A Cubical Set Approach to 2-Bundles with Connection and Wilson Surfaces

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Abstract

In the context of non-abelian gerbes we define a cubical version of categorical group 2-bundles with connection over a smooth manifold and consider their two-dimensional parallel transport with the aim of defining non-abelian Wilson surface functionals.

Key words and phrases: *cubical set; non-abelian gerbe; 2-bundle; two-dimensional holonomy; two-dimensional parallel transport; crossed module; categorical group; double groupoid; Higher Gauge Theory; Wilson surface; Wilson sphere; knotted sphere*
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1 Introduction

The aim of this paper is to address the differential geometry of (categorical group) 2-bundles over a smooth manifold M and their two dimensional parallel transport with a minimal use of two dimensional category theory, the ultimate goal being to define Wilson surface observables. The only categorical notion needed is that of an (edge symmetric, strict) double groupoid (with thin structure), which is equivalent to a crossed module or to a categorical group; see [BH1, BHS, BH6, BL, BS]. We also use the concept of a cubical set [BH2, J1, GM], a cubical analogue of a simplicial set, familiar in algebraic topology; see for example [Ma].

Our definition of a 2-bundle with connection will be given in the framework of cubical sets. Given a crossed module of Lie groups $\mathcal{G} = (\partial\colon E \to G, \triangleright)$, where $\triangleright$ is a left action of G on E by automorphisms, the definition of a cubical $\mathcal{G}$-2-bundle with connection $\mathcal{B}$ over a manifold M is an almost exact cubical analogue of the simplicial version considered in [H, BS1, BS2, BrMe]. Following [H, MP], we will consider a coordinate neighbourhood description of 2-bundles with connection. For a discussion of the total space of a 2-bundle see [RS, Bar].

We also define the thin homotopy double groupoid of a smooth manifold M. An advantage of the cubical setting over the simplicial setting is that subdivision is very easy to understand. In a 2-bundle with connection, all connection forms are in principle only locally defined. Therefore, given a smooth map $[0,1]^2 \to M$, to define its holonomy (for brevity we will use the term holonomy, instead of the more accurate term, parallel transport), one needs to subdivide $[0,1]^2$ into smaller squares, consider all the locally defined holonomies (which we will define and analyse carefully) and patch it all together by using the 1- and 2-transition functions of the 2-bundle, and the transition data of the connection. A double groupoid provides a convenient context for doing this type of calculations, and is easier to handle than the decomposition of $[0,1]^2$ into regions by means of a trivalent embedded graph of [P]. Citing [BHS, BH1], double groupoids trivially have an algebraic inverse to subdivision. This was the motivation for our cubical set approach to 2-bundles with connection and their holonomy.

Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a Lie crossed module. The $\mathcal{G}$-2-bundle holonomy which we define can be associated to oriented embedded 2-spheres $\Sigma \subset M$ yielding an element $\mathcal{W}(\mathcal{B}, \Sigma) \in \ker \partial \subset E$ (the Wilson sphere observable) independent of the parametrisation of the sphere and the chosen coordinate neighbourhoods, up to acting by elements of G. This follows from the invariance of 2-bundle holonomy under thin homotopy and the fact that the mapping class group of the sphere S^2 is $\{\pm 1\}$. This Wilson sphere observable depends only on the equivalence class of the 2-bundle with connection $\mathcal{B}$. For surfaces other than the sphere embedded in M, a holonomy can still be defined but it will a priori (since the mapping class group is more complicated) depend on the isotopy type of the parametrisation. We will illustrate this point with the case of Wilson tori.

An important problem that follows on from this construction is the definition of a gauge invariant action in the space of all 2-bundles with connection over a smooth closed 4-dimensional manifold, analogous to the Chern-Simons action for principal bundles with connection over a 3-dimensional closed manifold - see [B]. Given that a gauge invariant sphere holonomy was defined, this would permit a physical definition of invariants of knotted spheres in S^4 analogue to the Jones polynomial; see for example [W, Ko, AF].

Contents

2 Preliminaries

2.1 The Box Category and cubical sets

2.1.1 Cubical sets

The box category $\mathcal{B}$, see [J1, BH2, BH3, BHS, GM], is defined as the category whose set of objects is the set of standard n-cubes $D^n \doteq I^n$, where $I \doteq [0, 1]$, and whose set of morphisms is the set of maps generated by the cellular maps $\delta^{\pm}_{i,n} \colon D^n \to D^{n+1}$, where $i = 1, \ldots, n+1$ and $\sigma_{i,n} \colon D^{n+1} \to D^n$, $i = 1, \ldots, n+1$. We have put:

$$\delta^{-}_{i,n}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$$

$$\delta^{+}_{i,n}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = (x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$$

$$\sigma_{i,n+1}(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

We will usually abbreviate $\delta_{i,n} = \delta_i$ and $\sigma_{i,n} = \sigma_i$.

Definition 1 (Cubical set) *A cubical set K is a functor $\mathcal{B}^{\rm op} \to \mathrm{Sets}$, the category of sets; see [BH3, J1, GM]. Here $\mathcal{B}^{\rm op}$ is the opposite category of the box category $\mathcal{B}$.*

Unpacking this definition, we can see that a cubical set K is defined as being an assignment of a set K_n (the set of n-cubes) to each $n \in \mathbb{N}$, together with face maps $\partial^{\pm}_i \colon K_n \to K_{n-1}$ and degeneracy maps $\epsilon_i \colon K_{n-1} \to K_n$, where $i \in \{1, \ldots, n\}$ satisfying the cubical relations:

$$\begin{aligned} \partial^{\alpha}_i \partial^{\beta}_j &= \partial^{\beta}_{j-1} \partial^{\alpha}_i \qquad (i < j) \\ \epsilon_i \epsilon_j &= \epsilon_{j+1} \epsilon_i \qquad (i \leq j) \end{aligned} \qquad \partial^{\alpha}_i \epsilon_j = \begin{cases} \epsilon_{j-1} \partial^{\alpha}_i & (i < j) \\ \epsilon_j \partial^{a}_{i-1} & (i > j) \\ \mathrm{id} & (i = j) \end{cases} \tag{1}$$

Here $\alpha, \beta \in \{-, +\}$. A degenerate cube is a cube in the image of some degeneracy map. A cubical set K for which K_i consists only of degenerate cubes if $i > n$ will be called n-truncated.

If a cubical set K has an action of the group of symmetries of the n-cube (the n-hyperoctahedral group) in each set K_n, compatible with the faces and degeneracies in the obvious way, it will be called a *dihedral cubical set*. These are called cubical sets with reversions and interchanges in [GM]. Note that the hyperoctahedral group is generated by reflections and interchanges of coordinates, and is therefore isomorphic to $\mathbb{Z}_2^n \rtimes S_n$.

Example 2 *Let M be a manifold. The smooth singular cubical set $C(M)$ of M is given by all smooth maps $D^n \to M$, where $D^n = [0,1]^n$ is the n-cube, with the obvious faces and degeneracies. This is a dihedral cubical set in the obvious way.*

Example 3 *Analogously, given a smooth manifold M, the restricted smooth singular cubical set $C_r(M)$ of M is given by all smooth maps $f\colon D^n \to M$ for which there exists an $\epsilon > 0$ such that $f(x_1, x_2, \ldots x_n) = f(0, x_2, \ldots x_n)$ if $x_1 \leq \epsilon$, and analogously for any other face of D^n, of any dimension. We will abbreviate this condition by saying that f has a product structure close to the boundary of the n-cube. This condition allows the composition of n-cubes to be defined, which we will be needing shortly. In the terminology of [BH3], this example is a cubical set with connections and compositions.*

2.2 Lie crossed modules

All Lie groups and Lie algebras are taken to be finite-dimensional. For details on (Lie) crossed modules see, for example, [B1, BM, FM, FMP, B, BL], and references therein.

Definition 4 (Lie crossed module) *A crossed module (of groups) $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by a group morphism $\partial\colon E \to G$ together with a left action $\triangleright$ of G on E by automorphisms, such that:*

1. $\partial(X \triangleright e) = X\partial(e)X^{-1}; \forall X \in G, \forall e \in E,$
2. $\partial(e) \triangleright f = efe^{-1}; \forall e, f \in E.$

If both G and E are Lie groups, $\partial\colon E \to G$ is a smooth morphism, and the left action of G on E is smooth then $\mathcal{G}$ will be called a Lie crossed module.

A morphism $\mathcal{G} \to \mathcal{G}'$ between the Lie crossed modules $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ and $\mathcal{G}' = (\partial'\colon E' \to G', \triangleright')$ is given by a pair of smooth morphisms $\phi\colon G \to G'$ and $\psi\colon E \to E'$ making the diagram:

$$\begin{array}{ccc} E & \xrightarrow{\partial} & G \\ \psi\downarrow & & \downarrow\phi \\ E' & \xrightarrow{\partial'} & G' \end{array}$$

commutative. In addition we must have $\psi(X \triangleright e) = \phi(X) \triangleright' \psi(e)$ for each $e \in E$ and each $X \in G$.

Given a Lie crossed module $\mathcal{G} = (\partial\colon E \to G, \triangleright)$, then the induced Lie algebra map $\partial\colon \mathfrak{e} \to \mathfrak{g}$, together with the derived action of $\mathfrak{g}$ on $\mathfrak{e}$ (also denoted by $\triangleright$) is a differential crossed module, in the sense of the following definition - see [BS1, BS2, B, BC].

Definition 5 (Differential crossed module) *A differential crossed module $\mathfrak{G} = (\partial\colon \mathfrak{e} \to \mathfrak{g}, \triangleright)$ is given by a Lie algebra morphism $\partial\colon \mathfrak{e} \to \mathfrak{g}$ together with a left action of $\mathfrak{g}$ on the underlying vector space of $\mathfrak{e}$, such that:*

1. *For any $X \in \mathfrak{g}$ the map $e \in \mathfrak{e} \mapsto X \triangleright e \in \mathfrak{e}$ is a derivation of $\mathfrak{e}$, in other words*
$$X \triangleright [e, f] = [X \triangleright e, f] + [e, X \triangleright f]; \forall X \in \mathfrak{g}, \forall e, f \in \mathfrak{e}.$$
2. *The map $\mathfrak{g} \to \mathrm{Der}(\mathfrak{e})$ from $\mathfrak{g}$ into the derivation algebra of $\mathfrak{e}$ induced by the action of $\mathfrak{g}$ on $\mathfrak{e}$ is a Lie algebra morphism. In other words:*
$$[X, Y] \triangleright e = X \triangleright (Y \triangleright e) - Y \triangleright (X \triangleright e); \forall X, Y \in \mathfrak{g}, \forall e \in \mathfrak{e}.$$
3. $\partial(X \triangleright e) = [X, \partial(e)]; \forall X \in \mathfrak{g}, \forall e \in \mathfrak{e}.$

4. $\partial(e) \triangleright f = [e, f]; \forall e, f \in \mathfrak{e}$.

Note that the map $(X, e) \in \mathfrak{g} \times \mathfrak{e} \mapsto X \triangleright e \in \mathfrak{e}$ is necessarily bilinear.

A very useful identity satisfied in any differential crossed module is the following:

$$\partial(e) \triangleright f = [e, f] = -[f, e] = -\partial(f) \triangleright e, \forall e, f \in \mathfrak{e}. \tag{2}$$

This will be used several times in this paper.

Given a Lie crossed module $\mathcal{G} = (\partial\colon E \to G, \triangleright)$, we will also denote the induced action of G on $\mathfrak{e}$ by $\triangleright$. Finally, given a differential crossed module, $\mathfrak{G} = (\partial\colon \mathfrak{e} \to \mathfrak{g}, \triangleright)$ there exists a unique crossed module of simply connected Lie groups $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ whose differential form is $\mathfrak{G}$, up to isomorphism. The proof of this result is standard Lie theory, together with the lift of the Lie algebra action to a Lie group action, which can be found in [K], Thm 1.102.

2.2.1 The edge symmetric double groupoid $\mathcal{D}(\mathcal{G})$ where $\mathcal{G}$ is a crossed module

The definition of an edge symmetric (strict) double groupoid $\mathcal{K}$ (with thin structure) can be found for example in [BH1, BHS, BHKP, BS]. These are 2-truncated cubical sets for which the set of 1-cubes $\mathcal{K}_1$ is a groupoid, with set of objects given by the set of 0-cubes, and also with two partial compositions, vertical and horizontal, in the set $\mathcal{K}_2$ of 2-cubes (squares), each defining groupoid structures for which the set of objects is the set of 1-cubes. These horizontal and vertical compositions should verify the interchange law:

$$\begin{matrix}(k_1k_2)\\ \\(k_3k_4)\end{matrix} = \begin{pmatrix}k_1\\ \\k_3\end{pmatrix}\begin{pmatrix}k_2\\ \\k_4\end{pmatrix}, \forall k_1, k_2, k_3, k_4 \in \mathcal{K}_2,$$

familiar in 2-dimensional category theory, and be compatible with faces and degeneracies, in the obvious way. In particular, the identity maps of the vertical and horizontal compositions are given by degenerate squares.

There is also an extra condition that should be verified, which is the existence of a thin structure, meaning that there exist, among the squares of $\mathcal{K}$, special elements called thin such that:

1. Degenerate squares are thin.
2. Given $a, b, c, d \in \mathcal{K}_1$ with $ab = cd$, there exists a unique thin square k whose boundary is:

$$\begin{array}{ccc} * & \xrightarrow{d} & * \\ {\scriptstyle c}\uparrow & & \uparrow{\scriptstyle b} \\ * & \xrightarrow[a]{} & * \end{array} \quad ;$$

   in other words such that $\partial_d(k) = a, \partial_r(k) = b, \partial_u(k) = d$ and $\partial_l(k) = c$, where we have put $\partial_d = \partial_2^-, \partial_r = \partial_1^+, \partial_u = \partial_2^+$ and $\partial_l = \partial_1^-$.
3. Any composition of thin squares is thin.

Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a crossed module. Given that the categories of crossed modules, categorical groups and double groupoids with a unique object $*$ are equivalent (see [BH1, BH6, BHS, BS, BL]), we can construct a double groupoid $\mathcal{D}(\mathcal{G})$ out of $\mathcal{G}$. The 1-cubes $\mathcal{D}^1(\mathcal{G})$ of $\mathcal{D}(\mathcal{G})$ are given by all elements of G, with product as composition, and the unique source and target maps to the set $\{*\}$. The 2-cubes $\mathcal{D}^2(\mathcal{G})$ of $\mathcal{D}(\mathcal{G})$, which we will also call squares in $\mathcal{G}$, have the form:

$$\begin{array}{ccc} * & \xrightarrow{W} & * \\ {\scriptstyle Z}\uparrow & e & \uparrow{\scriptstyle Y} \\ * & \xrightarrow[X]{} & * \end{array} \tag{3}$$

where $X, Y, Z, W \in G$ and $e \in E$ is such that $\partial(e)^{-1}XY = ZW$. The horizontal and vertical compositions are:

$$\begin{matrix} * & \xrightarrow{W} & * \\ {\scriptstyle Z}\big\uparrow & e & \big\uparrow{\scriptstyle Y} \\ * & \xrightarrow[X]{} & * \end{matrix} \quad \begin{matrix} * & \xrightarrow{W'} & * \\ {\scriptstyle Y}\big\uparrow & e' & \big\uparrow{\scriptstyle Y'} \\ * & \xrightarrow[X']{} & * \end{matrix} = \begin{matrix} * & \xrightarrow{WW'} & * \\ {\scriptstyle Z}\big\uparrow & (X \triangleright e')e & \big\uparrow{\scriptstyle Y'} \\ * & \xrightarrow[XX']{} & * \end{matrix} \quad \text{and} \quad \begin{matrix} * & \xrightarrow{W'} & * \\ {\scriptstyle Z'}\big\uparrow & e' & \big\uparrow{\scriptstyle Y'} \\ * & \xrightarrow[W]{} & * \\ * & \xrightarrow{W} & * \\ {\scriptstyle Z}\big\uparrow & e & \big\uparrow{\scriptstyle Y} \\ * & \xrightarrow[X]{} & * \end{matrix} = \begin{matrix} * & \xrightarrow{W'} & * \\ {\scriptstyle ZZ'}\big\uparrow & eZ \triangleright e' & \big\uparrow{\scriptstyle YY'} \\ * & \xrightarrow[X]{} & * \end{matrix}$$

The thin structure on $\mathcal{D}(\mathcal{G})$ is given by: a square is thin if the element of E assigned to it is 1_E.

Alternatively the thin structure can be given by introducing the following special degeneracies, usually called connection maps (not to be confused with differential geometric connections) $\ulcorner, \llcorner, \urcorner, \lrcorner \colon \mathcal{D}^1(\mathcal{G}) \to \mathcal{D}^2(\mathcal{G})$, whose images are thin:

$$\ulcorner\left(* \xrightarrow{X} *\right) = \begin{matrix} * & \xrightarrow{1_G} & * \\ {\scriptstyle 1_G}\big\uparrow & 1_E & \big\uparrow{\scriptstyle X^{-1}} \\ * & \xrightarrow[X]{} & * \end{matrix}, \qquad \llcorner\left(* \xrightarrow{X} *\right) = \begin{matrix} * & \xrightarrow{X} & * \\ {\scriptstyle 1_G}\big\uparrow & 1_E & \big\uparrow{\scriptstyle X} \\ * & \xrightarrow[1_G]{} & * \end{matrix}$$

$$\urcorner\left(* \xrightarrow{X} *\right) = \begin{matrix} * & \xrightarrow{1_G} & * \\ {\scriptstyle X}\big\uparrow & 1_E & \big\uparrow{\scriptstyle 1_G} \\ * & \xrightarrow[X]{} & * \end{matrix}, \qquad \lrcorner\left(* \xrightarrow{X} *\right) = \begin{matrix} * & \xrightarrow{X} & * \\ {\scriptstyle X^{-1}}\big\uparrow & 1_E & \big\uparrow{\scriptstyle 1_G} \\ * & \xrightarrow[1_G]{} & * \end{matrix}$$

Here we are using results of [BHS, BH1, BH2, BH3, Hi], where it is shown that the existence of special degeneracies, satisfying a set of axioms, is equivalent to the existence of a thin structure. Then an element of $\mathcal{D}^2(\mathcal{G})$ is thin if and only if it is the composition of degenerate squares and the images of special degeneracies; see [Hi, BHS].

The set $\mathcal{D}^2(\mathcal{G})$ is actually a D_4-space, where D_4 is the dihedral group of symmetries of the square. This can be inferred from the existence of a thin structure. Consider the following representative elements $\rho_{\pi/2}, r_x, r_y$ and r_{xy} of D_4, where $\rho_{\pi/2}$ denotes anticlockwise rotation by 90 degrees, and r_x, r_y, r_{xy} denote reflection in the $y = 0$, $x = 0$ and $x = y$ axis (recall that these last three elements are generators of $D_4 \cong \mathbb{Z}_2^2 \rtimes S_2$). Under the action of these elements of D_4, the square (3) is transformed into, respectively:

$$\begin{matrix} * & \xrightarrow{Y^{-1}} & * \\ {\scriptstyle W}\big\uparrow & Z^{-1} \triangleright e & \big\uparrow{\scriptstyle X} \\ * & \xrightarrow[Z^{-1}]{} & * \end{matrix}, \quad \begin{matrix} * & \xrightarrow{X} & * \\ {\scriptstyle Z^{-1}}\big\uparrow & Z \triangleright e^{-1} & \big\uparrow{\scriptstyle Y^{-1}} \\ * & \xrightarrow[W]{} & * \end{matrix}, \quad \begin{matrix} * & \xrightarrow{W^{-1}} & * \\ {\scriptstyle Y}\big\uparrow & X \triangleright e^{-1} & \big\uparrow{\scriptstyle Z} \\ * & \xrightarrow[X^{-1}]{} & * \end{matrix}, \quad \begin{matrix} * & \xrightarrow{Y} & * \\ {\scriptstyle X}\big\uparrow & e^{-1} & \big\uparrow{\scriptstyle W} \\ * & \xrightarrow[Z]{} & * \end{matrix}$$

In fact each element of D_4 acts on $\mathcal{D}^2(\mathcal{G})$ by automorphisms, though some times permuting the horizontal and vertical multiplications, or the order of multiplications.

The horizontal and vertical inverses $e^{-\mathrm{h}}$ and $e^{-\mathrm{v}}$ of an element $e \in \mathcal{D}^2(\mathcal{G})$ are given by $e^{-\mathrm{h}} = r_y(e)$ and $e^{-\mathrm{v}} = r_x(e)$; we will often identify an element of $\mathcal{D}^2(\mathcal{G})$ with the element of E assigned to it, whenever there is no ambiguity.

There are two particular maps $\Phi, \Phi'_g \colon \mathcal{D}^2(\mathcal{G}) \to \mathcal{D}^2(\mathcal{G})$, where $g \in G$, called folding maps, which we would like to make explicit. These are defined as:

$$\Phi\left(\begin{matrix} * & \xrightarrow{W} & * \\ {\scriptstyle Z}\big\uparrow & e & \big\uparrow{\scriptstyle Y} \\ * & \xrightarrow[X]{} & * \end{matrix}\right) = \begin{matrix} * & \xrightarrow{ZWY^{-1}X^{-1}} & * \\ {\scriptstyle 1_G}\big\uparrow & e & \big\uparrow{\scriptstyle 1_G} \\ * & \xrightarrow[1_G]{} & * \end{matrix} \quad \text{and} \quad \Phi'_g\left(\begin{matrix} * & \xrightarrow{W} & * \\ {\scriptstyle Z}\big\uparrow & e & \big\uparrow{\scriptstyle Y} \\ * & \xrightarrow[X]{} & * \end{matrix}\right) = \begin{matrix} * & \xrightarrow{ZWY^{-1}X^{-1}} & * \\ {\scriptstyle g}\big\uparrow & g \triangleright e & \big\uparrow{\scriptstyle g} \\ * & \xrightarrow[1_G]{} & * \end{matrix}$$

There also exists an action of G on $\mathcal{D}^2(\mathcal{G})$, which has the form:

$$g \triangleright \left(\begin{matrix} * & \xrightarrow{W} & * \\ Z\uparrow & e & \uparrow Y \\ * & \xrightarrow[X]{} & * \end{matrix} \right) \quad = \quad \begin{matrix} * & \xrightarrow{gWg^{-1}} & * \\ gZg^{-1}\uparrow & g \triangleright e & \uparrow gYg^{-1} \\ * & \xrightarrow[gXg^{-1}]{} & * \end{matrix}$$

2.2.2 Flat $\mathcal{G}$-colourings, the edge symmetric triple groupoid $\mathcal{T}(\mathcal{G})$ and the nerve $\mathcal{N}(\mathcal{G})$ of the crossed module $\mathcal{G}$

Going one dimension up, following [BHS, BH1, BH2, BH3, BH6], we can analogously define an edge symmetric triple groupoid $\mathcal{T}(\mathcal{G})$ of thin 3-cubes in $\mathcal{G}$, from the crossed module $\mathcal{G} = (\partial\colon E \to G, \triangleright)$.

The 1- and 2-cubes of $\mathcal{T}(\mathcal{G})$ are already defined, being $\mathcal{T}^1(\mathcal{G}) = \mathcal{D}^1(\mathcal{G})$ and $\mathcal{T}^2(\mathcal{G}) = \mathcal{D}^2(\mathcal{G})$, so let us define the set of thin 3-cubes $\mathcal{T}^3(\mathcal{G})$ of $\mathcal{T}(\mathcal{G})$. Consider the set of assignments ($\mathcal{G}$-colourings of D^3) of an element of G to each edge of the standard cube $D^3 = [0,1]^3$ in $\mathbb{R}^3$ and of an element of E to each face of D^3. Each of these assignments can be mapped to the set of $\mathcal{G}$-colourings of D^2, i.e. assignments of elements of G to the set of edges of the standard square D^2 in $\mathbb{R}^2$, and an element of E to its unique face in several different ways, by using the maps $\delta_i^{\pm}, i = 1,2,3$ of 2.1.

Given a $\mathcal{G}$-colouring $\mathbf{c}_2$ of D^2, we put $X_i^{\pm}(\mathbf{c}_2) = \partial_i^{\pm}(\mathbf{c}_2) \in G$ as being $\mathbf{c}_2 \circ \delta_i^{\pm}(D^1)$ where $i = 1,2$. We also put $e(\mathbf{c}_2) = \mathbf{c}_2(D^2)$. Analogously, if $\mathbf{c}_3$ is a $\mathcal{G}$-colouring of D^3, we put $e_i^{\pm}(\mathbf{c}_3) = \partial_i^{\pm}(\mathbf{c}_3)$ as being the colouring of D^2 given by $\mathbf{c}_3 \circ \delta_i^{\pm}$ where $i = 1,2,3$.

Definition 6 (Flat $\mathcal{G}$-colouring) *A $\mathcal{G}$-colouring $\mathbf{c}_2$ of D^2 is said to be flat if it yields an element of $\mathcal{D}^2(\mathcal{G})$, in the obvious way, in other words if*

$$\partial(e(\mathbf{c}_2))^{-1} X_2^-(\mathbf{c}_2) X_1^+(\mathbf{c}_2) = X_1^-(\mathbf{c}_2) X_2^+(\mathbf{c}_2).$$

Analogously, a $\mathcal{G}$-colouring $\mathbf{c}_3$ of D^3 is said to be flat if:

1. *Each restriction $\partial_j^{\pm}(\mathbf{c}_3)$ of $\mathbf{c}_3$ is a flat $\mathcal{G}$-colouring of D^2.*
2. *The following holds:*

$$e_3^+(\mathbf{c}_3) = \begin{matrix} \ulcorner(\partial_2^+\partial_1^-(\mathbf{c}_3)) & e_2^+(\mathbf{c}_3) & \urcorner(\partial_2^+\partial_1^+(\mathbf{c}_3)) \\ \rho_{\pi/2}(e_1^-(\mathbf{c}_3)) & e_3^-(\mathbf{c}_3) & r_{xy}(e_1^+(\mathbf{c}_3)) \\ \llcorner(\partial_2^-\partial_1^-(\mathbf{c}_3)) & r_y(e_2^-(\mathbf{c}_3)) & \lrcorner(\partial_2^-\partial_1^+(\mathbf{c}_3)) \end{matrix} . \tag{4}$$

We will call this the **homotopy addition equation**, *following the terminology adopted in [BH5].*

The set $\mathcal{T}^3(\mathcal{G})$ of (thin) 3-cubes in $\mathcal{G}$ is given by the set of flat $\mathcal{G}$-colourings of the 3-cube.

The set $\mathcal{T}^3(\mathcal{G})$ of thin 3-cubes in $\mathcal{G}$ has three interchangeable associative compositions (horizontal, vertical and upwards), as well as boundary maps, $\partial_i^{\pm}, i = 1,2,3$. These compositions are induced by the horizontal and vertical composition of squares in $\mathcal{G}$ in the unique way such that the boundary maps $\partial_i^{\pm}$ in the transverse directions are groupoid morphisms. By considering the obvious degeneracies $\epsilon^i\colon \mathcal{D}^1(\mathcal{G}) \to \mathcal{D}^2(\mathcal{G}), i = 1,2$ and $\epsilon^i\colon \mathcal{D}^2(\mathcal{G}) \to \mathcal{T}^3(\mathcal{G}), i = 1,2,3$, obtained by projecting in the i^{th} direction (see 2.1.1), we can see that we obtain a 3-truncated cubical set $\mathcal{T}(\mathcal{G})$, which is a strict triple groupoid.

By continuing this process, one gets a cubical set $\mathcal{N}(\mathcal{G})$, the cubical nerve of $\mathcal{G}$, whose geometric realisation is the cubical classifying space of $\mathcal{G}$; see [BHS, BH4] and [BH5] for the simplicial version. The n-cubes of $\mathcal{N}(\mathcal{G})$ are given by all $\mathcal{G}$-colourings of the n-cube D^n such that for each 2- and 3-dimensional face of D^n the restriction of the colouring to it is flat. In fact $\mathcal{N}(\mathcal{G})$ is an ω-groupoid; see [BH4, BH5, BHS].

Note that the homotopy addition equation (4) can be expressed in several different ways by using the D_4-symmetry, and applying the maps Φ, Φ'_g. In particular, we get the equivalent equation:

$$\Phi'_{\partial_2^-\partial_1^-(\mathbf{c}_3)}(e_3^+(\mathbf{c}_3)) = \begin{matrix} e_1^-(\mathbf{c}_3) & e_2^+(\mathbf{c}_3) & r_x(e_1^+(\mathbf{c}_3)) & r_x(e_2^-(\mathbf{c}_3)) \\ & & \Phi(e_3^-(\mathbf{c}_3)) & \end{matrix} \tag{5}$$

2.3 Construction of the thin fundamental double groupoid of a smooth manifold

Let M be a smooth manifold. We now construct the thin fundamental double groupoid $\mathcal{S}_2(M)$ of M. For the analogous construction of the fundamental thin categorical group of a smooth manifold see [FMP].

2.3.1 1-paths, 2-paths and 1-tracks

Definition 7 (1-path) *A 1-path is given by a smooth map $\gamma\colon [0,1] \to M$ such that there exists an $\epsilon > 0$ such that γ is constant in $[0,\epsilon] \cup [1-\epsilon, 1]$; in the terminology of [CP], this can be abbreviated by saying that each end point of γ has a sitting instant. Given a 1-path γ, define the source and target or initial and end point of γ as $\sigma(\gamma) = \gamma(0)$ and $\tau(\gamma) = \gamma(1)$, respectively.*

Given two 1-paths γ and ϕ with $\tau(\gamma) = \sigma(\phi)$, their concatenation $\gamma\phi$ is defined in the usual way:

$$(\gamma\phi)(t) = \begin{cases} \gamma(2t), & \text{if } t \in [0,1/2] \\ \phi(2t-1), & \text{if } t \in [1/2,1] \end{cases}$$

Note that the concatenation of two 1-paths is also a 1-path, and in particular is smooth due to the sitting instant condition.

Definition 8 (2-paths) *A 2-path Γ is given by a smooth map $\Gamma\colon [0,1]^2 \to M$ such that there exists an $\epsilon > 0$ for which:*

1. $\Gamma(t,s) = \Gamma(0,s)$ *if* $0 \le t \le \epsilon$ *and* $s \in [0,1]$,
2. $\Gamma(t,s) = \Gamma(1,s)$ *if* $1-\epsilon \le t \le 1$ *and* $s \in [0,1]$,
3. $\Gamma(t,s) = \Gamma(t,0)$ *if* $0 \le s \le \epsilon$ *and* $t \in [0,1]$,
4. $\Gamma(t,s) = \Gamma(t,1)$ *if* $1-\epsilon \le s \le 1$ *and* $t \in [0,1]$.

We abbreviate this by saying that Γ has a product structure close to the boundary of $[0,1]^2$.

Given a 2-path Γ, define the following 1-paths:

$$\partial_l(\Gamma)(s) = \Gamma(0,s), s \in [0,1], \qquad \partial_r(\Gamma)(s) = \Gamma(1,s), s \in [0,1],$$
$$\partial_d(\Gamma)(t) = \Gamma(t,0), t \in [0,1], \qquad \partial_u(\Gamma)(t) = \Gamma(t,1), t \in [0,1].$$

If Γ and Γ' are 2-paths such that $\partial_r(\Gamma) = \partial_l(\Gamma')$ their horizontal concatenation $\Gamma \circ_{\mathrm{h}} \Gamma'$ is defined in the obvious way, in other words:

$$\big(\Gamma \circ_{\mathrm{h}} \Gamma'\big)(t,s) = \begin{cases} \Gamma(2t,s), & \text{if } t \in [0,1/2] \text{ and } s \in [0,1] \\ \Gamma'(2t-1,s), & \text{if } t \in [1/2,1] \text{ and } s \in [0,1] \end{cases}$$

Similarly, if $\partial_u(\Gamma) = \partial_d(\Gamma')$ we can define a vertical concatenation $\Gamma \circ_{\mathrm{v}} \Gamma'$ as:

$$\big(\Gamma \circ_{\mathrm{v}} \Gamma'\big)(t,s) = \begin{cases} \Gamma(t,2s), & \text{if } s \in [0,1/2] \text{ and } t \in [0,1] \\ \Gamma'(t,2s-1), & \text{if } s \in [1/2,1] \text{ and } t \in [0,1] \end{cases}$$

Note that again both concatenations are smooth due to the product structure condition.

Definition 9 *Two 1-paths ϕ and γ are said to be rank-1 homotopic (and we write $\phi \cong_1 \gamma$) if there exists a 2-path Γ such that:*

1. $\partial_l(\Gamma)$ *and* $\partial_r(\Gamma)$ *are constant.*
2. $\partial_u(\Gamma) = \gamma$ *and* $\partial_d(\Gamma) = \phi$.
3. $\mathrm{Rank}(\mathcal{D}_v\Gamma) \le 1, \forall v \in [0,1]^2$.

Here $\mathcal{D}$ denotes the derivative.

Thus if γ and ϕ are rank-1 homotopic, they have the same initial and end-points. Note also that rank-1 homotopy is an equivalence relation. Given a 1-path γ, the equivalence class to which it belongs is denoted by $[\gamma]$. Rank-1 homotopy is one of a number of notions of "thin" equivalence between paths or loops, and was introduced in [CP], following a suggestion by A. Machado.

We denote the set of 1-paths of M by $S_1(M)$. The quotient of $S_1(M)$ by the relation of thin homotopy is denoted by $\mathcal{S}_1(M)$. We call the elements of $\mathcal{S}_1(M)$ 1-tracks. The concatenation of 1-tracks together with the source and target maps $\sigma, \tau\colon \mathcal{S}_1(M) \to M$, defines a groupoid $\mathcal{S}_1(M)$ whose set of morphisms is $\mathcal{S}_1(M)$ and whose set of objects is M.

2.3.2 2-Tracks

We recall the notation of 2.1.1.

Definition 10 *Two 2-paths Γ and Γ' are said to be rank-2 homotopic (and we write $\Gamma \cong_2 \Gamma'$) if there exists a smooth map $J\colon [0,1]^3 \to M$ such that:*

1. $J(t,s,0) = \Gamma(t,s)$, $J(t,s,1) = \Gamma'(t,s)$ *for $s,t \in [0,1]$. In other words $J \circ \delta_3^- = \Gamma$ and $J \circ \delta_3^+ = \Gamma'$.*
2. *$J \circ \delta_i^\pm$ is a rank-1 homotopy from $\Gamma \circ \delta_i^\pm$ to $\Gamma' \circ \delta_i^\pm$, where $i = 1, 2$.*
3. *There exists an $\epsilon > 0$ such that $J(t,s,x) = J(t,s,0)$ if $x \leq \epsilon$ and $s,t \in [0,1]$, and analogously for all the other faces of $[0,1]^3$. We will describe this condition by saying that J has a product structure close to the boundary of $[0,1]^3$.*
4. $\mathrm{Rank}(\mathcal{D}_v J) \leq 2$ *for any $v \in [0,1]^3$.*

Note that rank-2 homotopy is an equivalence relation. To prove transitivity we need to use the penultimate condition of the previous definition. We denote by $S_2(M)$ the set of all 2-paths of M. The quotient of $S_2(M)$ by the relation of rank-2 homotopy is denoted by $\mathcal{S}_2(M)$. We call the elements of $\mathcal{S}_2(M)$ 2-tracks. If $\Gamma \in S_2(M)$, we denote the equivalence class in $\mathcal{S}_2(M)$ to which Γ belongs by $[\Gamma]$.

2.3.3 Horizontal and vertical compositions of 2-tracks

Suppose that Γ and Γ' are 2-paths with $\partial_u(\Gamma) \cong_1 \partial_d(\Gamma')$. Choose a rank-1 homotopy J connecting $\partial_u(\Gamma)$ and $\partial_d(\Gamma')$. Then $[\Gamma] \circ_v [\Gamma']$ is defined as $[(\Gamma \circ_v J) \circ_v \Gamma']$. The fact that this composition is well defined in $\mathcal{S}_2(M)$ is not tautological (and was left as an open problem in [MP]). However this follows immediately from the following lemma proved in [FMP].

Lemma 11 *Let $f\colon \partial(D^3) \to M$ be a smooth map such that $\mathrm{Rank}(\mathcal{D}_v f) \leq 1, \forall v \in \partial(D^3)$. Here $D^3 = [0,1]^3$. Suppose that f is constant in a neighbourhood of each vertex of $\partial(D^3)$. In addition, suppose also that in a neighbourhood $I \times [-\epsilon, \epsilon]$ of each edge I of $\partial(D^3)$, $f(x,t) = \phi(x)$, where $(x,t) \in I \times [-\epsilon, \epsilon]$ and $\phi\colon I \to M$ is smooth. Then f can be extended to a smooth map $F\colon D^3 \to M$ such that $\mathrm{Rank}(\mathcal{D}_w F) \leq 2, \forall w \in D^3$. Moreover we can choose F so that it has a product structure close to the boundary of D^3.*

Remark 12 *This basically says that any smooth map $f\colon S^2 \to M$ for which the rank of the derivative is less than or equal to 1, for each point in S^2, can be extended to all of the unit 3-ball, in such a way that the rank of the derivative of the resulting map at each point is less than or equal to 2.*

Analogously the horizontal composition of 2-paths descends to $\mathcal{S}_2(M)$. These compositions are obviously associative, and admit units and inverses. Note that the interchange law is also verified.

Finally, a 2-track $[\Gamma]$ is thin if it admits a representative which is a thin map, in other words for which $\mathrm{Rank}(\mathcal{D}_x \Gamma) \leq 1, \forall x \in [0,1]^2$. Lemma 11 implies that if $a,b,c,d\colon [0,1] \to M$ are 1-paths with $[ab] = [cd]$ then there exists a unique 2-track $[\Gamma]$ for which $\partial_d([\Gamma]) = [a]$, $\partial_r([\Gamma]) = [b]$, $\partial_l([\Gamma]) = [c]$ and $\partial_u([\Gamma]) = [d]$.

Therefore the following theorem holds:

Theorem 13 *Let M be a smooth manifold. The horizontal and vertical compositions in $\mathcal{S}_2(M)$ together with the boundary maps $\partial_u, \partial_d, \partial_l, \partial_r\colon \mathcal{S}_2(M) \to \mathcal{S}_1(M)$ define a double groupoid $\mathcal{S}_2(M)$, whose set of objects is given by all points of M, set of 1-morphisms by the set $\mathcal{S}_1(M)$ of 1-tracks on M, and set of 2-morphisms by all 2-tracks in $\mathcal{S}_2(M)$. In addition, $\mathcal{S}_2(M)$ admits a thin structure given by: a 2-track is thin if it admits a representative whose derivative has rank less than or equal to 1 (in other words if it is thin as a smooth map).*

Remark 14 *Another possible argument to prove that the compositions of 2-tracks are well defined is to adapt the arguments in [BH1, BHS, BH2, BH3, BHKP], which lead to the construction of the fundamental double groupoid of a triple of spaces and of a Hausdorff space (and can be continued to define the homotopy ω-groupoid of a filtered space). The same technique therefore leads to the construction of the fundamental ω-groupoid of a smooth manifold. Details will appear elsewhere.*

This construction should be compared with [HKK, BHKP], where the thin strict 2-groupoid of a Hausdorff space was defined, using a different notion of thin equivalence (factoring through a graph). For analogous non-strict constructions see [M, BS1, MP].

2.4 Connections and categorical connections in principal fibre bundles

To approach non-abelian integral calculus based on a crossed module, it is convenient (since the proofs are slightly easier) to consider categorical connections in principal fibre bundles. For details of this approach see [FMP]. For a treatment of non-abelian integral calculus based on a crossed module, using forms on the base space of the principal bundle, see [SW1, SW2, SW3].

2.4.1 Differential crossed module valued forms

Let M be a smooth manifold with its Lie algebra of vector fields denoted by $\mathfrak{X}(M)$. Consider also a differential crossed module $\mathfrak{G} = (\partial\colon \mathfrak{e} \to \mathfrak{g}, \triangleright)$. In particular the map $(X, e) \in \mathfrak{g} \times \mathfrak{e} \mapsto X \triangleright e \in \mathfrak{e}$ is bilinear.

Let $a \in \mathcal{A}^n(M, \mathfrak{g})$ and $b \in \mathcal{A}^m(M, \mathfrak{e})$ be $\mathfrak{g}$- and $\mathfrak{e}$-valued (respectively) differential forms on M. We define $a \otimes^{\triangleright} b$ as being the $\mathfrak{e}$-valued covariant tensor field on M such that

$$(a \otimes^{\triangleright} b)(A_1, \ldots, A_n, B_1, \ldots B_m) = a(A_1, \ldots, A_n) \triangleright b(B_1, \ldots, B_m);\, A_i, B_j \in \mathfrak{X}(M).$$

We also define an alternating tensor field $a \wedge^{\triangleright} b \in \mathcal{A}^{n+m}(M, \mathfrak{e})$, being given by

$$a \wedge^{\triangleright} b = \frac{(n+m)!}{n!m!} \operatorname{Alt}(a \otimes^{\triangleright} b).$$

Here Alt denotes the natural projection from the vector space of $\mathfrak{e}$-valued covariant tensor fields on M onto the vector space of $\mathfrak{e}$-valued differential forms on M. For example, if $a \in \mathcal{A}^1(M, \mathfrak{g})$ and $b \in \mathcal{A}^2(M, \mathfrak{e})$, then $a \wedge^{\triangleright} b$ satisfies:

$$(a \wedge^{\triangleright} b)(X, Y, Z) = a(X) \triangleright b(Y, Z) + a(Y) \triangleright b(Z, X) + a(Z) \triangleright b(X, Y), \qquad (6)$$

where $X, Y, Z \in \mathfrak{X}(M)$.

2.4.2 Categorical connections in principal fibre bundles

Let M be a smooth manifold and G a Lie group with Lie algebra $\mathfrak{g}$. Let also $\pi\colon P \to M$ be a smooth principal G-bundle over M. Denote the fibre at each point $x \in M$ as $P_x \doteq \pi^{-1}(x)$.

Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a Lie crossed module, where $\triangleright$ is a Lie group left action of G on E by automorphisms. Let also $\mathfrak{G} = (\partial\colon \mathfrak{e} \to \mathfrak{g}, \triangleright)$ be the associated differential crossed module. A $\mathcal{G}$-categorical connection on P is a pair (ω, m), where ω is a connection 1-form on P, i.e. $\omega \in \mathcal{A}^1(P, \mathfrak{g})$ is a 1-form on P with values in $\mathfrak{g}$ (the Lie algebra of G) such that:

- $R_g^*(\omega) = g^{-1}\omega g, \forall g \in G$, (i.e. ω is G-equivariant)
- $\omega(A^\#) = A, \forall A \in \mathfrak{g}$;

and $m \in \mathcal{A}^2(P, \mathfrak{e})$ is a 2-form on P with values in $\mathfrak{e}$, the Lie algebra of E, such that:

- m is G-equivariant, in the sense that $R_g^*(m) = g^{-1} \triangleright m$ for each $g \in G$.
- m is horizontal, in other words:

  $$m(X, Y) = m(X^H, Y^H), \text{ for each } X, Y \in \mathfrak{X}(P).$$

  In particular $m(X_u, Y_u) = 0$ if either of the vectors $X_u, Y_u \in T_uP$ is vertical, where $u \in P$. Here the map $X \in \mathfrak{X}(P) \mapsto X^H \in \mathfrak{X}(P)$ denotes the horizontal projection of vector fields on P with respect to the connection 1-form ω.

Finally we require the "vanishing of the fake curvature condition" [BS1, BS2, BrMe]:

$$\partial(m) = \Omega, \tag{7}$$

where $\Omega = d\omega + \frac{1}{2}\omega \wedge^{\mathrm{ad}} \omega \in \mathcal{A}^2(P, \mathfrak{g})$ is the curvature 2-form of ω.

2.4.3 The categorical curvature 3-form of a $\mathcal{G}$-categorical connection

Let P be a principal G-bundle over M. Let $\omega \in \mathcal{A}^1(P, \mathfrak{g})$ be a connection 1-form on P. Given an n-form a on P, the exterior covariant derivative of a is given by

$$Da = da \circ (H \times H \ldots \times H).$$

Let $\Omega \in \mathcal{A}^2(P, \mathfrak{g})$ be the (G-equivariant) curvature 2-form of the connection ω. It can be defined as the exterior covariant derivative $D\omega$ of the connection 1-form ω and also by the Cartan structure equation $\Omega = d\omega + \frac{1}{2}\omega \wedge^{\mathrm{ad}} \omega$. It is therefore natural to define:

Definition 15 (Categorical curvature) *Let $\mathcal{G} = (\partial \colon E \to G, \triangleright)$ be a Lie crossed module, and let $P \to M$ be a smooth principal G-bundle. The categorical curvature 3-form or 2-curvature 3-form of a $\mathcal{G}$-categorical connection (ω, m) on P is defined as $\mathcal{M} = Dm$, where the exterior covariant derivative D is taken with respect to ω.*

The following equation is an analogue of Cartan's structure equation.

Proposition 16 (Categorical structure equation) *We have: $\mathcal{M} = dm + \omega \wedge^{\triangleright} m$. In particular the 2-curvature 3-form $\mathcal{M}$ is G-equivariant, in other words: $R_g^*(\mathcal{M}) = g^{-1} \triangleright \mathcal{M}$, for each $g \in G$.*

The categorical-structure equation follows directly from the following natural lemma, easy to prove; see [FMP]:

Lemma 17 *Let a be a G-equivariant horizontal n-form in P. Then $Da = da + \omega \wedge^{\triangleright} a$.*

Recall that the usual Bianchi identity can be written as $D\Omega = 0$, which is the same as saying that $d\Omega + \omega \wedge^{\mathrm{ad}} \Omega = 0$.

Corollary 18 *The 2-curvature 3-form of a categorical connection is $\mathfrak{k}$-valued, where $\mathfrak{k}$ is the Lie algebra of $K = \ker(\partial)$.*

Proof. We have $\partial(\mathcal{M}) = \partial(dm + \omega \wedge^{\triangleright} m) = d\Omega + \omega \wedge^{\mathrm{ad}} \Omega = 0$, by the Bianchi identity. ■

The 2-curvature 3-form of a categorical connections satisfies the following.

Proposition 19 (2-Bianchi identity) *Let $\mathcal{M} \in \mathcal{A}^3(P, \mathfrak{e})$ be the 2-curvature 3-form of (ω, m). Then the exterior covariant derivative $D\mathcal{M}$ of M vanishes, which by lemma 17 is the same as: $d\mathcal{M} + \omega \wedge^{\triangleright} \mathcal{M} = 0$.*

2.4.4 Local form

Let $P \to M$ be a G-principal fibre bundle with a categorical connection (ω, m). Let $\{U_i\}$ be an open cover of M, with local sections $\sigma_i \colon U_i \to P$ of P. The local form of (ω, m) is given by the forms (ω_i, m_i), where $\omega_i = \sigma_i^*(\omega)$ and $m_i = \sigma_i^*(m)$, and we have $\partial(m_i) = d\omega_i + \frac{1}{2}\omega_i \wedge^{\mathrm{ad}} \omega_i = \Omega_i = \sigma_i^*(\Omega)$, and also $\omega_j = g_{ij}^{-1}\omega_i g_{ij} + g_{ij}^{-1} dg_{ij}$ and $m_j = g_{ij}^{-1} \triangleright m_i$. Here $\sigma_i g_{ij} = \sigma_j$. Conversely, given forms $\{(\omega_i, m_i)\}$ satisfying these conditions then there exists a unique categorical connection (ω, m) in P whose local form (with respect to the given sections σ_i) is (ω_i, m_i).

Note that locally the 2-curvature 3-form of a categorical connection reads $\mathcal{M}_i = dm_i + \omega_i \wedge^{\triangleright} m_i$, with $\mathcal{M}_j = g_{ij}^{-1} \triangleright \mathcal{M}_i$ and the 2-Bianchi identity is $d\mathcal{M}_i + \omega_i \wedge^{\triangleright} \mathcal{M}_i = 0$.

2.5 Holonomy and categorical holonomy in a principal fibre bundle

Let P be a principal G-bundle over the manifold M. Let $\omega \in \mathcal{A}^1(P, \mathfrak{g})$ be a connection on P. Recall that ω determines a parallel transport along smooth curves. Specifically, given $x \in M$ and a smooth curve $\gamma\colon [0,1] \to M$, with $\gamma(0) = x$, then there exists a smooth map:

$$(t, u) \in [0,1] \times P_x \mapsto \mathcal{H}_\omega(\gamma, t, u) \in P,$$

uniquely defined by the conditions:

1. $\frac{d}{dt}\mathcal{H}_\omega(\gamma, t, u) = \left(\widetilde{\frac{d}{dt}\gamma(t)}\right)_{\mathcal{H}_\omega(\gamma,t,u)}; \forall t \in [0,1], \forall u \in P_x$, where $\widetilde{\ }$ denotes the horizontal lift,
2. $\mathcal{H}_\omega(\gamma, 0, u) = u; \forall u \in P_x$.

In particular this implies that $\mathcal{H}_\omega(\gamma, t)$, given by $u \mapsto \mathcal{H}_\omega(\gamma, t, u)$, maps P_x bijectively into $P_{\gamma(t)}$, for any $t \in [0,1]$. We will also use the notation $\mathcal{H}_\omega(\gamma, 1, u) \doteq u\gamma$. Therefore if γ and γ' are such that $\gamma(1) = \gamma'(0)$ we have: $(u\gamma)\gamma' = u(\gamma\gamma')$. Recall that the parallel transport is G-equivariant, in other words:

$$\mathcal{H}_\omega(\gamma, t, ug) = \mathcal{H}_\omega(\gamma, t, u)g, \forall g \in G, \forall u \in P_x.$$

2.5.1 A form of the Ambrose-Singer Theorem

Let M be a smooth manifold. Let $D^n \doteq [0,1]^n$ be the n-cube, where $n \in \mathbb{N}$. A map $f\colon D^n \to M$ is said to be smooth if its partial derivatives of any order exist and are continuous as maps $D^n \to M$.

The well known relation between curvature and parallel transport can be summarised in the following lemma, proved for instance in [FMP, SW2].

Lemma 20 *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let P be a smooth principal G-bundle over the manifold M. Consider a smooth map $\Gamma\colon [0,1]^2 \to M$. For each $s, t \in [0,1]$, define the curves $\gamma_s, \gamma^t\colon [0,1] \to M$ as $\gamma_s(t) = \gamma^t(s) = \Gamma(t,s)$. Consider a connection $\omega \in \mathcal{A}^1(P, \mathfrak{g})$. Choose $u \in P_{\gamma^0(0)}$, and let $u_s = \mathcal{H}_\omega(\gamma^0, s, u)$, and analogously $u^t = \mathcal{H}_\omega(\gamma_0, t, u)$ where $s, t \in [0,1]$. The following holds for each $s, t \in [0,1]$:*

$$\omega\left(\frac{\partial}{\partial s}\mathcal{H}_\omega(\gamma_s, t, u_s)\right) = \int_0^t \Omega\left(\widetilde{\frac{\partial}{\partial t'}\gamma_s(t')}, \widetilde{\frac{\partial}{\partial s}\gamma_s(t')}\right)_{\mathcal{H}_\omega(\gamma_s, t', u_s)} dt', \tag{8}$$

and by reversing the roles of s and t we also have:

$$\omega\left(\frac{\partial}{\partial t}\mathcal{H}_\omega(\gamma^t, s, u^t)\right) = -\int_0^s \Omega\left(\widetilde{\frac{\partial}{\partial t}\gamma_{s'}(t)}, \widetilde{\frac{\partial}{\partial s'}\gamma_{s'}(t)}\right)_{\mathcal{H}_\omega(\gamma^t, s', u^t)} ds'. \tag{9}$$

Continuing the notation of the previous lemma, define the elements $\overset{\omega}{g}_\Gamma(u, t, s)$ by the rule:

$$\mathcal{H}_\omega(\gamma^t, s, u^t)\overset{\omega}{g}_\Gamma(u, t, s) = \mathcal{H}_\omega(\gamma_s, t, u_s).$$

Therefore

$$u\overset{\omega}{g}_\Gamma(u, t, s) = \mathcal{H}_\omega(\hat{\gamma}, 1, u)$$

where $\hat{\gamma}$ is the curve $\hat{\gamma} = \partial\Gamma'$, starting in $\Gamma(0,0)$ and oriented clockwise, and Γ' is the truncation of Γ such that $\Gamma'(t', s') = \Gamma(t't, s's)$, for $0 \leq s', t' \leq 1$.

By using the fact that $\frac{\partial}{\partial t}\mathcal{H}_\omega(\gamma_s, t, u_s)$ is horizontal it follows that:

$$\omega\left(\frac{\partial}{\partial t}\left(\mathcal{H}_\omega(\gamma^t, s, u^t)\overset{\omega}{g}_\Gamma(u, t, s)\right)\right) = 0.$$

Thus, by using the Leibniz rule together with the fact that ω is a connection 1-form,

$$(\overset{\omega}{g}_\Gamma(u, t, s))^{-1}\omega\left(\frac{\partial}{\partial t}\mathcal{H}_\omega(\gamma^t, s, u^t)\right)\overset{\omega}{g}_\Gamma(u, t, s) + (\overset{\omega}{g}_\Gamma(u, t, s))^{-1}\frac{\partial}{\partial t}\overset{\omega}{g}_\Gamma(u, t, s) = 0.$$

Therefore:

$$\frac{\partial}{\partial t}\overset{\omega}{g}_\Gamma(u,t,s) = \left(\int_0^s \Omega\left(\widetilde{\frac{\partial}{\partial t}\gamma_{s'}(t)}, \widetilde{\frac{\partial}{\partial s'}\gamma_{s'}(t)}\right)_{\mathcal{H}_\omega(\gamma^t,s',u^t)} ds'\right)\overset{\omega}{g}_\Gamma(u,t,s). \tag{10}$$

Analogously we have (since $\frac{\partial}{\partial s}\mathcal{H}_\omega(\gamma^t, s, u^t)$ is horizontal):

$$\frac{\partial}{\partial s}\overset{\omega}{g}_\Gamma(u,t,s) = \overset{\omega}{g}_\Gamma(u,t,s)\int_0^t \Omega\left(\widetilde{\frac{\partial}{\partial t'}\gamma_s(t')}, \widetilde{\frac{\partial}{\partial s}\gamma_s(t')}\right)_{\mathcal{H}_\omega(\gamma_s,t',u_s)} dt'. \tag{11}$$

2.5.2 Categorical holonomy in a principal fibre bundle

Let P be a principal fibre bundle with a $\mathcal{G}$-categorical connection (ω, m). Here $\mathcal{G} = (E \xrightarrow{\partial} G, \triangleright)$ is a Lie crossed module, where $\triangleright$ is a Lie group left action of G on E by automorphisms. Let also $\mathfrak{G} = (\partial\colon \mathfrak{e} \to \mathfrak{g}, \triangleright)$ be the associated differential crossed module.

As before, for each smooth map $\Gamma\colon [0,1]^2 \to M$, let $\gamma_s(t) = \gamma^t(s) = \Gamma(t,s)$. Let $a = \Gamma(0,0)$. Let also $u \in P_a$, $u_s = \mathcal{H}(\gamma^0, s, u)$ and $u^t = \mathcal{H}(\gamma_0, t, u)$. Define the function $\overset{(\omega,m)}{e_\Gamma}\colon P_a \times [0,1]^2 \to E$ as being the solution of the differential equation:

$$\frac{\partial}{\partial s}\overset{(\omega,m)}{e_\Gamma}(u,t,s) = \overset{(\omega,m)}{e_\Gamma}(u,t,s)\int_0^t m\left(\widetilde{\frac{\partial}{\partial t'}\gamma_s(t')}, \widetilde{\frac{\partial}{\partial s}\gamma_s(t')}\right)_{\mathcal{H}_\omega(\gamma_s,t',u_s)} dt', \tag{12}$$

with initial condition $\overset{(\omega,m)}{e_\Gamma}(u,t,0) = 1_E$, for each $t \in [0,1]$. Let $\overset{(\omega,m)}{e_\Gamma}(u) \doteq \overset{(\omega,m)}{e_\Gamma}(u,1,1)$. Compare with equations (10) and (11). The apparently non-symmetric way the horizontal and vertical directions are treated will be dealt with later.

Given a smooth map $\Gamma\colon [0,1]^2 \to M$, define:

$$\mathcal{X}_\Gamma = \gamma_0, \quad \mathcal{Y}_\Gamma = \gamma^1, \quad \mathcal{Z}_\Gamma = \gamma^0 \quad \text{and} \quad \mathcal{W}_\Gamma = \gamma_1.$$

Theorem 21 (Non-abelian Green theorem, bundle form) *For any $u \in P_a$ we have:*

$$\mathcal{H}_\omega(\mathcal{X}_\Gamma\mathcal{Y}_\Gamma, 1, u)\partial\Big(\overset{(\omega,m)}{e_\Gamma}(u)\Big) = \mathcal{H}_\omega(\mathcal{Z}_\Gamma\mathcal{W}_\Gamma, 1, u),$$

or, in the other notation of section 2.5,

$$u\mathcal{X}_\Gamma\mathcal{Y}_\Gamma\partial\Big(\overset{(\omega,m)}{e_\Gamma}(u)\Big) = u\mathcal{Z}_\Gamma\mathcal{W}_\Gamma.$$

Proof. Let $k_x = \mathcal{H}_\omega(\gamma^1, x, u^1)$ and $l_x = \mathcal{H}_\omega(\gamma_x, 1, u_x)$. Let $x \mapsto g_x \in G$ be defined as $k_x g_x = l_x$. We have, since $(\frac{d}{dx}k_x)g_x$ is horizontal:

$$\omega\left(\frac{d}{dx}(k_x g_x)\right) = \omega\left(k_x\frac{d}{dx}g_x\right) = \omega\left(k_x g_x g_x^{-1}\frac{d}{dx}g_x\right) = g_x^{-1}\frac{d}{dx}g_x.$$

On the other hand:

$$\omega\left(\frac{d}{dx}(k_x g_x)\right) = \omega\left(\frac{d}{dx}l_x\right) = \int_0^1 \Omega\left(\widetilde{\frac{\partial}{\partial t}\gamma_x(t)}, \widetilde{\frac{\partial}{\partial x}\gamma_x(t)}\right)_{\mathcal{H}_\omega(\gamma_x,t,u_x)} dt.$$

Therefore

$$\frac{d}{dx}g_x = g_x\int_0^1 \Omega\left(\widetilde{\frac{\partial}{\partial t}\gamma_x(t)}, \widetilde{\frac{\partial}{\partial x}\gamma_x(t)}\right)_{\mathcal{H}_\omega(\gamma_x,t,u_x)} dt. \tag{13}$$

This is a differential equation satisfied also by $x \mapsto \partial(\overset{(\omega,m)}{e_\Gamma}(u,x,1))$, by the vanishing of the fake curvature condition $\partial(m) = \Omega$, and both have the same initial conditions. ■

Note that it follows from the non-abelian Green theorem that:

$$\mathcal{H}_\omega(\gamma^t, s, u^t)\partial\Big(\overset{(\omega,m)}{e_\Gamma}(u,t,s)\Big) = \mathcal{H}_\omega(\gamma_s, t, u_s), \text{ for each } t, s \in [0,1]. \tag{14}$$

Lemma 22 (Vertical multiplication) *We have:*

$$\overset{(\omega,m)}{e_{\Gamma\circ_{\mathrm{v}}\Gamma'}}(u) = \overset{(\omega,m)}{e_{\Gamma}}(u)\,\overset{(\omega,m)}{e_{\Gamma'}}(u\mathcal{Z}_\Gamma).$$

Here $\Gamma, \Gamma' \colon [0,1]^2 \to M$ *are smooth maps such that* $\partial_u(\Gamma) = \partial_d(\Gamma')$ *and moreover* $\Gamma\circ_{\mathrm{v}}\Gamma'$ *is smooth.*

Proof. Obvious from the definition. ■

Lemma 23 (Vertical inversion) *We have:*

$$\overset{(\omega,m)}{e_{\Gamma}}(u)\,\overset{(\omega,m)}{e_{\Gamma^{-\mathrm{v}}}}(u\mathcal{Z}_\Gamma) = 1_E.$$

Here $\Gamma^{-\mathrm{v}}$ *denotes the obvious vertical reversion of* $\Gamma \colon [0,1]^2 \to M$.

Proof. Obvious from the definition. ■

Lemma 24 (Horizontal multiplication) *We have:*

$$\overset{(\omega,m)}{e_{\Phi\circ_{\mathrm{h}}\Psi}}(u) = \overset{(\omega,m)}{e_{\Psi}}(u\mathcal{X}_\Phi)\,\overset{(\omega,m)}{e_{\Phi}}(u).$$

Here $\Phi, \Psi' \colon [0,1]^2 \to M$ *are smooth maps such that* $\partial_r(\Phi) = \partial_l(\Psi)$ *and moreover* $\Phi\circ_{\mathrm{h}}\Psi$ *is smooth.*

Proof. Let $\Gamma = \Phi\circ_{\mathrm{h}}\Psi$. As before put $\phi_s(t) = \phi^t(s) = \Phi(t,s)$ and $\psi_s(t) = \psi^t(s) = \Psi(t,s)$. We have:

$$\begin{aligned}
&\frac{\partial}{\partial s}\left(\overset{(\omega,m)}{e_{\Psi}}(u\mathcal{X}_\Phi,1,s)\overset{(\omega,m)}{e_{\Phi}}(u,1,s)\right)\\
&\quad= \overset{(\omega,m)}{e_{\Psi}}(u\mathcal{X}_\Phi,1,s)\overset{(\omega,m)}{e_{\Phi}}(u,1,s)\left(\int_0^1 m\left(\widetilde{\frac{\partial}{\partial t}\phi_s(t)}, \widetilde{\frac{\partial}{\partial s}\phi_s(t)}\right)_{\mathcal{H}_\omega(\phi_s,t,u_s)} dt\right)\\
&\quad\quad+ \overset{(\omega,m)}{e_{\Psi}}(u\mathcal{X}_\Phi,1,s)\left(\int_0^1 m\left(\widetilde{\frac{\partial}{\partial t}\psi_s(t)}, \widetilde{\frac{\partial}{\partial s}\psi_s(t)}\right)_{\mathcal{H}_\omega(\psi_s,t,(u\mathcal{X}_\Phi)_s)} dt\right)\overset{(\omega,m)}{e_{\Phi}}(u,1,s)\\
&\quad Q + W.
\end{aligned}$$

Here $(u\mathcal{X}_\Phi)_s = \mathcal{H}_\omega(\mathcal{Z}_\Psi, s, u\mathcal{X}_\Phi)$. Let us analyse each term separately. We have:

$$Q = \overset{(\omega,m)}{e_{\Psi}}(u\mathcal{X}_\Phi,1,s)\overset{(\omega,m)}{e_{\Phi}}(u,1,s)\left(\int_0^{\frac{1}{2}} m\left(\widetilde{\frac{\partial}{\partial t}\gamma_s(t)}, \widetilde{\frac{\partial}{\partial s}\gamma_s(t)}\right)_{\mathcal{H}_\omega(\gamma_s,t,u_s)} dt\right)$$

where $\gamma_s(t) = \Phi\circ_{\mathrm{h}}\Psi(t,s)$. On the other hand:

$$\begin{aligned}
W &= \overset{(\omega,m)}{e_{\Psi}}(u\mathcal{X}_\Phi,1,s)\overset{(\omega,m)}{e_{\Phi}}(u,1,s)\left(\partial(\overset{(\omega,m)}{e_{\Phi}}(u,1,s))^{-1} \triangleright \left(\int_0^1 m\left(\widetilde{\frac{\partial}{\partial t}\psi_s(t)}, \widetilde{\frac{\partial}{\partial s}\psi_s(t)}\right)_{\mathcal{H}_\omega(\psi_s,t,(u\mathcal{X}_\Phi)_s)} dt\right)\right)\\
&= \overset{(\omega,m)}{e_{\Psi}}(u\mathcal{X}_\Phi,1,s)\overset{(\omega,m)}{e_{\Phi}}(u,1,s)\left(\int_0^1 m\left(\widetilde{\frac{\partial}{\partial t}\psi_s(t)}, \widetilde{\frac{\partial}{\partial s}\psi_s(t)}\right)_{\mathcal{H}_\omega(\psi_s,t,(u\mathcal{X}_\Phi)_s\partial(\overset{(\omega,m)}{e_{\Phi}}(u,1,s)))} dt\right)\\
&= \overset{(\omega,m)}{e_{\Psi}}(u\mathcal{X}_\Phi,1,s)\overset{(\omega,m)}{e_{\Phi}}(u,1,s)\left(\int_0^1 m\left(\widetilde{\frac{\partial}{\partial t}\psi_s(t)}, \widetilde{\frac{\partial}{\partial s}\psi_s(t)}\right)_{\mathcal{H}_\omega(\psi_s,t,u_s\phi_s)} dt\right)\\
&= \overset{(\omega,m)}{e_{\Psi}}(u\mathcal{X}_\Phi,1,s)\overset{(\omega,m)}{e_{\Phi}}(u,1,s)\left(\int_{\frac{1}{2}}^1 m\left(\widetilde{\frac{\partial}{\partial t}\gamma_s(t)}, \widetilde{\frac{\partial}{\partial s}\gamma_s(t)}\right)_{\mathcal{H}_\omega(\gamma_s,t,u_s)} dt\right).
\end{aligned}$$

Therefore both sides of the equation of the lemma satisfy the same differential equation, and they have the same initial condition. ■

Lemma 25 (Horizontal inversion) *We have:*

$$\overset{(\omega,m)}{e_{\Gamma^{-\mathrm{h}}}}(u\mathcal{X}_\Gamma)\overset{(\omega,m)}{e_\Gamma}(u) = 1_E,$$

where $\Gamma^{-\mathrm{h}}$ denotes the obvious horizontal reversion of $\Gamma\colon [0,1]^2 \to M$.

Proof. Analogous to the proof of the previous result. ■

Lemma 26 (Gauge transformations) *We have:*

$$\overset{(\omega,m)}{e_\Gamma}(ug) = g^{-1} \triangleright \overset{(\omega,m)}{e_\Gamma}(u).$$

Proof. Analogous to the proof of the previous result. ■

2.5.3 The Non-Abelian Fubini Theorem

We continue with the notation of 2.5.2. Again let $\Gamma\colon [0,1]^2 \to M$ be a smooth map, $a = \Gamma(0,0)$ and $u \in P_a$. Define $\overset{(\omega,m)}{f_\Gamma}(u,t,s)$ by the differential equation:

$$\frac{\partial}{\partial t}\overset{(\omega,m)}{f_\Gamma}(u,t,s) = \overset{(\omega,m)}{f_\Gamma}(u,t,s)\int_0^s m\left(\widetilde{\frac{\partial}{\partial s'}\gamma^t(s')}, \widetilde{\frac{\partial}{\partial t}\gamma^t(s')}\right)_{\mathcal{H}_\omega(\gamma^t,s',u^t)} ds', \tag{15}$$

with initial condition $\overset{(\omega,m)}{f_\Gamma}(u,0,s) = 1_E$, for each $s \in [0,1]$. Note that the differential equation for $\overset{(\omega,m)}{f_\Gamma}$ is obtained from the differential equation for $\overset{(\omega,m)}{e_\Gamma}$, equation (12), by reversing the roles of s and t. Let $\overset{(\omega,m)}{f_\Gamma}(u,1,1) \doteq \overset{(\omega,m)}{f_\Gamma}(u)$. The following holds.

Theorem 27 (Non-abelian Fubini Theorem, bundle form)

$$\overset{(\omega,m)}{e_\Gamma}(u)\overset{(\omega,m)}{f_\Gamma}(u) = 1.$$

Proof. In fact we show for every $t, s \in [0,1]$:

$$\overset{(\omega,m)}{e_\Gamma}(u,t,s)\overset{(\omega,m)}{f_\Gamma}(u,t,s) = 1. \tag{16}$$

In the following put $\overset{(\omega,m)}{e_\Gamma}(u,t,s) = e(t,s)$. Let θ be the canonical left invariant 1-form in E (the Maurer-Cartan 1-form); see 2.6.1. Taking the t derivative of (12), we obtain:

$$\frac{\partial}{\partial t}\theta\left(\frac{\partial}{\partial s}e(t,s)\right) = m\left(\widetilde{\frac{\partial}{\partial t}\gamma_s(t)}, \widetilde{\frac{\partial}{\partial s}\gamma_s(t)}\right)_{\mathcal{H}_\omega(\gamma_s,t,u_s)}.$$

By (14) and the G-equivariance of m:

$$\partial(e(t,s)) \triangleright \frac{\partial}{\partial t}\theta\left(\frac{\partial}{\partial s}e(t,s)\right) = m\left(\widetilde{\frac{\partial}{\partial t}\gamma_s(t)}, \widetilde{\frac{\partial}{\partial s}\gamma_s(t)}\right)_{\mathcal{H}_\omega(\gamma^t,s,u^t)}.$$

We also have:

$$\begin{aligned}
\frac{\partial}{\partial s}\left(\partial(e(t,s)) \triangleright \theta\left(\frac{\partial}{\partial t}e(t,s)\right)\right) &= \partial(e(t,s)) \triangleright \left(\partial\left(\theta\left(\frac{\partial}{\partial s}e(t,s)\right)\right) \triangleright \theta\left(\frac{\partial}{\partial t}e(t,s)\right)\right)\\
&\quad + \partial(e(t,s)) \triangleright \frac{\partial}{\partial s}\left(\theta\left(\frac{\partial}{\partial t}e(t,s)\right)\right)\\
&= \partial(e(t,s)) \triangleright \left(\left[\theta\left(\frac{\partial}{\partial s}e(t,s)\right), \theta\left(\frac{\partial}{\partial t}e(t,s)\right)\right] + \frac{\partial}{\partial s}\theta\left(\frac{\partial}{\partial t}e(t,s)\right)\right)\\
&= \partial(e(t,s)) \triangleright \frac{\partial}{\partial t}\theta\left(\frac{\partial}{\partial s}e(t,s)\right).
\end{aligned}$$

The second equation follows from the definition of a differential crossed module, and the third from the fact $d\theta(X,Y) = -[X,Y]$ for each $X, Y \in \mathfrak{e}$. Combining the two equations and integrating in s, with $\overset{(\omega,m)}{e_\Gamma}(u,t,0) = 1_E$, we obtain:

$$\frac{\partial}{\partial t}\overset{(\omega,m)}{e_\Gamma}(u,t,s) = \left(\int_0^s m\left(\widetilde{\frac{\partial}{\partial t}\gamma_{s'}(t)}, \widetilde{\frac{\partial}{\partial s'}\gamma_{s'}(t)}\right)_{\mathcal{H}_\omega(\gamma^t,s',u^t)} ds'\right)\overset{(\omega,m)}{e_\Gamma}(u,t,s),$$

with initial condition $\overset{(\omega,m)}{e_\Gamma}(u,0,s) = 1_E$, (set $t = 0$ in (12)), from which (16) follows as an immediate consequence. ■

Note that by using the non-abelian Fubini theorem, lemmas 24 and 25 follow directly from lemmas 22 and 23.

2.6 Dependence of the categorical holonomy on a smooth family of squares

In this subsection we prove a fundamental result giving the variation of the 2-holonomy of a smooth family of 2-paths in terms of the 2-curvature, analogous to equation (13) for the variation of the 1-holonomy of a smooth family of 1-paths in terms of the curvature. Let $P \to M$ be a principal G-bundle over the smooth manifold M with a $\mathcal{G}$-categorical connection (ω, m). Here $\mathcal{G} = (E \xrightarrow{\partial} G, \triangleright)$ is a Lie crossed module, where $\triangleright$ is a Lie group left action of G on E by automorphisms. Let $\mathfrak{G} = (\partial\colon \mathfrak{e} \to \mathfrak{g}, \triangleright)$ be the associated differential crossed module.

Consider a smooth map $J\colon [0,1]^3 \to M$. Put $J(t,s,x) = \Gamma^x(t,s)$, where $x,t,s \in [0,1]$. Define $q(x) = J(0,0,x)$, for each $x \in [0,1]$. Choose $u \in P_{q(0)}$ and let $u(x) = \mathcal{H}_\omega(q,x,u)$. We want to analyse the dependence on x of the categorical holonomy $\overset{(\omega,m)}{e_{\Gamma^x}}(u(x),t,s)$, see equation (12). To this end, we now prove the following well known technical lemma, also appearing in [FMP].

2.6.1 A well-known lemma

Let G be a Lie group. Consider a $\mathfrak{g}$-valued smooth function $V(s,x)$ defined on $[0,1]^2$. Consider the following differential equation in G:

$$\frac{\partial}{\partial s}a(s,x) = a(s,x)V(s,x),$$

with initial condition $a(0,x) = 1_G, \forall x \in [0,1]$. We want to know $\frac{\partial}{\partial x}a(s,x)$.

Let θ be the canonical $\mathfrak{g}$-valued 1-form on G. Thus θ is left invariant and satisfies $\theta(A) = A, \forall A \in \mathfrak{g}$, being defined uniquely by these properties. Also $d\theta(A,B) = -\theta([A,B])$, where $A, B \in \mathfrak{g}$. We have:

$$\frac{\partial}{\partial x}\theta\left(\frac{\partial}{\partial s}a(s,x)\right) = \frac{\partial}{\partial x}\theta\big(a(s,x)V(s,x)\big) = \frac{\partial}{\partial x}V(s,x).$$

On the other hand:

$$\begin{aligned}\frac{\partial}{\partial x}\theta\left(\frac{\partial}{\partial s}a(s,x)\right) &= da^*(\theta)\left(\frac{\partial}{\partial x},\frac{\partial}{\partial s}\right) + \frac{\partial}{\partial s}a^*(\theta)\left(\frac{\partial}{\partial x}\right) + a^*(\theta)\left(\left[\frac{\partial}{\partial x},\frac{\partial}{\partial s}\right]\right)\\ &= d\theta\left(\frac{\partial}{\partial x}a(s,x),\frac{\partial}{\partial s}a(s,x)\right) + \frac{\partial}{\partial s}\theta\left(\frac{\partial}{\partial x}a(s,x)\right).\end{aligned}$$

Therefore:

$$\theta\left(\frac{\partial}{\partial x}a(s,x)\right) = \int_0^s \left(-d\theta\left(\frac{\partial}{\partial x}a(s',x),\frac{\partial}{\partial s'}a(s',x)\right) + \frac{\partial}{\partial x}V(s',x)\right)ds' + \theta\left(\frac{\partial}{\partial x}a(0,x)\right).$$

Since $\frac{\partial}{\partial x}a(0,x) = 0$ (due to the initial conditions) we have the following:

Lemma 28

$$\frac{\partial}{\partial x}a(s,x) = a(s,x)\int_0^s \left(-d\theta\left(\frac{\partial}{\partial x}a(s',x),\frac{\partial}{\partial s'}a(s',x)\right) + \frac{\partial}{\partial x}V(s',x)\right)ds',$$

for each $x, s \in [0,1]$.

2.6.2 The relation between 2-curvature and categorical holonomy

The following main theorem is more general than the analogous result in [FMP, SW2] since it is valid for any smooth homotopy J connecting two 2-paths Γ and Γ', and in particular the basepoints of the 2-paths may vary with the parameter x. For this reason the proof is considerably longer, forcing several integrations by parts.

Theorem 29 *Let M be a smooth manifold. Let $\mathcal{G}=(\partial\colon E\to G,\triangleright)$ be a Lie crossed module. Let $P\to M$ be a principal G-bundle over M. Consider a $\mathcal{G}$-categorical connection (ω,m) on P. Let $J\colon[0,1]^3\to M$ be a smooth map. Let $J(t,s,x)=\Gamma^x(t,s)=\gamma_s^x(t)=\gamma^{x,t}(s);\forall t,s,x\in[0,1]$. Define $q(x)=\Gamma^x(0,0)$. Choose $u\in P_{q(0)}$, the fibre of P at $q(0)$. Let $u(x)=\mathcal{H}_\omega(q,x,u)$ and $u(x,s)=\mathcal{H}_\omega(\gamma^{x,0},s,u(x))$, where $s,x\in[0,1]$.*

Consider the map $(s,x)\in[0,1]^2\mapsto e_{\Gamma^x}(s)\in E$ defined by:

$$\frac{\partial}{\partial s}e_{\Gamma^x}(s)=e_{\Gamma^x}(s)\int_0^1 m\left(\widetilde{\frac{\partial}{\partial t}\gamma_s^x(t)},\widetilde{\frac{\partial}{\partial s}\gamma_s^x(t)}\right)_{\mathcal{H}_\omega(\gamma_s^x,t,u(x,s))}dt, \tag{17}$$

with initial condition:

$$e_{\Gamma^x}(0)=1_E,\forall x\in[0,1], \tag{18}$$

Let $e_{\Gamma^x}=e_{\Gamma^x}(1)$. For each $x\in[0,1]$, we have:

$$\begin{aligned}\frac{d}{dx}e_{\Gamma^x}=e_{\Gamma^x}&\int_0^1\int_0^1\mathcal{M}\left(\widetilde{\frac{\partial}{\partial x}\gamma_s^x(t)},\widetilde{\frac{\partial}{\partial t}\gamma_s^x(t)},\widetilde{\frac{\partial}{\partial s}\gamma_s^x(t)}\right)_{\mathcal{H}_\omega(\gamma_s^x,t,u(x,s))}dtds\\&+e_{\Gamma^x}\int_0^1 m\left(\widetilde{\frac{\partial}{\partial n}\hat{\gamma}^x(n)},\widetilde{\frac{\partial}{\partial x}\hat{\gamma}^x(n)}\right)_{\mathcal{H}_\omega(\hat{\gamma}^x,n,u(x))}dn,\end{aligned}$$

where $\hat{\gamma}^x=\partial\Gamma^x$, starting at $\Gamma^x(0,0)$ and oriented clockwise. Here $\mathcal{M}\in\mathcal{A}^3(P,\mathfrak{e})$ is the categorical curvature 3-form of (ω,m); see 2.4.3.

Proof. Consider the smooth map $f\colon[0,1]^3\to P$ such that $f(x,s,t)=\mathcal{H}_\omega(\gamma_s^x,t,u(x,s))$, for each $x,s,t\in[0,1]$. By definition we have: $\frac{\partial}{\partial t}f(x,s,t)=\widetilde{\frac{\partial}{\partial t}\gamma_s^x(t)}_{\mathcal{H}_\omega(\gamma_s^x,t,u(x,s))}$ and therefore $\omega(\frac{\partial}{\partial t}f(x,s,t))=0$. We also have: $\left(\frac{\partial}{\partial s}f(x,s,t)\right)^H=\widetilde{\frac{\partial}{\partial s}\gamma_s^x(t)}_{\mathcal{H}_\omega(\gamma_s^x,t,u(x,s))}$ and $\left(\frac{\partial}{\partial x}f(x,s,t)\right)^H=\widetilde{\frac{\partial}{\partial x}\gamma_s^x(t)}_{\mathcal{H}_\omega(\gamma_s^x,t,u(x,s))}$. Note also that $m(X,Y)$, $\Omega(X,Y)$ and $\mathcal{M}(X,Y,Z)$ vanish if either X, Y or Z is vertical.

By the 2-structure equation, see Proposition 16, and equation (6) it follows that (since $\mathcal{M}$ is horizontal):

$$\begin{aligned}&\int_0^1\int_0^1\mathcal{M}\left(\widetilde{\frac{\partial}{\partial x}\gamma_s^x(t)},\widetilde{\frac{\partial}{\partial t}\gamma_s^x(t)},\widetilde{\frac{\partial}{\partial s}\gamma_s^x(t)}\right)_{\mathcal{H}_\omega(\gamma_s^x,t,u(x,s))}dtds\\&=\int_0^1\int_0^1\mathcal{M}\left(\frac{\partial}{\partial x}f(x,s,t),\frac{\partial}{\partial t}f(x,s,t),\frac{\partial}{\partial s}f(x,s,t)\right)dtds\\&=\int_0^1\int_0^1 dm\left(\frac{\partial}{\partial x}f(x,s,t),\frac{\partial}{\partial t}f(x,s,t),\frac{\partial}{\partial s}f(x,s,t)\right)dtds\\&\quad+\int_0^1\int_0^1\omega\left(\frac{\partial}{\partial x}f(x,s,t)\right)\triangleright m\left(\frac{\partial}{\partial t}f(x,s,t),\frac{\partial}{\partial s}f(x,s,t)\right)dtds\\&\quad-\int_0^1\int_0^1\omega\left(\frac{\partial}{\partial s}f(x,s,t)\right)\triangleright m\left(\frac{\partial}{\partial t}f(x,s,t),\frac{\partial}{\partial x}f(x,s,t)\right)dtds.\end{aligned}$$

Using lemma 20 and integration by parts, we rewrite the integral in the last term:

$$\begin{aligned}
&\int_0^1 \omega\left(\frac{\partial}{\partial s}f(x,s,t)\right) \triangleright m\left(\frac{\partial}{\partial t}f(x,s,t), \frac{\partial}{\partial x}f(x,s,t)\right) dt \\
&= \int_0^1 \int_0^t \Omega\left(\frac{\partial}{\partial t'}f(x,s,t'), \frac{\partial}{\partial s}f(x,s,t')\right) dt' \triangleright m\left(\frac{\partial}{\partial t}f(x,s,t), \frac{\partial}{\partial x}f(x,s,t)\right) dt \\
&= \int_0^1 \Omega\left(\frac{\partial}{\partial t'}f(x,s,t'), \frac{\partial}{\partial s}f(x,s,t')\right) dt' \triangleright \int_0^1 m\left(\frac{\partial}{\partial t'}f(x,s,t'), \frac{\partial}{\partial x}f(x,s,t')\right) dt' \\
&\quad - \int_0^1 \Omega\left(\frac{\partial}{\partial t}f(x,s,t), \frac{\partial}{\partial s}f(x,s,t)\right) \triangleright \left(\int_0^t m\left(\frac{\partial}{\partial t'}f(x,s,t'), \frac{\partial}{\partial x}f(x,s,t')\right) dt'\right) dt.
\end{aligned}$$

Using $\partial(m) = \Omega$ and $\partial(e) \triangleright f = [e,f] = -[f,e] = -\partial(f) \triangleright e; \forall e, f \in \mathfrak{e}$, we have for the final term:

$$\begin{aligned}
&\int_0^1 \Omega\left(\frac{\partial}{\partial t}f(x,s,t), \frac{\partial}{\partial s}f(x,s,t)\right) \triangleright \left(\int_0^t m\left(\frac{\partial}{\partial t'}f(x,s,t'), \frac{\partial}{\partial x}f(x,s,t')\right) dt'\right) dt \\
&= -\int_0^1 \int_0^t \Omega\left(\frac{\partial}{\partial t'}f(x,s,t'), \frac{\partial}{\partial x}f(x,s,t')\right) dt' \triangleright m\left(\frac{\partial}{\partial t}f(x,s,t), \frac{\partial}{\partial s}f(x,s,t)\right) dt \\
&= -\int_0^1 \omega\left(\frac{\partial}{\partial x}f(x,s,t)\right) \triangleright m\left(\frac{\partial}{\partial t}f(x,s,t), \frac{\partial}{\partial s}f(x,s,t)\right) dt \\
&\quad + \int_0^1 \int_0^s \Omega\left(\frac{\partial}{\partial s'}f(x,s',0), \frac{\partial}{\partial x}f(x,s',0)\right) ds' \triangleright m\left(\frac{\partial}{\partial t}f(x,s,t), \frac{\partial}{\partial s}f(x,s,t)\right) dt \\
&= -\int_0^1 \omega\left(\frac{\partial}{\partial x}f(x,s,t)\right) \triangleright m\left(\frac{\partial}{\partial t}f(x,s,t), \frac{\partial}{\partial s}f(x,s,t)\right) dt \\
&\quad + \int_0^1 \omega\left(\frac{\partial}{\partial x}f(x,s,0)\right) \triangleright m\left(\frac{\partial}{\partial t}f(x,s,t), \frac{\partial}{\partial s}f(x,s,t)\right) dt.
\end{aligned}$$

where we have used Lemma 20 twice. Combining the previous equations, two terms cancel and we obtain:

$$\begin{aligned}
&\int_0^1 \int_0^1 \mathcal{M}\left(\widetilde{\frac{\partial}{\partial x}\gamma_s^x(t)}, \widetilde{\frac{\partial}{\partial t}\gamma_s^x(t)}, \widetilde{\frac{\partial}{\partial s}\gamma_s^x(t)}\right)_{\mathcal{H}_\omega(\gamma_s^x,t,u(x,s))} dtds \\
&= \int_0^1 \int_0^1 dm\left(\frac{\partial}{\partial x}f(x,s,t), \frac{\partial}{\partial t}f(x,s,t), \frac{\partial}{\partial s}f(x,s,t)\right) dtds \\
&\quad - \int_0^1 \int_0^1 \Omega\left(\frac{\partial}{\partial t'}f(x,s,t'), \frac{\partial}{\partial s}f(x,s,t')\right) dt' \triangleright \int_0^1 m\left(\frac{\partial}{\partial t'}f(x,s,t'), \frac{\partial}{\partial x}f(x,s,t')\right) dt'ds \\
&\quad - \int_0^1 \int_0^1 \omega\left(\frac{\partial}{\partial x}f(x,s,0)\right) \triangleright m\left(\frac{\partial}{\partial t}f(x,s,t), \frac{\partial}{\partial s}f(x,s,t)\right) dtds. \qquad (19)
\end{aligned}$$

For the second term on the right hand side in the theorem, we obtain:

$$\begin{aligned}
&\int_0^1 m\left(\widetilde{\frac{\partial}{\partial n}\hat{\gamma}^x(n)}, \widetilde{\frac{\partial}{\partial x}\hat{\gamma}^x(n)}\right)_{\mathcal{H}_\omega(\hat{\gamma}^x,n,u(x))} dn \\
&= \int_0^1 m\left(\frac{\partial}{\partial s}f(x,s,0), \frac{\partial}{\partial x}f(x,s,0)\right) ds + \int_0^1 m\left(\frac{\partial}{\partial t}f(x,1,t), \frac{\partial}{\partial x}f(x,1,t)\right) dt \\
&\quad - g(x,1)^{-1} \triangleright \left(\int_0^1 m\left(\frac{\partial}{\partial s}f'(x,s,1), \frac{\partial}{\partial x}f'(x,s,1)\right) ds - \int_0^1 m\left(\frac{\partial}{\partial t}f(x,0,t), \frac{\partial}{\partial x}f(x,0,t)\right) dt\right), \qquad (20)
\end{aligned}$$

where we have introduced $g(x,s) = \partial(e_{\Gamma^x}(s))$, as well as $f'(x,s,t) = \mathcal{H}_\omega\left(\gamma^{x,t}, s, \mathcal{H}_\omega(\gamma_0^x, t, u(x))\right)$. Therefore $f(x,s,1) = f'(x,s,1)\partial(e_{\Gamma^x}(s))$ by the non-abelian Green theorem. Note that $f'(x,0,t) = f(x,0,t)$. We will be using the function f' again shortly.

Thus it remains to prove that $e_{\Gamma^x}^{-1}\frac{d}{dx}e_{\Gamma^x}$ is equal to the sum of the right hand sides of (19) and (20).

By lemma 28, we have

$$\frac{d}{dx} e_{\Gamma^x} = e_{\Gamma^x}(A_x - B_x), \tag{21}$$

where

$$A_x = \int_0^1 \int_0^1 \frac{\partial}{\partial x} \left(m \left(\widetilde{\frac{\partial}{\partial t} \gamma_s^x(t)}, \widetilde{\frac{\partial}{\partial s} \gamma_s^x(t)} \right)_{\mathcal{H}_\omega(\gamma_s^x, t, u(x,s))} \right) dtds$$

$$B_x = \int_0^1 d\theta \left(\frac{\partial}{\partial x} e_{\Gamma^x}(s), \frac{\partial}{\partial s} e_{\Gamma^x}(s) \right) ds.$$

Let us analyse A_x and B_x separately. Using the well known equation:

$$d\alpha(X,Y,Z) = X\alpha(Y,Z) + Y\alpha(Z,X) + Z\alpha(X,Y) + \alpha(X,[Y,Z]) + \alpha(Y,[Z,X]) + \alpha(Z,[X,Y]),$$

valid for any smooth 2-form α in a manifold, and any three vector fields X, Y, Z in M, we obtain for A_x:

$$\begin{aligned} A_x &= \int_0^1 \int_0^1 \frac{\partial}{\partial x} m \left(\frac{\partial}{\partial t} f(x,s,t), \frac{\partial}{\partial s} f(x,s,t) \right) dtds \\ &= \int_0^1 \int_0^1 dm \left(\frac{\partial}{\partial x} f(x,s,t), \frac{\partial}{\partial t} f(x,s,t), \frac{\partial}{\partial s} f(x,s,t) \right) dtds \\ &\quad - \int_0^1 \int_0^1 \frac{\partial}{\partial t} m \left(\frac{\partial}{\partial s} f(x,s,t), \frac{\partial}{\partial x} f(x,s,t) \right) dtds - \int_0^1 \int_0^1 \frac{\partial}{\partial s} m \left(\frac{\partial}{\partial x} f(x,s,t), \frac{\partial}{\partial t} f(x,s,t) \right) dtds \\ &= \int_0^1 \int_0^1 dm \left(\frac{\partial}{\partial x} f(x,s,t), \frac{\partial}{\partial t} f(x,s,t), \frac{\partial}{\partial s} f(x,s,t) \right) dtds \\ &\quad + \int_0^1 m \left(\frac{\partial}{\partial s} f(x,s,0), \frac{\partial}{\partial x} f(x,s,0) \right) ds + \int_0^1 m \left(\frac{\partial}{\partial t} f(x,1,t), \frac{\partial}{\partial x} f(x,1,t) \right) dt \\ &\quad - \int_0^1 m \left(\frac{\partial}{\partial t} f(x,0,t), \frac{\partial}{\partial x} f(x,0,t) \right) dt - \int_0^1 m \left(\frac{\partial}{\partial s} f(x,s,1), \frac{\partial}{\partial x} f(x,s,1) \right) ds. \end{aligned} \tag{22}$$

Recall from before the definitions $g(x,s) = \partial(e_{\Gamma^x}(s))$ and $f'(x,s,t) = \mathcal{H}_\omega\left(\gamma^{x,t}, s, \mathcal{H}_\omega(\gamma_0^x, t, u(x))\right)$, and the relation $f(x,s,1) = f'(x,s,1)\partial(e_{\Gamma^x}(s))$. We thus have:

$$\omega \left(\frac{\partial}{\partial s} f'(x,s,1) \right) = \omega \left(\frac{\partial}{\partial s} \left(f(x,s,1) g^{-1}(x,s) \right) \right),$$

which since $\frac{\partial}{\partial s} f'(x,s,1)$ is horizontal implies, by using the Leibniz rule and the fact that ω is a connection 1-form, that:

$$g(x,s)\omega \left(\frac{\partial}{\partial s} f(x,s,1) \right) g^{-1}(x,s) + g(x,s) \frac{\partial}{\partial s} g^{-1}(x,s) = 0.$$

Analogously (this will be used later):

$$g^{-1}(x,s)\omega \left(\frac{\partial}{\partial x} \left(\mathcal{H}_\omega(\gamma^{x,1}, s, u(x)\gamma_0^x) \right) \right) g(x,s) = -g^{-1}(x,s) \frac{\partial}{\partial x} g(x,s) + \omega \left(\frac{\partial}{\partial x} \left(u(x,s)\gamma_s^x \right) \right),$$

which is the same as:

$$\frac{\partial}{\partial x} g(x,s) = g(x,s)\omega \left(\frac{\partial}{\partial x} f(x,s,1) \right) - \omega \left(\frac{\partial}{\partial x} f'(x,s,1) \right) g(x,s).$$

The very last term of (22) can be simplified as follows (since m is horizontal and G-equivariant):

$$
\begin{aligned}
&-\int_0^1 m\left(\frac{\partial}{\partial s}f(x,s,1),\frac{\partial}{\partial x}f(x,s,1)\right)ds\\
&\quad=-\int_0^1 g^{-1}(x,s)\triangleright m\left(\frac{\partial}{\partial s}f'(x,s,1),\frac{\partial}{\partial x}f'(x,s,1)\right)ds\\
&\quad=-g^{-1}(x,1)\triangleright\int_0^1 m\left(\frac{\partial}{\partial s}f'(x,s,1),\frac{\partial}{\partial x}f'(x,s,1)\right)ds\\
&\qquad+\int_0^1\int_0^s\frac{\partial}{\partial s}g^{-1}(x,s)\triangleright m\left(\frac{\partial}{\partial s'}f'(x,s',1),\frac{\partial}{\partial x}f'(x,s',1)\right)ds'ds\\
&\quad=-g^{-1}(x,1)\triangleright\int_0^1 m\left(\frac{\partial}{\partial s}f'(x,s,1),\frac{\partial}{\partial x}f'(x,s,1)\right)ds\\
&\qquad-\int_0^1\int_0^s\omega\left(\frac{\partial}{\partial s}f(x,s,1)\right)g^{-1}(x,s)\triangleright m\left(\frac{\partial}{\partial s'}f'(x,s',1),\frac{\partial}{\partial x}f(x,s',1)\right)ds'ds. \qquad (23)
\end{aligned}
$$

The penultimate equation follows from integrating by parts.

We now analyse B_x, for each $x\in[0,1]$. We have:

$$
\begin{aligned}
B_x&=d\theta\Big(e_{\Gamma^x}^{-1}(s)\frac{\partial}{\partial x}e_{\Gamma^x}(s),e_{\Gamma^x}^{-1}(s)\frac{\partial}{\partial s}e_{\Gamma^x}(s)\Big)\\
&=-\Big[e_{\Gamma^x}^{-1}(s)\frac{\partial}{\partial x}e_{\Gamma^x}(s),e_{\Gamma^x}^{-1}(s)\frac{\partial}{\partial s}e_{\Gamma^x}(s)\Big]\\
&=-\left(g^{-1}(x,s)\frac{\partial}{\partial x}g(x,s)\right)\triangleright\left(e_{\Gamma^x}^{-1}(s)\frac{\partial}{\partial s}e_{\Gamma^x}(s)\right)\\
&=-\omega\left(\frac{\partial}{\partial x}f(x,s,1)\right)\triangleright\int_0^1 m\left(\frac{\partial}{\partial t}f(x,s,t),\frac{\partial}{\partial s}f(x,s,t)\right)dt\\
&\quad+\left(g^{-1}(x,s)\omega\left(\frac{\partial}{\partial x}f'(x,s,1)\right)g(x,s)\right)\triangleright\int_0^1 m\left(\frac{\partial}{\partial t}f(x,s,t),\frac{\partial}{\partial s}f(x,s,t)\right)dt.
\end{aligned}
$$

By using lemma 20, this may be rewritten:

$$
\begin{aligned}
B_x=&-\int_0^1\Omega\left(\frac{\partial}{\partial t}f(x,s,t),\frac{\partial}{\partial x}f(x,s,t)\right)dt\triangleright\int_0^1 m\left(\frac{\partial}{\partial t}f(x,s,t),\frac{\partial}{\partial s}f(x,s,t)\right)dt\\
&-\int_0^s\Omega\left(\frac{\partial}{\partial s'}f(x,s',0),\frac{\partial}{\partial x}f(x,s',0)\right)ds'\triangleright\int_0^1 m\left(\frac{\partial}{\partial t}f(x,s,t),\frac{\partial}{\partial s}f(x,s,t)\right)dt\\
&+\left(g^{-1}(x,s)\left(\int_0^s\Omega\left(\frac{\partial}{\partial s'}f'(x,s',1),\frac{\partial}{\partial x}f'(x,s',1)\right)ds'\right)g(x,s)\right)\triangleright\int_0^1 m\left(\frac{\partial}{\partial t}f(x,s,t),\frac{\partial}{\partial s}f(x,s,t)\right)dt\\
&+\left(g^{-1}(x,s)\left(\int_0^1\Omega\left(\frac{\partial}{\partial t}f(x,0,t),\frac{\partial}{\partial x}f(x,0,t)\right)dt\right)g(x,s)\right)\triangleright\int_0^1 m\left(\frac{\partial}{\partial t}f(x,s,t),\frac{\partial}{\partial s}f(x,s,t)\right)dt,
\end{aligned}
$$

Again using $\partial(m)=\Omega$ and $\partial(e)\triangleright f=[e,f]=-[f,e]=-\partial(f)\triangleright e;\forall e,f\in\mathfrak{e}$, together with $\partial(m)=\Omega$ and lemma 20, for all but the second term of the right hand side of the previous equation, we obtain:

$$
\begin{aligned}
B_x=&\int_0^1\left(\int_0^1\Omega\left(\frac{\partial}{\partial t}f(x,s,t),\frac{\partial}{\partial s}f(x,s,t)\right)dt\triangleright\int_0^1 m\left(\frac{\partial}{\partial t}f(x,s,t),\frac{\partial}{\partial x}f(x,s,t)\right)dt\right)ds.\\
&-\int_0^1\int_0^1\omega\left(\frac{\partial}{\partial x}f(x,s,0)\right)\triangleright m\left(\frac{\partial}{\partial t}f(x,s,t),\frac{\partial}{\partial s}f(x,s,t)\right)dtds\\
&-\int_0^1\int_0^s\omega\left(\frac{\partial}{\partial s}f(x,s,1)\right)g^{-1}(x,s)\triangleright m\left(\frac{\partial}{\partial s'}f'(x,s',1),\frac{\partial}{\partial x}f'(x,s',1)\right)ds'ds\\
&-\int_0^1\int_0^1\omega\left(\frac{\partial}{\partial s}f(x,s,1)\right)g^{-1}(x,s)\triangleright m\left(\frac{\partial}{\partial t}f(x,0,t),\frac{\partial}{\partial x}f(x,0,t)\right)dtds \qquad (24)
\end{aligned}
$$

Finally, since (given that ω is a connection 1-form):

$$\begin{aligned}
\omega\left(\frac{\partial}{\partial s}f(x,s,1)\right)g^{-1}(x,s) &= g^{-1}(x,s)\omega\left(\frac{\partial}{\partial s}f(x,s,1)g^{-1}(x,s)\right)\\
&= g^{-1}(x,s)\omega\left(\frac{\partial}{\partial s}f'(x,s,1) - f(x,s,1)\frac{\partial}{\partial s}g^{-1}(x,s)\right)\\
&= -g^{-1}(x,s)\omega\left(f(x,s,1)\frac{\partial}{\partial s}g^{-1}(x,s)\right)\\
&= -\frac{\partial}{\partial s}g^{-1}(x,s),
\end{aligned}$$

the last term of the previous expression is rewritten as follows:

$$\begin{aligned}
-\int_0^1\int_0^1 \omega\left(\frac{\partial}{\partial s}f(x,s,1)\right)g^{-1}(x,s)\triangleright m\left(\frac{\partial}{\partial t}f(x,0,t),\frac{\partial}{\partial x}f(x,0,t)\right)dsdt\\
=\int_0^1\int_0^1 \frac{\partial}{\partial s}g^{-1}(x,s)\triangleright m\left(\frac{\partial}{\partial t}f(x,0,t),\frac{\partial}{\partial x}f(x,0,t)\right)dsdt\\
= g^{-1}(x,1)\triangleright\int_0^1 m\left(\frac{\partial}{\partial t}f(x,0,t),\frac{\partial}{\partial x}f(x,0,t)\right)dt - \int_0^1 m\left(\frac{\partial}{\partial t}f(x,0,t),\frac{\partial}{\partial x}f(x,0,t)\right)dt. \qquad (25)
\end{aligned}$$

Combining $A_x - B_x$ from equations (22), (23), (24), (25), four terms cancel and the remaining terms are equal to the sum of the right hand sides of (19) and (20). ■

2.6.3 Invariance under thin homotopy

From theorem 29 and the fact that the horizontal lift $X \mapsto \widetilde{X}$ of vector fields on M defines a linear map $\mathfrak{X}(M) \to \mathfrak{X}(P)$ we obtain the following:

Corollary 30 *Let M be a smooth manifold. Let also $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a Lie crossed module. Let $P \to M$ be a principal G-bundle over M, and consider a $\mathcal{G}$-categorical connection (ω, m) on P. If Γ and Γ' are rank-2 homotopic (see definition 10) 2-paths $[0,1]^2 \to M$ then $\overset{(\omega,m)}{e_\Gamma}(u,t,s) = \overset{(\omega,m)}{e_{\Gamma'}}(u,t,s)$, whenever $u \in P_{\Gamma(0,0)}$, the fibre of P at $\Gamma(0,0) = \Gamma'(0,0)$, and for each $t, s \in [0,1]$.*

2.6.4 A (dihedral) double groupoid map

Let P be a principal G bundle over M. We define a double groupoid $\mathcal{D}^2(P)$ whose set of objects is M, and whose set of morphisms $x \to y$ is given by all right G-equivariant maps $a\colon P_x \to P_y$. A 2-morphism is given by a square of the form:

$$\begin{array}{ccc}
P_z & \xrightarrow{\ d\ } & P_w\\
{\scriptstyle c}\uparrow & f & \uparrow{\scriptstyle b}\\
P_x & \xrightarrow[\ a\ ]{} & P_y
\end{array} \qquad (26)$$

where $x, y, z, w \in M$ and a, b, c, d are right G-equivariant maps. Finally $f\colon P_x \to E$ is a smooth map such that $f(ug) = g^{-1} \triangleright f(u)$ for each $u \in P_x$ and $g \in G$, satisfying $(b \circ a)(u)\partial(f(u)) = (d \circ c)(u)$, for each $u \in P_x$. The horizontal and vertical compositions are as in 2.5.2. We also have an action of the dihedral group of the 2-cube D_4 given by the horizontal and vertical reversions, and such that the interchange of coordinates is accomplished by the move $f \mapsto f^{-1}$. As a corollary of the discussion in the last two subsections it follows:

Theorem 31 *Whenever the principal G-bundle $P \to M$ is equipped with a categorical connection (ω, m), the holonomy and categorical holonomy maps $\mathcal{H}_\omega$ and $\overset{(\omega,m)}{e}$ define a double groupoid morphism $\overset{(\omega,m)}{\mathcal{H}}\colon \mathcal{S}_2(M) \to \mathcal{D}^2(P)$, where $\mathcal{S}_2(M)$ is the thin fundamental double groupoid of M. Given a dihedral group element $r \in D_4$ we have*

$$\overset{(\omega,m)}{\mathcal{H}}(\Gamma \circ r^{-1}) = r\left(\overset{(\omega,m)}{\mathcal{H}}(\Gamma)\right).$$

3 Cubical 2-bundles with connection

3.1 Definition of a cubical $\mathcal{G}$-2-bundle

Recall the conventions introduced in 2.1.1 and 2.2.2.

Let M be a smooth manifold. Let $\mathcal{U} = \{U_i\}_{i \in \mathcal{I}}$ be an open cover of M. From this we can define a cubical set $C(M, \mathcal{U})$. For each positive integer n the set $C^n(M, \mathcal{U})$ of n-cubes of $C(M, \mathcal{U})$ is given by all pairs (x, R), where R is an assignment of an element $U_v^R \in \mathcal{U}$ to each vertex of v of D^n, such that the intersection

$$U^R = \bigcap_{\text{vertices } v \text{ of } D^n} U_v^R$$

is non-empty, and $x \in U^R$. The face maps $\partial_i^{\pm} \colon C^n(M, \mathcal{U}) \to C^{n-1}(M, \mathcal{U})$ where $i \in \{1, \ldots, n\}$ and $n = 1, 2, \ldots$, are defined by

$$\partial_i^{\pm}(x, R) = (x, R \circ \delta_i^{\pm}).$$

Analogously, the degeneracies are given by:

$$\epsilon_i(x, R) = (x, R \circ \sigma_i).$$

Given an $x \in M$, the cubical set $C(M, \mathcal{U}, x)$ is given by all the cubes of $C(M, \mathcal{U})$ whose associated element of M is x.

Definition 32 (Cubical $\mathcal{G}$-2-bundle) *Let $\mathcal{G} = (\partial \colon E \to G, \triangleright)$ be a Lie crossed module. Let $\mathcal{N}(\mathcal{G})$ be the cubical nerve of $\mathcal{G}$; see 2.2.2. Let M be a smooth manifold and $\mathcal{U} = \{U_i\}_{i \in \mathcal{I}}$ be an open cover of M. A cubical $\mathcal{G}$-2-bundle over $(M, \mathcal{U})$ is given by a cubical map $C(M, \mathcal{U}) \to \mathcal{N}(\mathcal{G})$ such that all the maps involved are smooth.*

Unpacking this definition, we see that a cubical 2-bundle is specified by smooth maps $\phi_{ij} \colon U_i \cap U_j \to G$, where $U_i, U_j \in \mathcal{U}$ have a non-empty intersection, and also by smooth maps $\psi_{ijkl} \colon U_i \cap U_j \cap U_k \cap U_l \to E$, where $U_i, U_j, U_k, U_l \in \mathcal{U}$ have a non-empty intersection, such that:

1. *We have $\partial(\psi_{ijkl})^{-1}\phi_{ij}\phi_{jl} = \phi_{ik}\phi_{kl}$ in $U_{ijkl} \doteq U_i \cap U_j \cap U_k \cap U_l$. In other words, putting $\phi_{ij} = X_2^-(\mathbf{c}_2)$, $\phi_{ik} = X_1^-(\mathbf{c}_2)$, $\phi_{kl} = X_2^+(\mathbf{c}_2)$, $\phi_{jl} = X_1^+(\mathbf{c}_2)$ and $e(\mathbf{c}_2) = \psi_{ijkl}$ yields a flat $\mathcal{G}$-colouring $\mathbf{c}_2 = (\psi, \phi)_{ijkl}$ of D^2, for each $x \in U_{ijkl}$.*

2. *Given $i^\pm, j^\pm, k^\pm, l^\pm \in \mathcal{I}$ with $U_{i^-j^-k^-l^-} \cap U_{i^+j^+k^+l^+} \neq \emptyset$, and putting $e_3^{\pm}(\mathbf{c}_3) = (\psi, \phi)_{i^\pm j^\pm k^\pm l^\pm}$, $e_1^-(\mathbf{c}_3) = (\psi, \phi)_{i^-k^-i^+k^+}$, $e_1^+(\mathbf{c}_3) = (\psi, \phi)_{j^-l^-j^+l^+}$, $e_2^-(\mathbf{c}_3) = (\psi, \phi)_{i^-j^-i^+j^+}$, $e_2^+(\mathbf{c}_3) = (\psi, \phi)_{k^-l^-k^+l^+}$ yields a flat $\mathcal{G}$-colouring $\mathbf{c}_3$ of D^3 in $U_{i^-j^-k^-l^-} \cap U_{i^+j^+k^+l^+}$.*

3. *$\phi_{ii} = 1_G$ in U_i for all $i \in \mathcal{I}$.*

4. *$\psi_{iijj} = \psi_{ijij} = 1_E$ in U_{ij}*

See figure 1 for our conventions in labelling the vertices of D^2 and D^3.

The previous definition is therefore a cubical version of the simplicial version of a $\mathcal{G}$-2-bundle (and non-abelian gerbe) appearing for example in [BrMe, ACG, BS1, BS2, SW3].

Definition 33 (Dihedral $\mathcal{G}$-2-bundles, and multiplicative $\mathcal{G}$-2-bundles) *Recall that the cubical sets $C(M, \mathcal{U})$ and $\mathcal{N}(\mathcal{G})$ have a lot of extra structure, namely inversions and interchanges (therefore an action of the hyperoctahedral group), and also ω-groupoid structures. Therefore we can restrict our definition of a cubical $\mathcal{G}$-2-bundle and only allow cubical maps $C(M, \mathcal{U}) \to \mathcal{N}(\mathcal{G})$ preserving the structures above, therefore defining (respectively) dihedral $\mathcal{G}$-2-bundles and multiplicative $\mathcal{G}$-2-bundles.*

Explicitly, a cubical $\mathcal{G}$-2-bundle is said to be dihedral if the maps $\phi_{ij} \colon U_{ij} \to G$ and $\psi_{ijkl} \colon U_{ijkl} \to E$ satisfy the following extra conditions:

1. *We have $\phi_{ji} = \phi_{ij}^{-1}$ in U_{ij} for all $i, j \in \mathcal{I}$.*

2. *We have $\psi_{ikjl} = \psi_{ijkl}^{-1}$, $\psi_{jilk} = \phi_{ij} \triangleright \psi_{ijkl}^{-1}$ and $\psi_{klij} = \phi_{ik} \triangleright \psi_{ijkl}^{-1}$ in U_{ijkl}.*

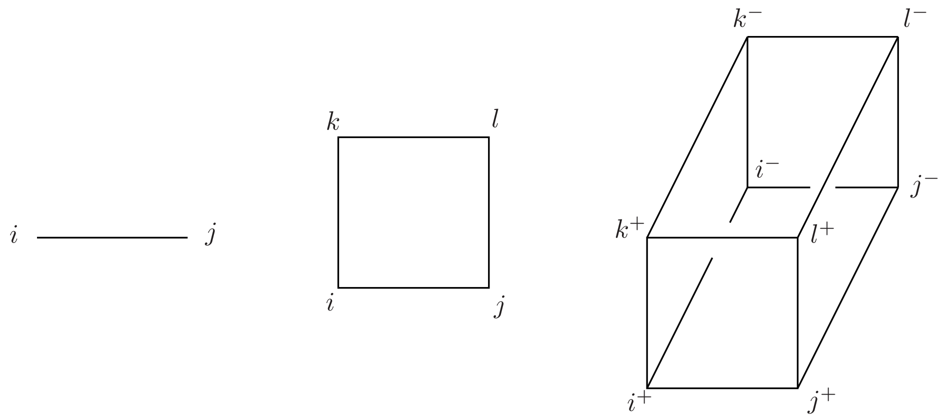


Figure 1: Label conventions in definition 32.

3.2 Connections in cubical $\mathcal{G}$-2-bundles

Let $\mathcal{G} = (\partial \colon E \to G, \triangleright)$ be a Lie crossed module, where $\triangleright$ is a Lie group left action of G on E by automorphisms. Let also $\mathfrak{G} = (\partial \colon \mathfrak{e} \to \mathfrak{g}, \triangleright)$ be the associated differential crossed module.

Definition 34 (Connection in a cubical $\mathcal{G}$-2-bundle) *Let M be a smooth manifold with an open cover $\mathcal{U} = \{U_i\}_{i \in \mathcal{I}}$. A connection in a cubical $\mathcal{G}$-2-bundle over $(M, \mathcal{U})$ is given by:*

- *For any $i \in \mathcal{I}$ a local connection pair (A_i, B_i) defined in U_i; in other words $A_i \in \mathcal{A}^1(U_i, \mathfrak{g})$, $B_i \in \mathcal{A}^2(U_i, \mathfrak{e})$ and $\partial(B_i) = dA_i + \frac{1}{2} A_i \wedge^{\mathrm{ad}} A_i = \Omega_{A_i}$.*
- *For any ordered pair (i, j) an $\mathfrak{e}$-valued 1-form η_{ij} in U_{ij}.*

The conditions that should hold are:

1. *For any $i \in \mathcal{I}$ we have $\eta_{ii} = 0$.*
2. *For any $i, j \in \mathcal{I}$ we have:*
$$A_j = \phi_{ij}^{-1} \left(A_i + \partial(\eta_{ij})\right) \phi_{ij} + \phi_{ij}^{-1} d\phi_{ij},$$
$$B_j = \phi_{ij}^{-1} \triangleright \left(B_i + d\eta_{ij} + \frac{1}{2} \eta_{ij} \wedge^{\mathrm{ad}} \eta_{ij} + A_i \wedge^{\triangleright} \eta_{ij} \right).$$
3. *For any $i, j, k, l \in \mathcal{I}$ we have:*
$$\eta_{ik} + \phi_{ik} \triangleright \eta_{kl} - \phi_{ik}\phi_{kl}\phi_{jl}^{-1} \triangleright \eta_{jl} - \phi_{ik}\phi_{kl}\phi_{jl}^{-1}\phi_{ij}^{-1} \triangleright \eta_{ij} = \psi_{ijkl}^{-1} d\psi_{ijkl} + \psi_{ijkl}^{-1} \left(A_i \wedge^{\triangleright} \psi_{ijkl}\right).$$

The equivalence of $\mathcal{G}$-2-bundles with connection will be dealt with in subsection 4.3.

Definition 35 (Dihedral connection) *If a $\mathcal{G}$-2-bundle is dihedral, then a connection in it is said to be dihedral if the following extra condition holds:*

$$\eta_{ji} = -\phi_{ij}^{-1} \triangleright \eta_{ij}, \text{ for each } i, j \in \mathcal{I};$$

therefore, condition 3 of the previous definition can be written as:

$$\eta_{ik} + \phi_{ik} \triangleright \eta_{kl} + \phi_{ik}\phi_{kl} \triangleright \eta_{lj} + \phi_{ik}\phi_{kl}\phi_{lj} \triangleright \eta_{ji} = \psi_{ijkl}^{-1} d\psi_{ijkl} + \psi_{ijkl}^{-1} \left(A_i \wedge^{\triangleright} \psi_{ijkl}\right).$$

4 Non-abelian integral calculus based on a crossed module

4.1 Path-ordered exponential and surface-ordered exponential

We continue with the notation and results of subsections 2.5 and 2.6. Alternative direct derivations of some of the following results appear in [BS1, SW1, SW2, SW3].

Let M be a manifold, and let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\gamma\colon [0,1] \to M$ be a piecewise smooth map. Let $A \in \mathcal{A}^1(M,\mathfrak{g})$ be a $\mathfrak{g}$-valued 1-form in M. We define, as is usual, the path ordered exponential $\overset{A}{g}\gamma(t) = \mathcal{P}\exp\left(\int_0^t A\left(\frac{d}{dt'}\gamma(t')\right)dt'\right)$ to be the solution of the differential equation:

$$\frac{d}{dt}\overset{A}{g}\gamma(t) = \overset{A}{g}\gamma(t)A\left(\frac{d}{dt}\gamma(t)\right),$$

with initial condition $\overset{A}{g}\gamma(0) = 1_G$; see [Ch]. Put $\overset{A}{g}\gamma \doteq \overset{A}{g}\gamma(1) = \mathcal{P}\exp\left(\int_0^1 A\left(\frac{d}{dt}\gamma(t)\right)dt\right)$. We immediately get that $\overset{A}{g}_{\gamma\gamma'} = \overset{A}{g}_\gamma \overset{A}{g}_{\gamma'}$, and also $\overset{A}{g}_{\gamma^{-1}} = (\overset{A}{g}_\gamma)^{-1}$. Here γ and γ' are piecewise smooth maps with $\gamma(1) = \gamma'(0)$.

Consider the trivial bundle $P = M \times G$ over M. Given $A \in \mathcal{A}^1(M,\mathfrak{g})$ there exists a unique connection 1-form ω_A in the trivial bundle P for which $A = \zeta^*(\omega_A)$, where $\zeta(x) = (x, 1_G)$ for each $x \in M$. We then have that:

$$\zeta(\gamma(t)) = \mathcal{H}_{\omega_A}(\gamma, t, \zeta(\gamma(0)))\mathcal{P}\exp\left(\int_0^t A\left(\frac{d}{dt'}\gamma(t')\right)dt'\right).$$

Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a Lie crossed module and let $\mathfrak{G} = (\partial\colon \mathfrak{e} \to \mathfrak{g}, \triangleright)$ be the associated differential crossed module. As before, if we have $B \in \mathcal{A}^2(M,\mathfrak{e})$ with $\partial(B) = \Omega_A = dA + \frac{1}{2}A \wedge^{\mathrm{ad}} A$ we define

$$\overset{(A,B)}{e_\Gamma}(t,s) = \mathcal{S}\exp\left(\int_0^s \int_0^t B\left(\frac{\partial}{\partial t'}\gamma_{s'}(t'), \frac{\partial}{\partial s'}\gamma_{s'}(t')\right)dt'ds'\right)$$

as being the solution of the differential equation:

$$\frac{\partial}{\partial s}\overset{(A,B)}{e_\Gamma}(t,s) = \overset{(A,B)}{e_\Gamma}(t,s)\int_0^t \left(\overset{A}{g}_{\gamma^0(s)}\overset{A}{g}_{\gamma_s(t')}\right) \triangleright B\left(\frac{\partial}{\partial t'}\gamma_s(t'), \frac{\partial}{\partial s}\gamma_s(t')\right)dt'$$

with initial conditions

$$\overset{(A,B)}{e_\Gamma}(t,0) = 1_E, \forall t \in [0,1].$$

Put $\overset{(A,B)}{e_\Gamma} = \overset{(A,B)}{e_\Gamma}(1,1)$. We can equivalently define the surface ordered exponential by the differential equation:

$$\frac{\partial}{\partial t}\overset{(A,B)}{e_\Gamma}(t,s) = \left(\int_0^s \left(\overset{A}{g}_{\gamma_0(t)}\overset{A}{g}_{\gamma^t(s')}\right) \triangleright B\left(\frac{\partial}{\partial t}\gamma_{s'}(t), \frac{\partial}{\partial s'}\gamma_{s'}(t)\right)ds'\right)\overset{(A,B)}{e_\Gamma}(t,s)$$

with initial conditions

$$\overset{(A,B)}{e_\Gamma}(0,s) = 1_E, \forall s \in [0,1];$$

see the proof of Theorem 27 and below.

As before, there exists a unique categorical connection $(\omega_A, m_{A,B})$ in the trivial bundle $P = M \times G$ for which $A = \zeta^*(\omega_A)$ and $B = \zeta^*(m_{A,B})$). We have that $\overset{(A,B)}{e_\Gamma}(t,s) = \overset{(\omega_A, m_{A,B})}{e_\Gamma}(\zeta(\Gamma(0,0)), t, s)$, see 2.5.3. The following follows immediately from the non-abelian Green theorem 21.

Theorem 36 (Non-abelian Green Theorem, elementary form) *Let $\Gamma\colon [0,1]^2 \to M$ be a 2-square. Put $\overset{A}{X}_\Gamma = \overset{A}{g}_{\mathcal{X}_\Gamma}$, $\overset{A}{Y}_\Gamma = \overset{A}{g}_{\mathcal{Y}_\Gamma}$, $\overset{A}{Z}_\Gamma = \overset{A}{g}_{\mathcal{Z}_\Gamma}$ and $\overset{A}{W}_\Gamma = \overset{A}{g}_{\mathcal{W}_\Gamma}$; see 2.5.2 for this notation. We have that:*

$$\partial\left(\overset{(A,B)}{e_\Gamma}\right)^{-1}\overset{A}{X}_\Gamma\overset{A}{Y}_\Gamma = \overset{A}{Z}_\Gamma\overset{A}{W}_\Gamma.$$

The following follows from theorems 22 and 24. See 2.1.1 and subsection 2.3.

Theorem 37 *Consider the map* $\overset{(A,B)}{\mathcal{H}} : C^2(M) \to \mathcal{D}^2(\mathcal{G})$ *such that:*

$$\overset{(A,B)}{\mathcal{H}}(\Gamma) = \begin{array}{ccc} * & \xrightarrow{\overset{A}{W}_\Gamma} & * \\ \overset{A}{Z}_\Gamma \Big\uparrow & \overset{(A,B)}{e}_\Gamma & \Big\uparrow \overset{A}{Y}_\Gamma \\ * & \xrightarrow[\overset{A}{X}_\Gamma]{} & * \end{array}$$

Then $\overset{(A,B)}{\mathcal{H}}(\Gamma \circ_{\mathrm{h}} \Gamma') = \overset{(A,B)}{\mathcal{H}}(\Gamma) \circ_{\mathrm{h}} \overset{(A,B)}{\mathcal{H}}(\Gamma')$ *and* $\overset{(A,B)}{\mathcal{H}}(\Gamma \circ_{\mathrm{v}} \Gamma') = \overset{(A,B)}{\mathcal{H}}(\Gamma) \circ_{\mathrm{v}} \overset{(A,B)}{\mathcal{H}}(\Gamma')$, *whenever the compositions of* $\Gamma, \Gamma' \colon [0,1]^2 \to M$ *are well defined.*

Passing to the quotient $\mathcal{S}_2(M)$ of $C_r^2(M)$ under thin homotopy it follows, by using theorem 29 and corollary 30, that:

Theorem 38 *The map* $\overset{(A,B)}{\mathcal{H}}$ *of the previous theorem yields a morphism* $\overset{(A,B)}{\mathcal{H}} : \mathcal{S}_2(M) \to \mathcal{D}^2(\mathcal{G})$ *of double groupoids with thin structure.*

The following result is a consequence of theorem 31.

Theorem 39 (Non-abelian Fubini Theorem) *The multiplicative map* $\overset{(A,B)}{\mathcal{H}} : C^2(M) \to \mathcal{D}^2(\mathcal{G})$ *preserves the action of the dihedral group* D_4 *of the square. Concretely for any element* r *of* D_4 *we have*

$$\overset{(A,B)}{\mathcal{H}}(\Gamma \circ r^{-1}) = r(\overset{(A,B)}{\mathcal{H}}(\Gamma)),$$

for each smooth map $\Gamma \colon [0,1]^2 \to M$.

This ultimately is a consequence of the fact that $\overset{(A,B)}{\mathcal{H}}$ preserves horizontal and vertical reversions and moreover interchanges of coordinates, which generate the dihedral group $D_4 \cong \mathbb{Z}_2^2 \rtimes S_2$ of the square.

We finish this subsection with the following important theorem:

Theorem 40 *Let* (A,B) *be a local connection pair in* M, *by which as usual we mean* $A \in \mathcal{A}^1(M,\mathfrak{g})$, $B \in \mathcal{A}^2(M,\mathfrak{e})$ *and* $\partial(B) = \Omega_A = dA + \frac{1}{2} A \wedge^{\mathrm{ad}} A$. *Let* $C = dB + A \wedge^{\triangleright} B$ *be the 2-curvature 3-form of* (A,B) *as in 2.4.3 and 2.4.4. Let* $J \colon [0,1]^3 \to M$ *be a smooth map such that* $J^*(C) = 0$. *Then the colouring* T *of* D^3 *such that:*

$$T \circ \delta_i^{\pm} = \overset{(A,B)}{\mathcal{H}}(\partial_i^{\pm} J)$$

is flat; see 2.2.2 and 2.1.1.

Proof. This follows from the construction in this subsection and theorem 29. Note the form (5) for the homotopy addition equation (4). ■

4.2 1-Gauge transformations

Let M be a smooth manifold. Let (A,B) and (A',B') be local connection pairs defined in M. For the time being we will drop the index i for the open cover and take A and B to be globally defined on M. We will return to the general case in the next section. In other words $A, A' \in \mathcal{A}^1(M,\mathfrak{g})$ and $B, B' \in \mathcal{A}^2(M,\mathfrak{e})$ are such that $\partial(B) = \Omega_A = dA + \frac{1}{2} A \wedge^{\mathrm{ad}} A$ and $\partial(B') = \Omega_{A'}$. Let $\eta \in \mathcal{A}^1(M,\mathfrak{e})$ be such that:

$$A' = A + \partial(\eta)$$

and

$$B' = B + d\eta + \frac{1}{2}\eta \wedge^{\mathrm{ad}} \eta + A \wedge^{\triangleright} \eta.$$

Given a smooth path $\gamma\colon [0,1] \to M$, define the following 2-square in $\mathcal{G}$:

$$\tau_A^{(1_G,\eta)}(\gamma) \quad = \quad \begin{array}{ccc} * & \xrightarrow{\overset{A'}{g_\gamma}} & * \\ 1_G\Big\uparrow & \overset{(A,\eta)}{f_\gamma} & \Big\uparrow 1_G \\ * & \xrightarrow[\overset{A}{g_\gamma}]{} & * \end{array} \quad \doteq \quad \begin{array}{ccc} * & \xrightarrow{\overset{A'}{g_\gamma}} & * \\ 1_G\Big\uparrow & \overset{(A_\eta,B_\eta)}{e_{\gamma\times I}} & \Big\uparrow 1_G \\ * & \xrightarrow[\overset{A}{g_\gamma}]{} & * \end{array}$$

Here $A_\eta = A + z\partial(\eta) \in \mathcal{A}^1(M\times I, \mathfrak{g})$ and

$$B_\eta = B + zd\eta + \frac{1}{2}z^2\eta \wedge^{\mathrm{ad}} \eta + zA \wedge^{\triangleright} \eta + dz\wedge\eta \in \mathcal{A}^2(M\times I, \mathfrak{e}),$$

where $I = [0,1]$, with coordinate z. It is an easy calculation to prove that $\partial(B_\eta) = \Omega_{A_\eta}$. In addition, $\gamma\times I\colon [0,1]^2 \to M\times I$ is the map $(\gamma\times I)(t,s) = (\gamma(t), s)$, where $s,t\in[0,1]$. We will see below (Remark 41) that $\overset{(A,\eta)}{e_\gamma} = \overset{(A_\eta,B_\eta)}{e_{\gamma\times I}}$ depends only on A,γ and η.

Let $h\colon M\to G$ be a smooth map. It is well known (and easy to prove) that if $A'' = h^{-1}A'h + h^{-1}dh$ then

$$\tau_{A'}^h(\gamma) = \begin{array}{ccc} * & \xrightarrow{\overset{A''}{g_\gamma}} & * \\ h(\gamma(0))\Big\uparrow & 1_E & \Big\uparrow h(\gamma(1)) \\ * & \xrightarrow[\overset{A'}{g_\gamma}]{} & * \end{array}$$

is a 2-square in $\mathcal{G}$. We also define the 2-squares:

$$\tau_A^{(h,\eta)}(\gamma) \doteq \begin{array}{c} \tau_{A'}^h(\gamma) \\ \tau_A^{(1_G,\eta)}(\gamma) \end{array} = \begin{array}{ccc} * & \xrightarrow{\overset{A''}{g_\gamma}} & * \\ h(\gamma(0))\Big\uparrow & \overset{(A,\eta)}{e_\gamma} & \Big\uparrow h(\gamma(1)) \\ * & \xrightarrow[\overset{A}{g_\gamma}]{} & * \end{array}$$

and

$$\hat{\tau}_A^{(h,\eta)}(\gamma) = r_{xy}\left(\tau_A^{(h,\eta)}(\gamma)\right) = \begin{array}{ccc} * & \xrightarrow{h(\gamma(1))} & * \\ \overset{A}{g_\gamma}\Big\uparrow & \left(\overset{(A,\eta)}{e_\gamma}\right)^{-1} & \Big\uparrow \overset{A''}{g_\gamma} \\ * & \xrightarrow[h(\gamma(0))]{} & * \end{array}$$

see 2.2.1. Put also

$$B'' = h^{-1} \triangleright (B + d\eta + A\wedge^{\triangleright}\eta + \frac{1}{2}\eta\wedge^{\mathrm{ad}}\eta).$$

We then say that (A'', B'') and (A,B) are related by the 1-gauge transformation (h,η).

Remark 41 *By the non-abelian Fubini's theorem,* $\overset{(A_\eta,B_\eta)}{e_{\gamma\times I}} = \overset{(A_\eta,B_\eta)}{e_{\gamma\times I}}(1,1)$*, where* $\overset{(A_\eta,B_\eta)}{e_{\gamma\times I}}(t,z)$ *can be defined by either of the following differential equations:*

$$\frac{\partial}{\partial z}\overset{(A_\eta,B_\eta)}{e_{\gamma\times I}}(t,z) = -\overset{(A_\eta,B_\eta)}{e_{\gamma\times I}}(t,z)\int_0^t \overset{A_z}{g_\gamma}(t')\triangleright\eta\left(\frac{\partial}{\partial t'}\gamma(t')\right)dt',$$

where $A_z = A + z\partial(\eta) \in \mathcal{A}^1(M,\mathfrak{g})$*, or*

$$\frac{\partial}{\partial t}\overset{(A_\eta,B_\eta)}{e_{\gamma\times I}}(t,z) = \left(-z\overset{A}{g_\gamma}(t)\triangleright\eta\left(\frac{\partial}{\partial t}\gamma(t)\right)\right)\overset{(A_\eta,B_\eta)}{e_{\gamma\times I}}(t,z)$$

with initial conditions:

$$\overset{(A_\eta,B_\eta)}{e_{\gamma\times I}}(\xi,0) = 1_E \text{ or } \overset{(A_\eta,B_\eta)}{e_{\gamma\times I}}(0,\xi) = 1_E, \text{ where } \xi\in[0,1],$$

in the first and second case, respectively. Therefore it follows that $\overset{(A_\eta,B_\eta)}{e_{\gamma\times I}}$ *depends only on* A, η *and* γ*, thus it can be written simply as* $\overset{(A,\eta)}{e_\gamma}$.

There is another setting for the 2-cubes τ *and* $\hat{\tau}$ *introduced here, which will be needed when we return to considering local connection pairs* (A_i, B_i) *(definition 34), namely*

$$\tau_{A_i}^{(\phi_{ij},\eta_{ij})}(\gamma)$$

where γ *is a 1-path whose image is contained in* U_{ij}*. We will refer to this 2-cube as a transition 2-cube for the 1-path* γ*. Note that the relation between* A_i *and* A_j *is identical to that between* A *and* A''*, replacing* h *by* ϕ_{ij} *and* η *by* η_{ij}*.*

4.2.1 The group of 1-gauge transformations

Let M be a smooth manifold. Let also $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a Lie crossed module with associated differential crossed module $\mathfrak{G} = (\partial\colon \mathfrak{e} \to \mathfrak{g}, \triangleright)$.

The group of 1-gauge transformations in M is the group of pairs (h, η), where $h\colon M \to G$ is smooth, and η is an $\mathfrak{e}$-valued 1-form in M. The product law will be given by the semidirect product: $(h, \eta)(h', \eta') = (hh', h \triangleright \eta' + \eta)$.

Recall that a local connection pair in M is given by a pair of forms $A \in \mathcal{A}^1(M, \mathfrak{g})$ and $B \in \mathcal{A}^2(M, \mathfrak{e})$ with $\partial(B) = \Omega_A = dA + \frac{1}{2}A \wedge^{\rm ad} A$. Then defining (as above):

$$(A, B) \triangleleft (h, \eta) = \left(h^{-1}Ah + \partial(h^{-1} \triangleright \eta) + h^{-1}dh, h^{-1} \triangleright (B + d\eta + A \wedge^{\triangleright} \eta + \frac{1}{2}\eta \wedge^{\rm ad} \eta)\right)$$

defines a right action of the group of 1-gauge transformations on the set of local connection pairs.

4.2.2 The coherence law for 1-gauge transformations

The following theorem expresses how the holonomy of a local connection pair changes under the group of 1-gauge transformations. We recall the notation of 2.1.1, 2.2.2 and 4.2.1. The notion of a flat $\mathcal{G}$-colouring appears in 2.2.2.

Theorem 42 (Coherence law for 1-gauge transformations) *Let* M *be a smooth manifold with a local connection pair* (A, B)*. Let also* (h, η) *be a 1-gauge transformation, and let* $(A'', B'') = (A, B) \triangleleft (h, \eta)$*. Let* $\Gamma\colon [0, 1]^2 \to M$ *be a smooth map. Define* $T^{(h,\eta)}_{(A,B)}(\Gamma) = T^{(h,\eta)}_{(A,B)}$ *as being the* $\mathcal{G}$*-colouring of the 3-cube* D^3 *such that:*

$$T^{(h,\eta)}_{(A,B)} \circ \delta_3^- = \overset{(A,B)}{\mathcal{H}}(\Gamma), \quad T^{(h,\eta)}_{(A,B)} \circ \delta_3^+ = \overset{(A'',B'')}{\mathcal{H}}(\Gamma)$$

and

$$T^{(h,\eta)}_{(A,B)} \circ \delta_i^\pm = \tau_A^{(h,\eta)}(\partial_i^\pm \Gamma), \quad i = 1, 2.$$

(Note that the colourings of the edges of D^3 *are determined from the colourings of the faces of it, given that they coincide in their intersections.) Then* $T^{(h,\eta)}_{(A,B)}$ *is flat.*

Proof. The colouring $T^{(h,0)}_{(A',B')}(\Gamma)$ is flat by lemma 26; here $(A', B') = (A, B) \triangleleft (1_G, \eta)$. Let us prove that the colouring $T^{(1_G,\eta)}_{(A,B)}(\Gamma)$ is flat. This follows from theorems 29 or 40 and the fact that if $\mathcal{M}_\eta = dB_\eta + A_\eta \wedge^\triangleright B_\eta \in \mathcal{A}^3(M \times \{z, z \in \mathbb{R}\}, \mathfrak{e})$ is the 2-curvature 3-form of (A_η, B_η) then the contraction of $\mathcal{M}_\eta$ with the vector field $\frac{\partial}{\partial z}$ vanishes. A more intricate calculation of this type appears in the proof of Theorem 54. The theorem follows from the fact that $\mathcal{T}(\mathcal{G})$, the set of flat $\mathcal{G}$-colourings of the 3-cube D^3, is a (strict) triple groupoid (see 2.2.2) and $T^{(h,\eta)}_{(A,B)} = T^{(1_G,\eta)}_{(A,B)} \circ_3 T^{(h,0)}_{(A',B')}$, where $\circ_3$ denotes upwards composition. ■

From remark 41 it follows:

Corollary 43 *Suppose* $\Gamma\colon [0, 1]^2 \to M$ *is such that* $\Gamma(\partial[0, 1]^2)) = x$*, where* $x \in M$*. Given a local connection pair* (A, B) *in* M *and a 1-gauge transformation* (h, η) *we then have:*

$$\overset{(A,B)\triangleleft(h,\eta)}{e_\Gamma} = h^{-1}(x) \triangleright \overset{(A,B)}{e_\Gamma}.$$

By construction we have:

Corollary 44 *Given a local connection pair (A, B) in M and a 1-gauge transformation $(h, 0)$ we then have for any smooth map $\Gamma\colon [0,1]^2 \to M$:*

$$\overset{(A,B)\triangleleft(h,0)}{e_\Gamma} = h^{-1}(\Gamma(0,0)) \triangleright \overset{(A,B)}{e_\Gamma} .$$

Theorem 42 may also be interpreted in a different way to give a relation between the holonomies for a 2-path Γ with image contained in U_{ij}, using local connection pairs (A_i, B_i) and (A_j, B_j). Note that $(A_j, B_j) = (A_i, B_i) \triangleleft (\phi_{ij}, \eta_{ij})$.

Theorem 45 (Transition 3-cube for a 2-path) *Given a connection on a cubical $\mathcal{G}$-2-bundle over a pair $(M, \mathcal{U})$, let $\Gamma\colon [0,1]^2 \to M$ be a smooth 2-path with image contained in U_{ij}. Define $T_{(A_i,B_i)}^{(\phi_{ij},\eta_{ij})}(\Gamma) = T_{(A_i,B_i)}^{(\phi_{ij},\eta_{ij})}$ as being the $\mathcal{G}$-colouring of the 3-cube D^3 such that:*

$$T_{(A_i,B_i)}^{(\phi_{ij},\eta_{ij})} \circ \delta_3^- = \overset{(A_i,B_i)}{\mathcal{H}}(\Gamma), \quad T_{(A_i,B_i)}^{(\phi_{ij},\eta_{ij})} \circ \delta_3^+ = \overset{(A_j,B_j)}{\mathcal{H}}(\Gamma)$$

and

$$T_{(A_i,B_i)}^{(\phi_{ij},\eta_{ij})} \circ \delta_k^\pm = \tau_{A_i}^{(\phi_{ij},\eta_{ij})}(\partial_i^\pm \Gamma), \quad k = 1, 2.$$

Then $T_{(A_i,B_i)}^{(\phi_{ij},\eta_{ij})}$ is flat.

4.2.3 Dihedral symmetry for 1-gauge transformations

Let M be a manifold with a local connection pair (A, B) and a 1-gauge transformation (h, η). Let $\gamma\colon [0,1] \to M$ be a smooth map.

Theorem 46 *We have:*

1. $\tau_A^{(h,\eta)}(\gamma^{-1}) = \tau_A^{(h,\eta)}(\gamma)^{-\mathrm{h}}$
2. *If $(A'', B'') = (A, B) \triangleleft (h, \eta)$ then $\tau_{A''}^{(h,\eta)^{-1}}(\gamma) = \left(\tau_A^{(h,\eta)}(\gamma)\right)^{-\mathrm{v}}$.*

Recall $e^{-\mathrm{h}} = r_x(e)$ and $e^{-\mathrm{v}} = r_y(e)$, where $e \in \mathcal{D}^2(\mathcal{G})$, denote the horizontal and vertical inversions of squares in $\mathcal{G}$.

Proof. The first statement is immediate. Let $h_0 = h(\gamma(0))$, $h_1 = h(\gamma(1))$ and $\eta' = -h^{-1} \triangleright \eta$. Let also $(A', B') = (A, B) \triangleleft (0, \eta)$. The second statement follows from:

$$\begin{array}{c} \tau_{A''}^{(h,\eta)^{-1}}(\gamma) \\ \tau_A^{(h,\eta)}(\gamma) \end{array} = \begin{array}{ccc} * & \xrightarrow{\overset{A}{g_\gamma}} & * \\ {\scriptstyle h_0^{-1}}\uparrow & \overset{(A'',\eta')}{e_\gamma} & \uparrow{\scriptstyle h_1^{-1}} \\ * & \xrightarrow[\overset{A''}{g_\gamma}]{} & * \\ * & \xrightarrow{\overset{A''}{g_\gamma}} & * \\ {\scriptstyle h_0}\uparrow & \overset{(A,\eta)}{e_\gamma} & \uparrow{\scriptstyle h_1} \\ * & \xrightarrow[\overset{A}{g_\gamma}]{} & * \end{array} = \begin{array}{ccc} * & \xrightarrow{\overset{A}{g_\gamma}} & * \\ {\scriptstyle 1_G}\uparrow & h_0 \triangleright \overset{(A'',\eta')}{e_\gamma} & \uparrow{\scriptstyle 1_G} \\ * & \xrightarrow[\overset{A'}{g_\gamma}]{} & * \\ * & \xrightarrow{\overset{A'}{g_\gamma}} & * \\ {\scriptstyle 1_G}\uparrow & \overset{(A,\eta)}{e_\gamma} & \uparrow{\scriptstyle 1_G} \\ * & \xrightarrow[\overset{A}{g_\gamma}]{} & * \end{array}$$

Now note

$$h_0 \triangleright \overset{(A'',\eta')}{e_\gamma} = \overset{(A',-\eta)}{e_\gamma} = \left(\overset{(A,\eta)}{e_\gamma}\right)^{-1};$$

the last equation can be inferred for example from the first equation of remark 41. ■

4.3 Equivalence of cubical 2-bundles with connection

Let M be a smooth manifold. Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a Lie crossed module and let $\mathfrak{G} = (\partial\colon \mathfrak{e} \to \mathfrak{g}, \triangleright)$ be the associated differential crossed module. We freely use the material of section 3.

4.3.1 A crossed module of groupoids of gauge transformations

We define a groupoid $M^1_{\mathcal{G}}$, whose set of objects $M^0_{\mathcal{G}}$ is given by the set of local connection pairs (A, B) in M, in other words $A \in \mathcal{A}^1(M, \mathfrak{g})$ and $B \in \mathcal{A}^2(M, \mathfrak{e})$ are smooth forms such that $\partial(B) = \Omega_A = dA + \frac{1}{2} A \wedge^{\rm ad} A$. The set of morphisms of $M^1_{\mathcal{G}}$ is given by all quadruples of the form (A, B, ϕ, η) where A and B are as above, $\phi\colon M \to G$ is a smooth map and $\eta \in \mathcal{A}^1(M, \mathfrak{e})$ is an $\mathfrak{e}$-valued smooth 1-form in M. The source of (A, B, ϕ, η) is (A, B) and its target is $(A, B) \triangleleft (\phi, \eta)$. The composition is given by the product of 1-gauge transformations; see 4.2.1. We also define a totally intransitive groupoid $M^2_{\mathcal{G}}$, consisting of all triples of the form (A, B, ψ), where (A, B) is a local connection pair in M and ψ is a smooth map $M \to E$. The source and target of (A, B, ψ) each are given by (A, B), and we define $(A, B, \psi)(A, B, \psi') = (A, B, \psi\psi')$.

The following lemma states that this gives rise to a crossed module of groupoids, a notion defined in [BH1, BHS, B1], for example. We follow the conventions of [FMPo].

Lemma 47 *The map $\partial\colon M^2_{\mathcal{G}} \to M^1_{\mathcal{G}}$ such that $(A, B, \psi) \mapsto \big(A, B, \partial\psi, \psi(d\psi^{-1}) + \psi(A \triangleright \psi^{-1})\big)$ is a groupoid morphism, and together with the left action:*

$$(A, B, \phi, \eta) \triangleright (A', B', \psi) = (A, B, \phi \triangleright \psi),$$

where $(A', B') = (A, B) \triangleleft (\phi, \eta)$, of the groupoid $M^1_{\mathcal{G}}$ on the totally intransitive groupoid $M^2_{\mathcal{G}}$ defines a crossed module of groupoids $M_{\mathcal{G}}$.

Proof. Much of this is straightforward calculations. One complicated bit is to prove that:

$$(A, B) \triangleright \big(\partial\psi, \psi(d\psi^{-1}) + \psi(A \triangleright \psi^{-1})\big) = (A, B)$$

It is easy to see that this is true at the level of 1-forms. At the level of the 2-forms we need to prove:

$$\begin{aligned} B = (\partial\psi)^{-1} \triangleright \Big(& B + d(\psi(d\psi^{-1})) + d(\psi(A \triangleright \psi^{-1})) + A \wedge^{\triangleright} (\psi(d\psi^{-1})) + A \wedge^{\triangleright} (\psi(A \triangleright \psi^{-1})) \\ &+ \frac{(\psi(d\psi^{-1}) \wedge^{\rm ad} (\psi(d\psi^{-1})}{2} + \frac{(\psi(A \triangleright \psi^{-1})) \wedge^{\rm ad} (\psi(A \triangleright \psi^{-1}))}{2} + (\psi(d\psi^{-1}) \wedge^{\rm ad} (\psi(A \triangleright \psi^{-1}))\Big). \end{aligned} \quad (27)$$

We can eliminate two terms by using:

$$d(\psi(d\psi^{-1})) + \frac{(\psi(d\psi^{-1}) \wedge^{\rm ad} (\psi(d\psi^{-1})}{2} = 0,$$

which follows from the fact $d\theta = \frac{1}{2}\theta \wedge^{\rm ad} \theta$, where θ is the Maurer-Cartan form. By using the Leibnitz rule it follows that:

$$A \wedge^{\triangleright} (\psi(A \triangleright \psi^{-1})) + \frac{(\psi(A \triangleright \psi^{-1})) \wedge^{\rm ad} (\psi(A \triangleright \psi^{-1}))}{2} = \psi\left(\left(\frac{A \wedge^{\rm ad} A}{2}\right) \triangleright \psi^{-1}\right).$$

Also we have

$$d(\psi(A \triangleright \psi^{-1})) + A \wedge^{\triangleright} (\psi d\psi^{-1})) + \psi(d\psi^{-1}) \wedge^{\rm ad} (\psi(A \triangleright \psi^{-1})) = \psi(dA \triangleright \psi^{-1}),$$

using $\psi(A \triangleright \psi^{-1}) = -(A \triangleright \psi)\psi^{-1}$ and $(d\psi)\psi^{-1} = -\psi d\psi^{-1}$.

Putting everything together, formula (27) reduces to:

$$\begin{aligned} \phi^{-1} \triangleright \left(B + \psi\left(\left(\frac{A \wedge^{\rm ad} A}{2}\right) \triangleright \psi^{-1}\right) + \psi(dA \triangleright \psi^{-1})\right) &= \phi^{-1} \triangleright \big(B + \psi\left(\partial(B)\right) \triangleright \psi^{-1})\big) \\ &= \phi^{-1} \triangleright \big(B + \psi B \psi^{-1} - B\big) \\ &= B. \end{aligned}$$

We have used the identity $\partial(V) \triangleright e = Ve - eV$ for each $V \in \mathfrak{e}$ and for each $e \in E$. This follows from the definition of a Lie crossed module.

We now prove the other difficult condition, namely:

$$\partial((A, B, \phi, \eta) \triangleright (A', B', \psi)) = (A, B, \phi, \eta)\partial((A', B', \psi)(A', B', \phi^{-1}, -\phi^{-1} \triangleright \eta)$$

or

$$\begin{aligned}(A, B, \partial(\phi \triangleright \psi), (\phi \triangleright \psi)d(\phi \triangleright \psi)^{-1} + (\phi \triangleright \psi)A \triangleright (\phi \triangleright \psi^{-1}))\\ = (A, B, \phi\psi\phi^{-1}, \eta + (\phi \triangleright \psi)(\phi \triangleright d\psi^{-1}) + (\phi \triangleright \psi)(\phi A' \triangleright \psi^{-1}) - \phi\partial(\psi)\phi^{-1} \triangleright \eta)\end{aligned} \quad (28)$$

Now use the fact that $A' = \phi^{-1}A\phi + \phi^{-1}d\phi + \partial(\phi^{-1} \triangleright \eta)$, and the terms involving η on the right hand side cancel. ■

Definition 48 *The crossed module of groupoids $M_{\mathcal{G}}$ of the previous lemma will be called the crossed module of gauge transformations in M.*

A very similar construction appears in [SW2].

4.3.2 Equivalence of 2-bundles with connection over a pair $(M, \mathcal{U})$

Definition 49 *Given a point $x \in M$, the crossed module $M_{\mathcal{G}}(x)$ of germs of gauge transformations is constructed in the following obvious way from $M_{\mathcal{G}}$. The set of objects $M^0_{\mathcal{G}}(x)$ of $M_{\mathcal{G}}(x)$ is given by the set of all triples (A, B, U) where U is open and $x \in U$, with the equivalence relation $(A, B, U) \cong (A', B', U')$ if $A = A'$ and $B = B'$ in some open neighbourhood of x. One proceeds analogously to define the morphisms $M^1_{\mathcal{G}}(x)$ and the 2-morphisms $M^2_{\mathcal{G}}(x)$ of $M_{\mathcal{G}}(x)$.*

Theorem 50 *Let $\mathcal{U}$ be an open cover of M. A cubical $\mathcal{G}$-2-bundle with connection over $(M, \mathcal{U})$ is given by a cubical map $C(M, \mathcal{U}, x) \to \mathcal{N}(M_{\mathcal{G}}(x))$, the cubical nerve of the crossed module of groupoids $M_{\mathcal{G}}(x)$ (see 2.2.2), for each $x \in M$, such that all involved maps are smooth (a more precise statement can be made in the language of sheaves).*

Proof. Easy calculations. ■

Definition 51 *We say that two cubical 2-bundles with connection $\mathcal{B}$ and $\mathcal{B}'$ over a pair $(M, \mathcal{U})$, say $(\phi_{ij}, \psi_{ijkl}, A_i, B_i, \eta_{ij})$ and $(\phi'_{ij}, \psi'_{ijkl}, A'_i, B'_i, \eta'_{ij})$, are equivalent (and we write $\mathcal{B} \cong_{\mathcal{U}} \mathcal{B}'$) if the associated cubical maps $C(M, \mathcal{U}, x) \to \mathcal{N}(M_{\mathcal{G}}(x))$, where $x \in M$, are homotopic, through a smooth homotopy.*

Since the nerve of a crossed module is a Kan cubical set, this is an equivalence relation.

Explicitly, $\mathcal{B} \cong_{\mathcal{U}} \mathcal{B}'$ if there exist smooth maps $\Phi_i : U_i \to G$ and $\Psi_{ij} : U_{ij} \to E$, as well as smooth forms $\mathcal{E}_i \in \mathcal{A}^1(U_i, \mathfrak{e})$ such that:

1. We have

$$\partial(A_i, B_i, \Psi_{ij}^{-1})(A_i, B_i, \Phi_i, \mathcal{E}_i)(A'_i, B'_i, \phi'_{ij}, \eta'_{ij}) = (A_i, B_i, \phi_{ij}, \eta_{ij})(A_j, B_j, \Phi_j, \mathcal{E}_j),$$

   where we suppose $(A'_i, B'_i) = (A_i, B_i) \triangleleft (\Phi_i, \mathcal{E}_i)$ and $(A_j, B_j) = (A_i, B_i) \triangleleft (\phi_{ij}, \eta_{ij})$.

2. The colouring T of D^3 such that $\partial_3^-(T) = (\phi, \psi)_{ijkl}$, $\partial_3^+(T) = (\phi', \psi')_{ijkl}$ (see subsection 4.4), and

$$T_1^- = \Phi_i\!\begin{array}{c} * \xrightarrow{\phi'_{ij}} * \\ \uparrow \quad \Psi_{ij} \quad \uparrow \\ * \xrightarrow[\phi_{ij}]{} * \end{array}\!\Phi_j, \quad T_1^+ = \Phi_k\!\begin{array}{c} * \xrightarrow{\phi'_{kl}} * \\ \uparrow \quad \Psi_{kl} \quad \uparrow \\ * \xrightarrow[\phi_{kl}]{} * \end{array}\!\Phi_l, \quad T_2^- = \Phi_i\!\begin{array}{c} * \xrightarrow{\phi'_{ik}} * \\ \uparrow \quad \Psi_{ik} \quad \uparrow \\ * \xrightarrow[\phi_{ik}]{} * \end{array}\!\Phi_k, \quad T_2^+ = \Phi_j\!\begin{array}{c} * \xrightarrow{\phi'_{jl}} * \\ \uparrow \quad \Psi_{jl} \quad \uparrow \\ * \xrightarrow[\phi_{jl}]{} * \end{array}\!\Phi_l$$

   is flat for each $x \in U_{ij}$ and any i, j; see 2.2.2. We have put $T_i^{\pm} = T \circ \delta_i^{\pm} = \partial_i^{\pm}(T)$.

4.3.3 Subdivisions of covers and the equivalence of cubical 2-bundles over a manifold

Let $\mathcal{U} = \{U_i\}_{i \in \mathcal{I}}$ be an open cover of M. A subdivision $\mathcal{V}$ of $\mathcal{U}$ is a map $i \in \mathcal{I} \mapsto S_i$, where S_i is a set, together with open sets $V_{a_i} \subset U_i$, where $a_i \in S_i$ such that $U_i = \cup_{a_i \in S_i} V_{a_i}$. If we are given a cubical 2-bundle with connection $\mathcal{B}$ over $C(M, \mathcal{U})$, we immediately have another one, $\mathcal{B}_{\mathcal{V}}$ over $\mathcal{V} = \{V_{a_i}\}$, provided by the obvious cubical map $C(M, \mathcal{V}) \to C(M, \mathcal{U})$. Its structure maps are such that $\phi_{a_i b_j} = \phi_{ij}$, and analogously for all the remaining information needed to specify a cubical 2-bundle with connection. For the same reason, it is easy to see that if $\mathcal{B} \cong_{\mathcal{U}} \mathcal{B}'$ then $\mathcal{B}_{\mathcal{V}} \cong_{\mathcal{V}} \mathcal{B}'_{\mathcal{V}}$ for any subdivision $\mathcal{V}$ of $\mathcal{U}$.

If $\mathcal{U} = \{U_i\}_{i \in \mathcal{I}}$ and $\mathcal{W} = \{W_j\}_{j \in \mathcal{J}}$ are open covers of M, then $\mathcal{U} \cap \mathcal{W}$ is the open cover $\{U_i \cap W_j\}_{(i,j) \in \mathcal{I} \times \mathcal{J}}$. It is a subdivision of both $\mathcal{U}$ and $\mathcal{W}$ in the obvious way.

Definition 52 (Equivalence of cubical 2-bundles with connection) *Two cubical 2-bundles with connection $\mathcal{B}$ and $\mathcal{B}'$ over the open covers $\mathcal{U} = \{U_i\}_{i \in \mathcal{I}}$ and $\mathcal{W} = \{W_j\}_{j \in \mathcal{J}}$ of M, respectively, are called equivalent if*

$$\mathcal{B}_{\mathcal{U} \cap \mathcal{W}} \cong_{\mathcal{U} \cap \mathcal{W}} \mathcal{B}'_{\mathcal{U} \cap \mathcal{W}}$$

The following follows from the previous discussion.

Theorem 53 *Equivalence of cubical 2-bundles with connection is an equivalence relation.*

4.4 Coherence law for transition 2-cubes

Let $\mathcal{B}$ be a cubical $\mathcal{G}$-2-bundle with connection over $(M, \mathcal{U})$ (definition 34). Suppose γ is a 1-path whose image is contained in the overlap U_{ijkl}. Recall the notation in 2.2.1, 2.2.2 and subsection 4.2, in particular the notion of transition 2-cube for the path γ. Let $(\psi, \phi)_{ijkl}$ be the 2-cube (for each $x \in M$):

$$(\psi, \phi)_{ijkl} = \begin{array}{ccc} * & \xrightarrow{\phi_{kl}} & * \\ {\scriptstyle \phi_{ik}}\uparrow & \psi_{ijkl} & \uparrow{\scriptstyle \phi_{jl}} \\ * & \xrightarrow[\phi_{ij}]{} & * \end{array}$$

Theorem 54 (Coherence law for transition 2-cubes) *Let $\gamma \colon [0,1] \to U_{ijkl} \subset M$ be a smooth map. We have:*

$$\begin{array}{c} \hat{\tau}_{A_i}^{(\phi_{ik}, \eta_{ik})}(\gamma) \quad \hat{\tau}_{A_k}^{(\phi_{kl}, \eta_{kl})}(\gamma) \quad \left(\hat{\tau}_{A_j}^{(\phi_{jl}, \eta_{jl})}\right)^{-\mathrm{h}}(\gamma) \quad \left(\hat{\tau}_{A_i}^{(\phi_{ij}, \eta_{ij})}\right)^{-\mathrm{h}}(\gamma) \\ \Phi((\psi, \phi)_{ijkl}(\gamma(0))) \end{array} = \Phi'_{\substack{A_i \\ g_\gamma}}((\psi, \phi)_{ijkl}(\gamma(1))), \qquad (29)$$

and therefore the $\mathcal{G}$-colouring T of D^3 such that:

$$T \circ \delta_2^- = (\psi, \phi)_{ijkl}(\gamma(0)), \quad T \circ \delta_2^+ = (\psi, \phi)_{ijkl}(\gamma(1))$$

and

$$T \circ \delta_1^- = \tau_{A_i}^{(\phi_{ik}, \eta_{ik})}(\gamma), \qquad T \circ \delta_3^+ = \hat{\tau}_{A_k}^{(\phi_{kl}, \eta_{kl})}(\gamma),$$
$$T \circ \delta_1^+ = \tau_{A_j}^{(\phi_{jl}, \eta_{jl})}(\gamma), \qquad T \circ \delta_3^- = \hat{\tau}_{A_i}^{(\phi_{ij}, \eta_{ij})}(\gamma),$$

is flat.

Proof. By theorem 46, the left hand side $F(\gamma)$ of (29) is (we omit the γ):

$$\begin{array}{c} \hat{\tau}_{A_i}^{(\phi_{ik}, \eta_{ik})} \quad \hat{\tau}_{A_k}^{(\phi_{kl}, \eta_{kl})} \quad \hat{\tau}_{A_l}^{(\phi_{jl}, \eta_{jl})^{-1}} \quad \hat{\tau}_{A_j}^{(\phi_{ij}, \eta_{ij})^{-1}} \\ \Phi((\psi, \phi)_{ijkl}(\gamma(0)) \end{array},$$

which can also be written as:

$$\begin{bmatrix} \hat{\tau}^{(1,\eta_{ik})}_{A_i} & \hat{\tau}^{(1,\phi_{ik}\triangleright\eta_{kl})}_{\phi_{ik}\triangleright A_k} & \hat{\tau}^{(1,-\phi_{ik}\phi_{kl}\phi_{jl}^{-1}\triangleright\eta_{jl})}_{\phi_{ik}\phi_{kl}\triangleright A_l} & \hat{\tau}^{(1,-\phi_{ik}\phi_{kl}\phi_{jl}^{-1}\phi_{ij}^{-1}\triangleright\eta_{ij})}_{\phi_{ik}\phi_{kl}\phi_{jl}^{-1}\triangleright A_j} \\ & \mathrm{id} & & \end{bmatrix} \circ_{\mathrm{h}}$$

$$\begin{bmatrix} \hat{\tau}^{\phi_{ik}}_{\phi_{ik}\phi_{kl}\phi_{jl}^{-1}\phi_{ij}^{-1}\triangleright A_i} & \hat{\tau}^{\phi_{kl}}_{\phi_{kl}\phi_{jl}^{-1}\phi_{ij}^{-1}\triangleright A_i} & \hat{\tau}^{\phi_{jl}^{-1}}_{\phi_{jl}^{-1}\phi_{ij}^{-1}\triangleright A_i} & \hat{\tau}^{\phi_{ij}^{-1}}_{\phi_{ij}^{-1}\triangleright A_i} \\ & \Phi((\psi,\phi)_{ijkl}(\gamma(0))) & & \end{bmatrix}$$

Here we have put $\phi \triangleright A = A \triangleleft \phi^{-1} = \phi A \phi^{-1} + \phi d\phi^{-1}$. Let $\gamma_t \colon [0,1] \to M$ be the path $\gamma_t(t') = \gamma(t't)$, where $t, t' \in [0,1]$. Let also $F'(\gamma_t) \in E$ be the element assigned to the square $F(\gamma_t)$. We then have (by using remark 41):

$$\begin{aligned} \frac{d}{dt}F'(\gamma_t) &= F'(\gamma_t)\ \overset{A_i}{g_{\gamma_t}} \triangleright \Big(\eta_{ik} + \phi_{ik} \triangleright \eta_{kl} - \phi_{ik}\phi_{kl}\phi_{jl}^{-1} \triangleright \eta_{jl} - \phi_{ik}\phi_{kl}\phi_{jl}^{-1}\phi_{ij}^{-1} \triangleright \eta_{ij}\Big)_{\frac{d}{dt}\gamma(t)} \\ &= F'(\gamma_t)\ \overset{A_i}{g_{\gamma_t}} \triangleright \Big(\psi_{ijkl}^{-1} d\psi_{ijkl} + \psi_{ijkl}^{-1}\left(A_i \triangleright \psi_{ijkl}\right)\Big)_{\frac{d}{dt}\gamma(t)} \end{aligned}$$

On the other hand:

$$\begin{aligned} \frac{d}{dt}\Big(\overset{A_i}{g_{\gamma_t}} \triangleright \psi_{ijkl}(\gamma(t))\Big) &= \Big(\overset{A_i}{g_{\gamma_t}} A_i \triangleright \psi_{ijkl} + \overset{A_i}{g_{\gamma_t}} \triangleright d\psi_{ijkl}\Big)_{\frac{d}{dt}\gamma(t)} \\ &= \Big(\overset{A_i}{g_{\gamma_t}} \triangleright \psi_{ijkl}\Big)\Big(\overset{A_i}{g_{\gamma_t}} \triangleright \psi_{ijkl}^{-1}\Big)\Big(\overset{A_i}{g_{\gamma_t}} A_i \triangleright \psi_{ijkl} + \overset{A_i}{g_{\gamma_t}} \triangleright d\psi_{ijkl}\Big)_{\frac{d}{dt}\gamma(t)} \\ &= \Big(\overset{A_i}{g_{\gamma_t}} \triangleright \psi_{ijkl}\Big)\overset{A_i}{g_{\gamma_t}} \triangleright \Big(\psi_{ijkl}^{-1} d\psi_{ijkl} + \psi_{ijkl}^{-1}\left(A_i \triangleright \psi_{ijkl}\right)\Big)_{\frac{d}{dt}\gamma(t)}. \end{aligned}$$

This proves that $F'(\gamma_t) = \overset{A_i}{g_{\gamma_t}} \triangleright \psi_{ijkl}(\gamma(t))$, which by taking $t = 1$ finishes the proof. ■

5 Wilson spheres and tori

5.1 Holonomy for an arbitrary 2-path in a smooth manifold

We recall the notation of subsections 4.1, 4.2 and 4.4.

5.1.1 Patching together local holonomies and transition functions

Let M be a smooth manifold. Let also $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a Lie crossed module with associated differential crossed module $\mathfrak{G} = (\partial\colon \mathfrak{e} \to \mathfrak{g}, \triangleright)$. Let $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ be an open cover of M. Let $\mathcal{B}\colon C(M,\mathcal{U}) \to \mathcal{N}(\mathcal{G})$ be a cubical $\mathcal{G}$-2-bundle over $(M,\mathcal{U})$. We therefore have maps $\phi_{ij}\colon U_{ij} \to G$ and $\psi_{ijkl}\colon U_{ijkl} \to E$ satisfying the conditions of definition 32.

Fix a connection $\{A_i, B_i, \eta_{ij}\}$ on $\mathcal{B}$, definition 34. Let $\Gamma\colon [0,1]^2 \to M$ be a 2-path. We want to define the holonomy

$$\overset{\{A_i,B_i,\eta_{ij}\}}{\mathcal{H}}(\Gamma) \doteq \mathcal{H}(\Gamma) \in \mathcal{D}^2(\mathcal{G}),$$

living in the set of squares of the double groupoid $\mathcal{D}^2(\mathcal{G})$; see 2.2.1.

We can use the Lebesgue covering lemma to break the unit square $Q = [0,1]^2$ into r^2 rectangles $\{Q_{ab}\}$, where $a, b = 1, \dots, r$, so that

$$Q = \begin{matrix} Q_{1r}Q_{2r}\dots Q_{rr} \\ \dots \\ Q_{12}Q_{22}\dots Q_{r2} \\ Q_{11}Q_{21}\dots Q_{r1} \end{matrix}, \tag{30}$$

with $\Gamma(Q_{ab}) \subset U_{i_{ab}}$, where $a, b \in \{1, \dots, r\}$ and $i_{ab} \in \mathcal{I}$.

For each $a, b \in \{1, \dots r\}$, let $\Gamma_{ab}\colon [0,1]^2 \to M$ be the restriction of Γ to Q_{ab}, rescaled and reparametrized to be a 2-path $[0,1]^2 \to M$.

Remark 55 *We require the reparametrization (defined up to thin homotopy), so as to be able to use the framework of 2-paths of Definition 8. The ϵ in that definition can be made arbitrarily small by using appropriate smooth bump functions in the reparametrization. Note that, had we chosen to work with piecewise smooth n-paths, instead of n-paths with sitting instants or with a product structure close to their boundary, we would not have had to deal with this minor technical issue. Henceforth, when it is necessary to subdivide n-paths, we will assume without further comment that the necessary reparametrizations have been made, so that the subdivided n-path can be regarded as a product of smaller n-paths.*

Put $\mathcal{X}_{ab} = \partial_d(\Gamma_{ab})$, $\mathcal{Y}_{ab} = \partial_r(\Gamma_{ab})$, $\mathcal{Z}_{ab} = \partial_l(\Gamma_{ab})$ and $\mathcal{W}_{ab} = \partial_u(\Gamma_{ab})$. Let also

$$\psi^{ad,cd}_{ab,cb} = (\psi, \phi)_{i_{ab}, i_{cb}, i_{ad}, i_{cd}}(\mathcal{Y}_{ab}(1)) \in \mathcal{D}^2(\mathcal{G})$$

(see section 4.2), whenever it makes sense. Finally let

$$\tau^{cd}_{ab} = \tau^{(\phi_{i_{ab}i_{cd}}, \eta_{i_{ab}i_{cd}})}_{A_{i_{ab}}}(\mathcal{W}_{ab}) \in \mathcal{D}^2(\mathcal{G}) \text{ and } \hat{\tau}^{cd}_{ab} = \hat{\tau}^{(\phi_{i_{ab}i_{cd}}, \eta_{i_{ab}i_{cd}})}_{A_{i_{ab}}}(\mathcal{Y}_{ab}) \in \mathcal{D}^2(\mathcal{G}).$$

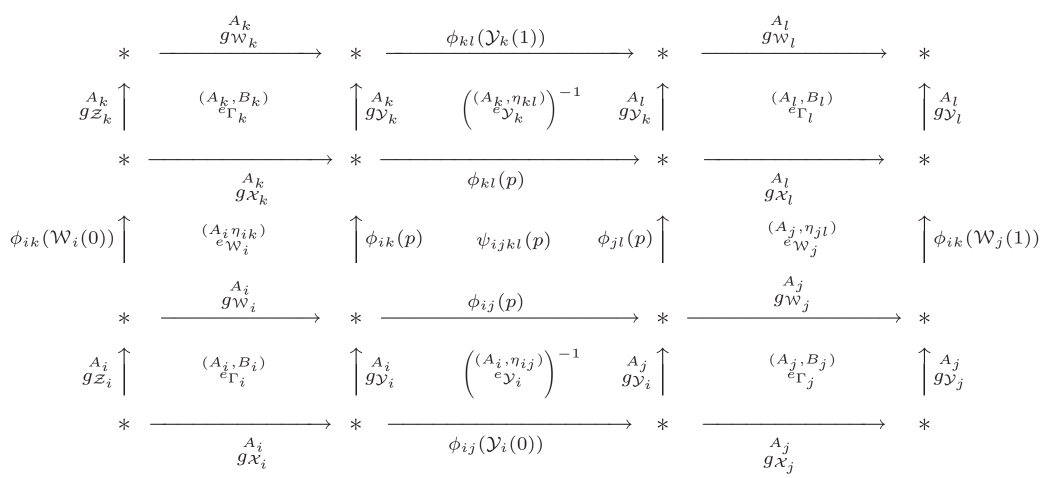
Then $\mathcal{H}(\Gamma) \in \mathcal{D}^2(\mathcal{G})$ is defined as:

$$\mathcal{H}(\Gamma) \quad = \quad \begin{matrix} \mathcal{H}(\Gamma_{1r}) & \hat{\tau}^{2r}_{1r} & \mathcal{H}(\Gamma_{2r}) & \hat{\tau}^{3r}_{2r} & \dots & \hat{\tau}^{rr}_{(r-1)r} & \mathcal{H}(\Gamma_{rr}) \\ \dots & \dots & \dots & \dots\ \dots & \dots & & \dots \\ \tau^{13}_{12} & \psi^{13,23}_{12,22} & \tau^{23}_{22} & \psi^{22,32}_{21,31} \dots & \psi^{(r-1)3,r3}_{(r-1)2,r2} & & \tau^{r3}_{r2} \\ \mathcal{H}(\Gamma_{12}) & \hat{\tau}^{22}_{12} & \mathcal{H}(\Gamma_{22}) & \hat{\tau}^{32}_{22} & \dots & \hat{\tau}^{r2}_{(r-1)2} & \mathcal{H}(\Gamma_{r2}) \\ \tau^{12}_{11} & \psi^{12,22}_{11,21} & \tau^{22}_{21} & \psi^{22,32}_{21,31} \dots & \psi^{(r-1)2,r2}_{(r-1)1,r1} & & \tau^{r2}_{r1} \\ \mathcal{H}(\Gamma_{11}) & \hat{\tau}^{21}_{11} & \mathcal{H}(\Gamma_{21}) & \hat{\tau}^{31}_{21} & \dots & \hat{\tau}^{r1}_{(r-1)1} & \mathcal{H}(\Gamma_{r1}) \end{matrix}, \qquad (31)$$

where we have put $\mathcal{H}(\Gamma_{ab}) = \overset{(A_{i_{ab}}, B_{i_{ab}})}{\mathcal{H}}(\Gamma_{ab})$. This is well defined due to the interchange law, and the associativity of the horizontal and vertical multiplications, which hold in a double-groupoid.

Let us look at the simple case when $r = 2$. Divide $Q = [0,1]^2$ as $Q = \begin{smallmatrix} Q_k Q_l \\ Q_i Q_j \end{smallmatrix}$. Set Γ_i to be the restriction of Γ to Q_i re-scaled to be a map $\Gamma_i \colon [0,1]^2 \to M$, and analogously for j, k, l. Let also $\mathcal{X}_i = \partial_d(\Gamma_i)$, $\mathcal{Y}_i = \partial_r(\Gamma_i)$, $\mathcal{Z}_i = \partial_l(\Gamma_i)$ and $\mathcal{W}_i = \partial_u(\Gamma_i)$, and analogously for j, k, l. Suppose that $\Gamma(Q_i) \subset U_i$, and analogously for j, k, l. Let p' be the central point of Q, the intersection of Q_i, Q_j, Q_k and Q_l, and $p = \Gamma(p')$. We will then have:

$$\mathcal{H}\begin{pmatrix} Q_k Q_l \\ Q_i Q_j \end{pmatrix} =$$



Note that we took the number of rows of the chosen partition of Q to coincide with its number of columns. However, we can obviously choose these to be different.

Of course, $\mathcal{H}(\Gamma)$ will a priori depend on the partition $\{Q_{ab}\}$ of Q and the open sets $\{U_{i_{ab}}\}$ chosen so that $\Gamma(Q_{ab}) \subset U_{i_{ab}}$. However, as in the case of principal bundles, this dependence is easy to control.

5.1.2 Independence under *subdividing* partitions

Suppose we subdivide one of the lines of (30). For instance put

$$Q_{1a}Q_{2a}\dots Q_{ra} = \begin{matrix} Q'_{1a}Q'_{2a}\dots Q'_{ra} \\ Q''_{1a}Q''_{2a}\dots Q''_{ra} \end{matrix}, \tag{32}$$

and take $\Gamma(Q'_{ka}), \Gamma(Q''_{ka}) \subset U_{i_{ka}}$. From conditions 3 and 4 of the definition of a cubical $\mathcal{G}$-2-bundle (definition 32) and condition 1 of the definition of a connection in a cubical $\mathcal{G}$-2-bundle (definition 34), it thus follows that:

$$\mathcal{H}\left(Q_{1a}Q_{2a}\dots Q_{ra}\right) = \mathcal{H}\begin{pmatrix} Q'_{1a}Q'_{2a}\dots Q'_{ra} \\ Q''_{1a}Q''_{2a}\dots Q''_{ra} \end{pmatrix}.$$

The same holds if we subdivide one of the columns of (30).

5.1.3 The case of paths

Let $\gamma\colon [0,1] \to M$ be a smooth path. Choose partitions $\{I_a\}_{a=1}^{r}$ and $\{I'_{a'}\}_{a'=1}^{r'}$ of $I = [0,1]$ together with elements $i_a, i'_{a'} \in \mathcal{I}$ such that $\gamma(I_a) \subset U_{i_a}$ and $\gamma(I'_{a'}) \subset U_{i'_{a'}}$. Let γ_a be the restriction of γ to I_a, rescaled to be a map $[0,1] \to M$. Let $x_a = \gamma_a(1)$.

Considering the partition $\{I_a\}$ of I, we define the holonomy $\mathcal{H}(\gamma) \in G = \mathcal{D}^1(\mathcal{G})$ in the following usual way:

$$\mathcal{H}(\gamma) = \overset{A_{i_1}}{g_{\gamma_1}}\ \phi_{i_1 i_2}(x_1)\ \overset{A_{i_2}}{g_{\gamma_2}}\ \phi_{i_2 i_3}(x_2)\dots \overset{A_{i_r}}{g_{\gamma_r}},$$

and analogously for the partition $\{I'_{a'}\}$ of I.

To compare the holonomies $\mathcal{H}(\gamma)$ and $\mathcal{H}'(\gamma)$ constructed from these two partitions of I, note that by condition 3 of the definition of a cubical 2-bundle, we can subdivide these partitions of I without affecting the value of the holonomy. Therefore we can suppose that the two partitions of I coincide, and that we took $\gamma(I_a) \subset U_{i_a}$ and $\gamma(I_a) \subset U_{i'_a}$, where i_a need not coincide with i'_a.

Theorem 56 (Coherence law for 1-holonomy) *Define:*

$$\tau(\gamma) = \tau_{i_1}^{i'_1}(\gamma_1)(\psi,\phi)_{i_1 i_2 i'_1 i'_2}(x_1)\tau_{i_2}^{i'_2}(\gamma_2)(\psi,\phi)_{i_2 i_3 i'_2 i'_3}(x_2)\dots\tau_{i_r}^{i'_r}(\gamma_r),$$

where

$$\tau_{i_a}^{i'_a} = \tau_{A_{i_a}}^{(\phi_{i_a i'_a},\eta_{i_a i'_a})} \in \mathcal{D}^2(\mathcal{G}),$$

thus

$$\partial_d \tau(\gamma) = \mathcal{H}(\gamma) \text{ and } \partial_u \tau(\gamma) = \mathcal{H}'(\gamma).$$

Then $\tau(\gamma)$ is a square in $\mathcal{G}$.

Proof. This is tautological and follows directly from the definitions; see subsections 4.2 and 4.4. Note that the set of squares in $\mathcal{G}$ forms a double groupoid, and therefore the horizontal composition of squares in $\mathcal{G}$ yields a square in $\mathcal{G}$. ■

5.1.4 The dependence of the holonomy on the chosen partition of $Q = [0,1]^2$

Continuing the notation of 5.1.1, suppose that we choose a different partition $\{Q'_{a'b'}\}_{a',b'=1}^{r'}$ of $Q' = [0,1]^2$, with the restriction of Γ to $Q'_{a'b'}$ verifying $\Gamma(Q'_{a'b'}) \subset U_{i'_{a'b'}}$ for each $a', b' \in \{1,\dots,r'\}$. We want to relate the holonomies $\mathcal{H}(\Gamma)$ and $\mathcal{H}'(\Gamma)$ obtained from these two partitions of $[0,1]^2$. Note that by using 5.1.2, we can suppose that the two partitions of $[0,1]^2$ coincide, in other words $Q_{ab} = Q'_{ab}$ for each a and b. However, i_{ab} need not coincide with i'_{ab}.

We have:

Theorem 57 (Coherence law for 2-holonomy) *Consider the colouring of T of D^3 such that:*

$$T \circ \delta_3^- = \mathcal{H}(\Gamma) \text{ and } T \circ \delta_3^+ = \mathcal{H}'(\Gamma)$$

and

$$T \circ \delta_i^{\pm} = \tau(\partial_i^{\pm}(\Gamma)), \quad i = 1,2;$$

see 5.1.3. This colouring is flat.

Proof. The proof is a three-dimensional analogue of the proof of theorem 56, making use of the fact that the set $\mathcal{T}^3(\mathcal{G})$ of flat colourings of D^3 (i.e. cubes in $\mathcal{G}$) is a strict triple groupoid; see 2.2.2. Specifically, looking at equation (31), each of its individual squares should be thickened to a cube in $\mathcal{G}$ such that the bottom face coincides with the square, and the top face with the corresponding square appearing in the calculation of $\mathcal{H}'(\Gamma)$. All these cubes can be chosen so that the lateral faces match. This follows from the transition 3-cubes for 2-paths Γ, subsection 4.2, the coherence law for transition 2-cubes for 1-paths, subsection 4.4, and condition 2 of the definition of a cubical $\mathcal{G}$-2-bundle, definition 32. ■

The following two corollaries follow immediately:

Corollary 58 *Let $\Gamma\colon [0,1]^2 \to M$ be a smooth map. Choose a partition of $\{Q_{ab}\}$ of $Q = [0,1]^2$ along with elements $i_{ab} \in \mathcal{I}$ such that $\Gamma(Q_{ab}) \subset U_{i_{ab}}$. Suppose that we choose another set of elements $i'_{ab} \in \mathcal{I}$ with $\Gamma(Q_{ab}) \subset U_{i'_{ab}}$, such that $i_{ab} = i'_{ab}$ on the boundary ∂Q of Q. Then the holonomies $\mathcal{H}(\Gamma)$ and $\mathcal{H}'(\Gamma)$ coincide.*

Proof. This follows from the previous theorem and the definition of $\tau(\gamma)$; theorem 56. We also need conditions 3 and 4 of the definition of a cubical $\mathcal{G}$-2-bundle and condition 1 of the definition of a connection in a cubical $\mathcal{G}$-2-bundle. ■

Corollary 59 *Let $\Gamma\colon Q \to M$ be a smooth map, where $Q = [0,1]^2$. Suppose $\Gamma(\partial Q) = x$, where $x \in M$. Choose some partition of Q, such that $\Gamma(Q_{ab}) \subset U_{i_{ab}}$ for each a, b. Suppose that we choose these $i_{ab} \in \mathcal{I}$ so that $i_{ab} = i_x$ on the boundary of Q; here $x \in U_{i_x}$. Then, given any other partition $\{Q'_{a'b'}\}$ of Q such that $\Gamma(Q'_{a'b'}) \subset U_{i'_{a'b'}}$ for each a', b' and, moreover, $i'_{a'b'} = i'_x$ on the boundary of Q, we have:*

$$\mathcal{H}'(\Gamma) = \left(\phi_{i_x i'_x}(x)\right)^{-1} \triangleright \mathcal{H}(\Gamma).$$

5.1.5 Invariance under (free) thin homotopy

Let M be a manifold with a local connection pair (A, B). It follows from theorem 40 that the two dimensional holonomy $\overset{(A,B)}{\mathcal{H}}(\Gamma)$, where $\Gamma\colon Q \to M$ is a smooth path ($Q = [0,1]^2$), is invariant under thin homotopy. Suppose that M is equipped with a cubical $\mathcal{G}$-2-bundle connection. Let us see how $\mathcal{H}(\Gamma)$ varies under thin homotopy. We will consider a slightly more general definition of thin homotopy (a generality that is needed to define Wilson spheres).

Definition 60 *Two smooth maps $\Gamma, \Gamma'\colon [0,1]^2 \to M$ are said to be freely thin homotopic if there exists a smooth map $L\colon [0,1]^2 \times [0,1] \to M$ such that $\operatorname{Rank}(\mathcal{D}_v L) \leq 2, \forall v \in [0,1]^3$, and so that $L_{[0,1]^2\times 0} = \Gamma$ and $L_{[0,1]^2\times 1} = \Gamma'$.*

Note that L is, in general, not a rank-2 homotopy since it lacks the conditions 1 and 2 of its definition; see 2.3.2.

Theorem 61 (Invariance under free thin homotopy) *Let $J\colon W \to M$, where $W \doteq [0,1]^3$, be a free thin homotopy connecting the smooth maps $\Gamma\colon Q \to M$ and $\Gamma'\colon Q' \to M$, where $Q, Q' = [0,1]^2$. Choose partitions $\{Q_{ab}\}$ and $\{Q'_{a'b'}\}$ of $[0,1]^2$ together with elements i_{ab} and $i'_{a'b'}$ of $\mathcal{I}$ such that $\Gamma(Q_{ab}) \subset U_{i_{ab}}$ and $\Gamma'(Q'_{a'b'}) \subset U_{i'_{a'b'}}$ for each a, b, a', b'. By subdividing if necessary (see 5.1.2) we can suppose that the two partitions coincide. In addition, we can suppose that they extend to a partition $\{W_{abc}\}_{a,b,c=1}^r$ of $W = [0,1]^3$ such that $J(Q_{abc}) \subset U_{i_{abc}}$, and, moreover, $i_{ab1} = i_{ab}$ and $i_{abr} = i'_{ab}$, for each a, b.*

The colouring T of D^3 such that

$$T \circ \delta_i^{\pm} = \mathcal{H}(\partial_i^{\pm} W); \quad i = 1, 2, 3$$

is flat.

Proof. The proof is analogous to the proof of theorem 57. The colouring T of D^3 is an upwards composition of the form $T_1 T'_{12} T_2 T'_{23} \dots T_n$, where $T'_{j(j+1)}$ are the elements of $\mathcal{T}^3(\mathcal{G})$ considered in theorem 57. On the other hand the $T_j \in \mathcal{T}^3(\mathcal{G})$ are obtained by looking at equation (31), where Γ is substituted by the j-th slice J^j of J, and thickening each individual element by using the homotopy J. In particular $\mathcal{H}(J^j_{ab})$ will be thickened to the colouring $\overset{(A_{i_{abj}}, B_{i_{abj}})}{\mathcal{H}}(\partial_v^{\pm} W_{abj})$ of D^3, which is flat by the following lemma. We can then thicken all the other elements of (31) to flat colourings of the 3-cube by using the transition 3-cubes for 2-paths Γ and the coherence law for transition 2-cubes for 1-paths, theorems 45 and 54 ■

Lemma 62 *Let M be a manifold with a local connection pair (A,B). Let $L\colon [0,1]^3 \to M$ be a map such that* $\mathrm{Rank}(\mathcal{D}_v L) \leq 2, \forall v \in [0,1]^3$. *Then the colouring T of D^3 such that*

$$T \circ \delta_i^{\pm} = \overset{(A,B)}{\mathcal{H}} (\partial_i^{\pm} L); \quad i = 1,2,3$$

is flat.

Proof. Follows directly from theorem 40. ■

The following analogue of corollary 59 holds.

Corollary 63 *Let $\Gamma\colon Q \to M$ and $\Gamma'\colon Q' \to M$ be smooth maps, where $Q, Q' = [0,1]^2$. Suppose $\Gamma(\partial Q) = x$ and $\Gamma'(\partial Q) = x'$, where $x, x' \in M$. Suppose that $J\colon Q \times [0,1] \to M$ is a free thin homotopy connecting Γ and Γ', such that $J(\partial Q \times \{t\}) = q(t)$, for some smooth map $q\colon [0,1] \to M$, thus $q(0) = x$ and $q(1) = x'$. Choose partitions $\{Q_{ab}\}$ and $\{Q'_{ab}\}$ (which we can suppose to coincide) of Q and Q' along with elements $i_{ab}, i'_{ab} \in \mathcal{J}$ (which need not coincide) such that $\Gamma(Q_{ab}) \subset U_{i_{ab}}$ and $\Gamma'(Q_{ab}) \subset U_{i'_{ab}}$ for each a, b. Suppose that we choose these $i_{ab}, i'_{ab} \in \mathcal{J}$ so that $i_{ab} = i_x$ and $i'_{ab} = i'_{x'}$ on the boundary of Q and Q'; here $x \in U_{i_x}$ and $x' \in U_{i'_{x'}}$. Then*

$$\mathcal{H}(\Gamma') = (\mathcal{H}(q))^{-1} \triangleright \mathcal{H}(\Gamma),$$

where $\mathcal{H}(q)$ is constructed in 5.1.3.

Proof. By subdividing if necessary (see 5.1.2) we can suppose that the two partitions extend to a partition $\{W_{abc}\}_{a,b,c=1}^r$ of $W = [0,1]^3$ such that $J(W_{abc}) \subset U_{i_{abc}}$, $i_{ab1} = i_{ab}$ and $i_{abr} = i'_{ab}$ and moreover, for fixed c we have $i_{abc} = j_c$ in the boundary of Q (in other words if $a \in \{1,r\}$ or $b \in \{1,r\}$.) Therefore by the previous theorem it follows that $\mathcal{H}(\Gamma') = (\mathcal{H}(q))^{-1} \triangleright \mathcal{H}(\Gamma)$, by using the obvious partition I_c of q, with $q(I_c) \subset U_{j_c}$. ■

5.1.6 Dihedral symmetry for the holonomy of general squares

Suppose that the $\mathcal{G}$-2-bundle $\mathcal{B}\colon C(M,\mathcal{U}) \to \mathcal{N}(\mathcal{G})$ is a dihedral cubical 2-bundle over $(M,\mathcal{U})$, and that it is provided with a dihedral cubical connection. Let $\Gamma\colon Q \to M$ be a smooth map. Let r be some element of the dihedral group D_4 of the square. Suppose we have a partition $\{Q_{ab}\}$ of $Q = [0,1]^2$, along with elements $i_{ab} \in \mathcal{J}$ satisfying $\Gamma(Q_{ab}) \subset U_{i_{ab}}$. Consider the obvious partition $\{Q'_{ab}\}$ of $\Gamma \circ r^{-1}$.

Theorem 64 *We have:*

$$\mathcal{H}(\Gamma \circ r^{-1}) = r(\mathcal{H}(\Gamma)).$$

Proof. This follows from theorems 39 and 46 and the definition of a dihedral cubical $\mathcal{G}$-2-bundle with a dihedral connection; definitions 33 and 35. Note that the action of r in $\mathcal{D}^2(\mathcal{G})$ is a double-groupoid morphism; see 2.2.1. ■

5.1.7 Dependence of the surface holonomy on the 2-bundle with connection equivalence class

We will now need the discussion in subsection 4.3. Suppose that $\mathcal{U}$ is an open cover of M. Let $\mathcal{V} = \{V_{t_i}\}_{i\in I, t_i \in S_i}$ be a subdivision of $\mathcal{U}$. Let $\mathcal{B}$ be a cubical $\mathcal{G}$-2-bundle with connection over $C(M,\mathcal{U})$, say $(\phi_{ij}, \psi_{ijkl}, A_i, B_i, \eta_{ij})$. Let $\Gamma\colon [0,1]^2 \to M$ be a smooth map. Let us choose a partition Q_{ab} of $Q = [0,1]^2$, together with elements $t^{ab}_{i_{ab}} \in S_{i_{ab}}$, where $i_{ab} \in \mathcal{J}$, such that $\Gamma(Q_{ab}) \subset V_{t^{ab}_{i_{ab}}}$ for each a, b. It is immediate that the value of the holonomy $H(\Gamma)$ does not depend on whether it is calculated by using $\mathcal{B}$ or $\mathcal{B}_{\mathcal{V}}$. Therefore, to analyse the dependence of 2-bundle holonomy on the equivalence class of the cubical $\mathcal{G}$-2-bundle with connection, it suffices to compare the holonomies $\mathcal{H}$ and $\mathcal{H}'$ of $\mathcal{B}$ and $\mathcal{B}'$, say $(\phi'_{ij}, \psi'_{ijkl}, A'_i, B'_i, \eta'_{ij})$, equivalent over $C(M,\mathcal{U})$, with the equivalence given by smooth maps $\Phi_i\colon U_i \to G$ and $\Psi_{ij}\colon U_{ij} \to E$, and smooth forms $\mathcal{E}_i \in \mathcal{A}^1(U_i, \mathfrak{e})$ - see subsection 4.3.2. Given a $\gamma\colon [0,1] \to M$, choose a partition $\{I_a\}_{a=1}^r$ of $I = [0,1]$, together with elements $i_a \in \mathcal{J}$, such that $\gamma(I_a) \subset U_{i_a}$. Let γ_a be the restriction of γ to I_a, rescaled to be a map $[0,1] \to M$. Let $x_a = \gamma_a(1)$.

We define the square $s(\gamma)$ in $\mathcal{G}$ in the following way:

$$s(\gamma) = \tau_{A_{i_1}}^{(\Phi_{i_1}, \mathcal{E}_{i_1})}(\gamma_1)\Psi'_{i_1 i_2}(x_1)\tau_{A_{i_2}}^{(\Phi_{i_2}, \mathcal{E}_{i_2})}(\gamma_2)\Psi'_{i_2 i_3}(x_2)\ldots$$

Here $\Psi'_{ij} = T_1^-$; see 4.3.2.

Suppose that we choose a partition $\{Q_{ab}\}$ of $Q = [0,1]^2$, with the restriction of Γ to Q_{ab} satisfying $\Gamma(Q_{ab}) \subset U_{i_{ab}}$ for each a, b. Let $\mathcal{H}(\Gamma)$ and $\mathcal{H}'(\Gamma)$ be the holonomies calculated in the equivalent 2-bundles with connection $\mathcal{B}$ and $\mathcal{B}'$ over $C(M, \mathcal{U})$. By using exactly the same argument as in the proof of theorem 57 we obtain the following:

Theorem 65 (Behaviour under $\mathcal{G}$-2-bundle equivalences) *Consider the colouring T of D^3 such that:*

$$T \circ \delta_3^- = \mathcal{H}(\Gamma) \text{ and } T \circ \delta_3^+ = \mathcal{H}'(\Gamma)$$

and

$$T \circ \delta_i^\pm = s(\partial_i^\pm(\Gamma)), \quad i = 1, 2.$$

This colouring is flat.

The following follows immediately.

Corollary 66 *Let $\Gamma\colon Q \to M$ be a smooth map, where $Q = [0,1]^2$. Suppose $\Gamma(\partial Q) = x$, where $x \in M$. Choose some partition of Q, such that $\Gamma(Q_{ab}) \subset U_{i_{ab}}$ for each a, b. Suppose that we choose these $i_{ab} \in \mathcal{I}$ so that $i_{ab} = i_x$ on the boundary of Q; here $x \in U_{i_x}$. We have*

$$\mathcal{H}'(\Gamma) = (\Phi_{i_x}(x))^{-1} \triangleright \mathcal{H}(\Gamma).$$

5.2 Associating holonomies to embedded surfaces

Let M be a smooth manifold. Let also $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a Lie crossed module with associated differential crossed module $\mathfrak{G} = (\partial\colon \mathfrak{e} \to \mathfrak{g}, \triangleright)$. Let $\mathcal{U} = \{U_i\}_{i \in \mathcal{I}}$ be an open cover of M. Let $\mathcal{B}\colon C(M, \mathcal{U}) \to \mathcal{N}(\mathcal{G})$ be a cubical $\mathcal{G}$-2-bundle over $(M, \mathcal{U})$. Fix a cubical $\mathcal{G}$-2-bundle connection $\{A_i, B_i, \eta_{ij}\}$ on $\mathcal{B}$, definition 34. Let Σ be an orientable surface embedded in M. Let us analyse how to define the holonomy $\mathcal{H}(\Sigma)$ of Σ. The construction will be a non-abelian analogue of [P].

5.2.1 The case of Wilson spheres

Suppose that $\Sigma \subset M$ is diffeomorphic to S^2. Choose an orientation of Σ. Consider an orientation preserving parametrisation $f\colon S^2 = D^2/\partial D^2 \to \Sigma \subset M$. We define the Wilson Sphere Functional as:

$$\mathcal{W}(\mathcal{B}, \Sigma) = \mathcal{H}(f) \in \ker \partial \subset E,$$

where a partition of D^2 is chosen as in corollary 59. Notice that the full form of $\mathcal{H}(f) \in \mathcal{D}^2(\mathcal{G})$ is:

$$\mathcal{H}(f) = \begin{array}{ccc} * & \xrightarrow{1_G} & * \\ {\scriptstyle 1_G}\big\uparrow & \mathcal{W}(\mathcal{B},\Sigma) & \big\uparrow{\scriptstyle 1_G} \\ * & \xrightarrow[1_G]{} & * \end{array}$$

Theorem 67 *The Wilson sphere functional $\mathcal{W}(\mathcal{B}, \Sigma)$ of an oriented embedded 2-sphere $\Sigma \subset M$ is independent of the parametrisation $f\colon S^2 \to \Sigma$ chosen, up to acting by elements of G, and is in particular independent of the chosen partition $\{Q_{ab}\}$ of $Q = [0,1]^2$ and the chosen elements $i_{ab} \in \mathcal{I}$. If the cubical $\mathcal{G}$-2-bundle and connection used are each dihedral then*

$$\mathcal{W}(\mathcal{B}, \Sigma^*) = (\mathcal{W}(\mathcal{B}, \Sigma))^{-1},$$

where Σ^* is obtained by reversing the orientation of Σ. If $\mathcal{B}$ and $\mathcal{B}'$ are equivalent $\mathcal{G}$-2-bundles with connection then $\mathcal{W}(\mathcal{B}', \Sigma)$ coincides with $\mathcal{W}(\mathcal{B}, \Sigma)$, up to acting by elements of G.

Proof. Let $f'\colon S^2 = D^2/\partial D^2 \to \Sigma \subset M$ be another orientation preserving parametrisation. Since the mapping class group of S^2 is $\{\pm 1\}$ there exists a smooth map $J\colon [0,1]^2 \times [0,1] \to \Sigma \subset M$ such that $J_{[0,1]^2 \times 0} = f$ and $J_{[0,1]^2 \times 1} = f'$, and moreover $J((\partial D^2) \times \{x\}) = h(x)$, where $x \in [0,1]$, for some smooth map $h\colon [0,1] \to M$. Certainly J is a free thin homotopy of the type appearing in corollary 63, which finishes the proof of the first statement. The second statement follows from theorem 64. The last statement follows from the discussion in 5.1.7. ■

Problem: Extend the previous theorem to immersed 2-spheres.

5.2.2 The case of tori with parametrisation

Suppose that the oriented surface $\Sigma \subset M$ is diffeomorphic to the torus $T^2 = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$. Let $f\colon D^2 \to \Sigma$ be an orientation preserving parametrisation of Σ. In other words f is the restriction to $[0,1]^2$ of an orientation preserving diffeomorphism $f'\colon \mathbb{R}^2/\mathbb{Z}^2 \to \Sigma$. Choose a partition $\{Q_{ab}\}_{a,b=1}^r$ of $Q = [0,1]^2$ as in subsection 5.1, and so that $i_{1b} = i_{rb}$ and $i_{a1} = i_{ar}$ for each a, b. Then the Wilson Torus Functional is defined as:

$$\mathcal{W}(\Sigma, f) = \mathcal{H}(f) \in \partial^{-1}(G^{(1)}) \subset E;$$

here $G^{(1)}$ is the commutator subgroup of G. Note that the full form of $\mathcal{H}(f)$ is

$$\mathcal{H}(f) = \begin{array}{ccc} * & \xrightarrow{\mathcal{H}(\partial_u f)} & * \\ {\scriptstyle \mathcal{H}(\partial_l f)}\Big\uparrow & \mathcal{W}(\Sigma,f) & \Big\uparrow{\scriptstyle \mathcal{H}(\partial_r f)} \\ * & \xrightarrow[\mathcal{H}(\partial_d f)]{} & * \end{array}$$

Since $i_{1b} = i_{rb}$, $i_{a1} = i_{ar}$ for each a, b and $\partial_u(f) = \partial_d(f)$, $\partial_l(f) = \partial_r(f)$ we can see that $\mathcal{W}(\Sigma, f) \in \partial^{-1}(G^{(1)})$.

Given that the mapping class group of the torus is $\mathrm{GL}(2, \mathbb{Z})$, theorem 67 does not hold for embedded tori. However the same argument will give:

Theorem 68 *The Wilson torus functional $\mathcal{W}(\Sigma, f)$ depends only on the isotopy type of the parametrisation $f\colon D^2 \to \Sigma$ and the equivalence class of the $\mathcal{G}$-2-bundle with connection up to changes of the form of the following simultaneous horizontal and vertical conjugation:*

$$\mathcal{H}(f) \mapsto \begin{array}{ccc} \ulcorner & e_2^{-\mathrm{v}} & \urcorner \\ e_1 & \mathcal{H}(f) & e_1^{-\mathrm{h}} \\ \llcorner & e_2 & \lrcorner \end{array},$$

where $\partial_r(e_1) = \partial_l(\mathcal{H}(f))$ and $\partial_u(e_2) = \partial_d(\mathcal{H}(f))$ and moreover $\partial_d(e_1) = \partial_l(e_2) = \partial_r(e_2) = \partial_u(e_1)$.

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