\Gamma_n(A, m) \supset \Gamma_n(A, \infty)$, we obtain,

$$-\int_{\Gamma_n(A,m)} \left(\Sigma_n(A, \cdot) - \frac{\Sigma_{n-1}(A, \cdot) + \Sigma_{n+1}(A, \cdot)}{2} \right) d\mu \leq -2J_n(A).$$

Or equivalently

$$\int_{\Gamma_n(A,m)} \left(\frac{\Sigma_{n-1}(A, \cdot) + \Sigma_{n+1}(A, \cdot)}{2} \right) d\mu \leq -2J_n(A) + \int_{\Gamma_n(A,m)} \Sigma_n(A, \cdot)d\mu,$$

and the lemma is proved.

5 End of the proof of the results

Now we will prove Proposition 2.1; We take $A \in \mathcal{G}$ and assume that A is a continuity point for the function $\mathcal{E}_n(\cdot)$ (see section 2.5) for all $n \in \{1, ..., d-1\}$. If follows that $J_n(A) = 0$ for each n, which is the same that $\lambda_n(A, p) = \lambda_{n+1}(A, p)$ for μ-generic points in $\Gamma_n(A, \infty)$ and each n. Now we take $p \in \mathcal{O}(A)$, if the spectrum if trivial we are done, otherwise for p such that $\lambda_n(A, p) > \lambda_{n+1}(A, p)$, and since the jump is zero, p can not be in $\Gamma_n(A, \infty)$. So if $p \notin \Gamma_n(A, \infty)$, then there exists $m \in \mathbb{N}$ such that $p \in \Lambda_n(A, m)$ and this m-dominated splitting has index n. Therefore we conclude that the Oseledets splitting is dominated and the prove of Proposition 2.1 is over.

Now, to prove Theorem 1.1, we note that the continuity points of an upper semi-continuous function is a residual set (see [19]), hence by Proposition 2.1 and the fact that $\mathcal{E}$ is upper semi-continuous we obtain the conclusion of Theorem 1.1.

The proof of Corollary 1.1 now follows easily, because if we start we $A \in \mathcal{T}$ (or $\mathcal{T}_a$) all perturbations we did remains inside the conservative setting. Take $A \in \mathcal{R}$ (the residual given by Theorem 1.1) and $p \in \mathcal{O}(A)$ with trivial spectrum. It is a consequence of Oseledets's Theorem that:

$$\lim_{t\to\pm\infty} \frac{1}{t} \log|\det(\Phi_A^t(p))| = \sum_{i=1}^{k(p)} \lambda_i(p).dim(E_p^i). \tag{5.14}$$

So if $A \in \mathcal{T}$, then $\det(\Phi_A^t(p)) = 1$, and we obtain $\sum_{i=1}^{k(p)} \lambda_i(p).dim(E_p^i) = 0$. Since all Lyapunov exponents are equal, they must all be zero. If $A \in \mathcal{T}_a$, then

$$\det(\Phi_A^t(p)) = \frac{a(p)}{a(\varphi^t(p))}.$$

We use again (5.14) and we obtain

$$\begin{aligned}
\lim_{t\to\pm\infty} \frac{1}{t} \log|\det(\Phi_A^t(p))| &= \lim_{t\to\pm\infty} \frac{1}{t} \log\left(\frac{a(p)}{a(\varphi^t(p))}\right) = \\
&= \lim_{t\to\pm\infty} \frac{1}{t} \log a(p) - \lim_{t\to\pm\infty} \frac{1}{t} \log a(\varphi^t(p)) = \\
&= -\lim_{t\to\pm\infty} \frac{1}{t} \log a(\varphi^t(p)) = \\
&= 0.
\end{aligned}$$

The last equality follows by the fact that $a(\cdot)$ is subexponential. Therefore all Lyapunov exponents are zero. Corollary 1.1 is now proved.

Remark 5.1 *Given $A \in \mathcal{T}$ (or $\mathcal{T}_a$), then if for μ-a.e. point $p \in X$, the Oseledets splitting of A is dominated or trivial at p, then A is a continuity point of $\mathcal{E}$. This follows from semi-continuity and also from the fact that if we perturb the system A a little bit we still have a dominated splitting. We leave the details to the reader.*

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