Error Bounds and Discretization Grids in the Solution of Weakly Singular Integral Equations

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Abstract

In the solution of weakly singular second kind Fredholm integral equations defined on the space of Lebesgue integrable complex valued functions by projection-type methods such as Kantorovitch method or Sloan method [7], the choice of the discretization grids is crucial. We will present the proof of an error bound in terms of the mesh size of the underlying discretization grid on which no regularity assumptions are made and compare it with other recently proposed error bounds [2]. This allows us to use non regular grids which is convenient when there are boundary layers or discontinuities in the right hand side function of the equation. We present some results using a simplified model of the radiative transfer in stellar atmospheres which illustrates the actual behaviour of the error in terms of the distribution of the points in the grid.

AMS Classification: 65J10, 65R20.
Keywords: Projection approximations in L^1, weakly singular integral operators, nonuniform grids, error bounds.

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[1]Financial support provided, in part, by *Fundação para a Ciência e a Tecnologia* (FCT) through *Centro de Matemática da Universidade do Porto* (CMUP) and also by the Research Centre of Mathematics of the University of Minho through the FCT Pluriannual Funding Program. This is an extended version of the abstract published in *2nd Dolomites Workshop on Constructive Approximation and Applications. Book of abstracts, pp.67-67 (2009).*

1 Introduction

We consider the Fredholm integral equation of the second kind

$$(T - zI)\varphi = f, \tag{1}$$

where $T : X \to X$ is a linear compact integral operator, X is a Banach space, z is in the resolvent set $re(T)$ and hence $z \neq 0$ since T is compact.

Equation (1) has a unique solution $\varphi \in X$, for any $f \in X$. It can be represented by means of the resolvent operator $R(z) := (T - zI)^{-1}$ by $\varphi = R(z)f$.

Let $X := L^1([0, \tau^*], \mathbb{C})$ be the space of complex-valued Lebesgue-integrable (classes of) functions on a closed interval $[0, \tau^*]$.

Let T be the operator defined on X by

$$(Tx)(s) := \int_0^{\tau^*} g(|s-t|)x(t)dt, \; s \in [0, \tau^*],$$

where $g :]0, +\infty[\to \mathbb{R}$ is a weakly singular function at 0 such that

$$g(0^+) = +\infty, \text{ and } g \in L^1([0, +\infty[, \mathbb{R}). \tag{2}$$

To make technical aspects simpler, we also assume that

$$g \in C^0(]0, +\infty[, \mathbb{R}), \; g \geq 0 \text{ in }]0, +\infty[, \text{ and} \tag{3}$$

$$g \text{ is a decreasing function in }]0, +\infty[. \tag{4}$$

Since the solution φ of (1) satisfies

$$\varphi = \frac{1}{z}(T\varphi - f), \tag{5}$$

we may expect boundary layers for φ at the end points of the domain and where f behaves in a similar way, for details see [5].

For this reason, the possibility of using non regular grids, thus allowing for better refinement in the sensitive areas, is important.

2 Projection Approximations

Let us consider $(\pi_n)_{n\geq 1}$, a sequence of bounded projections each one having finite rank and range $X_n \subset X$.

Then, the classical projection approximation methods for the solution of (1) use operators that may be denoted as follows

$$T_n^G := \pi_n T \pi_n, T_n^K := \pi_n T, T_n^S := T\pi_n,$$

where the upper label G refers to the Petrov Galerkin method, the upper label K refers to the Kantorovitch method and the upper label S to the Sloan or iterated Galerkin method.

For each of them, $T_n : X \to X$ is a bounded linear operator, and $(T_n)_{n\geq 1}$ is at least $\upsilon-$convergent to T, meaning that $(\|T_n\|)_{n\geq 1}$ is bounded, $\|(T_n - T)T\| \to 0$, and $\|(T_n - T)T_n\| \to 0$, (see [3]).

In the case of Kantorovitch method, which is the method that we will consider mainly here, the convergence is even uniform.

We use one of these approximating operators to set an approximate problem

$$(T_n - zI)\varphi_n = f \tag{6}$$

(or $(T_n - zI)\varphi_n = \pi_n f$, in the case of the Petrov-Galerkin approximation).

It is known that for $z \in re(T)$, and for n large enough, $z \in re(T_n)$ and

$$\varphi_n = R_n(z)f, \text{ where } R_n(z) := (T_n - zI)^{-1}.$$

In the next section we will consider the quality of φ_n as an approximation to φ by setting some bounds on the error.

3 Convergence of Approximate Solutions

Theorem 1 *Let $(\pi_n)_{n\geq 1}$ be a sequence of projections onto X_n, pointwise convergent to I. Then there exists n_0 such that for all $n \geq n_0$,*

$$\begin{aligned}
\|\varphi_n^G - \varphi\| &\leq \beta^G \|(I - \pi_n)\varphi\|, \\
\|\varphi_n^K - \varphi\| &\leq \beta^K \left(\|(I - \pi_n)\varphi\| + \frac{1}{|z|}\|(I - \pi_n)f\| \right), \\
\|\varphi_n^S - \varphi\| &\leq \beta^S \|(I - \pi_n)\varphi\|,
\end{aligned}$$

where the constants

$$\begin{aligned}
\beta^G &:= |z| \sup_{n\geq n_0} \|(\pi_n T - zI)^{-1}\| \\
\beta^K &:= |z| \sup_{n\geq n_0} \|(\pi_n T - zI)^{-1}\| \\
\beta^S &:= \sup_{n\geq n_0} \|(T\pi_n - zI)^{-1}\|\|T\|
\end{aligned}$$

are finite.

Proof:

We will prove the inequality for the Kantorovitch and Sloan cases, since the case of the Petrov-Galerkin method was addressed in [5].

The constants β^K and β^S are finite due to the compactness of T and consequent convergence of $(\pi_n T)$ in norm to T, in the Kantorovitch case, or in the $\upsilon-$convergence sense, in the Sloan method (see [3] and [7] or [8]).

Let us consider the following equalities based in equation (5) and its projection by π_n,

$$\begin{aligned} \varphi &= \frac{1}{z}(T\varphi - f), \\ \pi_n\varphi &= \frac{1}{z}(\pi_n T\varphi - \pi_n f), \\ \varphi_n^K &= \frac{1}{z}(\pi_n T\varphi_n^K - f) \\ \varphi_n^S &= \frac{1}{z}(T\pi_n\varphi_n^S - f), \end{aligned}$$

then , for the Kantorovitch case, we have

$$\begin{aligned} \varphi - \pi_n\varphi &= \varphi - \varphi_n^K + \varphi_n^K - \pi_n\varphi \\ &= \frac{1}{z}((\pi_n T - zI)(\varphi_n^K - \varphi) - (I - \pi_n)f) \end{aligned}$$

and

$$(\varphi_n^K - \varphi) = zR_n^K((I - \pi_n)\varphi + \frac{1}{z}(I - \pi_n)f).$$

Now, let us consider the case of the Sloan or iterated Galerkin method:

$$\begin{aligned} \varphi - \pi_n\varphi &= \varphi - \varphi_n^S + \varphi_n^S - \pi_n\varphi \\ &= \frac{1}{z}(T\pi_n - zI)(\varphi_n^S - \varphi) - \frac{1}{z}(T - zI)(I - \pi_n)\varphi \\ &= \frac{1}{z}(T\pi_n - zI)(\varphi_n^S - \varphi) - \frac{1}{z}T(I - \pi_n)\varphi + (I - \pi_n)\varphi \end{aligned}$$

and so

$$\varphi_n^S - \varphi = (T\pi_n - zI)^{-1}T(I - \pi_n)\varphi.$$

The norm is then

$$\|\varphi_n^S - \varphi\| \leq \|(T\pi_n - zI)^{-1}\|\|T\|\|(I - \pi_n)\varphi\| \leq \beta^S\|(I - \pi_n)\varphi\|.$$

∎

4 Discretization Grids and Error Bounds

Let us consider a general grid $\mathcal{G}_n := (\tau_j)_{j=0}^n$ set on $[0, \tau^*]$ such that

$$\tau_0 := 0,\ \tau_n := \tau^*,\ h_j := \tau_j - \tau_{j-1} > 0,$$
$$h_{\max} := \max_{1 \le j \le n} h_j \text{ and } h_{\min} := \min_{1 \le j \le n} h_j.$$

We associate to this grid the local mean functionals e_j^* defined by

$$\langle x\,,\, e_j^* \rangle \quad := \quad \frac{1}{h_j} \int_{\tau_{j-1}}^{\tau_j} x(t)\, dt,$$

and the piecewise constant canonical functions e_j given by

$$e_j(s) \quad := \quad \begin{cases} 1 & \text{for } s \in]\tau_{j-1}, \tau_j[, \\ 0 & \text{otherwise.} \end{cases}$$

We may then define the projections onto the subspace X_n, spanned by $\{e_j, j = 1, ..., n\}$, as

$$\pi_n x := \sum_{j=1}^n \langle x\,,\, e_j^* \rangle e_j \qquad \text{for } x \in L^1([0, \tau^*], \mathbb{C}).$$

In order to estimate the relative error of the Kantorovitch approximation φ_n^K in terms of the grid parameters, mainly $h_{\max}$, we have the following theorem :

Theorem 2 *For $z \neq 0$ and g satisfying (2) to (4), the relative error of the Kantorovitch approximation satisfies the following inequalities, in the subordinated operator norm,*

$$\text{(7)} \qquad \frac{\|\varphi_n^K - \varphi\|}{\|\varphi\|} \le 8\, C^K \int_0^{h_{\max}/2} g(\tau)\, d\tau$$

$$\text{(8)} \quad \frac{\|\varphi_n^S - \varphi\|}{\|\varphi\|} \le C^S \left(8 \int_0^{h_{\max}/2} g(\tau)\, d\tau + 2 \sum_{j=1}^n \omega_1(f_{|[\tau_{j-1}, \tau_j]}, h_j) / \|\varphi\| \right),$$

where the function ω_1 is the oscillation of f in L^1.
The constants are

$$C^K = \quad \sup_{n \ge n_0} \|(\pi_n T - zI)^{-1}\|$$
$$C^S = \quad \frac{1}{|z|} \sup_{n \ge n_0} \|(T\pi_n - zI)^{-1}\|\, \|T\|.$$

The oscillation is here taken in the sense given by [10]

$$\omega_1(x\big|_{[a,b]}, \delta) := \sup_{0\le h\le \delta} \int_a^{b-h} |x(s+h) - x(s)|\, ds.$$

Proof:

From Theorem 1 and considering equation (5), we have

$$\begin{aligned} \varphi_n^K - \varphi &= (\pi_n T - zI)^{-1} \left((I-\pi_n)(z\varphi + f)\right) \\ &= (\pi_n T - zI)^{-1} \left((I-\pi_n)T\varphi\right) \end{aligned}$$

and so

$$\frac{\|\varphi_n^K - \varphi\|}{\|\varphi\|} \le \sup_{n\ge n_0} \|R_n^K\| \|(I-\pi_n)T\| \tag{9}$$

The fact that $\|(I-\pi_n)T\| \le 8\int_0^{h_{\max}/2} g(\tau)\, d\tau$ is proved in [5], in the context of the Petrov-Galerkin Approximation.

Similarly, for the Sloan approximation φ_n^S, we have to bound

$$\|\varphi_n^s - \varphi\| \le \sup_{n\ge n_0} \|R_n^S\| \|T\| \|(I-\pi_n)\varphi\|. \tag{10}$$

Using equation (1) we have

$$(I-\pi_n)\varphi = \frac{1}{z}(I-\pi_n)(T\varphi - f)$$

and the norm is

$$\|(I-\pi_n)\varphi\| = \frac{1}{|z|}\left(\|(I-\pi_n)T\|\|\varphi\| + \|(I-\pi_n)f\|\right)$$

Here again we can see [5] to conclude that

$$\|(I-\pi_n)T\| \le 8\int_0^{h_{\max}/2} g(\tau)\, d\tau$$

and

$$\|(I-\pi_n)f\| \le 2\sum_{j=1}^{n} \omega_1(f\big|_{[\tau_{j-1},\tau_j]}, h_j).$$

∎

The error bounds given in this theorem will be compared to the following ones, on an example, in the next section.

In [6] the authors have proposed another bound for $\|(I-\pi_n)T\|$

$$\|(I-\pi_n)T\| \leq \tag{11}$$

$$2h_{\max}(g(h_{\min}/2)-g(\tau^*)) + 2h_{\max}(g(h_{\min})-g(\tau^*)) + \\ 4\int_0^{h_{\max}/2} g(\sigma)d\sigma + 4\int_0^{h_{\max}} g(\sigma)d\sigma + 4\int_0^{3h_{\max}/2} g(\sigma)d\sigma$$

Although this is less sharp than the one here proposed it may be interesting since it is set in terms of the maximum and minimum values of the amplitudes of the subintervals, and its proof is based on geometric considerations.

In [2], Ahues, Amosov and Largillier propose the following bound that requires the derivability of the kernel, which is not the case with the other bounds refered here.

$$\|(I-\pi_n)T\| \leq 4\left(\int_0^{h_{\max}/2} g(\sigma)d\sigma + h_{\max}\int_{h_{\max}/2}^{+\infty} |Dg(\sigma)|d\sigma\right). \tag{12}$$

5 Numerical Computations and Conclusions

The computations that we will show were done with an integral operator that comes from a simplified model of radiative transfer in stellar atmospheres. Its kernel is $g(s) := \frac{\varpi}{2}E_1(s)$, where E_1 is the first exponential integral function (see [1])

$$E_1(s) := \int_0^1 \frac{\exp(-s/\mu)}{\mu}d\mu, s > 0$$

with $s \in [0,\tau^*]$ representing the optical depth of the stellar atmosphere and $\tau^* \in]0,+\infty[$ the optical thickness. The albedo $\varpi \in]0,1[$ characterizes the scattering properties of the medium.

We take $z = 1$, $\tau^* = 100$, $\varpi = 0.75$ and the right hand side function of (6) as

$$f(s) := \begin{cases} -1 & \text{for} \quad 0 \leq s \leq \tau^*/2, \\ 0 & \text{for} \quad \tau^*/2 < s \leq \tau^* \end{cases}$$

(for details see [4] and [9]).

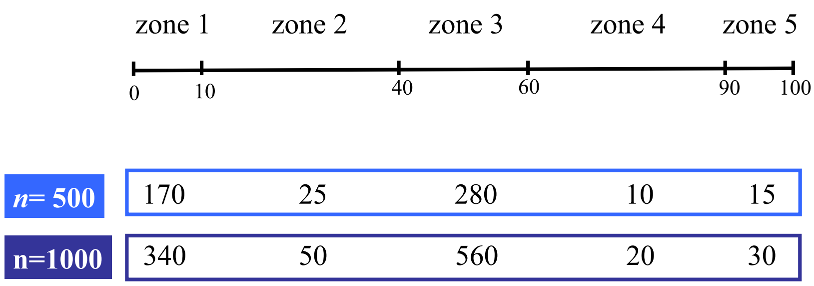


Figure 1: Number of nodes of the nonuniform grids in 5 zones of the interval $[0, \tau^*]$

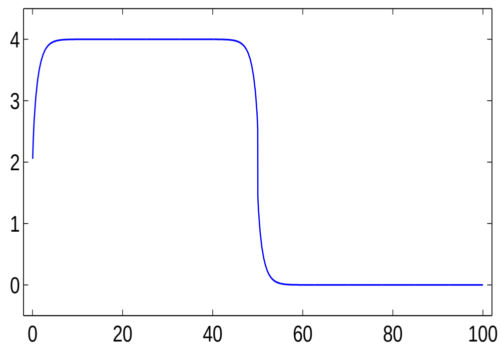


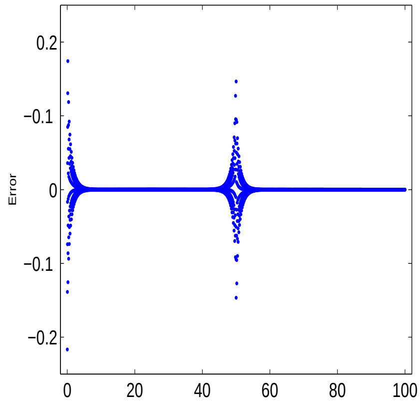
Figure 2: Reference solution φ^{ref}

The grids considered on $[0, 100]$ for this example were two uniform grids with $n = 500$ and $n = 1000$ and two nonuniform grids with $n = 500$ and $n = 1000$ as seen in Fig. 1.

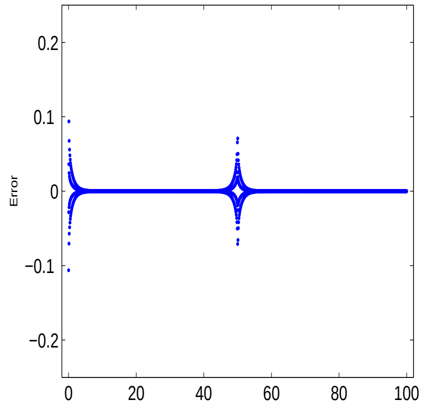
The computations have been performed with Matlab.

In this example we compare the error bounds refered in Section 4, but we also computed the relative error of the approximations, with respect to a reference solution φ^{ref}, obtained with a much finer grid (here with 4001 points) in order to see the usefulness of the nonuniform grids. Fig. 2 shows this reference solution.

In Fig.3 we compare the error of φ^K_{500} obtained with 500 equal subintervals (501 points) to the error of φ^K_{1000} obtained with 1000 equal subintervals (1001 points). The error is reduced by one half, approximately. But if we distribute the 501 points in a non uniform grid as described in Fig. 1, thus refining the grid more in the zone 1 where a boundary layer is expected, due to the singularity of the kernel, and in the middle of the



(a) Uniform grid with $n = 500$



(b) Uniform grid with $n = 1000$

Figure 3: Error ($\varphi^{ref} - \varphi_n$) vs grid points, Kantorovitch approximation with uniform grids

interval (zone 3) where the right hand side function has a discontinuity, and setting large subintervals in the zones 2 and 3, we have an overall error that is smaller than the error of the solution φ^K_{1000} obtained with an uniform grid twice as large.

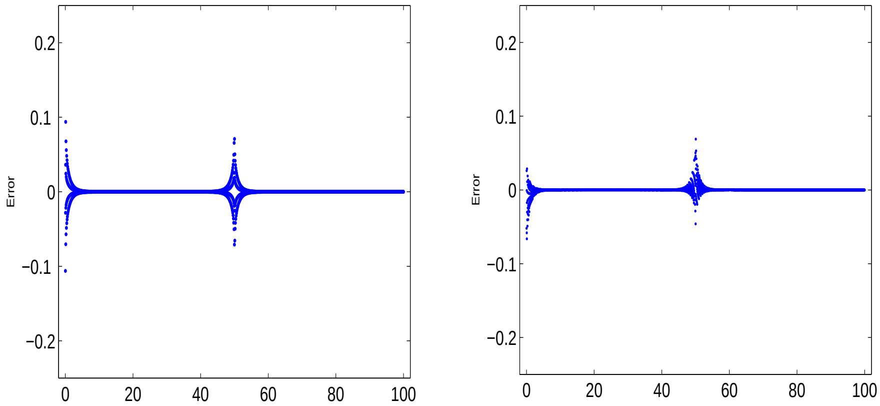
If we compute the solution with a nonuniform grid of $n = 1000$ subintervals distributed as in Fig. 1, the error (Fig. 5) reduces again, by approximately one half in relation to Fig. 4b and even better in relation to Fig. 4a.

Table 1 compares three error bounds for this example and the Kantorovitch approximation. Error bound A is given by (7), B is computed with equations (9) and (11) and error bound C with equations (9) and (12).

This table also contains the 1-norm of the relative error.

Similarly, Table 2 compares three error bounds for this example with the Sloan approximation. Error bound A is given by (8), B is computed with equations (10) and (11) and error bound C with equations (10) and (12). In this example the $\|(I - \pi_n)f$ is 0.

As we can see the error bounds A and C are comparable and the error bound C is a little larger. Comparing with the norm of the *a posteriori* error we can see that all the bounds are pessimistic. However they do not require the grid to be uniform. They are to be improved by analysing the error in the different zones separately, in future work.



(a) Uniform grid with $n = 1000$

(b) Non uniform grid with $n = 500$

Figure 4: Error $(\varphi^{ref} - \varphi_n)$ vs grid points, Kantorovitch approximation with uniform grid vs smaller nonuniform grid

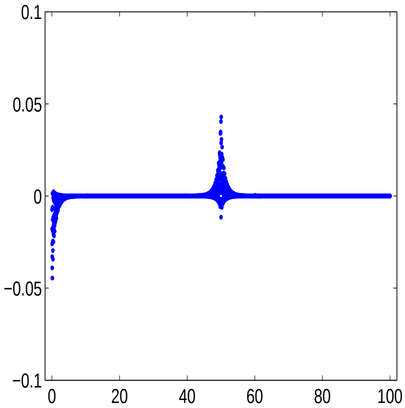


Figure 5: Error $(\varphi^{ref} - \varphi_n)$ Kantorovitch approximation with nonuniform grid of 1001 points

n	Grid	Error bound			Relative error
		A	B	C	
$n = 500$	Uniform $h_{\max} = \frac{1}{5}$	3.3E+0	9.3E+0	3.9E+0	1.2E-3
$n = 500$	Nonuniform $h_{\min} = \frac{1}{17}, h_{\max} = 3$	1.2E+1	6.6E+1	7.4E+0	4.6E-4
$n = 1000$	Uniform $h_{\max} = \frac{1}{10}$	2.1E+0	6.2E+0	2.6E+0	6.3E-4
$n = 1000$	Nonuniform $h_{\max} = \frac{1}{34}, h_{\max} = \frac{3}{2}$	9.4E+0	4.6E+1	7.8E0	3.0E-4

Table 1: Error bounds and relative error of the Kantorovitch approximation in L^1 norm

n	Grid	Error bound		
		A	B	C
$n = 500$	Uniform $h_{\max} = \frac{1}{5}$	2.5E+0	7.0E+0	2.9E+0
$n = 500$	Nonuniform $h_{\min} = \frac{1}{17}, h_{\max} = 3$	8.4E+0	4.9E+1	5.6E+0
$n = 1000$	Uniform $h_{\max} = \frac{1}{10}$	1.6E+0	4.7E+0	1.9E+0
$n = 1000$	Nonuniform $h_{\max} = \frac{1}{34}, h_{\max} = \frac{3}{2}$	7.1E+0	3.5E+1	5.9E+0

Table 2: Error bounds of the Sloan approximation

Acknowledgements
This research was partially supported by Fundação para a Ciência e a Tecnologia, through the Centro de Matemática da Universidade Porto (CMUP) and the Centro de Matemática da Universidade do Minho (CMat).

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