# On a Riemann-Liouville fractional analog of the Laplace operator with positive energy

M. Dalla Riva * S. Yakubovich*

September 30, 2010

**Abstract**

The purpose of this paper is the definition of a particular fractional analog of the Laplace operator in a rectangular domain in the plane by exploiting the Riemann-Liouville fractional derivatives. Such a definition allows the introduction of fractional boundary value problems which correspond to the classical Dirichlet, Neumann and mixed boundary value problems for the Laplace operator. By exploiting a suitable Integration by Parts Formula and the positiveness of the corresponding energy integral we verify some uniqueness results for the solutions of the boundary value problems and we show the existence of particular solutions which play the role of the affine functions.

**Keywords:** Fractional analog of Laplace operator, Riemann-Liouville integrals and derivative, Integration by Parts, Boundary value problems.
**2000 Mathematics Subject Classification:** 35J05, 26A33, 35A02.

## 1 Introduction

As is well known the fractional Laplace operator is usually defined by exploiting the Riesz transform (c.f. e.g. Samko, Kilbas and Marichev [8, Chap. 5].) Its properties and applications has been largely investigated in literature. We mention here as an example the work of Bogdan [1], Samko [7], Guan and Ma [4], Guan [3], Caffarelli, Salsa and Silvestre [2]. In particular, Guan has

*Work supported by *Fundação para a Ciência e a Tecnologia* (FCT, the programmes POCTI and POSI) through the *Centro de Matemática da Universidade do Porto* (CMUP). Available as a PDF file from http://www.fc.up.pt/cmup. The research of the first author has been also supported by FCT research grant SFRH/BPD/64437/2009.

shown in [3] the validity of a Integration by Parts Formula for the fractional Laplace operator. Due to the positiveness of the corresponding energy integral such a formula can be exploited to show uniqueness results for suitable boundary value problems. Here instead we consider a fractional analog of the Laplace operator in a rectangular domain in the plane defined by exploiting the Riemann-Liouville fractional derivatives in directions $x$ and $y$. We introduce the following notation.

$$\text{Let } x_0, x_1, y_0, y_1 \in \mathbb{R} \text{ with } x_0 < x_1,\ y_0 < y_1\,. \text{ Let } X \equiv ]x_0, x_1[\times]y_0, y_1[\,. \tag{1}$$

Let $\alpha, \beta \in ]0, 1[$. In [9] the second author has considered the operator $\Delta^{\alpha,\beta}$ defined by

$$\Delta^{\alpha,\beta} u(x, y) \equiv D^{1+\alpha}_{x_0+} u(x, y) + D^{1+\beta}_{y_0+} u(x, y) \quad \forall (x, y) \in \mathrm{cl}X \tag{2}$$

for all measurable function $u$ from $X$ to $\mathbb{R}$. Here $\mathrm{cl}X$ denotes the closure of $X$ and $D^{1+\alpha}_{x_0+}$, $D^{1+\beta}_{y_0+}$ denote the Riemann-Liouville fractional derivative by $x$ and $y$ of order $1+\alpha$ and $1+\beta$, respectively (cf. Samko, Kilbas and Marichev [8, Chap. 1 and 5].) In particular, by the results of [9] one can deduce a uniqueness result for the solution of the fractional boundary value problem

$$\begin{cases} \Delta^{\alpha,\beta} u(x, y) = 0 & & \text{for all } (x, y) \in X \\ I^{1-\beta}_{y_0+} u(x, y_0) = f_0(x)\,, & D^{\beta}_{y_0+} u(x, y_0) = f_1(x) & \text{for all } x \in ]x_0, x_1[\,, \\ I^{1-\alpha}_{x_0+} u(x_0, y) = g_0(y)\,, & D^{\alpha}_{x_0+} u(x_0, y) = g_1(y) & \text{for all } y \in ]y_0, y_1[\,, \end{cases} \tag{3}$$

where $I^{1-\alpha}_{x_0+}$ and $I^{1-\beta}_{y_0+}$ denote the Riemann-Liouville fractional integration by $x$ and $y$ of order $1-\alpha$ and $1-\beta$, respectively (cf. Samko et al. [8, Chap. 1 and 5].) We note that, for $\alpha$ and $\beta$ which tends to 1 the boundary value problem in (3) becomes

$$\begin{cases} \Delta u = 0 & & \text{in } X \\ u(x, y_0) = f_0(x)\,, & \frac{d}{dy} u(x, y_0) = f_1 & \text{for all } x \in [x_0, x_1]\,, \\ u(x_0, y) = g_0(y)\,, & \frac{d}{dx} u(x_0, y) = g_1 & \text{for all } y \in [y_0, y_1]\,. \end{cases} \tag{4}$$

The system of equations in (4) is a boundary value problem for the partial differential equation $\Delta u = 0$ with a Cauchy type condition on the edge $\{(x_0, y) : y \in [y_0, y_1]\} \cup \{(x, y_0) : x \in [x_0, x_1]\}$. As is well known, for the equation $\Delta u = 0$ it is more common to consider boundary conditions of Dirichlet, Neumann, Robin or mixed type. It is then natural to ask whether it would be possible to consider fractional boundary value problems like the one in (3) but with boundary condition which becomes of Dirichlet, Neumann, Robin or mixed type when $\alpha, \beta \to 1$ and in case to investigate the existence and the uniqueness properties of the corresponding solutions. In this paper we exploit a method based on a suitable Integration by Parts Formula and on the positiveness of the corresponding energy integral. Our aim is to prove uniqueness results for some fractional boundary value problems which correspond to the Dirichlet, Neumann, Robin or mixed classical boundary value problems. Unfortunately we could not succeed to apply this strategy to the operator $\Delta^{\alpha,\beta}$. For this reason we introduce the fractional partial differential operator $\Delta^{\alpha,\beta}_{-+} \equiv D^{\alpha}_{x_1-}D^{\alpha}_{x_0+} + D^{\beta}_{y_1-}D^{\beta}_{y_0+}$ (cf. definitions (12) here below.) For the operator $\Delta^{\alpha,\beta}_{-+}$ we can prove the validity of an Integration by Parts Formula which in turn implies uniqueness properties of the solutions of some boundary value problems for the fractional partial differential equations $\Delta^{\alpha,\beta}_{-+}u = F$ (cf. Thm. 4.2.) In particular, we consider in two examples mixed Dirichlet-Neumann and Robin type boundary conditions (cf. Examples 5.3 and 5.4.)

The paper is organize as follows. Section 2 is a section of preliminaries where we introduce some notation and we state some known properties of the fractional integral and derivatives. In Section 3 we introduce the space of functions $\mathcal{A}^{\alpha,\beta}(X)$ which is the natural domain of the operator $\Delta^{\alpha,\beta}_{-+}$. Then, we introduce the functions $C$, $L$ and $A$ which play the role of constant, linear and affine functions on a interval of $\mathbb{R}$. In Section 4 we prove the Integration by Parts formula for the operator $\Delta^{\alpha,\beta}_{-+}$ and for a couple of functions $u$, $v$ of $\mathcal{A}^{\alpha,\beta}(X)$. In Section 5 we deduce some uniqueness results for fractional boundary value problems for the operator $\Delta^{\alpha,\beta}_{-+}$. We also show in a example how to obtain existence and uniqueness results for those particular boundary value problems which admit solutions corresponding to the affine functions on $X$ (cf. Example 5.5.) In Section 6, we provide an equivalent formulation of the equation $\Delta^{\alpha,\beta}_{-+}u = F$ in terms of a fractional integral equation. In the last Section 7, we show the existence of solution $u$ of the equation $\Delta^{\alpha,\beta}_{-+}u = 0$ which do not correspond to the affine functions on $X$. To do so, we investigate equation $\Delta^{\alpha,\beta}_{-+}u = 0$ for $u(x, y) = f(x)g(y)$ (cf. Thm. 7.2 and 7.4.)

# 2 Preliminaries and notation

Let $a, b \in \mathbb{R}$, $a < b$. Let $\gamma \in ]0, 1[$. Let $f$ be a measurable function from the open interval $]a, b[$ to $\mathbb{R}$. Then $I^{\gamma}_{a+}f(x)$ and $D^{\gamma}_{a+}f(x)$ denote the left-sided Riemann-Liouville integral and derivative of $f$ in the point $x$ of $]a, b[$, respectively, and $I^{\gamma}_{b-}f(x)$, $D^{\gamma}_{b-}f(x)$ denote the right-sided Riemann-Liouville integral and derivative of $f$ in the point $x$ of $]a, b[$, respectively. Namely, we set

$$
\begin{aligned}
I^{\gamma}_{a+}f(x) &\equiv \frac{1}{\Gamma(\gamma)}\int_a^x (x-t)^{\gamma-1}f(t)\,dt\,, &(5)\\
D^{\gamma}_{a+}f(x) &\equiv \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_a^x (x-t)^{-\gamma}f(t)\,dt\,, &(6)\\
I^{\gamma}_{b-}f(x) &\equiv \frac{1}{\Gamma(\gamma)}\int_x^b (t-x)^{\gamma-1}f(t)\,dt\,, &(7)\\
D^{\gamma}_{b-}f(x) &\equiv -\frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_x^b (t-x)^{-\gamma}f(t)\,dt &(8)
\end{aligned}
$$

for all $x \in ]a,b[$. Here $\Gamma(\cdot)$ denotes the Euler Gamma Function. We define the Riemann-Liouville fractional integral and derivative in the endpoints $a$ and $b$ to be the limits as $x \to a$ and $x \to b$ of the expressions in (5)–(8). Namely, we set $I^{\gamma}_{a+}f(a) \equiv \lim_{x\to a^+} I^{\gamma}_{a+}f(x)$, $I^{\gamma}_{a+}f(b) \equiv \lim_{x\to b^-} I^{\gamma}_{a+}f(x)$, and $D^{\gamma}_{a+}f(a) \equiv \lim_{x\to a^+} D^{\gamma}_{a+}f(x)$, $D^{\gamma}_{a+}f(b) \equiv \lim_{x\to b^-} D^{\gamma}_{a+}f(x)$, $I^{\gamma}_{b-}f(a) \equiv \lim_{x\to a^+} I^{\gamma}_{b-}f(x)$, $I^{\gamma}_{b-}f(b) \equiv \lim_{x\to b^-} I^{\gamma}_{b-}f(x)$, $D^{\gamma}_{b-}f(a) \equiv \lim_{x\to a^+} D^{\gamma}_{b-}f(x)$, $D^{\gamma}_{b-}f(b) \equiv \lim_{x\to b^-} D^{\gamma}_{b-}f(x)$. We also find convenient to set

$$
I^{1}_{a+}f(x) \equiv \int_a^x f(t)\,dt\,, \qquad I^{1}_{b-}f(x) \equiv \int_x^b f(t)\,dt\,. \tag{9}
$$

Now let $x_0, x_1, y_0, y_1, X$ be as in (1). Let $u = u(\cdot,\cdot)$ be a function from $\mathrm{cl}X$ to $\mathbb{R}$. Let $(x,y)$ be a fixed point of $\mathrm{cl}X$. Then $u_x$ and $u^y$ denote the functions from $[y_0,y_1]$ to $\mathbb{R}$ and from $[x_0,x_1]$ to $\mathbb{R}$, respectively, defined by

$$
u_x(s) \equiv u(x,s) \quad \forall\, s \in [y_0,y_1]\,, \qquad u^y(t) \equiv u(t,y) \quad \forall\, t \in [x_0,x_1]\,.
$$

If $(x,y) \in \mathrm{cl}X$ then we denote by $I^{\gamma}_{x_0+}u(x,y)$ the $\gamma$ fractional integral in the point $x$ of the function $u^y$. Namely, we set $I^{\gamma}_{x_0+}u(x,y) \equiv I^{\gamma}_{x_0+}u^y(x)$ for all $(x,y) \in \mathrm{cl}X$. Similarly we define $D^{\gamma}_{x_0+}u(x,y) \equiv D^{\gamma}_{x_0+}u^y(x)$, and $I^{\gamma}_{y_0+}u(x,y) \equiv I^{\gamma}_{y_0+}u_x(y)$, and $D^{\gamma}_{y_0+}u(x,y) \equiv D^{\gamma}_{y_0+}u_x(y)$, and $I^{\gamma}_{x_1-}u(x,y) \equiv I^{\gamma}_{x_1-}u^y(x)$, and $D^{\gamma}_{x_1-}u(x,y) \equiv D^{\gamma}_{x_1-}u^y(x)$, and $I^{\gamma}_{y_1-}u(x,y) \equiv I^{\gamma}_{y_1-}u_x(y)$, and $D^{\gamma}_{y_1-}u(x,y) \equiv D^{\gamma}_{y_1-}u_x(y)$ for all $(x,y) \in \mathrm{cl}X$.

We collect in the following Lemmas 2.1, 2.2 and 2.3 some known boundedness and inversion properties of the operators $I^{\gamma}_{a+}$ and $I^{\gamma}_{b-}$ acting on $L_1(a,b)$ and on $L_1(X)$. In the sequel we denote by $L_p(a,b)$ and by $L_p(X)$ the classical Lebesgue spaces endowed with the usual norm $\|\cdot\|_p$, for all $p \in [1,+\infty[$. We denote by $AC[a,b]$ the set of absolutely continuous functions from the closed interval $[a,b]$ to $\mathbb{R}$. If $\lambda \in ]0,1[$, we denote by $H^{\lambda}[a,b]$ the space of the functions from $[a,b]$ to $\mathbb{R}$ which are Hölder continuous with exponent $\lambda$.

**Lemma 2.1.** *Let $a, b \in \mathbb{R}$, $a < b$. Let $\gamma \in ]0,1[$. Let $f \in L_1(a,b)$. Then the functions $I^{\gamma}_{a+}f$ and $I^{\gamma}_{b-}f$ belong to $L_r(a,b)$ for all $r \in [1, 1/(1-\gamma)[$. Moreover, we have $\|I^{\gamma}_{a+}f\|_r \le \frac{(b-a)^{\gamma-1+1/r}}{\Gamma(\gamma)(r(\gamma-1)+1)^{1/r}}\|f\|_1$ and $\|I^{\gamma}_{b-}f\|_r \le \frac{(b-a)^{\gamma-1+1/r}}{\Gamma(\gamma)(r(\gamma-1)+1)^{1/r}}\|f\|_1$ for all $r \in [1, 1/(1-\gamma)[$.*

*Proof.* The statement of the Lemma can be verified by means of a straightforward calculation based on the generalized Minkowski Inequality. □

**Lemma 2.2.** *Let $x_0, x_1, y_0, y_1, X$ be as in (1). Let $\gamma \in ]0,1[$, $p \in [1, 1/(1-\gamma)[$. Then the operators $I^{\gamma}_{x_0+}$, $I^{\gamma}_{x_1-}$, $I^{\gamma}_{y_0+}$, $I^{\gamma}_{y_1-}$ are bounded from $L_p(X)$ to itself.*

*Proof.* The statement of the Lemma can be verified by means of a straightforward calculation based on the generalized Minkowski Inequality. We include here a proof for the sake of completeness. Let $\phi \in L_p(X)$. Then we have

$$\begin{aligned}&\left\{\int_{x_0}^{x_1}\int_{y_0}^{y_1}\left|I^{\gamma}_{x_0+}\phi(x,y)\right|^p\,dxdy\right\}^{\frac{1}{p}} \qquad (10)\\ &\quad= \frac{1}{\Gamma(\gamma)}\left\{\int_{x_0}^{x_1}\int_{y_0}^{y_1}\left|\int_{x_0}^{x_1}\mathcal{X}_{t<x}\,(x-t)^{\gamma-1}\phi(t,y)\,dt\right|^p\,dxdy\right\}^{\frac{1}{p}}\end{aligned}$$

where $\mathcal{X}_{t<x} \equiv 1$ if $t < x$ and $\mathcal{X}_{t<x} \equiv 0$ if $t \ge x$. By the Minkowski Inequality the expression on the right hand side of (10) is less or equal than

$$\begin{aligned}&\frac{1}{\Gamma(\gamma)}\int_{x_0}^{x_1}\left\{\int_{x_0}^{x_1}\int_{y_0}^{y_1}\mathcal{X}_{t<x}\,(x-t)^{p(\gamma-1)}\,|\phi(t,y)|^p\,dxdy\right\}^{\frac{1}{p}}dt\\ &\quad= \frac{1}{\Gamma(\gamma)}\int_{x_0}^{x_1}\left\{\int_{x_0}^{x_1}\mathcal{X}_{t<x}\,(x-t)^{p(\gamma-1)}\,dx\right\}^{\frac{1}{p}}\left\{\int_{y_0}^{y_1}|\phi(t,y)|^p\,dy\right\}^{\frac{1}{p}}dt\,.\end{aligned}$$

Which in turn is less or equal than

$$\frac{(x_1-x_0)^{\gamma-1+1/p}}{\Gamma(\gamma)(p(\gamma-1)+1)^{1/p}}\int_{x_0}^{x_1}\left\{\int_{y_0}^{y_1}|\phi(t,y)|^p\,dy\right\}^{\frac{1}{p}}dt.$$

Finally, by the Hölder Inequality the integral right above is less or equal than $(x_1-x_0)^{1-1/p}\|\phi\|_p$. So that

$$\left\|I^{\gamma}_{x_0+}\phi\right\|_p \le \frac{(x_1-x_0)^{\gamma}}{\Gamma(\gamma)(p(\gamma-1)+1)^{1/p}}\|\phi\|_p\,.$$

The proof for $I^{\gamma}_{x_1-}$, $I^{\gamma}_{y_0+}$ and $I^{\gamma}_{y_1-}$ is similar and we omit it. □

**Lemma 2.3.** *Let $a, b \in \mathbb{R}$, $a < b$. Let $\gamma \in ]0,1[$. Let $f \in L_1(a,b)$. Then the following statements hold.*

*(i)* $f = D^{\gamma}_{a+} I^{\gamma}_{a+} f$.

*(ii)* $f = D^{\gamma}_{b-} I^{\gamma}_{b-} f$.

*(iii)* *If $I^{1-\gamma}_{a+} f \in AC[a,b]$, then*

$$I^{\gamma}_{a+} D^{\gamma}_{a+} f(x) = f(x) - \frac{r_a(x)^{\gamma-1}}{\Gamma(\gamma)} I^{1-\gamma}_{a+} f(a) \qquad \forall x \in [a,b]$$

*where $r_a(x) \equiv (x-a)$ for all $x \in [a,b]$.*

*(iv)* *If $I^{1-\gamma}_{b-} f \in AC[a,b]$, then*

$$I^{\gamma}_{b-} D^{\gamma}_{b-} f(x) = f(x) - \frac{r_b(x)^{\gamma-1}}{\Gamma(\gamma)} I^{1-\gamma}_{b-} f(b) \qquad \forall x \in [a,b]$$

*where $r_b(x) \equiv (b-x)$ for all $x \in [a,b]$.*

*Proof.* For a proof of (i) and (iii) we refer Samko et al. [8, Chap. 1, Thm. 2.4]. To prove (ii) we note that

$$I^{\gamma}_{b-} f(x) = Q\left(I^{\gamma}_{a+} Q f\right)(x), \quad D^{\gamma}_{b-} f(x) = Q\left(D^{\gamma}_{a+} Q f\right)(x) \qquad \forall x \in [a,b] \quad (11)$$

where $Q$ is the "reflection operator" which takes a function $g$ from $[a,b]$ to $\mathbb{R}$ to the function $Qg$ defined by $Qg(t) \equiv g(a+b-t)$ for all $t \in [a,b]$. Then we have $D^{\gamma}_{b-} I^{\gamma}_{b-} f = Q\left(D^{\gamma}_{a+} QQ\left(I^{\gamma}_{a+} Q f\right)\right) = Q\left(D^{\gamma}_{a+}\left(I^{\gamma}_{a+} Q f\right)\right)$ and statement (i) implies that $Q\left(D^{\gamma}_{a+}\left(I^{\gamma}_{a+} Q f\right)\right) = QQf = f$. Similarly we verify statement (iv). Indeed by (11) we have $I^{\gamma}_{b-} D^{\gamma}_{b-} f = Q\left(I^{\gamma}_{a+} QQ\left(D^{\gamma}_{a+} Q f\right)\right) = Q\left(I^{\gamma}_{a+}\left(D^{\gamma}_{a+} Q f\right)\right)$. Then statement (iii) implies that $Q\left(I^{\gamma}_{a+}\left(D^{\gamma}_{a+} Q f\right)\right) = QQf - \Gamma(\gamma)^{-1}(Q r_a^{\gamma-1}) Q(I^{1-\gamma}_{a+} Q f(a)) = f - \Gamma(\gamma)^{-1} r_b^{\gamma-1} I^{1-\gamma}_{b-} f(b)$. □

**Lemma 2.4.** *Let $a, b \in \mathbb{R}$, $a < b$. Let $\gamma, \theta \in ]0,1[$, $\gamma + \theta \leq 1$. Let $f \in L_1(a,b)$. Then $I^{\gamma}_{a+} I^{\theta}_{a+} f = I^{\gamma+\theta}_{a+} f$ and $I^{\gamma}_{b-} I^{\theta}_{b-} f = I^{\gamma+\theta}_{b-} f$.*

*Proof.* For a proof of $I^{\gamma}_{a+} I^{\theta}_{a+} f = I^{\gamma+\theta}_{a+} f$ we refer to Samko et al. [8, Chap. 1, Thm. 2.5]. Then, by equation $I^{\gamma}_{b-} I^{\theta}_{b-} f = Q I^{\gamma}_{a+} I^{\theta}_{a+} Q f$ we deduce the validity of $I^{\gamma}_{b-} I^{\theta}_{b-} f = I^{\gamma+\theta}_{b-} f$ (cf. proof of Lemma 2.3.) □

Now let $x_0, x_1, y_0, y_1, X$ be as in (1). Let $\alpha, \beta \in ]0,1[$. We set

$$\Delta^{\alpha,\beta}_{-+} u(x,y) \equiv -D^{\alpha}_{x_1-} D^{\alpha}_{x_0+} u(x,y) - D^{\beta}_{y_1-} D^{\beta}_{y_0+} u(x,y) \quad \forall (x,y) \in \mathrm{cl} X \quad (12)$$

and

$$\nabla^{\alpha,\beta}_{x_0,y_0+}u(x,y) \equiv \Big(D^{\alpha}_{x_0+}u(x,y), D^{\beta}_{y_0+}u(x,y)\Big) \quad \forall (x,y) \in \mathrm{cl}X \tag{13}$$

for all measurable function $u$ of $X$ to $\mathbb{R}$. We note that the expression on the right hand side of (12) is defined for almost all $(x,y) \in X$ if $D^{\alpha}_{x_0+}u^y \in I^{\alpha}_{x_1-}(L_1(x_0,x_1))$ and $D^{\beta}_{y_0+}u_x \in I^{\beta}_{y_1-}(L_1(y_0,y_1))$ for all fixed $(x,y) \in \mathrm{cl}X$. Moreover, if $u$ is a sufficiently smooth function on $X$ then the right hand side of (12) converges to $\Delta u \equiv \frac{d^2}{dx^2}u + \frac{d^2}{dy^2}u$ and the right hand side of (13) converges to $\nabla u \equiv (\frac{d}{dx}u\,,\ \frac{d}{dy}u)$ when $\alpha, \beta \to 1$ .

# 3 Some particular functions for $\Delta^{\alpha,\beta}_{-+}$

Let $a,b \in \mathbb{R}$, $a < b$. Let $\gamma \in ]0,1[$. Then we denote by $\mathcal{A}^{\gamma}(a,b)$ the space of functions from $]a,b[$ to $\mathbb{R}$ defined by

$$\mathcal{A}^{\gamma}(a,b) \equiv \left\{ f \in L_1(a,b)\ :\ I^{1-\gamma}_{a+}f \in AC[a,b] \text{ and } I^{1-\gamma}_{b-}D^{\gamma}_{a+}f \in AC[a,b] \right\}\ .$$

Let $x_0, x_1, y_0, y_1, X$ be as in (1). Let $\alpha, \beta \in ]0,1[$. We set

$$\mathcal{A}^{\alpha,\beta}(X) \equiv \Bigg\{ u \in L_1(X)\ : \tag{14}$$

$$u^y \in \mathcal{A}^{\alpha}(x_0,x_1)\,,\ u_x \in \mathcal{A}^{\beta}(y_0,y_1) \text{ for almost all } (x,y) \in \mathrm{cl}X \Bigg\}.$$

Let $c, d \in \mathbb{R}$. We denote by $C[\gamma,a,b,c]$, $L[\gamma,a,b,d]$ and $A[\gamma,a,b,c,d]$ the functions form $]a,b[$ to $\mathbb{R}$ defined by

$$C[\gamma,a,b,c](t) \equiv \frac{r_a(t)^{\gamma-1}}{\Gamma(\gamma)}c\,, \tag{15}$$

$$L[\gamma,a,b,d](t) \equiv \frac{I^{\gamma}_{a+}r_b^{\gamma-1}(t)}{\Gamma(\gamma)}d\,, \tag{16}$$

$$A[\gamma,a,b,c,d](t) \equiv \frac{1}{\Gamma(\gamma)}\Big\{ r_a(t)^{\gamma-1}c + [I^{\gamma}_{a+}r_b^{\gamma-1}](t)d \Big\} \tag{17}$$

for all $t \in ]a,b[$ and for all $c, d \in \mathbb{R}$ (see also Lemma 2.3 (iii) and (iv).) We note that for $\gamma \to 1$ the function $C[\gamma,a,b,c]$, $L[\gamma,a,b,d]$ and $A[\gamma,a,b,c,d]$ become constant, linear and affine on $]a,b[$, respectively.

For the space $\mathcal{A}^{\gamma}(a,b)$ the following statements hold.

**Proposition 3.1.** *Let $a,b \in \mathbb{R}$, $a < b$. Let $\gamma \in ]0,1[$. Then the following statements hold.*

(i) *A function $f$ from $]a,b[$ to $\mathbb{R}$ belongs to $\mathcal{A}^\gamma(a,b)$ if and only if there exist $\phi \in L_1(a,b)$ and $c,d \in \mathbb{R}$ such that $f = I^\gamma_{a+} I^\gamma_{b-}\phi + A[\gamma,a,b,c,d]$.*

(ii) *Let $f \in \mathcal{A}^\gamma(a,b)$. Let $\phi \in L_1(a,b)$. Let $c,d \in \mathbb{R}$. Then we have*

$$\begin{cases} D^\gamma_{b-} D^\gamma_{a+} f = \phi\,, \\ I^{1-\gamma}_{a+} f(a) = c\,, \\ I^{1-\gamma}_{b-} D^\gamma_{a+} f(b) = d \end{cases}$$

*if and only if $f = I^\gamma_{a+} I^\gamma_{b-}\phi + A[\gamma,a,b,c,d]$.*

(iii) *If $f \in \mathcal{A}^\gamma(a,b)$, then the functions $f$ and $D^\gamma_{a+} f$ belong to $L_r(a,b)$ for all $r \in [1,(1-\gamma)^{-1}[$.*

(iv) *If $f \in \mathcal{A}^\gamma(a,b)$ and $\gamma \le \frac{1}{2}$, then $f - C[\gamma,a,b,I^{1-\gamma}_{a+} f(a)] \in L_r(a,b)$ for all $r \in [1,(1-2\gamma)^{-1}[$.*

(v) *If $f \in \mathcal{A}^\gamma(a,b)$ and $\gamma > \frac{1}{2}$, then $f - C[\gamma,a,b,I^{1-\gamma}_{a+} f(a)] \in H^\lambda[a,b]$ for all $\lambda \in ]0,2\gamma-1[$.*

*Proof. (i)* Let $f \in \mathcal{A}^\gamma(a,b)$. Then $I^{1-\gamma}_{a+} f \in AC[a,b]$. Thus there exist $c \in \mathbb{R}$ and $\psi \in L_1(a,b)$ such that $I^{1-\gamma}_{a+} f = I^1_{a+}\psi + c$. By Lemma 2.3 (i), and by Lemma 2.4, and by the membership of $f$ in $L_1(a,b)$ we deduce that

$$f = D^{1-\gamma}_{a+}(I^1_{a+}\psi + c) = I^\gamma_{a+}\psi + D^{1-\gamma}_{a+} c\,. \tag{18}$$

Moreover, $D^\gamma_{a+} f = \frac{d}{dx}(I^1_{a+}\psi + c) = \psi$ and $I^{1-\gamma}_{b-} D^\gamma_{a+} f = I^{1-\gamma}_{b-}\psi$. Since we have $I^{1-\gamma}_{b-} D^\gamma_{a+} f \in AC[a,b]$, we deduce that there exists $d \in \mathbb{R}$ and $\phi \in L_1(a,b)$ such that $I^{1-\gamma}_{b-}\psi = I^1_{b-}\phi + d$. Thus Lemma 2.3 (ii) implies that $\psi = D^{1-\gamma}_{b-}(I^1_{b-}\phi + d)$. By equation (18) and by Lemma 2.4 we deduce that $f = I^\gamma_{a+} I^\gamma_{b-}\phi + D^{1-\gamma}_{a+} c + I^\gamma_{a+} D^{1-\gamma}_{b-} d$, which in turn implies that $f = I^\gamma_{a+} I^\gamma_{b-}\phi + \frac{1}{\Gamma(\gamma)} r_a^{1-\gamma} c + \frac{1}{\Gamma(\gamma)} I^\gamma_{a+} r_b^{1-\gamma} d = I^\gamma_{a+} I^\gamma_{b-}\phi + A[\gamma,a,b,c,d]$ (cf. Samko et al. [8, Chap. 1, § 2.5].)

Now let $f = I^\gamma_{a+} I^\gamma_{b-}\phi + A[\gamma,a,b,c,d]$. We show that $f \in \mathcal{A}^\gamma(a,b)$. Since $I^\gamma_{a+}$ and $I^\gamma_{b-}$ are bounded in $L_1(a,b)$ and since $r_a^{1-\gamma}$ and $r_b^{1-\gamma}$ belong to $L_1(a,b)$, we deduce that $f \in L_1(a,b)$ (cf. Lemma 2.1.) Then we have $I^{1-\gamma}_{a+} f = I^1_{a+} I^\gamma_{b-}\phi + c + \frac{1}{\Gamma(\gamma)} I^1_{a+} r_b^{1-\gamma} d$ and $I^{1-\gamma}_{b-} D^\gamma_{a+} f = I^1_{b-}\phi + d$ (cf. Lemma 2.4.) Thus $I^{1-\gamma}_{a+} f$ and $I^{1-\gamma}_{b-} D^\gamma_{a+} f$ belong to $AC[a,b]$. By definition (14) we conclude that $f \in \mathcal{A}^\gamma(a,b)$.

*(ii)* Since $f \in \mathcal{A}^\gamma(a,b)$, statement (i) implies that there exist $\phi' \in L_1(a,b)$, $c'$, $d' \in \mathbb{R}$ such that $f = I^\gamma_{a+} I^\gamma_{b-}\phi' + A[\gamma,a,b,c',d']$. By Lemma 2.3 (i) and (ii), and by a straightforward calculation we verify that $D^\gamma_{b-} D^\gamma_{a+} f = \phi'$ (see also

Samko et al. [8, Chap. 1, § 2.5].) By Lemma 2.4, $I^{1-\gamma}_{a+} f = I^1_{a+}(I^{1-\gamma}_{b-}\phi') + c' + \frac{d'}{\Gamma(\gamma)} I^1_{a+} r_b^{1-\gamma}$. By the membership of $I^{1-\gamma}_{b-}\phi'$ and of $r_b^{1-\gamma}$ in $L_1(a,b)$ we deduce that $I^{1-\gamma}_{a+} f(a) = c'$. Finally, $I^{1-\gamma}_{b-} D^{\gamma}_{a+} f = I^1_{b-}\phi' + d'$ and thus $I^{1-\gamma}_{b-} D^{\gamma}_{a+} f(b) = d'$. Hence the system of equations in statement (ii) holds if and only if $\phi' = \phi$, $c' = c$ and $d' = d$. The validity of the statement follows immediately.

*(iii)* Since $f \in L_1(a,b)$, we have $f = D^{1-\gamma}_{a+} I^{1-\gamma}_{a+} f$ (cf. Lemma 2.3 (i).) Since we have $I^{1-\gamma}_{a+} f \in AC[a,b]$ and since $D^{1-\gamma}_{a+}$ maps functions of $AC[a,b]$ to functions of $L_r(a,b)$ for all $r \in [1,(1-\gamma)^{-1}[$ (cf. Samko et al. [8, Chap. 1, Lemma 2.2]), we deduce that $f \in L_r(a,b)$ for all $r \in [1,(1-\gamma)^{-1}[$. To show that $D^{\gamma}_{a+} f \in L_r(a,b)$ for $r \in [1,(1-\gamma)^{-1}[$ we note that $D^{\gamma}_{a+} f = I^{\gamma}_{b-}\phi + \frac{r_b^{\gamma-1}}{\Gamma(\gamma)} d$ (cf. statement (i) and Lemma 2.3 (i), see also Samko et al. [8, Chap. 1, § 2.5].) Then the statement follows by Lemma 2.1 and by a straightforward calculation.

*(iv)* The operator $I^{\gamma}_{a+}$ maps functions of $L_p(a,b)$ with $p \in ]1,\gamma^{-1}[$ to functions of $L_q(a,b)$ with $q = p(1-\gamma p)^{-1}$ (cf. Samko et al. [8, Chap. 1, Thm. 3.5].) Then, by the equality $f - C[\alpha, a, b, I^{1-\gamma}_{a+} f(a)] = I^{\gamma}_{a+} D^{\gamma}_{a+} f$ and by statement (iii) we deduce the validity of statement (iv) (see also Lemma 2.3 (iii).)

(v) The operator $I^{\gamma}_{a+}$ maps functions of $L_p(a,b)$ with $p > \gamma^{-1}$ to functions of $H^{\gamma-\frac{1}{p}}[a,b]$ (cf. Samko et al. [8, Chap. 1, Thm. 3.6].) Then, by the equality $f - C[\alpha, a, b, I^{1-\gamma}_{a+} f(a)] = I^{\gamma}_{a+} D^{\gamma}_{a+} f$ and by statement (iii) we deduce the validity of statement (iv) (see also Lemma 2.3 (iii).) □

In the following Proposition 3.2 we emphasize the analogy between the role played by the functions $C$, $L$ and $A$ (cf. (15), (16) and (17)) as solutions of the fractional differential equation $D^{\gamma}_{b-} D^{\gamma}_{a+} f = 0$ and the corresponding constant, linear and affine solutions of the 'classical' differential equation $\frac{d^2}{dx^2} f = 0$. To do so, we find convenient to introduce the following notation. Let $\gamma \in ]0,1[$, $a, b \in \mathbb{R}$, $a < b$. Let $f \in \mathcal{A}^{\gamma}(a,b)$. Then we set

$$B^{\gamma}_{D} f \equiv I^{1-\gamma}_{a+} f(a)\,, \tag{19}$$

$$B^{\gamma}_{DD} f \equiv I^{\gamma}_{b-} D^{\gamma}_{a+} f(a)\,, \tag{20}$$

$$B^{\gamma}_{N} f \equiv I^{1-\gamma}_{b-} D^{\gamma}_{a+} f(b)\,, \tag{21}$$

$$B^{\gamma}_{NN} f \equiv -I^{\gamma}_{a+} D^{\gamma}_{b-} D^{\gamma}_{a+} f(b)\,. \tag{22}$$

Here the letter '$D$' stands for 'Dirichlet' and the letter '$N$' stands for 'Neumann'. We note that, for $f$ is sufficiently smooth the right hand sides of (19)–(22) converge to $f(a)$, $f(b) - f(a)$, $\frac{d}{dt} f(b)$ and $\frac{d}{dt} f(b) - \frac{d}{dt} f(a)$, respectively, as $\gamma \to 1$.

**Proposition 3.2.** *Let $\gamma \in ]1/2,1[$, $a, b \in \mathbb{R}$, $a < b$. Let $f \in \mathcal{A}^{\gamma}(a,b)$. Then the following statement hold.*

*(i) The following condition are equivalent.*

*(i.a)* $f = 0$,

*(i.b)* $D^{\gamma}_{b-}D^{\gamma}_{a+}f = 0$, $B^{\gamma}_{D}f = 0$ *and* $B^{\gamma}_{DD}f = 0$,

*(i.c)* $D^{\gamma}_{b-}D^{\gamma}_{a+}f = 0$, $B^{\gamma}_{D}f = 0$ *and* $B^{\gamma}_{N}f = 0$.

*(ii) The following condition are equivalent.*

*(ii.a)* $f = C[\gamma, a, b, B^{\gamma}_{D}f]$,

*(ii.a)* $D^{\gamma}_{a+}f = 0$,

*(ii.c)* $D^{\gamma}_{b-}D^{\gamma}_{a+}f = 0$ *and* $B^{\gamma}_{N}f = 0$,

*(ii.d)* $D^{\gamma}_{b-}D^{\gamma}_{a+}f = 0$ *and* $B^{\gamma}_{DD}f = 0$.

*(iii)* $f = L[\gamma, a, b, B^{\gamma}_{N}f]$ *if and only if* $D^{\gamma}_{b-}D^{\gamma}_{a+}f = 0$ *and* $B^{\gamma}_{D}f = 0$.

*(iv)* $f = A[\gamma, a, b, B^{\gamma}_{D}f, B^{\gamma}_{N}f]$ *if and only if* $D^{\gamma}_{b-}D^{\gamma}_{a+}f = 0$.

*(v) If* $f = A[\gamma, a, b, B^{\gamma}_{D}f, B^{\gamma}_{N}f]$ *then* $B^{\gamma}_{NN}f = 0$.

*Proof.* Since $f \in \mathcal{A}^{\gamma}(a, b)$ and $D^{\gamma}_{b-}D^{\gamma}_{a+}f = 0$, Proposition 3.1 (i), (ii) implies that $f = A[\gamma, a, b, B^{\gamma}_{D}f, B^{\gamma}_{N}f]$. Then the equivalence of (i.a) and (i.c) follows immediately. Now we prove that (i.b) implies (i.a). Since $B^{\gamma}_{D}f = 0$ we have $f = L[\gamma, a, b, B^{\gamma}_{N}f]$. Since $B^{\gamma}_{DD}f = 0$ we have $B^{\gamma}_{DD}L[\gamma, a, b, B^{\gamma}_{N}f] = I^{\gamma}_{b-}\frac{r_b^{\gamma-1}}{\Gamma(\gamma)}(a)B^{\gamma}_{N}f = 0$ (cf. Lemma 2.3 (i).) By a direct evaluation one can verify that $I^{\gamma}_{b-}\frac{r_b^{\gamma-1}}{\Gamma(\gamma)}(a) = \frac{(b-a)^{2\gamma-1}}{\Gamma(2\gamma-1)}$ (cf. Samko et al. [8, Chap. 1, (2.45)].) We deduce that $B^{\gamma}_{N}f = 0$ and thus $f = 0$. By a linearity argument one immediately verifies that (i.a) implies (i.c). Hence, the proof of statement (i) is completed. The proof of statements (ii)–(v) is similar and we omit it. □

In the following Theorems 3.4 and 3.5 we show that the analogy between the role played by the functions $C$, $L$ and $A$ and the constant, linear and affine functions (cf. Proposition 3.2) can be extended to the case of functions defined on the open rectangle $X$. To do so we need the following elementary Lemma 3.3.

**Lemma 3.3.** *Let* $x_0, x_1, y_0, y_1, X$ *be as in* (1). *Let* $\gamma \in ]0, 1[$. *Let* $f$ *be a function from* $X$ *to* $\mathbb{R}$. *If* $f = 0$ *almost everywhere on* $X$ *then* $I^{\gamma}_{x_0+}f = 0$, $I^{\gamma}_{x_1-}f = 0$, $D^{\gamma}_{x_0+}f = 0$ *and* $D^{\gamma}_{x_1-}f = 0$ *almost everywhere on* $X$.

*Proof.* By Lemma 2.2 we deduce that $I^{\gamma}_{x_0+}f = 0$, $I^{\gamma}_{x_1-}f = 0$, $I^{1-\gamma}_{x_0+}f = 0$ and $I^{1-\gamma}_{x_1-}f = 0$ a.e. on $X$. Then we also have $D^{\gamma}_{x_0+}f = \frac{d}{dx}I^{1-\gamma}_{x_0+}f = 0$ and $D^{\gamma}_{x_1-}f = -\frac{d}{dx}I^{1-\gamma}_{x_1-}f = 0$ a.e. on $X$. □

**Theorem 3.4.** *Let $x_0, x_1, y_0, y_1, X$ be as in* (1)*. Let $\alpha, \beta \in ]1/2, 1[$. Let $u$ be a function of $\mathcal{A}^{\alpha,\beta}(X)$. Then we have*

$$D^{\alpha}_{x_1-} D^{\alpha}_{x_0+} u = 0 \quad \text{and} \quad D^{\beta}_{y_1-} D^{\beta}_{y_0+} u = 0 \quad \text{a.e. on } X \tag{23}$$

*if and only if there exist four real constants $c_1$, $c_2$, $c_3$ and $c_4$ such that*

$$\begin{aligned} &u(x,y) \\ &\quad = c_1 C[\alpha, x_0, x_1, 1](x) C[\beta, y_0, y_1, 1](y) + c_2 C[\alpha, x_0, x_1, 1](x) L[\beta, y_0, y_1, 1](y) \\ &\qquad + c_3 L[\alpha, x_0, x_1, 1](x) C[\beta, y_0, y_1, 1](y) + c_4 L[\alpha, x_0, x_1, 1](x) L[\beta, y_0, y_1, 1](y) \end{aligned} \tag{24}$$

*for almost all $(x,y) \in X$.*

*Proof.* If $u$ is as in (24), then we deduce by Proposition 3.2 and a by a straightforward application of the Fubini Theorem that $u$ satisfies the condition in (23).

Now we assume that $u$ satisfies the condition in (23) and we show the validity of equation (24). By the membership of $u$ in $\mathcal{A}^{\alpha,\beta}(X)$ and by Proposition 3.1 (i), (ii) there exists a function $\phi$ from $X$ to $\mathbb{R}$ such that $\phi^y \in L_1(x_0, x_1)$ for a.e. $y \in ]y_0, y_1[$ and such that $u(x,y) = I^{\alpha}_{x_0+} I^{\alpha}_{x_1-} \phi(x,y) + A[\alpha, x_0, x_1, B^{\alpha}_D u^y, B^{\alpha}_N u^y](x)$ for a.e. $(x,y) \in X$. Equation $D^{\alpha}_{x_1-} D^{\alpha}_{x_0+} u = 0$ and Lemmas 2.3, 3.3 imply that $\phi = 0$ almost everywhere on $X$. Then, we have $I^{\alpha}_{x_0+} I^{\alpha}_{x_1-} \phi = 0$ almost everywhere on $X$. By Lemma 3.3 we deduce that

$$u(x,y) = A[\alpha, x_0, x_1, B^{\alpha}_D u^y, B^{\alpha}_N u^y](x) \text{ for a.e. } (x,y) \in X\,. \tag{25}$$

Similarly, we can show that $u(x,y) = A[\beta, y_0, y_1, B^{\beta}_D u_x, B^{\beta}_N u_x](y)$ for a.e. $(x,y) \in X$. So that

$$A[\alpha, x_0, x_1, B^{\alpha}_D u^y, B^{\alpha}_N u^y](x) = A[\beta, y_0, y_1, B^{\beta}_D u_x, B^{\beta}_N u_x](y) \tag{26}$$

for almost all $(x,y) \in X$. By multiplying both on the left and right hand side equation (26) by $C[\alpha, x_0, x_1, 1]^{-1} C[\beta, y_0, y_1, 1]^{-1} = \Gamma(\alpha)\Gamma(\beta) r^{1-\alpha}_{x_0} r^{1-\beta}_{y_0}$ we deduce that

$$\begin{aligned} &C[\beta, y_0, y_1, 1]^{-1}(y) B^{\alpha}_D u^y \\ &\quad + \left(C[\alpha, x_0, x_1, 1]^{-1}(x) L[\alpha, x_0, x_1, 1](x)\right) \left(C[\beta, y_0, y_1, 1]^{-1}(y) B^{\alpha}_N u^y\right) \\ &\quad = C[\alpha, x_0, x_1, 1]^{-1}(x) B^{\beta}_D u_x \\ &\qquad + \left(C[\beta, y_0, y_1, 1]^{-1}(y) L[\beta, y_0, y_1, 1](y)\right) \left(C[\alpha, x_0, x_1, 1]^{-1}(x) B^{\alpha}_N u_x\right) \end{aligned} \tag{27}$$

for almost all $(x,y) \in X$. We now note that the function $r^{1-\alpha}_{x_0} r^{1-\beta}_{y_0} u$ is continuous on $\mathrm{cl}X \setminus \{(x_i, y_j)\}_{i,j \in \{0,1\}}$ (cf. Proposition 3.1 (i) and (v).) Then both the left and the right hand side of equation (27) define a continuous function on $\mathrm{cl}X \setminus \{(x_i, y_j)\}_{i,j\in\{0,1\}}$. In particular, for $y \in ]y_0, y_1[$ fixed the functions defined by the left and the right hand side of (27) have finite limits as $x$ goes to $x_0$. Moreover, such limits coincide. We note that $L[\alpha, x_0, x_1, 1]$ is continuous on $[x_0, x_1]$ for $\alpha > 1/2$ (cf. e.g. Proposition 3.1 (v).) Thus $C[\alpha, x_0, x_1, 1]^{-1}(x)L[\alpha, x_0, x_1, 1](x)$ vanishes as $x \to x_0$. Then, by taking the limit as $x \to x_0$ in (27) we deduce that

$$\begin{aligned} &C[\beta, y_0, y_1, 1]^{-1}(y)B^\alpha_D u^y \\ &\quad = \lim_{x\to x_0} \Big\{ C[\alpha, x_0, x_1, 1]^{-1}(x)B^\beta_D u_x \\ &\qquad + \big(C[\beta, y_0, y_1, 1]^{-1}(y)L[\beta, y_0, y_1, 1](y)\big) \big(C[\alpha, x_0, x_1, 1]^{-1}(x)B^\alpha_N u_x\big) \Big\}. \end{aligned} \tag{28}$$

Since the function $C[\beta, y_0, y_1, 1]^{-1}(y)L[\beta, y_0, y_1, 1](y)$ of $y \in ]y_0, y_1[$ is non-constant and since the limit on the right hand side of (28) exists in $\mathbb{R}$, we deduce that both the limits $l_1 \equiv \lim_{x\to x_0} C[\alpha, x_0, x_1, 1]^{-1}(x)B^\beta_D u_x$ and $l_2 \equiv \lim_{x\to x_0} C[\alpha, x_0, x_1, 1]^{-1}(x)B^\beta_N u_x$ exist in $\mathbb{R}$. Then equation (28) and definition (17) imply that

$$B^\alpha_D u^y = C[\beta, y_0, y_1, 1](y)l_1 + L[\beta, y_0, y_1, 1](y)l_2 = A[\beta, y_0, y_1, l_1, l_2](y) \tag{29}$$

for all $y \in ]y_0, y_1[$. Now, by the definition in (17) and by (26) we deduce that

$$\begin{aligned} C[\alpha, x_0, x_1, 1](x)A[\beta, y_0, y_1, l_1, l_2](y) + L[\alpha, x_0, x_1, 1](x)B^\alpha_N u^y \\ = C[\beta, y_0, y_1, 1](y)B^\beta_D u_x + L[\beta, y_0, y_1, 1](y)B^\beta_N u_x \end{aligned} \tag{30}$$

for all $(x, y) \in X$. We observe that the function $L[\alpha, x_0, x_1, 1]$ is strictly positive on $[x_0, x_1]$. Then we can divide both the left and right hand side of (30) by $L[\alpha, x_0, x_1, 1]$ and we obtain

$$\begin{aligned} B^\alpha_N u^y &= -\frac{C[\alpha, x_0, x_1, 1](x)}{L[\alpha, x_0, x_1, 1](x)} A[\beta, y_0, y_1, l_1, l_2](y) \\ &\quad + C[\beta, y_0, y_1, 1](y)\frac{B^\beta_D u_x}{L[\alpha, x_0, x_1, 1](x)} + L[\beta, y_0, y_1, 1](y)\frac{B^\beta_N u_x}{L[\alpha, x_0, x_1, 1](x)} \\ &= C[\beta, y_0, y_1, 1](y)\frac{B^\beta_D u_x - C[\alpha, x_0, x_1, l_1](x)}{L[\alpha, x_0, x_1, 1](x)} \\ &\quad + L[\beta, y_0, y_1, 1](y)\frac{B^\beta_N u_x - C[\alpha, x_0, x_1, l_2](x)}{L[\alpha, x_0, x_1, 1](x)} \end{aligned} \tag{31}$$

for all $(x, y) \in X$. Since $C[\beta, y_0, y_1, 1]$ and $L[\beta, y_0, y_1, 1]$ are linearly independent elements of the space of real functions on $]y_0, y_1[$, we deduce that there exist $m_1, m_2 \in \mathbb{R}$ such that

$$\frac{B_D^\beta u_x - C[\alpha, x_0, x_1, l_1](x)}{L[\alpha, x_0, x_1, 1](x)} = m_1 \quad \text{and} \quad \frac{B_N^\beta u_x - C[\alpha, x_0, x_1, l_1](x)}{L[\alpha, x_0, x_1, 1](x)} = m_2 \, .$$

Hence

$$B_N^\alpha u^y = A[\beta, y_0, y_1, m_1, m_2] \, . \tag{32}$$

The validity of condition (24) now follows by the definition in (17), and by equations (25), (29) and (32). □

**Theorem 3.5.** *Let $x_0, x_1, y_0, y_1, X$ be as in (1). Let $\alpha, \beta \in ]1/2, 1[$. Let $u$ be a function of $\mathcal{A}^{\alpha,\beta}(X)$. Then we have*

$$D_{x_0+}^\alpha u = 0 \quad \textit{and} \quad D_{y_0+}^\beta u = 0 \quad \textit{a.e. on } X \tag{33}$$

*if and only if there exists a constant $c \in \mathbb{R}$ such that*

$$u(x,y) = c\, C[\alpha, x_0, x_1, 1](x) C[\beta, y_0, y_1, 1](y) \quad \textit{for a.e. } (x,y) \in X \, . \tag{34}$$

*Proof.* If $u$ is as in (34) then we deduce by Proposition 3.2 (ii) and by a straightforward application of Fubini Theorem that $u$ satisfies the condition in (33). Now assume that $u$ satisfies the condition in (33). Then $u$ satisfies the condition in (23) and Theorem 3.4 implies that $u$ is as in (24) with $c_1, c_2, c_3, c_4 \in \mathbb{R}$. By condition (33) we have $D_{x_0+}^\alpha D_{y_0+}^\beta u(x,y) = 0$. By a straightforward calculation we deduce that $c_4 (r_{x_1-}^{\alpha-1}/\Gamma(\alpha))(r_{y_1-}^{\beta-1}/\Gamma(\beta)) = 0$, which in turns implies that $c_4 = 0$ (see also Samko et al. [8, Chap. 1, §2.5].) Then $D_{x_0+}^\alpha u(x,y) = c_3 (r_{x_1-}^{\alpha-1}/\Gamma(\alpha)) C[\beta, y_0, y_1, 1](y) = 0$ and $D_{y_0+}^\beta u(x,y) = c_2 C[\alpha, x_0, x_1, 1](x)(r_{y_1-}^{\beta-1}/\Gamma(\beta)) = 0$ (see also Samko et al. [8, Chap. 1, §2.5].) Thus $c_2 = 0$ and $c_3 = 0$ and condition (34) holds with $c \equiv c_1$. □

# 4 Integration by Parts Formula for $\Delta_{-+}^{\alpha,\beta}$

In this Section we prove an Integration by Parts Formula for the operator $\Delta_{-+}^{\alpha,\beta}$. To do so we first verify in the following Proposition 4.1 an e Integration by Parts Formula for the operator $D_{b-}^\gamma D_{a+}^\gamma$ and for a couple of functions $f$, $g$ of $\mathcal{A}^\gamma(a,b)$, with $\gamma > \frac{1}{2}$.

**Proposition 4.1.** *Let $a, b \in \mathbb{R}$, $a < b$. Let $\gamma \in ]1/2, 1[$. Let $f$ , $g \in \mathcal{A}^\gamma(a,b)$. Then*

$$\int_a^b (D_{a+}^\gamma f)\,(D_{a+}^\gamma g)\,dt \tag{35}$$
$$= \int_a^b (D_{b-}^\gamma D_{a+}^\gamma f)\big(g - C[\gamma, a, b, B_D^\gamma g]\big)\,dt + (B_N^\gamma f)(B_{DD}^\gamma g) \, .$$

*Moreover,*

$$\int_a^b (D^\gamma_{a+}f)\,(D^\gamma_{a+}g)\,dt \tag{36}$$
$$= \int_a^b (D^\gamma_{b-}D^\gamma_{a+}f)g\,dt + (B^\gamma_{NN}f)(B^\gamma_D g) + (B^\gamma_N f)(B^\gamma_{DD}g)$$

*whenever* $\int_a^b (D^\gamma_{b-}D^\gamma_{a+}f)g\,dt$ *exists finite (cf. definitions* (19)–(22)*.)*

*Proof.* By Proposition 3.1 (i), (ii) and by the membership of $f$ and $g$ in $\mathcal{A}^\gamma(a,b)$ there exist two functions $\phi$, $\psi$ in $L_1(a,b)$ such that $f = I^\gamma_{a+}I^\gamma_{b-}\phi + A[\gamma,a,b,B^\gamma_D f, B^\gamma_N f]$ and $g = I^\gamma_{a+}I^\gamma_{b-}\psi + A[\gamma,a,b,B^\gamma_D g, B^\gamma_N g]$. Then Proposition 3.2 (ii) and Lemma 2.3 (i) imply that

$$D^\gamma_{a+}f = I^\gamma_{b-}\phi + \frac{B^\gamma_N f}{\Gamma(\gamma)} r_b^{\gamma-1}, \qquad D^\gamma_{a+}g = I^\gamma_{b-}\psi + \frac{B^\gamma_N f}{\Gamma(\gamma)} r_b^{\gamma-1}. \tag{37}$$

So, the integral in the left hand side of (35) equals

$$\int_a^b \left(I^\gamma_{b-}\phi + \frac{B^\gamma_N f}{\Gamma(\gamma)} r_b^{\gamma-1}\right)\left(I^\gamma_{b-}\psi + \frac{B^\gamma_N g}{\Gamma(\gamma)} r_b^{\gamma-1}\right)\,dt \tag{38}$$

We verify that $r_b^{\gamma-1}$, $I^\gamma_{b-}\phi$ and $I^\gamma_{b-}\psi$ belong to $L_r(a,b)$ for all $r \in [1, 1/(1-\gamma)[$ (cf. Lemma 2.1.) Then, by the Hölder Inequality and by the assumption $\gamma > 1/2$ we verify that the functions $(I^\gamma_{b-}\phi)(I^\gamma_{b-}\psi)$, $(I^\gamma_{b-}\phi)r_b^{\gamma-1}$, $(I^\gamma_{b-}\phi)r_a^{\gamma-1}$, $r_b^{2\gamma-2}$ are summable on $]a,b[$. It follows that the integral in (38) equals

$$\int_a^b \left(I^\gamma_{b-}\phi\right)\left(I^\gamma_{b-}\phi\right)\,dt + \frac{B^\gamma_N f}{\Gamma(\gamma)}\int_a^b \left(I^\gamma_{b-}\psi\right) r_b^{\gamma-1}\,dt \tag{39}$$
$$+\frac{B^\gamma_N g}{\Gamma(\gamma)}\int_a^b \left(I^\gamma_{b-}\phi\right) r_b^{\gamma-1}\,dt + \frac{B^\gamma_N f B^\gamma_N g}{\Gamma(\gamma)^2}\int_a^b r_b^{2\gamma-2}\,dt.$$

Now, by the membership of $\phi$ in $L_1(a,b)$ and of $I^\gamma_{b-}\psi$ in $L_r(a,b)$ for $r \in [1, 1/(1-\gamma)[$, and by the Integration by Parts Theorem for fractional integrals (cf. Samko et al. [8, Chap. 1, (2.20)]), and by Proposition 3.1 (ii) we have

$$\int_a^b \left(I^\gamma_{b-}\phi\right)\left(I^\gamma_{b-}\psi\right)\,dt \tag{40}$$
$$= \int_a^b \phi\left(I^\gamma_{a+}I^\gamma_{b-}\psi\right)\,dt = \int_a^b \phi\,\left(g - A[\gamma,a,b,B^\gamma_D g, B^\gamma_N g]\right)\,dt$$
$$= \int_a^b \phi\left(g - \frac{r_a^{\gamma-1}}{\Gamma(\gamma)}B^\gamma_D g - \frac{I^{1-\gamma}_{a+}r_b^{\gamma-1}}{\Gamma(\gamma)}B^\gamma_N g\right)dt$$

Since $g - \frac{r_a^{\gamma-1}}{\Gamma(\gamma)} B_D^\gamma g$ is a continuous function on $[a, b]$ (cf. Proposition 3.1 (v)) the last integral in (40) equals

$$\int_a^b \phi\big(g - \frac{r_a^{\gamma-1}}{\Gamma(\gamma)} B_D^\gamma g\big)\,dt - \frac{B_N^\gamma g}{\Gamma(\gamma)} \int_a^b \phi\, I_{a+}^{1-\gamma} r_b^{\gamma-1}\,dt \tag{41}$$

By the membership of $\phi$ in $L_1(a, b)$ and of $r_b^{\gamma-1}$ in $L_r(a, b)$ for $r \in [1, 1/(1-\gamma)[$, and by the Integration by Parts Theorem for fractional integral, we have

$$\frac{B_N^\gamma g}{\Gamma(\gamma)} \int_a^b \phi I_{a+}^{1-\gamma} r_b^{\gamma-1}\,dt = \frac{B_N^\gamma g}{\Gamma(\gamma)} \int_a^b (I_{b-}^\gamma \phi) r_b^{\gamma-1}\,dt \tag{42}$$

So that, the expression in (39) equals

$$\int_a^b \phi\big(g - \frac{r_a^{\gamma-1}}{\Gamma(\gamma)} B_D^\gamma g\big)\,dt + \frac{B_N^\gamma f}{\Gamma(\gamma)} \int_a^b \left(I_{b-}^\gamma \psi\right) r_b^{\gamma-1} dt + \frac{B_N^\gamma f B_N^\gamma g}{\Gamma(\gamma)^2} \int_a^b r_b^{2\gamma-2} dt. \tag{43}$$

We now note that $\psi = D_{b-}^\gamma D_{a+}^\gamma g$ and $B_N^\gamma g = I_{b-}^{1-\gamma} D_{a+}^\gamma g(b)$ (see Proposition 3.1 (ii).) Then, by Lemma 2.3 (iv) we have

$$\begin{aligned} &\frac{B_N^\gamma f}{\Gamma(\gamma)} \int_a^b \left(I_{b-}^\gamma \psi\right) r_b^{\gamma-1}\,dt \qquad (44) \\ &= \frac{B_N^\gamma f}{\Gamma(\gamma)} \int_a^b (D_{a+}^\gamma g) r_b^{\gamma-1}\,dt - \frac{B_N^\gamma f}{\Gamma(\gamma)^2} (I_{b-}^{1-\gamma} D_{a+}^\gamma g(b)) \int_a^b r^{2\gamma-2} dt \\ &= B_N^\gamma f\, I_{b-}^\gamma D_{a+}^\gamma g(a) - \frac{B_N^\gamma f B_N^\gamma g}{\Gamma(\gamma)^2} \int_a^b r^{2\gamma-2} dt\,. \end{aligned}$$

We deduce that the expression in (43) equals the expression on the right hand side of (35) (see also Proposition 3.1 (ii).) Thus the equation in (35) is proved. Equation (36) follows by (35) and by the definition of $I_{a+}^\gamma$. □

We note that for $\gamma \to 1$ the equation in (36) becomes the usual Integration by Parts Formula $\int_a^b (\frac{d}{dt} f)(\frac{d}{dt} g)\,dt = -\int_a^b (\frac{d^2}{dt^2} f) g\,dt + (\frac{d}{dt} f(b) - \frac{d}{dt} f(a)) g(a) + (\frac{d}{dt} f(b))(g(b) - g(a)) = -\int_a^b (\frac{d^2}{dt^2} f) g\,dt + (\frac{d}{dt} f(b)) g(b) - (\frac{d}{dt} f(a)) g(a)$. We are now ready to prove in the following Theorem 4.2 an Integration by Parts Formula for the operator $\Delta_{-+}^{\alpha,\beta}$ and for a couple of functions $u$, $v$ in $\mathcal{A}^{\alpha,\beta}(X)$ with $\alpha$, $\beta$ in $]1/2, 1[$.

**Theorem 4.2.** *Let $x_0, x_1, y_0, y_1, X$ be as in* (1). *Let $\alpha, \beta \in ]1/2, 1[$. Let $u$ and $v$ be functions of $\mathcal{A}^{\alpha,\beta}(X)$. If the functions which take $(x, y) \in X$ to $D_{x_0+}^\alpha u(x, y)\, D_{x_0+}^\alpha v(x, y)$, and to $D_{y_0+}^\beta u(x, y)\, D_{y_0+}^\beta v(x, y)$, and to*

$$(D_{x_1-}^\alpha D_{x_0+}^\alpha u(x, y))(v(x, y) - C[\gamma, x_0, x_1, B_D^\gamma v^y](x)),$$

*and to*

$$(D^{\beta}_{y_1-}D^{\beta}_{y_0+}u(x,y))(v(x,y) - C[\gamma, y_0, y_1, B^{\gamma}_{D} v_x](y))$$

*belong to* $L_1(X)$*, then*

$$\begin{aligned}
\int_{x_0}^{x_1}\int_{y_0}^{y_1} \langle \nabla^{\alpha,\beta}_{x_0,y_0+} u\,,\, \nabla^{\alpha,\beta}_{x_0,y_0+} v\rangle\, dxdy \qquad (45)\\
= -\int_{x_0}^{x_1}\int_{y_0}^{y_1} \Big\{ (\Delta^{\alpha,\beta}_{-+}u(x,y))v(x,y) \\
+(D^{\alpha}_{x_1-}D^{\alpha}_{x_0+}u(x,y))C[\gamma, x_0, x_1, B^{\gamma}_{D} v^y](x)\\
+(D^{\beta}_{y_1-}D^{\beta}_{y_0+}u(x,y))C[\gamma, y_0, y_1, B^{\gamma}_{D} v_x](y) \Big\} dxdy\\
+\int_{y_0}^{y_1} (B^{\alpha}_{N}u^y)(B^{\alpha}_{DD}v^y)dy + \int_{x_0}^{x_1} (B^{\beta}_{N}u_x)(B^{\beta}_{DD}v_x)dx\,.
\end{aligned}$$

*If we further assume that* $(D^{\beta}_{y_1-}D^{\beta}_{y_0+}u(x,y))v$ *and* $(D^{\alpha}_{x_1-}D^{\alpha}_{x_0+}u(x,y))v$ *belong to* $L_1(X)$ *then*

$$\begin{aligned}
\int_{x_0}^{x_1}\int_{y_0}^{y_1} \langle \nabla^{\alpha,\beta}_{x_0,y_0+} u\,,\, \nabla^{\alpha,\beta}_{x_0,y_0+} v\rangle\, dxdy \qquad (46)\\
= -\int_{x_0}^{x_1}\int_{y_0}^{y_1} (\Delta^{\alpha,\beta}_{-+}u)v\, dxdy\\
+\int_{y_0}^{y_1} (B^{\alpha}_{NN}u^y)(B^{\alpha}_{D}v^y)dy + \int_{x_0}^{x_1} (B^{\beta}_{NN}u_x)(B^{\beta}_{D}v_x)dx\\
+\int_{y_0}^{y_1} (B^{\alpha}_{N}u^y)(B^{\alpha}_{DD}v^y)dy + \int_{x_0}^{x_1} (B^{\beta}_{N}u_x)(B^{\beta}_{DD}v_x)dx\,.
\end{aligned}$$

*Proof.* By definition (13) we have

$$\begin{aligned}
\int_{x_0}^{x_1}\int_{y_0}^{y_1} \langle \nabla^{\alpha,\beta}_{x_0,y_0+} u\,,\, \nabla^{\alpha,\beta}_{x_0,y_0+} v\rangle\, dxdy \qquad (47)\\
= \int_{x_0}^{x_1}\int_{y_0}^{y_1} D^{\alpha}_{x_0+}u\, D^{\alpha}_{x_0+}v + D^{\beta}_{y_0+}u\, D^{\beta}_{y_0+}v\, dxdy\,.
\end{aligned}$$

Then, by the Fubini Theorem we deduce that the integral on the right hand side of (47) equals

$$\begin{aligned}
\int_{y_0}^{y_1}\left(\int_{x_0}^{x_1} D^{\alpha}_{x_0+}u^y(x)\, D^{\alpha}_{x_0+}v^y(x)\, dx\right) dy \qquad (48)\\
+\int_{x_0}^{x_1}\left(\int_{y_0}^{y_1} D^{\alpha}_{x_0+}u_x(y)\, D^{\alpha}_{x_0+}v_x(y)\, dy\right) dx\,.
\end{aligned}$$

Then the validity of (45) and (46) follows by Proposition (4.1) and by a straightforward calculation based on the Fubini Theorem. □

# 5 Some uniqueness results

In this Section we show some uniqueness results for fractional boundary value problems for the equation $\Delta^{\alpha,\beta}_{-+}u = F$ which correspond to mixed Dirichlet-Neumann boundary value problems and to Robin boundary value problems for the differential equation $\Delta u = F$ when $\alpha, \beta \to 1$ (cf. Examples 5.3 and 5.4 here below.) To do so, we exploit the following Lemmas 5.1, 5.2.

**Lemma 5.1.** *Let $x_0, x_1, y_0, y_1, X$ be as in* (1)*. Let $\alpha, \beta \in ]1/2, 1[$. Then the following conditions are equivalent.*

*(i) $u$ is a function of $\mathcal{A}^{\alpha,\beta}(X)$, the maps $|D^\alpha_{x_0+}u|^2$, $|D^\beta_{y_0+}u|^2$, $(D^\alpha_{x_1-}D^\alpha_{x_0+}u)u$ and $(D^\beta_{y_1-}D^\beta_{y_0+}u)u$ belong to $L_1(X)$, and we have*

$$\begin{cases} \Delta^{\alpha,\beta}_{-+}u = 0 & \text{in } X\,, \\ (B^\alpha_{NN}u^y)(B^\alpha_D u^y) \leq 0 & \text{for } y \in ]y_0, y_1[\,, \\ (B^\alpha_N u^y)(B^\alpha_{DD}u^y) \leq 0 & \text{for } y \in ]y_0, y_1[\,, \\ (B^\beta_{NN}u_x)(B^\beta_D u_x) \leq 0 & \text{for } x \in ]x_0, x_1[\,, \\ (B^\beta_N u_x)(B^\beta_{DD}u_x) \leq 0 & \text{for } x \in ]x_0, x_1[\,. \end{cases} \tag{49}$$

*(ii) There exits $c \in \mathbb{R}$ such that $u(x,y) = c\,C[\alpha, x_0, x_1, 1](x)C[\beta, y_0, y_1, 1](y)$ for almost all $(x,y) \in X$.*

*Proof.* If $u = c\,C[\alpha, x_0, x_1, 1]C[\beta, y_0, y_1, 1]$ then Proposition 3.2 (ii) and (v) imply that $D^\alpha_{x_0+}u^y = 0$, $D^\alpha_{x_1-}D^\alpha_{x_0+}u^y = 0$, $B^\alpha_{DD}u^y = 0$, $B^\alpha_{NN}u^y = 0$ for all $y \in ]y_0, y_1[$ and that $D^\beta_{y_0+}u_x = 0$, $D^\beta_{y_1-}D^\beta_{y_0+}u_x = 0$, $B^\beta_{DD}u_x = 0$, $B^\beta_{NN}u_x = 0$ for all $x \in ]x_0, x_1[$. Hence, $u$ satisfy the conditions in (i). Now let $u$ be as in (i). Then equation (46) holds with $v = u$. We deduce that

$$\begin{aligned} &\int_{x_0}^{x_1}\int_{y_0}^{y_1} \left|\nabla^{\alpha,\beta}_{-+}u\right|^2 dxdy \\ &\quad = \int_{x_0}^{x_1}\int_{y_0}^{y_1} \left(D^\alpha_{x_0+}u\right)^2 dxdy + \int_{x_0}^{x_1}\int_{y_0}^{y_1} \left(D^\beta_{y_0+}u\right)^2 dxdy \leq 0\,, \end{aligned}$$

which in turn implies that $D^\alpha_{x_0+}u = 0$ and $D^\beta_{y_0+}u = 0$ almost everywhere on $X$. Then the validity of statement (ii) follows by Theorem 3.5. □

**Lemma 5.2.** *Let $x_0, x_1, y_0, y_1, X$ be as in* (1)*. Let $\alpha, \beta \in ]1/2, 1[$. Let $u \in \mathcal{A}^{\alpha,\beta}(X)$ be as in condition (i) of Lemma 5.1. Then the following conditions are equivalent.*

*(i) $u = 0$.*

*(ii) There exists at least one point $x^* \in ]x_0, x_1[$ such that $B_D^\beta u_{x^*} = 0$.*

*(iii) There exists at least one point $y^* \in ]y_0, y_1[$ such that $B_D^\alpha u^{y^*} = 0$.*

*Proof.* By Lemma 5.1, we have $u = c\,C[\alpha, x_0, x_1, 1]C[\beta, y_0, y_1, 1]$. Then we can verify that $B_D^\beta u_{x^*} = c\,C[\alpha, x_0, x_1, 1](x^*)$, $B_D^\beta u^{y^*} = c\,C[\beta, y_0, y_1, 1](y^*)$ (cf. Proposition 3.2.) The equivalence of (i)–(iii) and condition $c = 0$ follows immediately. $\square$

**Example 5.3.** *Let $x_0, x_1, y_0, y_1, X$ be as in (1). Let $\alpha, \beta \in ]1/2, 1[$. Let $F$ be a function from $X$ to $\mathbb{R}$. Let $f_0$, $f_1$ be functions from $]x_0, x_1[$ to $\mathbb{R}$. Let $g_0$, $g_1$ be functions from $]y_0, y_1[$ to $\mathbb{R}$. Then there exist at most one function $u \in \mathcal{A}^{\alpha,\beta}(X)$ such that*

$$\begin{cases} \Delta_{-+}^{\alpha,\beta} u = F & \text{in } X\,, \\ B_D^\beta u_x = f_0(x) & \text{for } x \in ]x_0, x_1[\,, \\ B_{DD}^\beta u_x = f_1(x) & \text{for } x \in ]x_0, x_1[\,, \\ B_N^\alpha u^y = g_0(y) & \text{for } y \in ]y_0, y_1[\,, \\ B_{NN}^\alpha u^y = g_1(y) & \text{for } y \in ]y_0, y_1[ \end{cases}$$

*and such that $|D_{x_0+}^\alpha u|^2$, $|D_{y_0+}^\beta u|^2$, $(D_{x_1-}^\alpha D_{x_0+}^\alpha u)u$ and $(D_{y_1-}^\beta D_{y_0+}^\beta u)u$ belong to $L_1(X)$.*

*Proof.* By linearity it suffices to show that $u = 0$ for $F = 0$, $f_0 = f_1 = 0$ and $g_0 = g_1 = 0$. Then the validity of the statement follows by Lemmas 5.1, 5.2. $\square$

**Example 5.4.** *Let $x_0, x_1, y_0, y_1, X$ be as in (1). Let $\alpha, \beta \in ]1/2, 1[$. Let $F$ be a function from $X$ to $\mathbb{R}$. Let $f$ be a functions from the boundary $\partial X$ of $X$ to $\mathbb{R}$. Assume that $f \leq 0$ and that there exists at least one point $y^* \in ]y_0, y_1[$ such that $f(x_0, y^*) < 0$. Then there exist at most one function $u \in \mathcal{A}^{\alpha,\beta}(X)$ such that*

$$\begin{cases} \Delta_{-+}^{\alpha,\beta} u = F & \text{in } X\,, \\ B_{NN}^\beta u_x = f(x, y_0) B_D^\beta u_x & \text{for } x \in ]x_0, x_1[\,, \\ B_N^\beta u_x = f(x, y_1) B_{DD}^\beta u_x & \text{for } x \in ]x_0, x_1[\,, \\ B_{NN}^\alpha u^y = f(x_0, y) B_D^\alpha u^y & \text{for } y \in ]y_0, y_1[\,, \\ B_N^\alpha u^y = f(x_1, y) B_{DD}^\alpha u^y & \text{for } y \in ]y_0, y_1[ \end{cases} \tag{50}$$

*and such that $|D_{x_0+}^\alpha u|^2$, $|D_{y_0+}^\beta u|^2$, $(D_{x_1-}^\alpha D_{x_0+}^\alpha u)u$ and $(D_{y_1-}^\beta D_{y_0+}^\beta u)u$ belong to $L_1(X)$.*

*Proof.* By linearity it suffices to show that $u = 0$ for $F = 0$. By Lemma 5.1, there exists $c \in \mathbb{R}$ such that $u = cC[\alpha, x_0, x_1, 1]C[\beta, y_0, y_1, 1]$. Then Proposition 3.2 (v) implies that $B^\alpha_{NN} u^y = 0$ for all $y \in ]y_0, y_1[$. Since $f(x_0, y^*) < 0$, the third equation in (50) implies that $B^\alpha_D u^{y^*} = 0$. Then $u = 0$ by Lemma 5.2. □

By exploiting Theorem 3.4 and the uniqueness results in this Section, we can show existence and uniqueness results for those particular boundary value problems for the equation $\Delta^{\alpha,\beta}_{-+} u = 0$ which admit solutions corresponding to the affine functions on $X$ when $\alpha, \beta \to 1$ (cf. equation (24).) In the following Example 5.5 we consider the case of a mixed Dirichlet-Neumann boundary value problem.

**Example 5.5.** *Let $x_0, x_1, y_0, y_1, X$ be as in* (1). *Let $\alpha, \beta \in ]1/2, 1[$. Let $d_1, d_2, d_3, d_4 \in \mathbb{R}$. Then there exists a unique solution $u \in \mathcal{A}^{\alpha,\beta}(X)$ of*

$$\begin{cases} \Delta^{\alpha,\beta}_{-+} u = 0 & \textit{in } X\,, \\ B^\beta_D u_x = A[\alpha, x_0, x_1, d_1, d_3](x) & \textit{for } x \in ]x_0, x_1[\,, \\ B^\beta_{DD} u_x = \frac{(x_1 - x_0)^{2\gamma - 1}}{\Gamma(2\gamma - 1)} A[\alpha, x_0, x_1, d_2, d_4](x) & \textit{for } x \in ]x_0, x_1[\,, \\ B^\alpha_N u^y = A[\beta, y_0, y_1, d_3, d_4](y) & \textit{for } y \in ]y_0, y_1[\,, \\ B^\alpha_{NN} u^y = 0 & \textit{for } y \in ]y_0, y_1[ \end{cases} \tag{51}$$

*such that $|D^\alpha_{x_0+} u|^2$, $|D^\beta_{y_0+} u|^2$, $(D^\alpha_{x_1-} D^\alpha_{x_0+} u)u$ and $(D^\beta_{y_1-} D^\beta_{y_0+} u)u$ belong to $L_1(X)$. Moreover, the solution $u$ satisfies the equation in* (24) *with $c_1 = d_1$, $c_2 = d_2$, $c_3 = d_3$, $c_4 = d_4$.*

*Proof.* By Proposition 3.2 and by a straightforward calculation we verify that the function defined by the right hand side of (24) with $c_1 = d_1$, $c_2 = d_2$, $c_3 = d_3$, $c_4 = d_4$ is a solution of the system in (51) (see also Samko et al. [8, Chap. 1, § 2.5].) The uniqueness of such a solution follows by the result in Example 5.3. □

# 6 An integral equivalent for $\Delta^{\alpha,\beta}_{-+} u = F$

In this Section we introduce a fractional integral equation equivalent to the fractional differential equation $\Delta^{\alpha,\beta}_{-+} u = F$. To do so, we need the following elementary Lemmas 6.1 and 6.2.

**Lemma 6.1.** *Let $x_0, x_1, y_0, y_1, X$ be as in* (1). *Let $\alpha, \beta \in ]0, 1[$. Let $f \in L_1(X)$. Then we have $I^\alpha_{x_0+} I^\beta_{y_0+} f = I^\beta_{y_0+} I^\alpha_{x_0+} f$, and $I^\alpha_{x_1-} I^\beta_{y_1-} f = I^\beta_{y_1-} I^\alpha_{x_1-} f$.*

*Proof.* Let $f_+(x,y) \equiv \sup\{f(x,y),0\}$ and $f_-(x,y) \equiv \sup\{-f(x,y),0\}$ for all $(x,y) \in X$. So that $f = f_+ - f_-$. Then we have $f_+, f_- \in L_1(X)$ and $f_+, f_- \geq 0$. By Lemma 2.2 we deduce that

$$I^{\alpha}_{x_0+} I^{\beta}_{y_0+} f = I^{\alpha}_{x_0+} I^{\beta}_{y_0+} f_+ - I^{\alpha}_{x_0+} I^{\beta}_{y_0+} f_- \,.$$

By the Fubini Theorem we have $I^{\alpha}_{x_0+} I^{\beta}_{y_0+} f_+ = I^{\beta}_{y_0+} I^{\alpha}_{x_0+} f_+$ and $I^{\alpha}_{x_0+} I^{\beta}_{y_0+} f_- = I^{\beta}_{y_0+} I^{\alpha}_{x_0+} f_-$. Then $I^{\alpha}_{x_0+} I^{\beta}_{y_0+} f = I^{\beta}_{y_0+} I^{\alpha}_{x_0+} f$. The proof for $I^{\alpha}_{x_1-}$ and $I^{\beta}_{y_1-}$ is similar and we omit it. □

**Lemma 6.2.** *Let $x_0, x_1, y_0, y_1, X$ be as in* (1). *Let $\gamma \in ]0,1[$. Let $f \in L_1(X)$. If $I^{\gamma}_{x_0+} f = 0$ (or $I^{\gamma}_{x_1-} f = 0$, or $I^{\gamma}_{y_0+} f = 0$, or $I^{\gamma}_{x_1-} f = 0$) a.e. on $X$, then $f = 0$.*

*Proof.* Let $I^{\gamma}_{x_0+} f = 0$ a.e. on $X$. Then $D^{\gamma}_{x_0+} I^{\gamma}_{x_0+} f(x,y) = D^{\gamma}_{x_0+} I^{\gamma}_{x_0+} f^y(x) = 0$ for a.e. $(x,y) \in X$ (cf. Lemma 3.3.) By the membership of $f$ in $L_1(X)$ and by the Fubini Theorem we have $f^y \in L_1(x_0,x_1)$ for a.e. $y \in ]y_0,y_1[$. Then, by Lemma 2.3 (i), $f(x,y) = f^y(x) = D^{\gamma}_{x_0+} I^{\gamma}_{x_0+} f^y(x) = 0$ for a.e. $x \in ]x_0,x_1[$ and a.e. $y \in ]y_0,y_1[$. □

**Theorem 6.3.** *Let $x_0, x_1, y_0, y_1, X$ be as in* (1). *Let $\alpha, \beta \in ]0,1[$. Let $F \in L_1(X)$. Let $u \in \mathcal{A}^{\alpha,\beta}(X)$ be such that $D^{\alpha}_{x_1-} D^{\alpha}_{x_0+} u$ and $D^{\beta}_{y_1-} D^{\alpha}_{y_0+} u$ belong to $L_1(X)$. Then the following equations* (52) *and* (53) *are equivalent.*

$$\Delta^{\alpha,\beta}_{-+} u = F \qquad \text{a.e. in } X\,, \tag{52}$$

$$\begin{aligned} &-I^{\alpha}_{x_0+} I^{\alpha}_{x_1-} u(x,y) - I^{\beta}_{y_0+} I^{\beta}_{y_1-} u(x,y) \\ &\quad + \frac{1}{\Gamma(\alpha)} I^{\beta}_{y_0+} I^{\beta}_{y_1-} A\big[\alpha, x_0, x_1, B^{\alpha}_{D} u^y, B^{\alpha}_{N} u^y\big](x) \\ &\quad + \frac{1}{\Gamma(\beta)} I^{\alpha}_{x_0+} I^{\alpha}_{x_1-} A\big[\beta, y_0, y_1, B^{\beta}_{D} u_x, B^{\beta}_{N} u_x\big](y) \\ &\qquad = I^{\alpha}_{x_0+} I^{\beta}_{y_0+} I^{\alpha}_{x_1-} I^{\beta}_{y_1-} F(x,y) \qquad \text{for a.e. } (x,y) \in X\,. \end{aligned} \tag{53}$$

*Proof.* By the membership of $\Delta^{\alpha,\beta}_{-+} u$ and $F$ in $L_1(X)$, and by the boundedness of the operators $I^{\alpha}_{x_0+}$, $I^{\beta}_{y_0+}$, $I^{\alpha}_{x_1-}$, $I^{\beta}_{y_1-}$ from $L_1(X)$ to itself (cf. Lemma 2.2), and by Lemma 6.2 the equation in (52) is equivalent to

$$I^{\alpha}_{x_0+} I^{\beta}_{y_0+} I^{\alpha}_{x_1-} I^{\beta}_{y_1-} \Delta^{\alpha,\beta}_{-+} u = I^{\alpha}_{x_0+} I^{\beta}_{y_0+} I^{\alpha}_{x_1-} I^{\beta}_{y_1-} F\,. \tag{54}$$

By definition (12), and by the membership of $I^{1-\alpha}_{x_1-} D^{\alpha}_{x_0+} u^y$ in $AC[x_0,x_1]$ and of $I^{1-\beta}_{y_1-} D^{\beta}_{y_0+} u_x$ in $AC[y_0,y_1]$, and by statements (iii) and (iv) of Lemma 2.3, and by Lemma 6.1 we deduce that

$$I^{\alpha}_{x_1-}I^{\beta}_{y_1-}\Delta^{\alpha,\beta}_{-+}u(x,y) = -I^{\alpha}_{x_1-}D^{\beta}_{y_0+}u(x,y) - I^{\beta}_{y_1-}D^{\alpha}_{x_0+}u(x,y) \tag{55}$$
$$+I^{\beta}_{y_1-}\left\{\frac{r_{x_0}(x)^{\alpha-1}}{\Gamma(\alpha)}I^{1-\alpha}_{x_1-}D^{\alpha}_{x_0+}u(x_1,y)\right\} + I^{\alpha}_{x_1-}\left\{\frac{r_{y_0}(y)^{\beta-1}}{\Gamma(\beta)}I^{1-\beta}_{y_1-}D^{\beta}_{y_0+}u(x,y_1)\right\}$$

for almost all $(x,y) \in X$. Now, by (55), and by the membership of $I^{1-\alpha}_{x_0+}u^y$ in $AC[x_0,x_1]$ and of $I^{1-\beta}_{x_0+}u_x$ in $AC[y_0,y_1]$, and by (iii) and (iv) of Lemma 2.3, and by Lemma 6.1, and by definitions (17), (19), (21) we deduce that the function on the left hand side of (54) equals the function on left hand side of (53) almost everywhere on $X$. □

# 7 Solutions in the form $u(x,y) = f(x)g(y)$

The functions $u \in \mathcal{A}^{\alpha,\beta}(X)$ which satisfy the equation in (24) for some $c_1,c_2,c_3,c_4 \in \mathbb{R}$ are solutions of the equation $\Delta^{\alpha,\beta}_{-+}u = 0$. Indeed we have $D^{\alpha}_{x_1-}D^{\alpha}_{x_0+}u = 0$ and $D^{\beta}_{y_1-}D^{\beta}_{y_0+}u = 0$. Besides, every function $u \in \mathcal{A}^{\alpha,\beta}(X)$ which satisfies both the equations $D^{\alpha}_{x_1-}D^{\alpha}_{x_0+}u = 0$ and $D^{\beta}_{y_1-}D^{\beta}_{y_0+}u = 0$, is a solution of the equation in (24) for some $c_1,c_2,c_3,c_4 \in \mathbb{R}$ (cf. Theorem 3.4.) Our purpose in this Section is to show the existence of solutions $u \in \mathcal{A}^{\alpha,\beta}(X)$ of $\Delta^{\alpha,\beta}_{-+}u = 0$ which are not solution of both the equations $D^{\alpha}_{x_1-}D^{\alpha}_{x_0+}u = 0$ and $D^{\beta}_{y_1-}D^{\beta}_{y_0+}u = 0$. To do so we consider equation $\Delta^{\alpha,\beta}_{-+}u = 0$ for functions $u(x,y) = f(x)g(y)$ and we need the following Lemma.

**Lemma 7.1.** *Let $a,b,c,d,\omega \in \mathbb{R}$, $a < b$. Let $\gamma \in ]0,1[$. Let $f \in L_1(a,b)$. If $f + \omega I^{\gamma}_{a+}I^{\gamma}_{b-}f = A[\gamma,a,b,c,d]$, then $f \in \mathcal{A}^{\gamma}(a,b)$ and $B^{\gamma}_{D}f = c$, $B^{\gamma}_{N}f = d$.*

*Proof.* By the boundedness of $I^{\gamma}_{b-}$ from $L_1(a,b)$ to itself and by the membership of $f$ in $L_1(a,b)$, we have $I^{\gamma}_{b-}f \in L_1(a,b)$. Then $I^{1-\gamma}_{a+}f = -\omega I^{1}_{a+}I^{\gamma}_{b-}f + c + \frac{d}{\Gamma(\gamma)}I^{1}_{a+}r_b^{\gamma-1}$ and $I^{1-\gamma}_{b-}D^{\gamma}_{a+}f = -\omega I^{1}_{b-}f + d$ (cf. Lemma 2.4 and Lemma 2.3 (i) and (ii).) By the membership of $I^{\gamma}_{b-}f$, $r_b^{\gamma-1}$ and $f$ in $L_1(a,b)$ we deduce that $I^{1-\gamma}_{a+}f$ and $I^{1-\gamma}_{b-}D^{\gamma}_{a+}f$ are absolutely continuous on $[a,b]$ and that $I^{1-\gamma}_{a+}f(a) = c$ and $I^{1-\gamma}_{b-}D^{\gamma}_{a+}f(b) = d$. Thus the validity of the Lemma follows by the definition of $\mathcal{A}^{\gamma}(a,b)$ and by the definitions in (19), (21). □

In the following Theorem 7.2 we show that equation $\Delta^{\alpha,\beta}_{-+}u = 0$ with $u(x,y) \equiv f(x)g(y)$ is equivalent to a system of fractional integral equations for the functions $f$ and $g$.

**Theorem 7.2.** *Let $x_0,x_1,y_0,y_1,X$ be as in (1). Let $f \in L_1(x_0,x_1)$, $g \in L_1(y_0,y_1)$. Assume that $\int_{x_0}^{x_1}|f|dx > 0$ and $\int_{y_0}^{y_1}|g|dy > 0$. Let $u \in L_1(X)$*

*be defined by $u(x,y) \equiv f(x)g(y)$ for all $(x,y) \in X$. Then the following conditions are equivalent.*

*(i)* $u \in \mathcal{A}^{\alpha,\beta}(X)$ *and* $\Delta^{\alpha,\beta}_{-+}u = 0$ *a.e. on* $X$.

*(ii) There exists* $\omega, c_f, d_f, c_g, d_g \in \mathbb{R}$ *such that*

$$\begin{cases} f(x) + \omega I^{\alpha}_{x_0+} I^{\alpha}_{x_1-} f(x) = A\big[\alpha, x_0, x_1, c_f, d_f\big](x) & \text{for a.e. } x \in ]x_0, x_1[\,, \\ g(y) - \omega I^{\beta}_{y_0+} I^{\beta}_{y_1-} g(y) = A\big[\beta, y_0, y_1, c_g, d_g\big](y) & \text{for a.e. } y \in ]y_0, y_1[\,. \end{cases}$$

*Proof.* First we show that the condition in (i) implies the condition in (ii). By the membership of $u$ in $\mathcal{A}^{\alpha,\beta}(X)$, we have $(I^{1-\alpha}_{x_0+}f)g(y) \in AC[x_0,x_1]$ for a.e. $y \in ]y_0,y_1[$. Since the function $g$ is not identically equal to 0, we deduce that $I^{1-\alpha}_{x_0+}f \in AC[x_0,x_1]$. Similarly, we prove that $I^{1-\alpha}_{x_1-}D^{\alpha}_{x_0+}f \in AC[x_0,x_1]$, $I^{1-\beta}_{y_0+}g$, $I^{1-\beta}_{y_1-}D^{\beta}_{y_0+}g \in AC[x_0,x_1]$. Thus $f \in \mathcal{A}^{\alpha}(x_0,x_1)$ and $g \in \mathcal{A}^{\beta}(y_0,y_1)$. Now, by Theorem 6.3 and by Fubini Theorem we deduce that

$$\begin{aligned} &\big\{I^{\alpha}_{x_0+}I^{\alpha}_{x_1-}f(x)\big\}\big\{g(y) - A\big[\beta, y_0, y_1, B^{\beta}_{D}g, B^{\beta}_{N}g\big](y)\big\} \\ &\qquad + \big\{I^{\beta}_{y_0+}I^{\beta}_{y_1-}g(y)\big\}\big\{f(x) - A\big[\alpha, x_0, x_1, B^{\alpha}_{D}f, B^{\alpha}_{N}f\big](x)\big\} = 0 \end{aligned} \tag{56}$$

for a.e. $(x,y)$ in $X$. Since $I^{\beta}_{y_0+}$ and $I^{\beta}_{y_1-}$ are bounded in $L_1(y_0,y_1)$, we have $I^{\beta}_{y_0+}I^{\beta}_{y_1-}g \in L_1(y_0,y_1)$. Moreover, $\int_{y_0}^{y_1} |I^{\beta}_{y_0+}I^{\beta}_{y_1-}g| dy > 0$. Indeed if $I^{\beta}_{y_0+}I^{\beta}_{y_1-}g = 0$ then $g = D^{\beta}_{y_1-}D^{\beta}_{y_0+}(I^{\beta}_{y_0+}I^{\beta}_{y_1-}g) = 0$, which contradicts the assumption $\int_{y_0}^{y_1}|g|\,dy > 0$. Then, by multiplying the left and right hand side of equation (56) by the function

$$\operatorname{sign}(I^{\beta}_{y_0+}I^{\beta}_{y_1-}g)(y) \equiv \begin{cases} +1 & \text{if } I^{\beta}_{y_0+}I^{\beta}_{y_1-}g(y) > 0\,, \\ -1 & \text{if } I^{\beta}_{y_0+}I^{\beta}_{y_1-}g(y) < 0\,, \\ 0 & \text{if } I^{\beta}_{y_0+}I^{\beta}_{y_1-}g(y) = 0\,, \end{cases}$$

and by integrating over $y \in ]y_0,y_1[$, we deduce that

$$\omega I^{\alpha}_{x_0+}I^{\alpha}_{x_1-}f(x) + f(x) - A\big[\alpha, x_0, x_1, B^{\alpha}_{D}f, B^{\alpha}_{N}f\big](x) = 0 \tag{57}$$

for a.e. $x \in ]x_0,x_1[$, where

$$\omega \equiv \frac{\int_{y_0}^{y_1} \operatorname{sign}(I^{\beta}_{y_0+}I^{\beta}_{y_1-}g)(y)\big\{g(y) - A\big[\beta, y_0, y_1, B^{\beta}_{D}g, B^{\beta}_{N}g\big](y)\big\}dy}{\int_{y_0}^{y_1} |I^{\beta}_{y_0+}I^{\beta}_{y_1-}g| dy}\,. \tag{58}$$

Then, by equations (56) and (57) we deduce that

$$\begin{aligned} &\big\{I^{\alpha}_{x_0+}I^{\alpha}_{x_1-}f(x)\big\}\big\{g(y) - A\big[\beta, y_0, y_1, B^{\beta}_{D}g, B^{\beta}_{N}g\big](y)\big\} \\ &\qquad -\omega\big\{I^{\alpha}_{x_0+}I^{\alpha}_{x_1-}f(x)\big\}\big\{I^{\beta}_{y_0+}I^{\beta}_{y_1-}g(y)\big\} = 0 \qquad \text{for a.e. } (x,y) \in X\,. \end{aligned} \tag{59}$$

Now we observe that $I^{\alpha}_{x_0+}I^{\alpha}_{x_1-}f \in L_1(x_0,x_1)$ and $\int_{x_0}^{x_1} |I^{\alpha}_{x_0+}I^{\alpha}_{x_1-}f|dx > 0$. Thus equation (59) implies that

$$g(y) - A\big[\beta, y_0, y_1, B^{\beta}_{D}g, B^{\beta}_{N}g\big](y) - \omega I^{\beta}_{y_0+}I^{\beta}_{y_1-}g(y) = 0 \text{ for a.e. } y \in ]y_0, y_1[\,.$$

Then condition (ii) holds with $\omega$ as in (58) and $c_f \equiv B^{\alpha}_{D}f$, $d_f \equiv B^{\alpha}_{N}f$, $c_g \equiv B^{\beta}_{D}g$, $d_g \equiv B^{\beta}_{N}g$.

Now we assume that (ii) holds and we show the validity of (i). By Lemma 7.1 we deduce that $f \in \mathcal{A}^{\alpha}(x_0,x_1)$ and $g \in \mathcal{A}^{\beta}(y_0,y_1)$. By a straightforward computation we verify that $u \in \mathcal{A}^{\alpha,\beta}(X)$. By the membership of $f$ in $\mathcal{A}^{\alpha}(x_0,x_1)$ and by Proposition 3.1 we deduce that there exists a function $\phi \in L_1(x_0,x_1)$ such that $D^{\alpha}_{x_1-}D^{\alpha}_{x_0+}f = \phi$. Then $D^{\alpha}_{x_1-}D^{\alpha}_{x_0+}u = (D^{\alpha}_{x_1-}D^{\alpha}_{x_0+}f)g = \phi\, g$ belongs to $L_1(X)$. Similarly we can show that $D^{\beta}_{y_1-}D^{\beta}_{y_0+}u \in L_1(X)$. Then Theorem 6.3 implies that equation $\Delta^{\alpha,\beta}_{-+}u = 0$ is equivalent to the integral equation in (53) with $F = 0$. Since $u(x,y) = f(x)g(y)$, the integral equation in (53) with $F = 0$ is equivalent to the integral equation in (56). By exploiting Lemma 7.1 we verify that that the equation in (56) holds for $f$ and $g$ as in condition (ii). The Theorem is thus proved. □

In the following Theorem 7.4 we show the existence of functions $f \neq 0$ and $g \neq 0$ which satisfy the condition in (ii) of Theorem 7.2 with $\omega \neq 0$. Accordingly, the corresponding function $u(x,y) \equiv f(x)g(y)$ is a solution of $\Delta^{\alpha,\beta}_{-+}u = 0$ which is not simultaneously solution of $D^{\alpha}_{x_1-}D^{\alpha}_{x_0+}u = 0$ and $D^{\beta}_{y_1-}D^{\beta}_{y_0+}u = 0$. Indeed, $D^{\alpha}_{x_1-}D^{\alpha}_{x_0+}u = -\omega u$, and $D^{\beta}_{y_1-}D^{\beta}_{y_0+}u = \omega u$, and $u \neq 0$. To prove Theorem 7.4 we need the following Lemma 7.3.

**Proposition 7.3.** *Let $a, b \in \mathbb{R}$, $a < b$. Let $\gamma \in ]0,1[$. Let $p \in ]1,+\infty[$. Then there exists a discrete subset $\Omega[\gamma,p,a,b]$ of $\mathbb{R}$ such that the operator $I + \omega I^{\gamma}_{a+}I^{\gamma}_{b-}$ from $L_p(a,b)$ to itself which takes $f$ to $f + \omega I^{\gamma}_{a+}I^{\gamma}_{b-}f$ is an isomorphism for all $\omega \in \mathbb{R} \setminus \Omega[\gamma,p,a,b]$.*

*Proof.* We note that the operators $I^{\gamma}_{a+}$ and $I^{\gamma}_{b-}$ are compact from $L_p(a,b)$ to itself as operators with a weakly singular kernel (cf. e.g. Mikhlin and Prössdorf [6, Chap. II, Th. 4.1], see also Krasnosel′skiĭ, Zabreĭko, Pustyl′nik and Sobolevskiĭ [5, Th. 5.6].) Then, the operator $I^{\gamma}_{a+}I^{\gamma}_{b-}$ is compact from $L_p(a,b)$ to itself. By the known properties of compact operators there exists a subset $\sigma$ of $\mathbb{R}$ such that the operator $\rho I + I^{\gamma}_{a+}I^{\gamma}_{b-}$ has a bounded inverse for all $\rho \in \mathbb{R} \setminus \sigma$. Moreover, $\sigma$ is bounded, contains 0 and may have only 0 as an accumulation point. We set $\Omega[\gamma,p,a,b] \equiv \{\omega \in \mathbb{R} : 1/\omega \in \sigma\}$. Then $\Omega[\gamma,p,a,b]$ is a discrete subset of $\mathbb{R}$ and $I + \omega I^{\gamma}_{a+}I^{\gamma}_{b-}$ is an isomorphism for all $\omega \in \mathbb{R} \setminus \Omega[\gamma,p,a,b]$. □

**Theorem 7.4.** *Let $x_0, x_1, y_0, y_1, X$ be as in* (1). *Let $\alpha, \beta \in ]0,1[$. Then there exist a discrete subset $\Omega$ of $\mathbb{R}$ such that the system of fractional integral equations*

$$\begin{cases} f + \omega I^{\alpha}_{x_0+} I^{\alpha}_{x_1-} f = A\big[\alpha, x_0, x_1, c_f, d_f\big] , \\ g - \omega I^{\beta}_{y_0+} I^{\beta}_{y_1-} g = A\big[\beta, y_0, y_1, c_g, d_g\big] \end{cases} \tag{60}$$

*has a unique solution $(f, g)$ in $L_1(x_0, x_1) \times L_1(y_0, y_1)$ for all $c_f$, $d_f$, $c_g$, $d_g \in \mathbb{R}$ and all $\omega \in \mathbb{R} \setminus \Omega$.*

*Proof.* Let $p_\alpha \in ]1, 1/(1-\alpha)[$, $p_\alpha \in ]1, 1/(1-\beta)[$. Let $\Omega \equiv \Omega[\alpha, p_\alpha, x_0, x_1] \cup (-\Omega[\beta, p_\beta, y_0, y_1])$. Then $\Omega$ is a discrete subset of $\mathbb{R}$ and the operator which takes a couple of functions $(f, g)$ to $(f + \omega I^{\alpha}_{x_0+} I^{\alpha}_{x_1-} f \, , \, g - \omega I^{\beta}_{y_0+} I^{\beta}_{y_1-} g)$ is an isomorphism from $L_{p_\alpha}(x_0, x_1) \times L_{p_\beta}(y_0, y_1)$ to itself (cf. Lemma 7.3.) By Proposition 3.1 (i), (iii) we have $A\big[\alpha, x_0, x_1, c_f, d_f\big] \in L_{p_\alpha}(x_0, x_1)$ and $A\big[\beta, y_0, y_1, c_g, d_g\big] \in L_{p_\beta}(y_0, y_1)$. Thus, for $\omega \in \mathbb{R} \setminus \Omega$ there exists a unique pair of functions $(f, g) \in L_{p_\alpha}(x_0, x_1) \times L_{p_\beta}(y_0, y_1)$ which satisfies the system in (60). Since $L_{p_\alpha}(x_0, x_1) \subset L_1(x_0, x_1)$ and $L_{p_\beta}(y_0, y_1) \subset L_1(y_0, y_1)$, the statement of the Theorem is now proved. $\square$

# References


[1] Krzysztof Bogdan. The boundary Harnack principle for the fractional Laplacian. *Studia Math.*, 123(1):43–80, 1997.

[2] Luis A. Caffarelli, Sandro Salsa, and Luis Silvestre. Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian. *Invent. Math.*, 171(2):425–461, 2008.

[3] Qing-Yang Guan. Integration by parts formula for regional fractional Laplacian. *Comm. Math. Phys.*, 266(2):289–329, 2006.

[4] Qing-Yang Guan and Zhi-Ming Ma. Boundary problems for fractional Laplacians. *Stoch. Dyn.*, 5(3):385–424, 2005.

[5] M. A. Krasnosel′skiĭ, P. P. Zabreĭko, E. I. Pustyl′nik, and P. E. Sobolevskiĭ. *Integral operators in spaces of summable functions.* Noordhoff International Publishing, Leiden, 1976.

[6] Solomon Grigorevich Mikhlin and Siegfried Prössdorf. *Singular integral operators*, volume 68 of *Mathematische Lehrbücher und Monographien, II. Abteilung: Mathematische Monographien [Mathematical Textbooks and Monographs, Part II: Mathematical Monographs].* Akademie-Verlag, Berlin, 1986.

[7] Stefan G. Samko. *Hypersingular integrals and their applications*, volume 5 of *Analytical Methods and Special Functions.* Taylor & Francis Ltd., London, 2002.

[8] Stefan G. Samko, Anatoly A. Kilbas, and Oleg I. Marichev. *Fractional integrals and derivatives.* Gordon and Breach Science Publishers, Yverdon, 1993.

[9] Semyon Yakubovich. Eigenfunctions and fundamental solutions of the fractional two-parameter laplacian. *International Journal of Mathematics and Mathematical Sciences*, 2010:18, 2010.