A convolution operator related to the generalized Mehler-Fock and Kontorovich-Lebedev transforms

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Abstract

In this paper we study a generalization of an index integral involving the product of modified Bessel functions and associated Legendre functions. It is applied to a convolution construction associated with this integral, which is related to the classical Kontorovich-Lebedev and generalized Mehler-Fock transforms. Mapping properties and norm estimates in weighted L_p-spaces, $1 \leq p \leq 2$ are investigated. An application to a class of convolution integral equations is considered. Necessary and sufficient conditions are found for the solvability of these equations in L_2.

Keywords: Kontorovich-Lebedev transform; generalized Mehler-Fock transform; Modified Bessel function; Associated Legendre functions; Convolution integral equations; Index integrals.

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1 Introduction and preliminary results

In this paper we consider the following generalization of the index integral from [2] (see also [9, 10])

$$e^{-yz} J_m(yR) = \sqrt{\frac{2}{\pi y}} \int_{\mathbb{R}_+} \frac{\tau \tanh(\pi\tau)}{Z_\tau^m} K_{i\tau}(y)\ P^m_{-\frac{1}{2}+i\tau}(\mu)\ P^m_{-\frac{1}{2}+i\tau}(\eta)\ d\tau,\ m \in \mathbb{N}_0, \tag{1}$$

where $y > 0$, $R = \sqrt{(\eta^2 - 1)(1 - \mu^2)}$, $z = \mu\eta$, $\mu \in [0, 1]$, $\eta \in [1, +\infty[$. It involves the product of the modified Bessel functions and associated Legendre functions. Formula (1) seems to be new and interesting from the pure mathematical point of view. In (1) we denote by

$$Z_\tau^m = \frac{\Gamma\left(\frac{1}{2} + m + i\tau\right)}{\Gamma\left(\frac{1}{2} - m + i\tau\right)},$$

$J_m(\omega), K_{i\tau}(y)$ are Bessel's and modified Bessel's functions, respectively, $P^m_{-\frac{1}{2}+i\tau}(\omega)$, $\mathrm{Re}(\omega) > 0$, $m \in \mathbb{N}_0$ is the associated Legendre function and $P^0_{-\frac{1}{2}+i\tau}(\omega) \equiv P_{-\frac{1}{2}+i\tau}(\omega)$ is the Legendre or conical function.

As it is known (see [1]-Vol. II, [8]) the modified Bessel function $K_{i\tau}(x)$ can be represented by the Fourier integral

$$K_{i\tau}(x) = \int_{\mathbb{R}_+} e^{-x\cosh(u)} \cos(xu)\ du, \qquad x > 0.$$

Hence, when $\tau \in \mathbb{R}$, it is real-valued and even with respect to the pure imaginary index $i\tau$. Furthermore, this integral can be extended to the strip $\delta \in [0, \frac{\pi}{2}[$ in the upper half-plane, i.e.,

$$K_{i\tau}(x) = \frac{1}{2} \int_{i\delta-\infty}^{i\delta+\infty} e^{-t\cosh(z)+i\tau z}\ dz,$$

and leads for each $x > 0$ to the uniform estimate [14]

$$|K_{i\tau}(x)| \le \sqrt{\frac{\pi}{2x\cos(\delta)}} e^{-\delta\tau - x\cos(\delta)}, \qquad 0 \le \delta < \frac{\pi}{2}. \tag{2}$$

Moreover, we have

$$\int_{\mathbb{R}_+} \tau \tanh(\pi\tau) K_{i\tau}(y)\ d\tau = \sqrt{\frac{\pi y}{2}}\ e^{-y}, \qquad y > 0, \tag{3}$$

which is the limit case of relation (2.16.48.15) in [6] (see [10]).

We note also its asymptotic behavior at infinity [1]

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \left[1 + O\left(\frac{1}{z}\right)\right], \quad z \to +\infty,$$

and near the origin

$$z^{\nu}\ K_{\nu}(z) = 2^{\nu-1}\Gamma(\nu) + o(1), \quad z \to 0,$$
$$K_0(z) = -\log(z) + O(1), \quad z \to 0.$$

When x is fixed we have the following behavior of the modified Bessel function $K_{i\tau}(x)$ with respect to the index $\tau \to +\infty$ [14]

$$K_{i\tau}(x) = \sqrt{\frac{2\pi}{\tau}}\ e^{-\frac{\pi\tau}{2}}\ \left[\sin\left(\tau\ln(\tau) - \tau - \tau\ln\left(\frac{x}{2}\right) + \frac{\pi}{4}\right) + O\left(\tau^{-1}\right)\right]. \tag{4}$$

By $L_p(\Omega; w(x)\ dx)$, $1 < p < +\infty$ we denote the weighted L_p-space with the norm

$$||f||_{L_p(\Omega);w(x)\ dx} = \left(\int_{\Omega} |f(x)|^p\ w(x)\ dx\right)^{\frac{1}{p}}$$
$$||f||_{L_p(\Omega);w(x)\ dx} = \text{ess sup}_{x\in\Omega}|f(x)|.$$

The modified Bessel function $K_{i\tau}(x)$ is the kernel of the Kontorovich-Lebedev transform (see [8, 14])

$$\mathcal{K}_{i\tau}[f] = \lim_{N\to 0}\int_{\frac{1}{N}}^{+\infty} K_{i\tau}(x)\ f(x)\ \frac{dx}{\sqrt{x}}, \tag{5}$$

which is an isometric isomorphism [12]

$$\mathcal{K}_{i\tau} : L_2(\mathbb{R}_+; dx) \to L_2(\mathbb{R}_+; \tau\sinh(\pi\tau)\ d\tau), \tag{6}$$

and the convergence of integral (5) is in the mean-square sense with respect to the norm of the space $L_2(\mathbb{R}_+; \tau\sinh(\pi\tau)\ d\tau)$. Moreover, the Parseval identity

$$\frac{2}{\pi^2}\int_0^{+\infty} \tau\sinh(\pi\tau)|\mathcal{K}_{i\tau}[f]|^2\ d\tau = \int_0^{+\infty} |f(x)|^2\ dx \tag{7}$$

holds and the inverse operator is defined by the formula

$$f(x) = \lim_{N\to+\infty}\frac{2}{\pi^2}\int_0^N \tau\sinh(\pi\tau)\frac{K_{i\tau}(x)}{\sqrt{x}}\mathcal{K}_{i\tau}[f]\ d\tau, \tag{8}$$

where the convergence is in mean-square with respect to the norm of $L_2(\mathbb{R}_+; dx)$.

We will also employ the modified Bessel function $I_m(\omega)$, $m \in \mathbb{N}_0$, which is related to the special function $J_m(\omega)$ by the formula

$$I_m(\omega) = i^{-m/2} J_m\left(i\ \omega\right),\ m \in \mathbb{N}_0. \tag{9}$$

We recall here a Fourier type expansion (see [5]) for the generating function of the modified Bessel functions $I_m(\omega)$, which will be used below

$$e^{\omega\cos\varphi} = I_0(\omega) + 2\sum_{m=1}^{\infty} I_m(\omega)\cos m\varphi,\ -\pi < \varphi < \pi. \tag{10}$$

Formula (1) involves the product of associated Legendre functions of different parameters (see [3] and [1]-Vol.I). Function $P_\nu(z)$ is the associated Legendre function of the first kind, which is analytic in the half-plane $\mathrm{Re}(z) > -1$ and entire with respect to ν. The following representations and relations will be useful in sequel (see [1, 3, 6])

$$P^m_{-\frac{1}{2}+i\tau}(\cos(\beta)) = \sqrt{\frac{2}{\pi}} \frac{(-1)^m Z^m_\tau}{\Gamma\left(m+\frac{1}{2}\right)\sin^m(\beta)} \int_0^\beta \frac{\cosh(\tau\theta)\ d\theta}{(\cos(\theta)-\cos(\beta))^{\frac{1}{2}-m}}, \quad 0<\beta<\pi,\ q>0,\ m\in\mathbb{N}_0. \tag{11}$$

$$P^m_{-\frac{1}{2}+i\tau}(\cosh(\alpha)) = \sqrt{\frac{2}{\pi}} \frac{\sinh^{-m}(\alpha)}{\Gamma\left(\frac{1}{2}+m\right)} \int_0^\alpha \frac{\cos(\xi\tau)}{(\cosh(\alpha)-\cosh(\xi))^{\frac{1}{2}-m}}\ d\xi, \quad \alpha>0,\ m\in\mathbb{N}_0. \tag{12}$$

$$P_{-\frac{1}{2}+i\tau}(\cosh(\alpha)) = \sqrt{\frac{2}{\pi}} \frac{\cosh(\pi\tau)}{\pi} \int_{\mathbb{R}_+} e^{-y\cosh(\alpha)} K_{i\tau}(y)\ \frac{dy}{\sqrt{y}}, \quad \alpha\ge 0. \tag{13}$$

$$P^m_{-\frac{1}{2}+i\tau}(\eta) = (\eta^2-1)^{\frac{m}{2}} \frac{d^m P_{-\frac{1}{2}+i\tau}(\eta)}{d\eta^m}, \quad m\in\mathbb{N}. \tag{14}$$

$$P^m_{-\frac{1}{2}+i\tau}(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m P_{-\frac{1}{2}+i\tau}(x)}{dx^m}, \quad -1<x<1. \tag{15}$$

$$\left|P^m_{-\frac{1}{2}+i\tau}(x)\right| \le \left(\frac{\sqrt{x^2+1}}{x+1}\right)^m \frac{\Gamma\left(m+\frac{1}{2}\right)}{\sqrt{\pi}} \cosh(\pi\tau)\ P^0_{-\frac{1}{2}}(x). \tag{16}$$

The proof of (1) will be based on the following addition theorem for the associated Legendre functions (see [1]-Vol.I)

$$P_{-\frac{1}{2}+i\tau}\left(z_1 z_2 - \left[(z_1^2-1)(z_2^2-1)\right]^{\frac{1}{2}}\cos\varphi\right) = P_{-\frac{1}{2}+i\tau}(z_1)\ P_{-\frac{1}{2}+i\tau}(z_2) + 2\sum_{m=1}^\infty (-1)^m \frac{P^m_{-\frac{1}{2}+i\tau}(z_1)\ P^m_{-\frac{1}{2}+i\tau}(z_2)}{Z^m_\tau} \cos(m\varphi), \tag{17}$$

for any $\varphi\in(-\pi,\pi),\ \mathrm{Re}(z_i)>0,\ i=1,2$. It has the relation [1]-Vol. I

$$P^{-m}_{-\frac{1}{2}+i\tau}(x) = \frac{P^m_{-\frac{1}{2}+i\tau}(x)}{Z^m_\tau}, \tag{18}$$

and the following uniform asymptotic expansion by the index τ at infinity

$$P^m_{-\frac{1}{2}+i\tau}(\cos(\theta)) = \frac{\Gamma\left(\frac{1}{2}+m+i\tau\right)}{\Gamma(1+i\tau)} \left(\frac{\pi}{2}\sin(\theta)\right)^{-\frac{1}{2}} \left(\cos\left(i\tau\theta-\frac{\pi}{4}+\frac{m\pi}{2}\right)+O(\tau^{-1})\right), \quad \tau\to\infty. \tag{19}$$

The associated Legendre function of second kind is denoted by $Q_\nu(z)$ and it is analytic in the half-plane $\mathrm{Re}(z)>1$. It has the following uniform asymptotic behavior at infinity (see [1]-Vol.I)

$$Q_\nu(z) = O\left(\frac{\sqrt{2}}{2^{\nu+1}} \frac{\Gamma(1+\nu)}{\Gamma\left(\nu+\frac{3}{2}\right)} z^{-\nu-1}\right), \qquad z\to\infty,$$

which can be easily obtained from its representation in terms of the Gauss hypergeometric function (see [3]). We will appeal to the following integral representation

$$Q_{\nu-\frac{1}{2}}(\cosh(\alpha)) = \sqrt{\frac{\pi}{2}} \int_{\mathbb{R}_+} e^{-y\cosh(\alpha)} I_\nu(y)\ \frac{dy}{\sqrt{y}}; \qquad \mathrm{Re}(\nu) > -\frac{1}{2},\ \alpha > 0, \tag{20}$$

where $I_\nu(z)$ is the modified Bessel function of the third kind (see [1]-Vol.I).

The generalized Mehler-Fock transform (see [11, 13, 14]) is defined by

$$MF_\mu[f](t) = \int_1^{+\infty} P^\mu_{-\frac{1}{2}+i\tau}(x) f(x)\ dx, \tag{21}$$

where $t = \sigma + i\tau$, $\tau \in \mathbb{R}$, $|\sigma| < \frac{1}{2} - \mu$, $\mu < 1$. Its inverse operator has the form

$$f(x) = \frac{1}{\pi} \int_{\mathbb{R}_+} \tau \sinh(\pi\tau) \left| \Gamma\left(\frac{1}{2} - \mu - i\tau\right) \right|^2 P^\mu_{-\frac{1}{2}+i\tau}(x)\ MF_\mu[f](t)\ dt, \tag{22}$$

where t, σ and τ satisfy the same conditions as (21).

2 Convergence properties and validity of the index integral (1)

The aim of this section is to study the convergence properties of the index integral (1). We start with the following

Lemma 2.1 *Let $y > 0$, $\hat{R} = \sqrt{(\eta^2 - 1)(\mu^2 - 1)}, z = \mu\eta,\ \mu, \eta \in [1, +\infty[,\ m \in \mathbb{N}_0$. Then*

$$e^{-yz} I_m(y\hat{R}) = (-1)^m \sqrt{\frac{2}{\pi y}} \int_{\mathbb{R}_+} \frac{\tau \tanh(\pi\tau)}{Z^m_\tau}\ K_{i\tau}(y)\ P^m_{-\frac{1}{2}+i\tau}(\mu)\ P^m_{-\frac{1}{2}+i\tau}(\eta)\ d\tau, \tag{23}$$

where the integral converges absolutely and uniformly by $\mu, \eta \in [1, \infty)$.

Proof. Since $\mu, \eta \geq 1$ we correspond $\mu = \cosh(\alpha_1)$, $\eta = \cosh(\alpha_2)$, $\alpha_i > 0$, $i = 1, 2$. Hence substituting these values in (23) and taking into account relation (18), we denote by $F_m(y, \alpha_1, \alpha_2)$ its right-hand side to have

$$\begin{aligned} &F_m(y, \alpha_1, \alpha_2) \\ &= (-1)^m \sqrt{\frac{2}{\pi y}} \int_{\mathbb{R}_+} \tau \tanh(\pi\tau)\ Z^m_\tau\ K_{i\tau}(y)\ P^{-m}_{-\frac{1}{2}+i\tau}(\cosh(\alpha_1))\ P^{-m}_{-\frac{1}{2}+i\tau}(\cosh(\alpha_2))\ d\tau. \end{aligned} \tag{24}$$

Meanwhile, representation (12) gives the following straightforward uniform estimate by $\alpha \geq 0$ for the modulus of the associated Legendre function $P^{-m}_{-\frac{1}{2}+i\tau}(\cosh(\alpha))$

$$\begin{aligned}\left|P^{-m}_{-\frac{1}{2}+i\tau}(\cosh(\alpha))\right| &\leq \frac{2}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}+m\right)}\left(\frac{\cosh(\alpha)-1}{\sinh(\alpha)}\right)^m \int_0^\alpha \frac{d\xi}{(\cosh(\alpha)-\cosh(\xi))^{\frac{1}{2}}} \\ &\leq \frac{2\tanh^m\left(\frac{\alpha}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}+m\right)}\int_0^\alpha \frac{d\xi}{(\alpha^2-\xi^2)^{\frac{1}{2}}} \\ &= \frac{\sqrt{\pi}\ \tanh^m\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{1}{2}+m\right)} \\ &\leq \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}+m\right)}. \end{aligned} \tag{25}$$

Since via Stirling's asymptotic formula for gamma-functions we have $Z^m_\tau = O(\tau^{2m})$, $\tau \to \infty$, we find the uniform estimate of the product

$$\left|Z^m_\tau\ P^{-m}_{-\frac{1}{2}+i\tau}(\cosh(\alpha_1))\ P^{-m}_{-\frac{1}{2}+i\tau}(\cosh(\alpha_2))\right| \leq C\ \frac{\tau^{2m}}{\Gamma^2\left(\frac{1}{2}+m\right)}, \tag{26}$$

where $C > 0$ is an absolute constant.

Next, we call equalities (10), (9) adjusting it to our notations, then multiply both sides on $\tau\tanh(\pi\tau)\ K_{i\tau}(y)$ and integrate with respect to τ. Hence, appealing to relation (3) we come out after simple manipulations with the result

$$\begin{aligned} &e^{y\hat{R}\cos(\varphi)} \\ &= I_0(y\hat{R}) + e^{yz}\sqrt{2\pi y}\int_{\mathbb{R}_+}\tau\tanh(\pi\tau)\ K_{i\tau}(y)\sum_{m=1}^{\infty}(-1)^m\frac{P^m_{-\frac{1}{2}+i\tau}(\mu)\ P^m_{-\frac{1}{2}+i\tau}(\eta)}{Z^m_\tau}\ \cos(m\varphi)\ d\tau. \end{aligned} \tag{27}$$

The change of the order of integration and summation in (27) is guaranteed by inequality (2), inequality (25) and the estimate for sufficiently big positive A and each $y > 0$

$$\begin{aligned} &\int_A^{+\infty}\tau\tanh(\pi\tau)|K_{i\tau}(y)|\sum_{m=1}^{\infty}\left|\frac{P^m_{-\frac{1}{2}+i\tau}(\mu)\ P^m_{-\frac{1}{2}+i\tau}(\eta)}{Z^m_\tau}\right|\ d\tau \\ &\leq C_1(y)\sum_{m=1}^{+\infty}\frac{1}{\Gamma^2\left(\frac{1}{2}+m\right)}\int_A^{+\infty}\tau^{2m+1}e^{-\delta\tau}\ d\tau \\ &< C_2(y)\sum_{m=1}^{+\infty}\frac{\delta^{-2(m+1)}\ \Gamma(2(m+1))}{\Gamma^2\left(\frac{1}{2}+m\right)} < +\infty, \end{aligned}$$

where $C_i(y) > 0,\ i = 1, 2$ are constants and $1 < \delta < \frac{\pi}{2}$. Consequently, inverting in (26) the order of integration and summation we use notation (24) to obtain the equality

$$e^{y\hat{R}\cos(\varphi)} = I_0(y\hat{R}) + 2e^{yz}\sum_{m=1}^{\infty}F_m(y, \operatorname{arccosh}(\mu), \operatorname{arccosh}(\eta))\ \cos(m\varphi). \tag{28}$$

Hence comparing with (10) we use the uniqueness of this expansion and get immediately the relation

$$e^{yz} F_m(y, \operatorname{arccosh}(\mu), \operatorname{arccosh}(\eta)) = I_m(y\hat{R}),$$

which yields (23) and completes the proof.

$\blacksquare$

Letting $\mu \in [0,1]$ and using relations (14), (9), $\hat{R} = \pm\, iR$, together with formula (15), we substitute it in (23) to obtain formally

$$\begin{aligned}
&e^{-yz} J_m(yR) \\
&= (-1)^m (1-\mu)^{\frac{m}{2}} \sqrt{\frac{2}{\pi y}} \int_{\mathbb{R}_+} \frac{\tau \tanh(\pi\tau)}{Z_\tau^m}\, K_{i\tau}(y) \frac{d^m P_{-\frac{1}{2}+i\tau}(\mu)}{d\mu^m}\, P^m_{-\frac{1}{2}+i\tau}(\eta)\, d\tau \\
&= \sqrt{\frac{2}{\pi y}} \int_{\mathbb{R}_+} \frac{\tau \tanh(\pi\tau)}{Z_\tau^m}\, K_{i\tau}(y)\, P^m_{-\frac{1}{2}+i\tau}(\mu)\, P^m_{-\frac{1}{2}+i\tau}(\eta)\, d\tau, \qquad 0 < \mu \leq 1, \quad \eta \geq 1.
\end{aligned}$$

This yields (1). Now we will prove that this procedure is indeed possible, i.e. integral (1) converges absolutely for any $(\mu, \eta) \in (0,1] \times [1, +\infty[$ and uniformly by $\mu \in [0,1]$ for any $\eta \geq 1$.

The absolute convergence can be proved as follows. Taking (11) we have the inequality for $\beta \in \left[0, \frac{\pi}{2}\right[$

$$\begin{aligned}
\frac{\left| P^m_{-\frac{1}{2}+i\tau}(\cos(\beta)) \right|}{Z_\tau^m} \quad &\leq \quad \sqrt{\frac{2}{\pi}}\, e^{\beta\tau} \frac{(1-\cos(\beta))^m}{\Gamma\left(m+\frac{1}{2}\right)\, \sin^m(\beta)} \int_0^\beta \frac{d\theta}{(\cos(\theta)-\cos(\beta))^{\frac{1}{2}}} \\
&= \quad \sqrt{\frac{2}{\pi}}\, e^{\beta\tau} \frac{\tan^m\left(\frac{\beta}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)} \int_0^\beta \frac{d\theta}{(\cos(\theta)-\cos(\beta))^{\frac{1}{2}}} \leq C_\beta\, e^{\beta\tau},
\end{aligned}$$

where $C_\beta > 0$ is a constant. Hence, corresponding $\mu = \cos(\beta)$, $\beta \in \left[0, \frac{\pi}{2}\right[$, we estimate the rest of the integral in the right-hand side of (1). Precisely, invoking (18), (25) and asymptotic formula (4), we obtain for sufficiently big $A > 0$

$$\begin{aligned}
&\int_A^{+\infty} \frac{\tau \tanh(\pi\tau)}{Z_\tau^m} \left| K_{i\tau}(y)\, P^m_{-\frac{1}{2}+i\tau}(\cos(\beta))\, P^m_{-\frac{1}{2}+i\tau}(\eta) \right|\, d\tau \\
&\qquad\qquad \leq C \int_A^{+\infty} \tau^{2m+1} e^{\beta\tau} \left| K_{i\tau}(y) \right|\, d\tau < \infty,\ C > 0.
\end{aligned}$$

This proves the absolute convergence of (1). Moreover, it gives the uniform convergence by $\mu \in [\varepsilon, 1]$, $\varepsilon > 0$. Finally we will prove the uniform convergence by $\mu \in [0, \varepsilon]$. This means that for $\mu = \cos(\beta)$ we consider $\beta \in \left[\frac{\pi}{2} - \delta,\ \frac{\pi}{2}\right]$ for a small positive δ. To do this we appeal to the Stirling asymptotic formula for gamma-functions, asymptotic formulas (4) and (19). Thus we find

$$\int_A^{+\infty} \frac{\tau \tanh(\pi\tau)}{Z_\tau^m} K_{i\tau}(y)\ P^m_{-\frac{1}{2}+i\tau}(\cos(\beta))\ P^m_{-\frac{1}{2}+i\tau}(\eta)\ d\tau$$

$$= O\left(\int_A^{+\infty} \tau^m e^{\left(\beta-\frac{\pi}{2}\right)\tau} P^{-m}_{-\frac{1}{2}+i\tau}(\eta) \sin\left(\tau \ln\left(\frac{2\tau}{y}\right) - \tau + \frac{\pi}{4}\right)\ d\tau\right), \qquad \beta \in \left[\frac{\pi}{2} - \delta,\ \frac{\pi}{2}\right].$$

In order to finish the proof we should verify the uniform convergence by β of the latter integral for any $y > 0$, $\eta \geq \eta_0 > 1$, $m \in \mathbb{N}$. In fact, it will follow from the Abel test if we prove the convergence of the integral

$$\int_A^{+\infty} \sin\left(\tau \ln\left(\frac{2y}{\tau}\right) - \tau + \frac{\pi}{4}\right)\ \tau^m\ P^{-m}_{-\frac{1}{2}+i\tau}(\eta)\ d\tau. \tag{29}$$

In order to do this we first call representation (12) and put $\eta = \cosh(\alpha)$, $\alpha \geq \alpha_0 = \operatorname{arccosh}(\eta_0) > 0$. Hence, for $m \in \mathbb{N}$, we deduce

$$\begin{aligned} \tau^m P^{-m}_{-\frac{1}{2}+i\tau}(\cosh(\alpha)) &= \tau^m \sqrt{\frac{1}{2\pi}} \frac{\sinh^{-m}(\alpha)}{\Gamma\left(\frac{1}{2}+m\right)} \int_{-\alpha}^{\alpha} \frac{e^{i\xi\tau}}{(\cosh(\alpha) - \cosh(\xi))^{\frac{1}{2}-m}}\ d\xi \\ &= i^m \sqrt{\frac{1}{2\pi}} \frac{\sinh^{-m}(\alpha)}{\Gamma\left(\frac{1}{2}+m\right)} \int_{-\alpha}^{\alpha} \frac{\left(e^{i\xi\tau}\right)^{(m)}}{(\cosh(\alpha) - \cosh(\xi))^{\frac{1}{2}-m}}\ d\xi \\ &= i^{-m} \sqrt{\frac{1}{2\pi}} \frac{\sinh^{-m}(\alpha)}{\Gamma\left(\frac{1}{2}+m\right)} \int_{-\alpha}^{\alpha} e^{i\xi\tau}\ \left[(\cosh(\alpha) - \cosh(\xi))^{m-\frac{1}{2}}\right]^{(m)}\ d\xi. \end{aligned} \tag{30}$$

Then, using representation (30) in (29), we prove its convergence. Thus we have proved the uniform convergence by $\mu \in [0, 1]$ for each $k > 0$ and $\eta > 1$. Using this fact we may put in (1) $\mu = 0$ to get its important particular case

$$J_m\left(y\sqrt{\eta^2 - 1}\right) = \sqrt{\frac{2}{\pi y}} \int_{\mathbb{R}_+} \frac{\tau \tanh(\pi\tau)}{Z_\tau^m}\ K_{i\tau}(y)\ P^m_{-\frac{1}{2}+i\tau}(0)\ P^m_{-\frac{1}{2}+i\tau}(\eta)\ d\tau.$$

3 A convolution operator and its mapping properties

We begin with

Definition 3.1 *Let f, g be functions from $]1, +\infty[$ into $\mathbb{C}$. Then the function $f * g$ defined on $\mathbb{R}_+$ by*

$$(f * g)(x) = \frac{\sqrt{2}}{\pi\sqrt{\pi}} \int_1^{+\infty} \int_1^{+\infty} e^{-xuv} I_m\left(x\sqrt{(u^2 - 1)(v^2 - 1)}\right) f(u)g(v)\ du\ dv,\ m \in \mathbb{N}_0, \tag{31}$$

is called the convolution related to the Kontorovich-Lebedev and the generalized Mehler-Fock transforms (6) and (21), respectively (provided that it exists).

Theorem 3.2 *Let $f, g \in L_p(]1,+\infty[;dx)$, $1 < p < 2$. Then the convolution $(f * g)(x)$ exists for almost all $x > 0$ and belongs to $L_2(\mathbb{R}_+;dx)$. The convolution is commutative and*

$$||f * g||_{L_2(\mathbb{R}_+;dx)} \leq C\,||f||_{L_2(]1,+\infty[;dx)}\,||g||_{L_2(]1,+\infty[;dx)}\,, \tag{32}$$

where $C > 0$ is an absolute constant.

Proof. From Definition 3.1 it follows that $f * g$ is a commutative operation. Further, by virtue of Fubini's theorem with the use of the generalized Minkowski inequality there exists

$$||f * g||_{L_2(\mathbb{R}_+;dx)}$$
$$\leq \frac{\sqrt{2}}{\pi\sqrt{\pi}} \int_1^{+\infty}\int_1^{+\infty} \left(\underbrace{\int_{\mathbb{R}_+} e^{-2xuv} I_m^2\left(x\sqrt{(u^2-1)(v^2-1)}\right)\, dx}_{\mathbf{I}} \right)^{\frac{1}{2}} |f(u)g(v)|\, du\, dv. \tag{33}$$

The interior integral $\mathbf{I}$ is calculated by relation (2.15.20.1) in [6]. Consequently, we obtain

$$\mathbf{I} = \frac{1}{\pi\sqrt{(u^2-1)(v^2-1)}}\, Q_{m-\frac{1}{2}}\left(\frac{2u^2v^2}{(u^2-1)(v^2-1)} - 1\right).$$

Substituting this value in (33) and using the Hölder inequality for double integrals it becomes

$$||f * g||_{L_2(\mathbb{R}_+;dx)}$$
$$\leq \frac{\sqrt{2}}{\pi^2}\left(\int_1^{+\infty}\int_1^{+\infty} \left((u^2-1)(v^2-1)\right)^{-\frac{q}{4}} Q^{\frac{1}{2}}_{m-\frac{1}{2}}\left(\frac{2u^2v^2}{(u^2-1)(v^2-1)} - 1\right) du\, dv\right)^{\frac{1}{q}}$$
$$\times\, ||f||_{L_p(]1,+\infty[;dx)}\,||g||_{L_p(]1,+\infty[;dx)}\,, \qquad q = \frac{p}{p-1}. \tag{34}$$

Meanwhile, calling representation (20) of the Legendre function $Q_{\nu-\frac{1}{2}}(\cosh(\alpha))$, and using relation (8.4.22.3) in [7]

$$e^{-x} I_m(x) = \frac{1}{2\pi i\sqrt{\pi}} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s+m)\Gamma\left(\frac{1}{2}-s\right)}{\Gamma(1+m-s)} (2x)^{-s}\, ds, \qquad 0 < \gamma < \frac{1}{2},$$

where the latter integral is evidently convergent for $q \in \left]\frac{1}{\gamma}, \frac{2}{\gamma}\right[\subset]2,+\infty[$ because $0 < \gamma < \frac{1}{2}$. Since γ is arbitrary from this interval, inequality (34) is true for any $2 < q = \frac{p}{p-1}$.

Hence appealing to relation (2.4.4.7) in [5] and putting

$$C = \frac{\left(\Gamma\left(1-\frac{q\gamma}{2}\right)\ \Gamma\left(\frac{q\gamma-1}{2}\right)\right)^{\frac{2}{q}}}{2^{\frac{4+q}{2q}}\ \pi^{\frac{2+5q}{2q}}} \left(\int_{\gamma-i\infty}^{\gamma+i\infty} \left|\frac{\Gamma(s+m)\Gamma\left(\frac{1}{2}-s\right)}{\Gamma(1+m-s)}\right| ds\right)^{\frac{1}{2}}, \quad 0<\gamma<\frac{1}{2}$$

we get (32), which completes the proof.

■

Theorem 3.3 *Let $f, g \in L_p(]1, +\infty[; dx)$, $1 \le p < 2$. Then for all $x > 0$ the following generalized Parseval equality takes place*

$$(f * g)(x) = \frac{2\ e^{-\frac{m\pi i}{4}}}{\pi^2} \int_{\mathbb{R}_+} \frac{\tau\ \tanh(\pi\tau)}{Z_\tau^m} \frac{K_{i\tau}(x)}{\sqrt{x}}\ MF_m[f](\tau)\ MF_m[g](\tau)\ d\tau, \tag{35}$$

where the integral is absolutely convergent.

Proof. In fact, we employ integral (1) and substitute it in (31). The change of the order of integration is guaranteed by Theorem 3.2 and Fubini's theorem. Finally, the definition of the generalized Mehler-Fock transform (21) leads to (35).

■

Corollary 3.4 *Under the conditions of Theorem 3.2 the product*

$$MF_m[f](\tau)\ MF_m[g](\tau) \in L_2\left(\mathbb{R}_+; \frac{\tau\ \tanh(\pi\tau)}{Z_\tau^m \cosh(\pi\tau)}\ d\tau\right).$$

Moreover, the factorization identity (see (5))

$$\mathcal{K}_{i\tau}[f * g] = \frac{e^{-\frac{m\pi i}{4}}}{Z_\tau^m \cosh(\pi\tau)} MF_m[f](\tau)\ MF_m[g](\tau) \tag{36}$$

and the Parseval equality hold

$$\int_{\mathbb{R}_+} |(f * g)(x)|^2\ dx = \frac{2}{\pi^2} \int_{\mathbb{R}_+} \frac{\tau\ \tanh(\pi\tau)}{(Z_\tau^m)^2 \cosh(\pi\tau)}\ |MF_m[f](\tau)\ MF_m[g](\tau)|^2\ d\tau.$$

Proof. Via Theorem 3.2 $f * g \in L_2(\mathbb{R}_+; dx)$ the statement is an immediate consequence of the L_2-theory for the Kontorovich-Lebedev transform (5) by virtue os equalities (7), (8).

■

Theorem 3.5 *Let $f \in L_p(]1,+\infty[;dx)$, $1 \le p < 2$. The generalized Mehler-Fock transform (21) is the composition of the Kontorovich-Lebedev transform (5) and the following Laplace transform*

$$(Lf)(x) = \int_1^{+\infty} e^{-xt} f(t)\ dt, \qquad x > 0. \tag{37}$$

Namely, we have the equality

$$MF_m[f](\tau) = \frac{1}{\pi}\sqrt{\frac{2}{\pi}}\ e^{-\frac{m\pi i}{4}}\ Z_\tau^m \cosh(\pi\tau)\ \mathcal{K}_{i\tau}[Lf](\tau), \tag{38}$$

where all involved integrals are absolutely convergent.

Proof. In fact, (38) takes place due to (5), (14), (13), (21) and Fubini's theorem. The latter fact can be verified employing the following estimate.

$$\begin{aligned}
&\int_1^{+\infty} \left| P^m_{-\frac{1}{2}+i\tau}(x)\ f(x)\right|\ dx \\
&\le \left(\int_1^{+\infty} \left| P^m_{-\frac{1}{2}+i\tau}(x)\right|^q\ dx\right)^{\frac{1}{q}} \left(\int_1^{+\infty} |f(x)|^p\ dx\right)^{\frac{1}{p}} \\
&\le \frac{\Gamma\left(m+\frac{1}{2}\right)}{\sqrt{\pi}}\ \cosh(\pi\tau) \left(\int_1^{+\infty} \left|P_{-\frac{1}{2}}(x)\right|^q\ dx\right)^{\frac{1}{q}} ||f||_{L_p(]1,+\infty[;dx)} \\
&\le \frac{\sqrt{2}\ \Gamma\left(m+\frac{1}{2}\right)}{\pi^2}\ \cosh(\pi\tau) \int_{\mathbb{R}_+} \left(\int_1^{+\infty} e^{-qxy}\ dx\right)^{\frac{1}{q}} K_0(y)\ \frac{dy}{\sqrt{y}}\ ||f||_{L_p(]1,+\infty[;dx)} \\
&= \frac{\sqrt{2}\ \Gamma\left(m+\frac{1}{2}\right)}{\pi^2}\ \cosh(\pi\tau) q^{-\frac{1}{q}} \int_{\mathbb{R}_+} e^{-y}\ K_0(y)\ y^{-\left(\frac{1}{q}+\frac{1}{2}\right)}\ dy \quad < +\infty; \quad q = \frac{p}{p-1}
\end{aligned}$$

■

4 Convolution integral equations

This section will be devoted to the class of integral equations of the first kind related with the convolution operator (31). Namely, we will examine a solvability of the following integral equations

$$\int_1^{+\infty} K(x,y) f(y)\ dy \ = \ g(x), \quad x > 0 \tag{39}$$

$$\int_1^{+\infty} \left[\lambda e^{-xy} + K(x,y)\right] f(y)\ dy \ = \ g(x), \quad \lambda \in \mathbb{C}, x > 0, \tag{40}$$

where the kernel $K(x,y)$ is defined by the integral

$$K(x,y) \equiv K_h(x,y) = \frac{\sqrt{2}}{\pi\sqrt{\pi}} \int_1^{+\infty} e^{-xyu}\ I_m(x\sqrt{(y^2-1)(u^2-1)})\ h(u)\ du; \qquad m \in \mathbb{N}_0,$$

h, g are given functions and f is to be determined.

Definition 4.1 *Let $1 < p < 2$. We call by*

$$\mathcal{M}F_{p,2} \equiv \{\psi(\tau) \in L_2(\mathbb{R}_+; \tau \tanh(\pi\tau)\ d\tau) : \ \psi(\tau) = MF_m[f](\tau), \\ f \in L_2(]1,+\infty[; dx) \cap L_p(]1,+\infty[; dx)\}$$

a class of images of $f \in L_2(]1,+\infty[; dx) \cap L_p(]1,+\infty[; dx)$ under the generalized Mehler-Fock transform (21), considering the restriction of this map to

$$MF_m : L_2(]1,+\infty[; dx) \cap L_p(]1,+\infty[; dx) \to \mathcal{M}F_{p,2}.$$

We note that $\mathcal{M}F_{p,2}$ is a subspace of $L_2(\mathbb{R}_+; \tau \tanh(\pi\tau)\ d\tau)$.

Theorem 4.2 *Let $1 < p < 2$, $g \in L_2(\mathbb{R}_+; dx)$ and $h(x) \in L_p(]1,+\infty[)$. Then for the solvability of equation (39) in $L_2(]1,+\infty[; dx) \cap L_p(]1,+\infty[; dx)$ it is necessary and sufficient that*

$$\frac{Z_\tau^m\ \cosh(\pi\tau)}{e^{-\frac{m\pi i}{4}}\ MF_m[h](\tau)} \in \mathcal{M}F_{p,2}.$$

Moreover, the corresponding solution $f(x)$ is unique and given by the formula

$$f(x) = \frac{1}{\pi} \int_{\mathbb{R}_+} \tau \sinh(\pi\tau) \left| \Gamma\left(\frac{1}{2} - m - i\tau\right) \right|^2 P^m_{-\frac{1}{2}+i\tau}(x)\ \frac{Z_\tau^m\ \cosh(\pi\tau)}{e^{-\frac{m\pi i}{4}}\ MF_m[h](\tau)}\ \mathcal{K}_{i\tau}[g](\tau)\ d\tau, \quad (41)$$

where the convergence is with respect to the norm in $L_2(]1,+\infty[; dx)$.

Proof. *Necessity.* Under conditions of the theorem equation (39) is satisfied, then the convolution (31) exists and by (36)

$$\mathcal{K}_{i\tau}[g](\tau) = \frac{e^{-\frac{m\pi i}{4}}}{Z_\tau^m \cosh(\pi\tau)}\ MF_m[f](\tau)\ MF_m[h](\tau).$$

However, $MF_m[f] \in \mathcal{M}F_{p,2}$. Hence

$$\frac{Z_\tau^m \cosh(\pi\tau)}{e^{-\frac{m\pi i}{4}}\ MF_m[h](\tau)} \in \mathcal{M}F_{p,2} \quad (42)$$

and the corresponding solution in $L_2(\mathbb{R}_+; dx)$ is given by (41) via inversion formula (22) for the generalized Mehler-Fock transform (21).

Sufficiency. Now assuming

$$\frac{Z_\tau^m \cosh(\pi\tau)\ \mathcal{K}_{i\tau}[g](\tau)}{e^{-\frac{m\pi i}{4}}\ MF_m[h](\tau)} \in \mathcal{M}_{p,2} \quad (43)$$

we get correspondingly via (41) and Definition 4.1 that $f \in L_2(]1,+\infty[; dx) \cap L_p(]1,+\infty[; dx)$. Further, owing to conditions of the theorem the left hand-side of (39) is the convolution like (31) $(f*h)(x)$, which belongs to $L_2(\mathbb{R}_+; dx)$. Therefore, due to the factorization identity (31) we obtain

$$\mathcal{K}_{i\tau}[f * h] = \frac{e^{-\frac{m\pi i}{4}}}{Z_\tau^m \cosh(\pi\tau)}\ MF_m[f](\tau)\ MF_m[h](\tau). \quad (44)$$

But, by 22 and (41),

$$MF_m[f](\tau) = \frac{Z_\tau^m \cosh(\pi\tau)}{e^{-\frac{m\pi i}{4}} \; MF_m[h](\tau)} \; \mathcal{K}_{i\tau}[g](\tau).$$

Substituting this expression into (40) we find

$$\mathcal{K}_{i\tau}[f * g] = \frac{e^{-\frac{m\pi i}{4}}}{Z_\tau^m \cosh(\pi\tau)} \; MF_m[h](\tau) \; \frac{Z_\tau^m \cosh(\pi\tau)}{e^{-\frac{m\pi i}{4}} \; MF_m[h](\tau)} \; \mathcal{K}_{i\tau}[g](\tau) = \mathcal{K}_{i\tau}[g](\tau).$$

So by the uniqueness property for the Kontorovich-Lebedev transform equation (39) is satisfied and (41) is the unique solution from the class $L_2(]1, +\infty[; dx) \cap L_p(]1, +\infty[; dx)$.

■

Equation (40) can be treated similarly by using the composition representation (36). Indeed, under conditions of Theorem 4.2 after applying the Kontorovich-Lebedev transform to both sides of (40) and taking into account the factorization identity (44) we get the following algebraic equality

$$\frac{\lambda}{\pi}\sqrt{\frac{2}{\pi}} \; MF_m[f](\tau) + MF_m[f](\tau) \; e^{-\frac{m\pi i}{4}} \; MF_m[h](\tau) = Z_\tau^m \cosh(\pi\tau) \; \mathcal{K}_{i\tau}[g](\tau), \quad (45)$$

which can be solved with respect to $MF_m[f](\tau)$ if

$$\frac{\lambda}{\pi}\sqrt{\frac{2}{\pi}} + e^{-\frac{m\pi i}{4}} \; MF_m[h](\tau) \neq 0, \qquad \tau \in \mathbb{R}_+.$$

Hence

$$MF_m[f](\tau) = Z_\tau^m \cosh(\pi\tau) \; \mathcal{K}_{i\tau}[g](\tau) \left[\frac{\lambda}{\pi}\sqrt{\frac{2}{\pi}} + e^{-\frac{m\pi i}{4}} \; MF_m[h](\tau)\right]^{-1}, \quad (46)$$

and we come out with the following result.

Theorem 4.3 *Under conditions of Theorem 4.2 for the solvability of equation (40) in* $L_2(]1, +\infty[; dx) \cap L_p(]1, +\infty[; dx)$, $1 < p < 2$ *it is necessary and sufficient that the right-hand side of (46) belongs to* $\mathcal{M}F_{p,2}$. *Then the corresponding solution* $f(x)$ *is unique and given by the formula*

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_{\mathbb{R}_+} \tau \sinh(\pi\tau) \left|\Gamma\left(\frac{1}{2} - m + i\tau\right)\right|^2 P^m_{-\frac{1}{2}+i\tau}(x) \\ &\quad \times Z_\tau^m \cosh(\pi\tau) \; \mathcal{K}_{i\tau}[g](\tau) \left[\frac{\lambda}{\pi}\sqrt{\frac{2}{\pi}} + e^{-\frac{m\pi i}{4}} \; MF_m[h](\tau)\right]^{-1} d\tau, \;\; x > 1, \end{aligned} \quad (47)$$

where the convergence is with respect to the norm in $L_2(]1, +\infty[; dx)$.

Let us now indicate the special case of the equation (40) when its solution (39) can be represented in the resolvent form. Suppose that $g(x)$ is the modified Laplace transform (37) of some function $\varphi(t) \in L_2(]1,+\infty[;dx) \cap L_p(]1,+\infty[;dx)$, $1 < p < 2$, i.e.,

$$g(x) = \int_1^{+\infty} e^{-xt}\varphi(t)\ dt.$$

A class of such functions g belongs to $L_2(\mathbb{R}_+;dx)$. In fact, by virtue of the generalized Minkowski and Hölder inequalities we have the estimate

$$\begin{aligned}
||g||_{L_2(\mathbb{R}_+;dx)} &= \left(\int_{\mathbb{R}_+} |g(x)|^2\ dx\right)^{\frac{1}{2}} \\
&\leq \int_1^{+\infty} \left(\int_{\mathbb{R}_+} e^{-2xt}\ dx\right)^{\frac{1}{2}} |\varphi(t)|\ dt \\
&\leq \frac{1}{\sqrt{2}}\ ||\varphi||_{L_p(]1,\infty[;dt)} \left(\int_{\mathbb{R}_+} \frac{dt}{t^{\frac{q}{2}}}\right)^{\frac{1}{q}} \\
&= \frac{2^{\frac{1}{q}-\frac{1}{2}}}{(q-2)^{\frac{1}{q}}}\ ||\varphi||_{L_p(]1,+\infty[;dt)} \quad < +\infty, \qquad q = \frac{p}{p-1}.
\end{aligned}$$

Therefore, by composition representation (38) and inversion formula (22) for the generalized Mehler-Fock transform solution (47) becomes in the form

$$\begin{aligned}
f(x) = \sqrt{\frac{\pi}{2}}\ e^{-\frac{m\pi i}{4}} \int_{\mathbb{R}_+} \tau \sinh(\pi\tau) \left|\Gamma\left(\frac{1}{2} - m + i\tau\right)\right|^2 \\
\times P^m_{-\frac{1}{2}+i\tau}(x)\ MF_m[\varphi](\tau) \left[\frac{\lambda}{\pi}\sqrt{\frac{2}{\pi}} + e^{-\frac{m\pi i}{4}}\ MF_m[h](\tau)\right]^{-1} d\tau
\end{aligned}$$

$$\begin{aligned}
&= \frac{\pi^2}{2\lambda}\ e^{-\frac{m\pi i}{4}} \int_{\mathbb{R}_+} \tau \sinh(\pi\tau) \left|\Gamma\left(\frac{1}{2} - m + i\tau\right)\right|^2 P^m_{-\frac{1}{2}+i\tau}(x)\ MF_m[\varphi](\tau)\ d\tau \\
&\quad - \frac{\pi^2}{2\lambda}\ e^{-\frac{m\pi i}{4}} \int_{\mathbb{R}_+} \tau \sinh(\pi\tau) \left|\Gamma\left(\frac{1}{2} - m + i\tau\right)\right|^2 P^m_{-\frac{1}{2}+i\tau}(x)\ MF_m[\varphi](\tau) \\
&\qquad \times MF_m[h](\tau) \left[\frac{\lambda}{\pi}\sqrt{\frac{2}{\pi}} + e^{-\frac{m\pi i}{4}}\ MF_m[h](\tau)\right]^{-1} d\tau \\
&= \frac{\pi^2}{2\lambda}\ e^{-\frac{m\pi i}{4}} \left[\varphi(x) - \int_{\mathbb{R}_+} \tau \sinh(\pi\tau) \left|\Gamma\left(\frac{1}{2} - m + i\tau\right)\right|^2 \right. \\
&\qquad \left. \times P^m_{-\frac{1}{2}+i\tau}(x)\ MF_m[\varphi](\tau) MF_m[h](\tau) \left[\frac{\lambda}{\pi}\sqrt{\frac{2}{\pi}} + e^{-\frac{m\pi i}{4}}\ MF_m[h](\tau)\right]^{-1} d\tau\right],
\end{aligned}$$

with $\lambda \neq 0$.

Finally, let us consider an example of equation (40), letting

$$MF_m[h](\tau) = \left|\Gamma\left(\frac{1}{2} - m + i\tau\right)\right|^2 .$$

In order to find an original we use the inversion formula (22) to obtain

$$h(x) \equiv h_m(x) = \frac{1}{\pi} \int_{\mathbb{R}_+} \tau \sinh(\pi\tau) \left|\Gamma\left(\frac{1}{2} - m + i\tau\right)\right|^4 P^m_{-\frac{1}{2}+i\tau}(x)\ d\tau. \qquad (48)$$

As far as we know integral (48) is absent in classical references on integrals and special functions (see [1, 7]). However, one can treat integral (48) involving negative integers $-m$. Taking into account relations (18), (14), representation (13) and inverting the order of integration and differentiation via absolute and uniform convergence of the correspondent integrals, we get

$$\begin{aligned} h_{-m}(x) &= \frac{\sqrt{2}\ (-1)^m}{\pi\sqrt{\pi}} (x^2-1)^{\frac{m}{2}} \frac{d^m}{dx^m} \int_{\mathbb{R}_+} \frac{e^{-yx}}{\sqrt{y}} \\ &\quad \times \int_{\mathbb{R}_+} \tau \sinh(\pi\tau) \left|\Gamma\left(\frac{1}{2} + m + i\tau\right)\right|^2 K_{i\tau}(y)\ d\tau\ dy. \qquad (49) \end{aligned}$$

Hence by the definition of the Kontorovich-Lebedev transform (5) and its inversion formula (7) together with relation (2.16.6.4) in [7], the inner integral with respect to τ has the value

$$\int_{\mathbb{R}_+} \tau \sinh(\pi\tau) \left|\Gamma\left(\frac{1}{2} + m + i\tau\right)\right|^2 K_{i\tau}(y)\ d\tau = \pi\sqrt{\pi}\ m!\ 2^{m-\frac{1}{2}}\ y^{m+\frac{1}{2}}\ e^{-y}.$$

Therefore, we derive the value of the index integral

$$\begin{aligned} &\int_{\mathbb{R}_+} \tau \sinh(\pi\tau) \left|\Gamma\left(\frac{1}{2} + m + i\tau\right)\right|^4 P^{-m}_{-\frac{1}{2}+i\tau}(x)\ d\tau \\ &\qquad = \pi 2^m\ (m!)^2\ \frac{(x^2-1)^{\frac{m}{2}}}{(1+x)^{2m+1}} \prod_{k=1}^{m} (m+k). \end{aligned}$$

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