Fixed points of endomorphisms of graph groups

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ABSTRACT

It is shown, for a given graph group G, that the fixed point subgroup $\mathrm{Fix}\,\varphi$ is finitely generated for every endomorphism φ of G if and only if G is a free product of free abelian groups. The same conditions hold for the subgroup of periodic points. Similar results are obtained for automorphisms, if the dependence graph of G is a transitive forest.

1 Introduction

Gersten proved in the eighties that the fixed point subgroup of an automorphism of a free group of finite rank is always finitely generated [8]. Using a different approach, Cooper gave an alternative topological proof [7]. This result was generalized to further classes of groups and endomorphisms in subsequent years. Goldstein and Turner extended it to monomorphisms of free groups [9], and later to arbitrary endomorphisms [10]. Collins and Turner extended it to automorphisms of free products of freely indecomposable groups [6] (see the survey by Ventura [22]). With respect to automorphisms, the widest generalization is to hyperbolic groups and is due to Paulin [16].

These results inspired Bestvina and Handel to develop a new line of research through their innovative train track techniques, bounding the rank of the fixed point subgroup for the free group automorphism case [3]. These results were generalized by the third author by considering symmetric endomorphisms of free products and using the concept of Kurosh rank [19, 20, 21]. These results are general enough to imply finiteness theorems for arbitrary endomorphisms of finitely generated virtually free groups.

In [17], in the sequence of previous work developped with Cassaigne [4, 5], the second author used automata-theoretic methods to obtain finiteness results for monomorphisms of monoids defined by special confluent rewriting systems, and in [18], followed a different approach that also implied the virtually free group endomorphism case.

In the present paper, we discuss which graph groups (also widely known as right angled Artin groups) admit finitely generated fixed point (and also periodic point) subgroups for every endomorphism/automorphism. A complete solution is reached in the endomorphism case, a partial solution in the automorphism case.

The fact that not all graph groups satisfy these properties (Theorems 3.1 and 4.1) shows that the behaviour of fixed point subgroups imposes strong restrictions on the structure of the ambient group.

2 Preliminaries

Given a group G, we denote by $\mathrm{Aut}\, G$ (respectively $\mathrm{End}\, G$) the automorphism group (respectively endomorphism monoid) of G. Given $\varphi \in \mathrm{End}\, G$, we say that $g \in G$ is a *fixed point* of φ if $g\varphi = g$. If $g\varphi^n = g$ for some $n \geq 1$, we say that g is a *periodic point* of φ. Let $\mathrm{Fix}\, \varphi$ (respectively $\mathrm{Per}\, \varphi$) denote the set of all fixed points (respectively periodic points) of φ. Clearly,

$$\mathrm{Per}\, \varphi = \bigcup_{n \geq 1} \mathrm{Fix}\, \varphi^n. \tag{1}$$

The *free group* on an alphabet A is denoted by F_A. We denote by R_A the set of all reduced words on $\widetilde{A} = A \cup A^{-1}$. For every $g \in F_A$, we denote by $\overline{g}$ the unique reduced word representing g.

We need to introduce *rational subsets* for an arbitrary monoid M. Given $X \subseteq M$, let X^* denote the submonoid of M generated by X. We denote by $\mathrm{Rat}\, M$ the smallest class $\mathcal{X}$ of subsets of M containing the finite sets and satisfying

$$X, Y \in \mathcal{X} \;\Rightarrow\; X \cup Y, XY, X^* \in \mathcal{X}.$$

The following classical result, known as Benois Theorem, relates through reduction the rational subsets of the free group F_A with the rational subsets of the free monoid $\widetilde{A}^*$:

Theorem 2.1 [1] *Let $X \subseteq F_A$. Then $X \in \mathrm{Rat}\, F_A$ if and only if $\overline{X} \in \mathrm{Rat}\, \widetilde{A}^*$.*

As a consequence, we have:

Corollary 2.2 [1] $\mathrm{Rat}\, F_A$ *is closed under the boolean operations.*

A (finite) *independence alphabet* is an ordered pair of the form (A, I), where A is a (finite) set and I is a symmetric anti-reflexive relation on A. We can view (A, I) as an undirected graph without loops or multiple edges, denoted by $\Gamma(A, I)$, by taking A as the vertex set and I as the edge set. Conversely, every such graph determines an independence alphabet.

Let $N(A, I)$ denote the normal subgroup of F_A generated by the commutators

$$\{[a, b] \mid (a, b) \in I\}.$$

The *graph group* $G(A, I)$ is the quotient $F_A/N(A, I)$, i.e the group defined by the group presentation

$$\langle A \mid [a, b] \ ((a, b) \in I)\rangle.$$

Such groups are also known as *right angled Artin groups*, *free partially commutative groups* or even *trace groups*.

If $I = \emptyset$, then $G(A, I)$ is the free group on A. Let $\Delta_A = \{(a, a) \mid a \in A\}$. If $I = (A \times A) \setminus \Delta_A$ (i.e $\Gamma(A, I)$ is a complete graph), then $G(A, I)$ is the free abelian group on A. It is well known that the class of (finitely generated) graph groups is closed under free product and direct product: the free product $G(A_1, I_1) * G(A_2, I_2)$ is realized by the disjoint union of the independence graphs (A_1, I_1) and (A_2, I_2), and the direct product $G(A_1, I_1) \times G(A_2, I_2)$ is realized by the graph obtained from the disjoint union of (A_1, I_1) and (A_2, I_2) by connecting every vertex in A_1 to every vertex in A_2. However, not all graph groups can be built from infinite cyclic groups using these two operators, the simplest example being given by the graph

$$a \text{ --- } b \text{ --- } c \text{ --- } d$$

We remark also that, for every $a \in A$, we may define a homomorphism $\pi_a : G(A, I) \to \mathbb{Z}$ by $a\pi_a = 1$ and $b\pi_a = 0$ for $b \in A \setminus \{a\}$.

Finally, we need some concepts and results involving free products. Assume that $G = G_1 * G_2 * \ldots * G_n$. By the Kurosh Subgroup Theorem [12], every subgroup of G is a free product of the form $F_X * H_1 * \ldots * H_m$, where each H_j is a conjugate of a subgroup of some G_i. The *Kurosh rank* of H is the sum $|X| + m$. In 2007, the third author proved the following result:

Theorem 2.3 [21, Theorem 7] *Let $G = G_1 * \ldots * G_n$ be a free product of finitely generated nilpotent and finite groups and let $\varphi \in \operatorname{End} G$. Then $\operatorname{Fix} \varphi$ has Kurosh rank at most n.*

We say that a group G satisfies the *maximal condition on subgroups* if every subgroup of G is finitely generated. The following result, proved by the third author in 2005, will be used in the discussion of periodic points:

Theorem 2.4 [20, Corollary 6.3] *Let $G = G_1 * \ldots * G_n$ be a free product of groups satisfying the maximal condition on subgroups. If $H_1 \subseteq H_2 \subseteq \ldots$ is an ascending chain of subgroups whose Kurosh ranks are bounded by a natural number N, then there exists an index m such that $H_i = H_m$ for every $i \geq m$.*

3 Endomorphisms

We can reach a complete answer in the case of fixed/periodic points of endomorphisms:

Theorem 3.1 *Let (A, I) be a finite independence alphabet. Then the following conditions are equivalent:*

(i) $\operatorname{Fix} \varphi$ *is finitely generated for every* $\varphi \in \operatorname{End} G(A, I)$*;*

(ii) $\operatorname{Per} \varphi$ *is finitely generated for every* $\varphi \in \operatorname{End} G(A, I)$*;*

(iii) $I \cup \Delta_A$ *is transitive;*

(iv) $\Gamma(A,I)$ *is a disjoint union of complete graphs;*

(v) $G(A,I)$ *is a free product of finitely many free abelian groups of finite rank.*

Proof. (i) $\Rightarrow$ (iii). Suppose that $I \cup \Delta_A$ is not transitive. Then there exist $a,b,c \in A$ such that $(a,b),(b,c) \in I \cup \Delta_A$ and $(a,c) \notin I \cup \Delta_A$. Note that this implies that a,b,c are all distinct.

Write $G = G(A,I)$. We define $\varphi \in \operatorname{End} G$ by $a\varphi = ab$, $b\varphi = b$, $c\varphi = b^{-1}c$ and $x\varphi = 1$ for $x \in A \setminus \{a,b,c\}$. We show that

$$u \in \operatorname{Fix}\varphi \;\Rightarrow\; u\pi_a = u\pi_c. \tag{2}$$

Indeed, we have $u\varphi\pi_b = u\pi_a + u\pi_b - u\pi_c$, hence $u\varphi = u$ yields $u\pi_a = u\pi_c$ and (2) holds.

Write $A_0 = \{a,c\}$ and let $\Psi : G \to F_{A_0}$ be the homomorphism defined by $a\Psi = a$, $c\Psi = c$ and $x\Psi = 1$ for $x \in A \setminus \{a,c\}$. We claim that

$$a^i c^j \in (\operatorname{Fix}\varphi)\Psi \;\Leftrightarrow\; i = j \tag{3}$$

holds for all $i,j \geq 0$.

Indeed, the direct implication follows from (2). On the other hand, we have $(a^ic^i)\varphi = (ab)^i(b^{-1}c)^i = a^ic^i$, hence $a^ic^i = (a^ic^i)\Psi \in (\operatorname{Fix}\varphi)\Psi$ for every $i \geq 0$. Thus (3) holds.

Suppose that $\operatorname{Fix}\varphi$ is finitely generated. Then $(\operatorname{Fix}\varphi)\Psi$ is a finitely generated subgroup of F_{A_0} and therefore $(\operatorname{Fix}\varphi)\Psi \in \operatorname{Rat} F_{A_0}$. But $a^*c^* \in \operatorname{Rat} F_{A_0}$ as well and so by Corollary 2.2 also $L = (\operatorname{Fix}\varphi)\Psi \cap a^*c^* \in \operatorname{Rat} F_{A_0}$. Now (3) yields $L = \{a^nc^n \mid n \geq 0\}$. Since $L \subseteq R_{A_0}$, it follows from Theorem 2.1 that $L \in \operatorname{Rat} A_0^*$, a contradiction since this is a famous example of a non rational language (it is easy to check that it fails the *Pumping Lemma* for rational languages [2, Lemma I.4.5]). Therefore $\operatorname{Fix}\varphi$ is not finitely generated.

(ii) $\Rightarrow$ (iii). We adapt the proof of (i) $\Rightarrow$ (iii) taking the same endomorphism φ and showing that we may replace $\operatorname{Fix}\varphi$ by $\operatorname{Per}\varphi$ in both (2) and (3).

Indeed, for every $n \geq 1$, we have $a\varphi^n = ab^n$, $b\varphi^n = b$, $c\varphi^n = b^{-n}c$ and $x\varphi^n = 1$ for $x \in A \setminus \{a,b,c\}$. Since $u\varphi^n\pi_b = n(u\pi_a) + u\pi_b - n(u\pi_c)$, then $u\varphi^n = u$ yields $u\pi_a = u\pi_c$ and (1) yields

$$u \in \operatorname{Per}\varphi \;\Rightarrow\; u\pi_a = u\pi_c. \tag{4}$$

Now, similarly to the fixed point case, we show that

$$a^i c^j \in (\operatorname{Per}\varphi)\Psi \;\Leftrightarrow\; i = j \tag{5}$$

holds for all $i,j \geq 0$. Finally, we use (4) and (5) in similar fashion to reach the desired contradiction.

(iii) $\Rightarrow$ (iv) $\Rightarrow$ (v). Immediate.

(v) $\Rightarrow$ (i). Since abelian groups are nilpotent, G is the free product of finitely many finitely generated nilpotent groups. It follows from Theorem 2.3 that $\operatorname{Fix}\varphi$ has finite Kurosh rank. Since each of the free factors in the Kurosh decomposition of $\operatorname{Fix}\varphi$ is itself finitely generated, it follows that $\operatorname{Fix}\varphi$ is finitely generated.

(v) $\Rightarrow$ (ii). Assume that $G(A,I)$ is a free product of n free abelian groups of finite rank. It is well known that every finitely generated abelian group satisfies the maximal chain condition on subgroups. Consider the chain of subgroups of $G(A,I)$

$$\operatorname{Fix}\varphi \subseteq \operatorname{Fix}\varphi^{2!} \subseteq \operatorname{Fix}\varphi^{3!} \subseteq \ldots \tag{6}$$

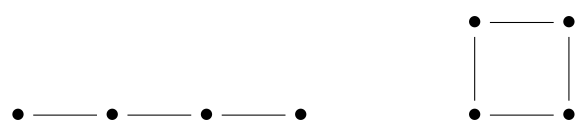
By Theorem 2.3, each $\operatorname{Fix}\varphi^{k!}$ has Kurosh rank at most n, hence by Theorem 2.4, there exists some m such that $\operatorname{Fix}\varphi^{k!} = \operatorname{Fix}\varphi^{m!}$ for every $k \geq m$. By (1), and since $\operatorname{Fix}\varphi^{r} \subseteq \operatorname{Fix}\varphi^{s}$ whenever $r|s$, we get

$$\operatorname{Per}\varphi = \bigcup_{k\geq 1} \operatorname{Fix}\varphi^{k} = \bigcup_{k\geq 1} \operatorname{Fix}\varphi^{k!} = \operatorname{Fix}\varphi^{m!},$$

therefore $\operatorname{Per}\varphi$ is finitely generated by the implication (v) $\Rightarrow$ (i). $\square$

4 Automorphisms

If we restrict our attention to automorphisms, we can prove an analogue of Theorem 3.1, but only for *transitive forests*. We say that a graph is a transitive forest if it has no induced subgraph of one of the following forms:



Transitive forests constitute an important class of graphs in the context of graph groups since they often establish the territory of positive algorithmic properties. For instance, results of Lohrey and Steiberg show that a graph group $G(A, I)$ has solvable submonoid membership problem (or solvable rational subset membership problem) if and only if $\Gamma(A, I)$ is a transitive forest [13].

Theorem 4.1 *Let (A, I) be a finite independence alphabet such that $\Gamma(A, I)$ is a transitive forest. Then the following conditions are equivalent:*

(i) $\operatorname{Fix}\varphi$ *is finitely generated for every* $\varphi \in \operatorname{Aut} G(A, I)$*;*

(ii) $\operatorname{Per}\varphi$ *is finitely generated for every* $\varphi \in \operatorname{Aut} G(A, I)$*;*

(iii) $I \cup \Delta_A$ *is transitive;*

(iv) $\Gamma(A, I)$ *is a disjoint union of complete graphs;*

(v) $G(A, I)$ *is a free product of free abelian groups.*

Proof. In view of Theorem 3.1, it suffices to show that (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iii).

(i) $\Rightarrow$ (iii). Suppose that $I \cup \Delta_A$ is not transitive. Then there exist (distinct) $a, b, c \in A$ such that $(a, b), (b, c) \in I \cup \Delta_A$ and $(a, c) \notin I \cup \Delta_A$.

Write $G = G(A, I)$. We define $\varphi \in \operatorname{End} G$ by $a\varphi = ab$, $c\varphi = b^{-1}c$ and $x\varphi = x$ for $x \in A \setminus \{a, c\}$. We must show that φ is well defined, so take $d \in A \setminus \{a, b, c\}$. All the other cases being easily checked, it suffices to show that $(a, d) \in I$ implies $abd = dab$ in G (the implication $(c, d) \in I \Rightarrow b^{-1}cd = db^{-1}c$ in G is analogous). Indeed, $(a, d) \in I$ implies $(b, d) \in I$ because (A, I) is a transitive forest: since $(a, c) \notin I$, this is the only way of avoiding the forbidden configurations. Therefore φ is well defined.

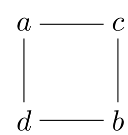
Now φ is clearly an epimorphism. Green's theorem on graph products [11, Corollary 5.4] implies that G is residually finite, and therefore hopfian by a theorem of Mal'cev [15]. This implies that the epimorphism φ is indeed an automorphism.

Now we use the same argument as in the proof of Theorem 3.1 to show that $\mathrm{Fix}\,\varphi$ is not finitely generated.

(ii) $\Rightarrow$ (iii). Analogous. $\square$

The obvious question is now what happens in the automorphism case if $\Gamma(A, I)$ is not a transitive forest. Note that this excludes $\Gamma(A, I)$ from being a disjoint union of complete graphs! The following examples show that it can go either way.

Example 4.2 *Let* $\Gamma(A, I)$ *be the graph*



Then $\mathrm{Fix}\,\varphi$ *and* $\mathrm{Per}\,\varphi$ *are both finitely generated for every* $\varphi \in \mathrm{Aut}\,G(A, I)$*.*

Indeed, we may write $G = G(A, I) = F_{a,b} \times F_{c,d}$. Given $\varphi_1 \in \mathrm{Aut}\,F_{a,b}$ and $\varphi_2 \in \mathrm{Aut}\,F_{c,d}$, let $(\varphi_1, \varphi_2) \in \mathrm{Aut}\,G$ be given by $(u, v)(\varphi_1, \varphi_2) = (u\varphi_1, v\varphi_2)$ (type I automorphism). Let $\psi : F_{a,b} \to F_{c,d}$ be the isomorphism defined by $a\psi = c$ and $b\psi = d$. Finally, let $\sigma : G \to F_{c,d} \times F_{a,b}$ be the isomorphism given by $(u, v)\sigma = (v, u)$. It is immediate that $(\varphi_1, \varphi_2)\sigma(\psi^{-1}, \psi) \in \mathrm{Aut}\,G$ as well (type II automorphism). We claim that all the automorphisms of G must be of type I or type II.

Let $H = (F_{a,b} \times \{1\}) \cup (\{1\} \times F_{c,d})$. If F is a free group of rank > 1 and $u \in F$, it is well known [14] that the centralizer C_u satisfies

$$C_u \cong \begin{cases} \mathbb{Z} & \text{if } u \neq 1 \\ F & \text{if } u = 1 \end{cases}$$

It follows that, for every $(u, v) \in G$,

$$C_{(u,v)} \cong \begin{cases} \mathbb{Z} \times \mathbb{Z} & \text{if } u, v \neq 1 \\ G & \text{if } u = v = 1 \\ \mathbb{Z} \times F_{a,b} & \text{otherwise} \end{cases}$$

and so H, the set of elements with nonabelian centralizer, must be fixed by every $\varphi \in \mathrm{Aut}\,G$.

Suppose first that $a\varphi \in F_{a,b} \times \{1\}$. Since $(ab)\varphi \neq (ba)\varphi$, we see that $b\varphi \in F_{a,b} \times \{1\}$. It follows easily that both $F_{a,b} \times \{1\}$ and $\{1\} \times F_{c,d}$ must be fixed by φ. Thus φ must be of type I.

Assume now that $a\varphi \in \{1\} \times F_{c,d}$. Similar arguments show that φ admits as restrictions isomorphisms $\varphi' : F_{a,b} \times \{1\} \to \{1\} \times F_{c,d}$ and $\varphi'' : \{1\} \times F_{c,d} \to F_{a,b} \times \{1\}$. Now $\varphi'(1, \psi^{-1})\sigma \in \mathrm{Aut}\,(F_{a,b} \times \{1\})$, hence $\varphi'(1, \psi^{-1})\sigma = (\varphi_1, 1)$ for some $\varphi_1 \in \mathrm{Aut}\,F_{a,b}$. Similarly, $\varphi''(\psi, 1)\sigma = (1, \varphi_2)$ for some $\varphi_2 \in \mathrm{Aut}\,F_{c,d}$. Hence

$$\begin{aligned} (u, v)\varphi &= ((u, 1)\varphi')((1, v)\varphi'') = ((u, 1)(\varphi_1, 1)\sigma(1, \psi))((1, v)(1, \varphi_2)\sigma(\psi^{-1}, 1)) \\ &= ((u, 1)(\varphi_1, \varphi_2)\sigma(\psi^{-1}, \psi))((1, v)(\varphi_1, \varphi_2)\sigma(\psi^{-1}, \psi)) \\ &= (u, v)(\varphi_1, \varphi_2)\sigma(\psi^{-1}, \psi) \end{aligned}$$

holds for every $(u, v) \in G$ and so φ is type II.

We show next that $\mathrm{Fix}\,\varphi$ is finitely generated for all type I and type II automorphisms. If $\varphi = (\varphi_1, \varphi_2)$ is type I, then $\mathrm{Fix}\,\varphi = \mathrm{Fix}\,\varphi_1 \times \mathrm{Fix}\,\varphi_2$ is finitely generated in view of Gersten's theorem (see also Theorem 4.1). Hence we may assume that $\varphi = (\varphi_1, \varphi_2)\sigma(\psi^{-1}, \psi)$ is type II.

Given $(u, v) \in G$, we have $(u, v)\varphi = (u, v)$ if and only if $v\varphi_2\psi^{-1} = u$ and $u\varphi_1\psi = v$. This is equivalent to

$$u\varphi_1\psi\varphi_2\psi^{-1} = u \quad \wedge \quad u\varphi_1\psi = v.$$

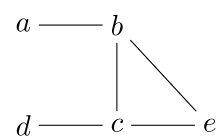
Let $K = \mathrm{Fix}\,\varphi_1\psi\varphi_2\psi^{-1}$. By Gersten's theorem, K is finitely generated, say $K = \langle z_1, \ldots, z_r \rangle$. Hence

$$\mathrm{Fix}\,\varphi = \{(u, u\varphi_1\psi) \mid u \in K\} = \langle (z_1, z_1\varphi_1\psi), \ldots, (z_r, z_r\varphi_1\psi) \rangle$$

is also finitely generated in this case.

Regarding $\mathrm{Per}\,\varphi$, the type I case is analogous and the type II case may be reduced to the type case I in view of $\mathrm{Per}\,\varphi = \mathrm{Per}\,\varphi^2$.

Example 4.3 *Let $\Gamma(A, I)$ be the graph*



Then there exists some $\varphi \in \mathrm{Aut}\,G(A, I)$ such that neither $\mathrm{Fix}\,\varphi$ nor $\mathrm{Per}\,\varphi$ are finitely generated.

We adapt the proof of Theorem 3.1. Define $\varphi \in \mathrm{End}\,G$ by $a\varphi = ab^{-1}$, $b\varphi = b$, $c\varphi = c$, $d\varphi = dc^{-1}$ and $e\varphi = ebc$. This is a well-defined homomorphism which turns out to be an epimorphism, and therefore an automorphism since G is hopfian. Applying π_b and π_c to fixpoints yields

$$u \in \mathrm{Fix}\,\varphi \;\Rightarrow\; u\pi_a = u\pi_d = u\pi_e. \tag{7}$$

Let $A_0 = \{a, d, e\}$ and let $\Psi : G \to F_{A_0}$ be the homomorphism defined by $a\Psi = a$, $d\Psi = d$, $e\Psi = e$ and $x\Psi = 1$ for $x \in A \setminus A_0$. Then we use (7) to show that

$$a^i e^j d^k \in (\mathrm{Fix}\,\varphi)\Psi \;\Leftrightarrow\; i = j = k$$

holds for all $i, j, k \geq 0$. Intersecting $\mathrm{Fix}\,\varphi$ with $a^*e^*d^*$ yields a non rational language (not even *context-free*, actually – see [2]) and a straightforward adaptation of the argument used previously proves that $\mathrm{Fix}\,\varphi$ is not finitely generated. The discussion of $\mathrm{Per}\,\varphi$ is also adapted as in the proof of Theorem 3.1.

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