DYNAMICS NEAR THE PRODUCT OF PLANAR HETEROCLINIC ATTRACTORS

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Abstract. Motivated by problems in equivariant dynamics and connection selection in heteroclinic networks, Ashwin and Field investigated the product of planar dynamics where one at least of the factors was a planar homoclinic attractor. However, they were only able to obtain partial results in the case of a product of two planar homoclinic attractors. We give general results for the product of planar homoclinic and heteroclinic attractors. We show that the likely limit set of the basin of attraction of the product of two planar heteroclinic attractors is always the unique one-dimensional heteroclinic network which covers the heteroclinic attractors in the factors. The method we use is general and likely to apply to products of higher dimensional heteroclinic attractors as well as to situations where the product structure is broken but the cycles are preserved.

1. Introduction

One way of analyzing the dynamics of coupled dynamical systems is to first understand the dynamics of product systems (that is, uncoupled systems) and then perturb by adding coupling. In the case of two uncoupled dynamical systems, the perturbation could be to a skew product system where the base system weakly forces the second system. Motivated by problems in equivariant dynamics and coupled systems, Ashwin and Field [4] made a preliminary study of product dynamics when one of the factors was a (planar) homoclinic attractor. Strong results were obtained when the other factor was an attracting limit cycle — perhaps not so surprisingly, the main result (that the product was a minimal Milnor attractor [4, Theorem 1.2]) depended on non-trivial results from metric number theory. Strong results were also

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found when the second factor was a basic hyperbolic set. However, if both factors were planar homoclinic attractors, results were only obtained for a very restricted model [4, §6, Theorem 6.13].

In this paper we give quite general results about the *likely limit sets* of products of two-dimensional homoclinic and heteroclinic attractors. Although we restrict to products of two-dimensional systems, we believe that the methods we use have much wider applicability both to higher dimensional systems and to systems which are perturbations of a product.

Before stating our main result and giving examples, we need to recall the definition of the 'likely limit set' (details and references are given in the next section). Let M be a compact manifold, possibly with boundary, with measure ℓ which we assume is locally equivalent to Lebesgue measure. Suppose that X is a compact indecomposable attractor for the flow $\Phi_t : M \to M$ and that X has basin of attraction $\mathcal{B}(X)$ which we assume has strictly positive measure. It may be the case that ℓ-almost all points in $\mathcal{B}(X)$ are forward asymptotic to a proper subset of X. We capture this idea by defining the likely limit set of $\mathcal{B}(X)$ to be the smallest compact flow invariant subset Z of X with the property that for ℓ-almost all points $x \in \mathcal{B}(X)$, the omega limit set $\omega(x) \subset Z$.

Suppose that Φ_t is a C^1 flow on the compact surface M, possibly with boundary. A *heteroclinic network* for the flow ϕ_t consists of a closed connected 1-dimensional Φ_t-invariant subset Σ of M which is the union of a finite set $\mathcal{E}(\Sigma)$ of hyperbolic saddle points and a finite set of Φ_t-trajectories connecting equilibria in $\mathcal{E}_\Sigma$ such that the graph defined by equilibria (vertices) and trajectories (edges) is strongly connected (given two equilibria $\mathbf{p}, \mathbf{q} \in \mathcal{E}(\Sigma)$, there exists a finite chain of trajectories in Σ joining $\mathbf{p}$ to $\mathbf{q}$). The simplest example of a heteroclinic network is a homoclinic loop consisting of one equilibrium and one trajectory (see below). We say that the heteroclinic network Σ is a *heteroclinic attractor* if the basin of attraction of Σ is a neighbourhood of Σ in M (below we allow for one-sided attractors, such as a homoclinic loop).

Theorem 1.1. *Let M_1, M_2 be compact surfaces and $\Sigma_1 \subset M_1$, $\Sigma_2 \subset M_2$ be heteroclinic attractors for C^2 flows $\phi_t^i : M_i \to M_i$, $i = 1, 2$. The likely limit set of $\mathcal{B}(\Sigma_1 \times \Sigma_2)$ for the product flow $\phi_t^1 \times \phi_t^2$ is the heteroclinic network*

$$(\Sigma_1 \times \mathcal{E}(\Sigma_2)) \cup (\mathcal{E}(\Sigma_1) \times \Sigma_2) \subset M_1 \times M_2.$$

In figure 1 we show a number of examples of homoclinic and heteroclinic attractors.

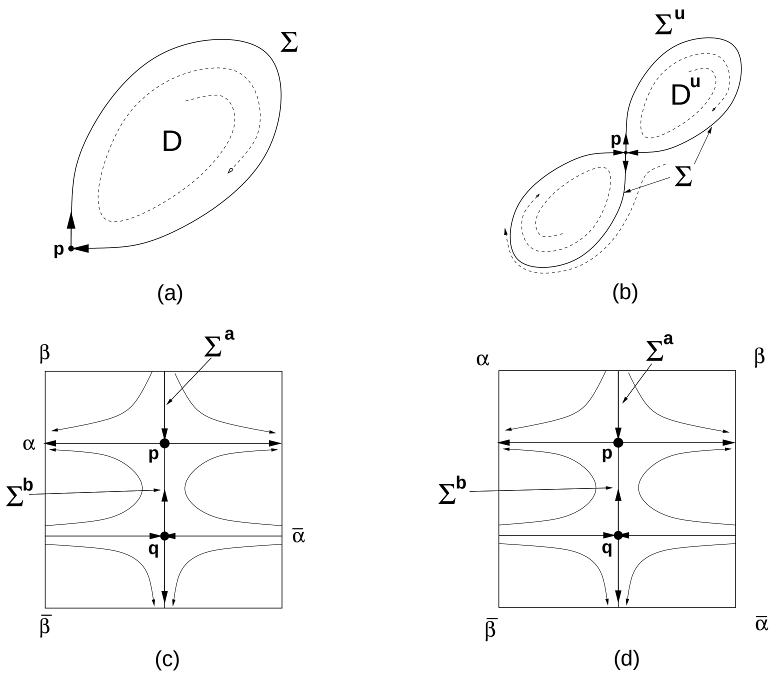

Figure 1. Homoclinic attractors on surfaces. (a) One sided homoclinic attractor, (b) figure-of-eight homoclinic attractor, (c) heteroclinic attractor on the Klein bottle, (d) heteroclinic attractor on projective space

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The attracting planar homoclinic loop $\Sigma \subset D$ shown in figure 1(a) is the simplest example of a planar homoclinic attractor. In this case the single equilibrium point $\mathbf{p}$ on the loop is a saddle point with contraction dominating the expansion. This loop is a one-sided attractor — nothing is said about the dynamics on the complement of the region D enclosed by the loop. If we take two such attracting planar homoclinic loops $\Sigma_1 \subset D_1$, $\Sigma_2 \subset D_2$, with corresponding equilibria $\mathbf{p}_1$, $\mathbf{p}_2$, then it is a consequence of results in [4] that the only possibilities for the likely limit set of $\mathcal{B}(\Sigma_1 \times \Sigma_2) \subset D_1 \times D_2$ are either $\Sigma_1 \times \Sigma_2$ or the homoclinic network $(\Sigma_1 \times \{\mathbf{p}_2\}) \cup (\{\mathbf{p}_1\} \times \Sigma_2)$. It follows from theorem 1.1 that the second case is the only possibility.

The figure of eight homoclinic attractor Σ shown in figure 1(b) is attracting on both sides provided that the single equilibrium point $\mathbf{p}$ on the loop is a saddle point with contraction dominating the expansion. We consider the product of two loops of this type or the product with a one-sided homoclinic attractor of the type shown in figure 1(a). For example, if we take the product of figure of eight loops Σ_1, Σ_2, then theorem 1.1 implies that the likely limit set of $\mathcal{B}(\Sigma_1 \times \Sigma_2)$ is always the homoclinic network $(\Sigma_1 \times \{\mathbf{p}_2\}) \cup (\{\mathbf{p}_1\} \times \Sigma_2)$. In this case, there is a richer structure present as if we look at the product of $\mathcal{B}(\Sigma_1^u)$ (see figure 1) with that part of the basin of attraction of Σ_2 exterior to the loop, then the likely limit set is $(\Sigma_1^u \times \{\mathbf{p}_2\}) \cup (\{\mathbf{p}_1\} \times \Sigma_2)$.

In figure 1(c,d), we show two simple examples of heteroclinic attractors defined on the Klein bottle (c) and 2-dimensional real projective space (d). In both figures we identify opposite edges of the square so that α is identified with $\bar{\alpha}$, β with $\bar{\beta}$. The vector field on the Klein bottle has two saddle point equilibria $\mathbf{p}, \mathbf{q}$ on the heteroclinic network $\Sigma = \Sigma^a \cup \Sigma^b$, where in figure 1(c) we have indicated the unique edge of the heteroclinic cycle Σ^b (respectively, Σ^a), which is not common to Σ^a (respectively, Σ^b).

The vector field on projective space also has two saddle point equilibria $\mathbf{p}, \mathbf{q}$ on the heteroclinic network $\Sigma = \Sigma^a \cup \Sigma^b$. In this case the complement of the network has *three* connected regions.

If we take the product of the heteroclinic attractor Σ_1 on the Klein bottle with that the heteroclinic attractor Σ_2 on projective space, then it follows from theorem 1.1 that the likely limit set of $\mathcal{B}(\Sigma_1 \times \Sigma_2)$ is the heteroclinic network $\Sigma = (\Sigma_1 \times \{\mathbf{p}_2, \mathbf{q}_2\}) \cup (\{\mathbf{p}_1, \mathbf{q}_1\} \times \Sigma_2)$.

Remarks 1.2. (1) For general attracting heteroclinic networks, we expect the presence of *essentially asymptotically stable* subnetworks (see Brannath [6] and Melbourne [18]). More precisely, if Σ is an attracting heteroclinic network associated to a finite number of equilibria with only real eigenvalues, then the connections associated to the strongest expanding eigenvalues determine a possibly smaller attractor. If the attractor is a heteroclinic cycle then it is essentially asymptotically stable (in a neighbourhood of the network, almost all orbits converge to the cycle). An explicit example of this phenomenon was given by Kirk and Silber [15]. Given an asymptotically stable heteroclinic network Σ, define the principal out-connection of a saddle to be the heteroclinic connection corresponding to the most positive expanding eigenvalue of the linearization at the node. Ashwin and Chossat [5] conjectured the existence of an essentially asymptotically stable subnetwork $\Sigma^\star \subset \Sigma$ containing the principal connections that forms part of the attractor Σ. A general proof of essential asymptotic stability may be done based on the Strong Lambda Lemma of Deng [8] but this lies beyond the scope of this paper. By contrast, products of heteroclinic attractors generally lead to heteroclinic networks without proper essentially asymptotically stable subnetworks. This is irrespective of the relative strength of eigenvalues at the equilibria. In all of the examples described above,

the likely limit sets contain no proper essentially asymptotically stable cycles.
(2) For a product of planar heteroclinic cycles, the invariant subspaces forced by the product structure are least codimension two. In the absense of proper essentially asymptotically stable cycles, it is reasonable to ask whether switching (in the sense described by Aguiar *et al.* [2] and Homburg *et al.* [12]) can occur. We address the study of this phenomenon at the end of section 3.

We describe the contents of the paper by section. In section 2, we review basic definitions and results on Milnor attractors and the likely limit set. We also establish some conventions for our subsequent analysis of planar and surface attracting homoclinic and heteroclinic cycles. In section 3, we establish notational conventions and outline the strategy of the proof of the main result for the case of the product of two planar attracting homoclinic loops. The key and new ingredient is the use of the Borel-Cantelli lemma. In section 4 we give the proof for the product of two planar attracting homoclinic loops subject only to a restriction on the connection maps — we assume they are linear. In section 5, we show how our methods easily extend to general products of heteroclinic attractors, including figure eight homoclinic cycles and attracting heteroclinic cycles. We continue to assume the linearity restriction on connection maps. In section 6, we remove the linearity assumption on connection maps. The resulting analysis is surprisingly delicate, especially in the resonant case where we assume that both heteroclinic cycles have the same asymptotic attractivity (for a product of homoclinic loops, this amounts to the ratio of the eigenvalues at the equilibria being equal). We present the details only in the case of a product of attracting homoclinic loops but the extension of our methods and results to the general case is clear. Overall these three sections illustrate how we use our argument based on the Borel-Cantelli lemma in a technically simple situation (section 4); how we extend to more general networks (section 5) and finally how we handle the arguments in the technically more demanding case when we make no simplifying assumptions on connection maps (section 6).

In sections 7 and 8, we show the results of numerical simulations of product systems as well as consider cases where we break homoclinic connections but preserve the product struction. We conclude with a brief discussion of possible generalizations and extensions of our results.

2. Preliminaries

2.1. **Milnor attractors and the likely limit set.** Let M be a differential manifold, possibly with boundary, and let ℓ denote a measure on M locally equivalent to the Lebesgue measure on charts (for example, if M is an orientable Riemannian manifold, then ℓ can be the measure defined by the Riemannian volume form). If Z is a measurable subset of M with $\ell(M) \neq 0$, we let $\mathcal{F}(Z)$ denote the set of measurable subsets Z' of Z such that $\ell(Z \smallsetminus Z') = 0$.

Suppose that $\Phi_t : M \to M$ is a C^1 flow (or semi-flow) on M. Given $x \in M$, let

$$\omega(x) = \cap_{T>0} \overline{\{\Phi_t(x) \mid t \geq T\}}$$

denote the ω-limit set of the trajectory through x.

If X is a compact invariant subset of M, we let $\mathcal{B}(X) = \{x \in M \mid \omega(x) \subset X\}$ denote the *basin of attraction* of X. We recall the definitions of a Milnor attractor and minimal Milnor attractor (for more details we refer to Milnor [19]).

Definition 2.1 (Milnor [19])**.** A compact invariant subset X of M is a *Milnor attractor* if

(1) $\ell(\mathcal{B}(X)) > 0$;
(2) for any proper compact invariant subset Y of X, $\ell(\mathcal{B}(X) \smallsetminus \mathcal{B}(Y)) > 0$.

We say X is a *minimal (Milnor) attractor* if for all proper compact invariant subsets Y of X, $\ell(\mathcal{B}(Y)) = 0$.

Remark 2.2. A Milnor attractor X is minimal iff there is a full measure subset B of $\mathcal{B}(X)$ such that $\omega(x) = X$ for all $x \in B$.

Definition 2.3 (cf. Milnor [19])**.** Let $Z \subset M$ be measurable with $\ell(Z) > 0$, and Z forward Φ_t-invariant. The *likely limit set* $\mathcal{L}(Z)$ *of* Z is the smallest closed Φ_t-invariant subset of Z that contains all ω-limit sets except for a subset of Z of zero measure. That is,

$$\mathcal{L}(Z) = \cap_{Z' \in \mathcal{F}(Z)} \overline{\{\omega(x) \mid x \in Z'\}}.$$

Remarks 2.4. (1) If Z is relatively compact, then $\mathcal{L}(Z)$ is a non-empty, compact Φ_t-invariant subset of M.
(2) The definition of the likely limit set applies to measurable subsets $Z \subset M$ with $\ell(Z) > 0$ that are not necessarily forward Φ_t-invariant: since the flow is assumed C^1, it preserves measure zero sets and from this it follows straightforwardly that $\mathcal{L}(Z) = \mathcal{L}(\cup_{t\geq 0}\Phi_t(Z))$. (Of course, $\mathcal{L}(Z)$ may be empty if $\cup_{t\geq 0}\Phi_t(Z)$ is not relatively compact.)

We recall two results from [4] about likely limit sets.

Lemma 2.5 (Ashwin & Field [4, Lemma 2.3]). *Let $Z \subset M$ be measurable with $\ell(Z) > 0$, and Z forward Φ_t-invariant.*

(1) *$x \in \mathcal{L}(Z)$ iff for all $\varepsilon > 0$ and all $Z' \in \mathcal{F}(Z)$ there exists $a \in Z'$ such that $B_\varepsilon(x) \cap \omega(a) \neq \emptyset$ ($B_\varepsilon(x)$ denotes the ε ball about x);*

(2) *$\mathcal{L}(Z)$ is a minimal Milnor attractor iff for all $x \in \mathcal{L}(Z)$ and all $\varepsilon > 0$ and all $H \subset Z$ with $\ell(H) > 0$, we have*

$$\ell(\{a \in H \mid B_\varepsilon(x) \cap \omega(a) \neq \emptyset\}) > 0.$$

Theorem 2.6 (cf. Ashwin & Field [4, Theorem 1.1]). *Let M_1, M_2 be compact manifolds, possibly with boundary and $\Phi_t = (\phi_t^1, \phi_t^2)$ be a product of C^1 flows on $M_1 \times M_2$. Suppose that $X_i \subset M_i$ are forward ϕ_t^i-invariant measurable subsets of M_i of strictly positive measure, $i = 1, 2$. Then $\mathcal{L}(X_1 \times X_2)$ is invariant under the $\mathbb{R}^2$ action defined by (ϕ_t^1, ϕ_s^2), $t, s \in \mathbb{R}^2$.*

Proof. It is no loss of generality to assume that the X_i are ϕ_t^i-invariant subsets of M_i (replace X_i by $\cup_{t\in\mathbb{R}}\phi_t^i(X_i)$). The proof given in [4] for the case $X_i = M_i$, $i = 1, 2$, extends trivially to the case where the $X_i \subset M_i$ are ϕ_t^i-invariant measurable subsets of M_i of strictly positive measure – the crucial point is that the flows ϕ_t^i preserve sets of measure zero. □

2.2. **Planar homoclinic attractors.** Suppose that we are given a C^2 semi-flow Φ_t defined on some region $D^\star$ of $\mathbb{R}^2$ containing the origin. We assume that the origin is a hyperbolic saddle with associated eigenvalues $-\mu < 0 < \lambda$ and eigen-directions the x- and y-axes of $\mathbb{R}^2$ respectively. We also assume that there is a homoclinic cycle $\Sigma \subset D^\star$ connecting the origin (see figure 2).

If $\mu > \lambda$, then it is well known that Σ is a (one-sided) attracting homoclinic cycle. More precisely, if we let $D \subset D^\star$ be the compact region in $\mathbb{R}^2$ with boundary Σ, then there exists an open neighbourhood N of Σ in D such that for all $x \in N \setminus \Sigma$, $\omega(x) = \Sigma$. We may choose N so that N is forward Φ_t-invariant with smooth interior boundary ∂N and such that the trajectories of Φ_t intersect ∂N transversally (see figure 3 and note that if $x \notin D$, then $\omega(x) \not\subset \Sigma$).

Given the setup shown in figure 3, N is a subset of the basin of attraction $\mathcal{B}(\Sigma)$ of Σ and, of course, $\mathcal{L}(N) = \mathcal{L}(\mathcal{B}(\Sigma)) = \Sigma$. In future, we always assume that a homoclinic cycle Σ of this type is the boundary of a compact region $D \subset \mathbb{R}^2$ and comes with a choice of neighbourhood N satisfying the properties described above. We denote by the quadruple (Σ, Φ_t, D, N).

Remark 2.7. Noting remarks 2.4(2), if A is an open interior neighbourhood of $(0, 0)$ (see figure 3), then $\mathcal{L}(A) = \Sigma$.

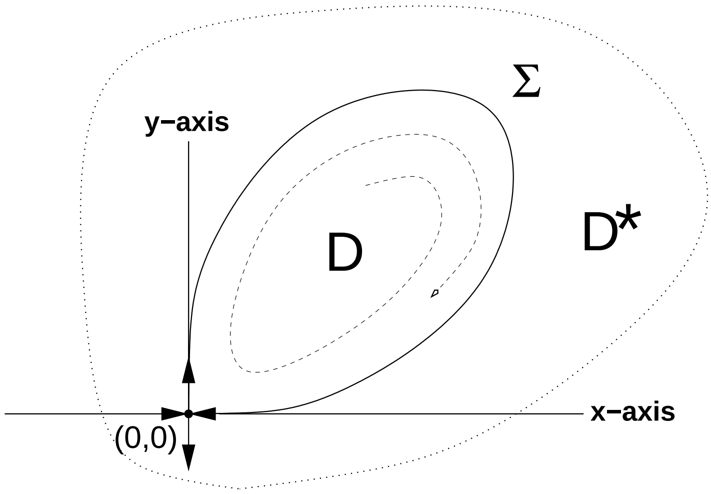

FIGURE 2. Attracting homoclinic loop in the plane. Inside the region bounded by the homoclinic cycle Σ, trajectories approach the cycle. Outside this region, trajectories eventually go away from the cycle.

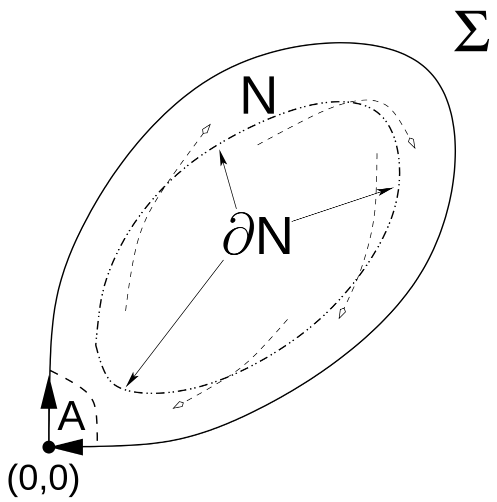

FIGURE 3. Interior neighbourhood N of Σ. Note that N is constructed in such a way that it is forward flow invariant and trajectories of Φ_t intersect ∂N transversely.

We may carry out a similar process for an attracting 'figure of eight' homoclinic cycle in the plane or, more generally an attracting planar heteroclinic cycle such as the Guckenheimer-Holmes cycle [11], [10,

§5.2] or the heteroclinic cycles on the Klein bottle and projective space shown in figure 1.

3. Products of planar attracting homoclinic loops

For the next two sections we assume that $(\Sigma_i, \phi^i_t, D_i, N_i)$, $i = 1, 2$, are planar attracting homoclinic loops ('homoclinic attractors'). Recall that we assume $\partial D_i = \Sigma_i$, ϕ^t_i is a C^2 flow on D_i, and $N_i \subset \mathcal{B}(\Sigma_i)$ is a forward ϕ^t_i-invariant open interior neighbourhood of Σ_i. We assume that both homoclinic attractors have a saddle point at $(0, 0)$, and corresponding eigenvalues $-\mu_i < 0 < \lambda_i$, where $\rho_i = \mu_i/\lambda_i > 1$, $i = 1, 2$. Set $\Sigma = (\Sigma_1 \times \{(0, 0)\}) \cup (\{0, 0\} \times \Sigma_2) \subset D_1 \times D_2 \subset \mathbb{R}^4$ and let $\mathbf{0} = (0, 0, 0, 0) \in \Sigma$ denote the unique equilibrium for the product flow in $N_1 \times N_2$. Let $\Phi_t = (\phi^1_t, \phi^2_t)$ denote the product flow on $D_1 \times D_2$.

Using theorem 2.6, it was shown in Ashwin & Field [4, Theorem 6.1] that either $\mathcal{L}(N_1 \times N_2) = \Sigma$ or $\mathcal{L}(N_1 \times N_2) = \Sigma_1 \times \Sigma_2$. Our aim is to prove that we always have $\mathcal{L}(N_1 \times N_2) = \Sigma$. For the remainder of this section, we outline the strategy that we use for the proof as well as establish notational conventions.

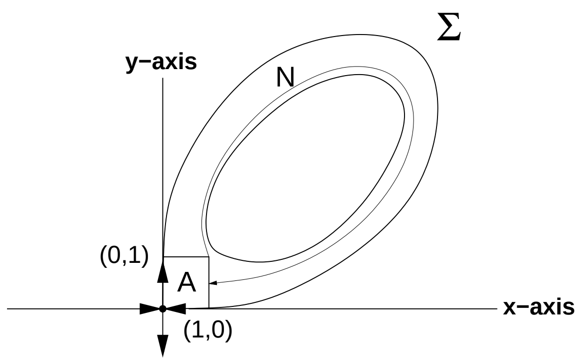

Figure 4. Linearizing coordinates at the origin.

Since $(0, 0)$ is a hyperbolic saddle point of a C^2 *planar* flow, we can always choose a C^1-linearization of ϕ^i_t on some closed neighbourhood A_i of $(0, 0)$, $i = 1, 2$ (see Samovol [21] and note that C^2 regularity of the flow will play a key role in section 6). Linearly rescaling coordinates, we may assume that $A_i = [0, 1] \times [0, 1] \subset N$. Let $A^\circ_i = [0, 1) \times [0, 1)$ denote the interior of A_i in D_i. We may choose the A_i so that the forward ϕ^i_t-trajectory through $(1, 1)$ meets $\{1\} \times [0, 1] \subset \partial A_i$ after one circuit of the loop (see figure 4).

Given $i \in \{1,2\}$, let $U_i = \{1\} \times [0,1]$, $V_i = [0,1] \times \{1\} \subset \partial A_i$ denote the vertical and horizontal interior boundaries of A_i (see figure 5).

We adopt the convention that if $y \in [0,1] \approx U_i$, then $\phi_t^i(y)$ is defined to be $\phi_t^i(1,y)$. We similarly identify a point $(x,1) \in V_i$ with $x \in [0,1]$.

We have a C^1 time of first return map $\xi_i : V_i \to \mathbb{R}$ and associated C^1 connection map $C_i : V_i \to U_i$ defined by $C_i(x) = \phi^i_{\xi_i(x)}(x)$, $x \in V_i$.

Let $\tau_i^+ = \sup_{x\in V_i} \xi_i(x)$, $\tau_i^- = \inf_{x \in V_i} \xi_i(x)$. Obviously, $\tau_i^+ \geq \tau_i^- > 0$. Since the time it takes for a trajectory starting at $y \in U_i$ to exit A_i though V_i grows without bound as $y \to 0^+$, it is easy to verify that every trajectory in $N_1 \times N_2$ passes through $A_1 \times A_2$. From this it follows that $\mathcal{L}(A_1 \times A_2) = \mathcal{L}(N_1 \times N_2)$.

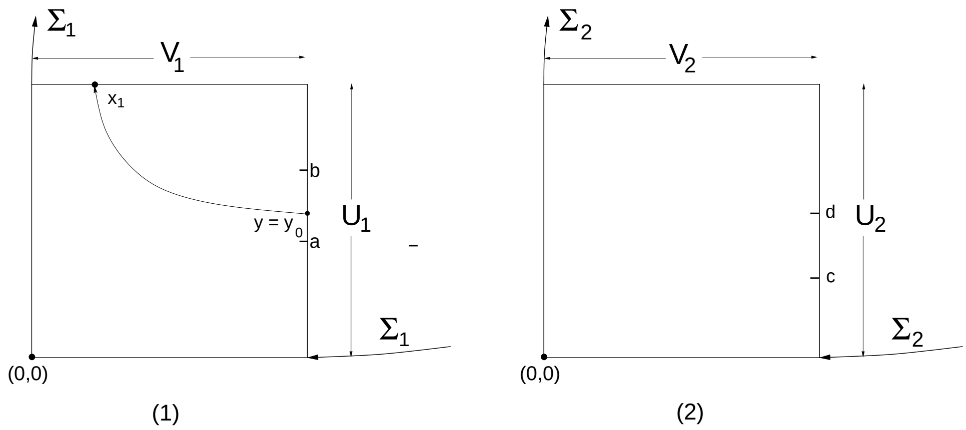

FIGURE 5. Notation and setup for the sets A_1, A_2: all trajectories with initial conditions on $U_i \setminus \{(1,1)\}$ (respectively $V_i \setminus \{(1,1)\}$), are inside A_i for small strictly positive (respectively, negative) time.

Given $b \in U_1$, $b > 0$, let $a < b$ denote the first point of intersection of the forward ϕ_t^1-trajectory through b with U_1. Similarly, given $d \in U_2$, $d > 0$, define $c < d$ to be the first point of intersection of the forward ϕ_t^2-trajectory through d with U_2. (See figure 5.)

Given $y \in [a,b]$, let $0 < s_1^y(1) < s_1^y(2) < \cdots$ denote the sequence of positive times t at which $\phi_t^1(y) \in V_1$ and $0 = t_1^y(0) < t_1^y(1) < t_1^y(2) < \cdots$ denote the sequence of positive times t at which $\phi_t^1(y) \in U_1$. Observe that $t_1^y(n) - s_1^y(n) \in [\tau_1^-, \tau_1^+]$.

Let $(x_n^1) \subset V_1$ be the decreasing sequence of points of intersection of the forward trajectory through $y \in [a,b]$ with V_1 and (y_n^1) be the corresponding sequence of points of intersection with U_1. We have

$$\begin{aligned}
y_n^1 &= \phi^1_{\xi_1(x_n^1)}(x_n^1), \\
x_n^1 &= \phi^1_{s_1^y(n)}(y) = \phi^1_{s_1^y(n)-s_1^y(1)}(x_1^1), \\
y_n^1 &= \phi^1_{t_1^y(n)}(y) = \phi^1_{s_1^y(n)+\xi_1(x_n^1)}(y), \\
s_1^y(n+1) - t_1^y(n) &= \lambda_1^{-1}\log(y_n^1)^{-1}, \\
x_{n+1}^1 &= (y_n^1)^{\mu_1/\lambda_1} = (y_n^1)^{\rho_1},
\end{aligned}$$

where the last two statements use the linearity of the flow ϕ_t^1 on A_1 and hold for $n \geq 0$. Similar statements hold for ϕ_t^2.

Definition 3.1. Given $n \in \mathbb{N}$, $y \in [a,b]$, we say $t > 0$ is *n-singular* for A_1 if $t \in [s_1^y(n), t_1^y(n)]$. If $t \geq s_1^y(n)$, we say the trajectory $\phi_t^1(y)$ has made at least n-turns about the homoclinic cycle Σ_1.

Remarks 3.2. (1) If t is n-singular for A_1, then $\phi_t^1(y) \notin A_1^\circ$.
(2) The definition of making at least n-terms about Σ_1 implicitly depends on A_1.

Let $\mathcal{C} \subset \Sigma_1 \times \Sigma_2$ consist of all pairs $(p,q) \in \Sigma_1 \times \Sigma_2$ such that $p, q \notin [0,1) \times \{0\} \cup \{0\} \times [0,1)$. If we let $\kappa_i \subset \Sigma_i$ be the closed arcs defined by $\kappa_i = \Sigma_i \setminus A_i^\circ$, then $\mathcal{C} = \kappa_1 \times \kappa_2$.

Definition 3.3. Given A_1, A_2 as defined above, a point $(X,Y) \in A_1 \times A_2$ is a *bad point* if $\omega(X,Y) \cap \mathcal{C}^\circ \neq \emptyset$ ($\mathcal{C}^\circ$ denotes the interior of $\mathcal{C}$ in $\Sigma_1 \times \Sigma_2$).

Remark 3.4. We shall show that the set of bad points in $A_1 \times A_2$ has measure zero. It then follows immediately from theorem 2.6 that $\mathcal{L}(A_1 \times A_2) = \Sigma$.

Let $E = [a,b] \times [c,d] \subset U_1 \times U_2$. For $n \in \mathbb{N}$, define $E_n \subset E$ by

$$E_n = \{(y,z) \in E \mid \exists t \in [s_1^y(n), t_1^y(n)] \text{ such that } \phi_t^2(z) \notin A_2^\circ\}.$$

We refer to points of E_n as the *n-bad points of E*. Observe that $(y,z) \in E_n$ if and only if there exists $t > 0$, $m \in \mathbb{N}$ such that t is n-singular for A_1 and m-singular for A_2. In particular, $\phi_t^2(z) \notin A_2^\circ$. If, for $m \in \mathbb{N}$, we define

$$E_{n,m} = \{(y,z) \in E \mid [s_1^y(n), t_1^y(n)] \cap [s_2^z(m), t_2^z(m)] \neq \emptyset\},$$

then $E_n = \cup_{m \geq 1} E_{n,m}$.

Lemma 3.5. *If $(y,z) \in E$, then a point $(p,q) \in \mathcal{C}^\circ \cap \omega(y,z)$ only if there exists an infinite increasing sequence $n_1 < n_2 < \cdots$ ... such that $(y,z) \in \cap_{j \geq 1} E_{n_j}$. Conversely, if there exists an infinite increasing* *sequence* $n_1 < n_2 < \cdots ...$ *such that* $(y,z) \in \cap_{j\geq 1} E_{n_j}$*, then* $(p,q) \in \mathcal{C} \cap \omega(y,z)$.

Proof. If there exists $N \in \mathbb{N}$ such that $(y,z) \notin E_n$, $n \geq N$, then all the limit points of the Φ_t-trajectory through (y,z) must lie in $(A_1 \times \Sigma_2) \cup (\Sigma_1 \times A_2)$ which is disjoint from $\mathcal{C}^\circ$. Conversely, suppose $(y,z) \in \cap_{j\geq 1} E_{n_j}$. Using the compactness of $\Sigma_1 \times \Sigma_2$, we can pick a subsequence (m_j) of (n_j) and sequence $t_{m_1} < t_{m_2} < \cdots$ with $t_{m_j} \in [s_1^y(m_j), t_1^y(m_j)]$ and $\phi^2_{t_{m_j}}(z) \notin A_2^\circ$, such that $(\Phi_{t_{m_j}}(y,z))$ converges to a point in $\mathcal{C}$. □

Applying the Borel-Cantelli lemma. Let E_∞ denote the subset of E consisting of points (y,z) such that there exists an infinite increasing sequence (n_j) such that $(y,z) \in \cap_{j\geq 1} E_{n_j}$. We have $E_\infty = \cap_{N\geq 1} \cup_{n \geq N} E_n$. It follows from lemma 3.5 that the bad points of E are a subset of E_∞. Take Lebesgue measure ℓ_2 on E. Our main work will be to show $\sum_{n=1}^\infty \ell_2(E_n) < \infty$. It then follows from the Borel-Cantelli lemma that $\ell_2(E_\infty) = 0$ and so there is a full measure subset E' of E such that $\omega(y,z) \cap \mathcal{C}^\circ = \emptyset$ for all $(y,z) \in E'$.

The proof we give for the convergence of $\sum_{n=1}^\infty \ell_2(E_n)$ applies to all products $E = [a,b] \times [c,d] \subset (0,1] \times (0,1] \subset U_1 \times U_2$, where a, c are the points of first return. Since $(0,1] \times (0,1]$ can written as a countable union of such products, it follows by the σ-additivity of ℓ_2 that the set of bad points in $U_1 \times U_2$ has zero measure.

Furthermore, we are able to show that our arguments do not depend on the particular choice of linearizing neighbourhood $A_1 \times A_2$: we can replace $E = (\{1\} \times [a,b]) \times (\{1\} \times [c,d])$ by any product of vertical intervals from $A_1 \times A_2$. That is, suppose $x, x', b, d \in (0,1]$, and define corresponding intervals $I_x = \{x\} \times [a,b] \subset A_1$, $I_{x'} = \{x'\} \times [c,d] \subset A_2$ such that a is the first point of intersection of the forward ϕ^1_t-orbit through (x,b) with $\{x\} \times [0,1]$, and similarly for c. Set $E = I_x \times I_{x'}$. Now linearly rescale the horizontal coordinates to obtain a new box linearizing neighbourhood $A_1^\star \times A_2^\star \subset A_1 \times A_2$ such that in the new coordinates $A_1^\star, A_2^\star = [0,1] \times [0,1]$ and $I_x = [a,b] \subset \{1\} \times (0,1]$, $I_{x'} = [c,d] \subset \{1\} \times (0,1]$ (note that the vertical coordinates are unchanged). By the original argument, it follows there is a full measure subset E' of E such that $\omega(y,z) \cap \mathcal{C}^\circ = \emptyset$ for all $(y,z) \in E'$ ($\mathcal{C}$ is expanded when we use the smaller box $A_1^\star \times A_2^\star$ so we can safely take the *original* $\mathcal{C}$ for this argument). Using the argument of the previous paragraph, we deduce that the bad points are a measure zero subset of $(\{x\} \times (0,1]) \times (\{x'\} \times [0,1]) \subset A_1 \times A_2$ for all $x, x' \in (0,1]$. Since the set of bad points in $A_1 \times A_2$ is easily seen to be measurable, it follows by Fubini's theorem that the set of bad points form a measure zero subset of $A_1 \times A_2$. Therefore there is a full-measure subset A' of $A_1 \times A_2$ for which $\omega(y,z) \cap \mathcal{C}^\circ = \emptyset$ for all $(y,z) \in A'$ proving that $\mathcal{L}(A_1 \times A_2) \neq \Sigma_1 \times \Sigma_2$. Hence, $\mathcal{L}(N_1 \times N_2) = \Sigma$.

Remark 3.6. The final step uses the $\mathbb{R}^2$-invariance of the likely limit set (theorem 2.6). However, we could have avoided this by observing that our arguments show the existence of a full-measure subset A' of $A_1 \times A_2$ for which $\omega(y,z) \cap \mathcal{C}^\circ = \emptyset$ for all $(y,z) \in A'$. Since this holds for a base of box neighbourhoods $A_1 \times A_2$ of $\mathbf{0}$, we deduce that $\mathcal{L}(N_1 \times N_2) \subset \Sigma$. Of course, for product dynamics we can infer equality. However, in other situations where the product structure is broken but the attracting homoclinic loops persist, the inclusion $\mathcal{L}(N_1 \times N_2) \subset \Sigma$ may be strict (this is of particular interest for products of planar 'figure of eight' attracting homoclinic loops.)

4. Products of homoclinic loops: proof of a special case

In this section, we present the proof for products of planar attracting homoclinic loops under a simplifying assumption on the connection maps $C_i : V_i \to U_i$. We assume that for both flows ϕ^i_t, there exist strictly positive constants τ_1, τ_2, m_1, m_2 such that

$$\xi_i(x) = \tau_i,\ C_i(x) = m_i x,\ x \in U_i. \tag{4.1}$$

We give the proof for general connection maps in section 6.

Remarks 4.1. (1) Since we are assuming $A_i = [0,1] \times [0,1] \subset N_i$, we always have $m_i < 1$, $i = 1, 2$ (else N_i would contain a limit cycle).
(2)

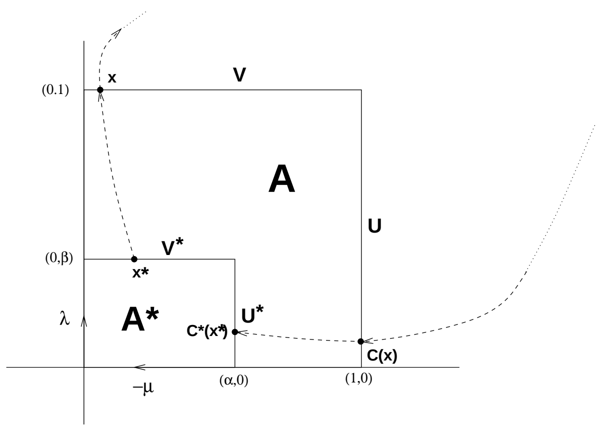

Figure 6. Rescaling coordinates and the connection map.

Observe that (4.1) continues to hold, though generally with different constants τ_1, τ_2, m_1, m_2, if we replace A_1, A_2 by smaller rectangular neighbourhoods $A_1^\star, A_2^\star$ (relative to the same choice of linearizing coordinates). Referring to figure 6, suppose we linearly rescale coordinates on $A^\star$ so that, in the rescaled coordinates, $A^\star = [0,1] \times [0,1]$ (we omit the subscript i). We may choose unique $T, S \geq 0$ such that $\alpha = e^{-\mu S}$, $\beta = e^{-\lambda T}$. In coordinates $(x^\star, y^\star)$ on $A^\star = [0,1] \times [0,1]$, the connection map $C^\star : V^\star \to U^\star$ is given by

$$C^\star(x^\star) = e^{(T+S)\lambda} C(e^{-\mu(T+S)} x^\star) \tag{4.2}$$

In particular, if $C(x) = mx$, then in rescaled coordinates, we have

$$C^\star(x^\star) = e^{(T+S)(\lambda-\mu)} m x^\star.$$

Since $\lambda - \mu < 0$, $e^{(T+S)(\lambda-\mu)} m < m$, if $T + S > 0$.

As in the previous section, we choose $E = [a,b] \times [c,d] \subset U_1 \times U_2$.

Lemma 4.2. *Let $(y,z) \in E$, $n \in \mathbb{N}$. We have*

1. $t_1^y(n) = s_1^y(n) + \tau_1$ *and* $t_2^z(n) = s_2^z(n) + \tau_2$.
2. $t_1^y(n) = n\tau_1 - \frac{1}{\lambda_1}(\alpha_n \log y + \pi_n \log m_1)$, *where* $\alpha_n = \frac{\rho_1^n - 1}{\rho_1 - 1}$, *and* $\pi_n = \frac{\rho_1^n - n\rho_1 + (n-1)}{(\rho_1 - 1)^2}$.
3. $t_2^z(n) = n\tau_2 - \frac{1}{\lambda_2}(\beta_n \log z + \theta_n \log m_2)$, *where* $\beta_n = \frac{\rho_2^n - 1}{\rho_2 - 1}$, *and* $\theta_n = \frac{\rho_2^n - n\rho_2 + (n-1)}{(\rho_2 - 1)^2}$.

Proof. (1) Immediate from (4.1) and the definitions of $t_1^y(n)$, $s_1^y(n)$.
(2,3) We prove (2), the proof of (3) is identical. Let $y_0 = y, y_1, \cdots$ and $x_1, x_2, \cdots$ denote the successive points of intersection of the forward trajectory through y with U_1 and V_1 respectively. We have $s_1^y(n) - t_1^y(n) = -\frac{1}{\lambda_1} \log y_n$, $x_n = y_n^{\rho_1}$, and $y_{n+1} = m_1 x_n = m_1 y_n^{\rho_1}$. Substituting and summing the finite sums gives the result. □

We will need the estimate on the ratios π_n/α_n, θ_n/β_n given by the next lemma.

Lemma 4.3. *(Notation as above.) For all $n \in \mathbb{N}$ we have*

$$0 \leq \frac{\pi_n}{\alpha_n} \leq \frac{1}{\rho_1 - 1}, \quad 0 \leq \frac{\theta_n}{\beta_n} \leq \frac{1}{\rho_2 - 1}.$$

Proof. Computing we find that

$$\frac{\pi_n}{\alpha_n} = \frac{\rho_1^n - n\rho_1 + (n-1)}{(\rho_1 - 1)(\rho_1^n - 1)} = \frac{1}{\rho_1 - 1}\left(1 - \frac{n}{\sum_{j=1}^{n-1} \rho_1^j}\right).$$

Since $\rho_1^j \geq 1$, $j \geq 0$, the result follows. □

Definition 4.4. Given $n, m \in \mathbb{N}$ and $y \in [a, b]$, we define the closed subinterval $[Z^1_m(y, n), Z^2_m(y, n)]$ of $[c, d]$ by

$$[Z^1_m(y, n), Z^2_m(y, n)] = \{z \in [c, d] \mid (y, z) \in E_{n,m}\}.$$

We refer to $[Z^1_m(y, n), Z^2_m(y, n)]$ as a *bad subinterval.*

Lemma 4.5. *If $n, m \in \mathbb{N}$, $y \in [a, b]$ and $[Z^1_m(y, n), Z^2_m(y, n)] \neq \emptyset$, then $Z^1_m(y, n)$ is given as the maximum of c and the unique solution of $s^z_2(m) = s^y_1(n) + \tau_1$ and $Z^2_m(y, n)$ as the minimum of d and the unique solution of $s^z_2(m) = s^y_1(n) - \tau_2$.*

Proof. The result follows by noting that $s^z_2(m)$ is a decreasing function of $z \in [c, d]$ and that the endpoints of the bad interval are given as the intersection of $[c, d]$ with the interval with endpoints determined by the equations $s^z_2(m) = s^y_1(n) - \tau_2, s^y_1(n) + \tau_1$. □

Proposition 4.6. *Every bad subinterval $[Z^1_m(y, n), Z^2_m(y, n)]$ of $[c, d]$ is contained in the interval $[\overline{Z}^1_m(y, n), \overline{Z}^2_m(y, n)]$ where,*

$$\begin{aligned} \overline{Z}^2_m(y, n) &= e^{\frac{\lambda_2}{\beta_m}(m\tau_2-(n-1)\tau_1)} y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m}} m_1^{\frac{\lambda_2\pi_n}{\lambda_1\theta_m}} m_2^{-\frac{\theta_m}{\beta_m}} \\ \overline{Z}^1_m(y, n) &= e^{\frac{\lambda_2}{\beta_m}((m-1)\tau_2-n\tau_1)} y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m}} m_1^{\frac{\lambda_2\pi_n}{\lambda_1\theta_m}} m_2^{-\frac{\theta_m}{\beta_m}} < \overline{Z}^2_m(y, n) \end{aligned}$$

Proof. By lemma 4.5, $[Z^1_m(y, n), Z^2_m(y, n)]$ is contained in an interval with end points given by the solution of $s^z_2(m) = s^y_1(n) + \tau_1, s^y_1(n) - \tau_2$. Solving for z, using lemma 4.2, gives the result. □

Remark 4.7. We have $\overline{Z}^1_m(y, n) = e^{\frac{-\lambda_2}{\beta_m}(\tau_1+\tau_2)}\overline{Z}^2_m(y, n)$.

Let $y \in [a, b]$, $n \in \mathbb{N}$ and $t \in [s^y_1(n), s^y_1(n) + 1]$. The number of turns of $\phi^2_t(z)$ is an increasing function of $z \in [c, d]$. In particular, the maximum number of turns is made when $z = d$. If we let $M_E(n, y)$ denote the number of turns taken by d around Σ_2 in time $s^y_1(n) + 1$, we have

$$M_E(n, y) = \min\{m \mid s^d_2(m + 1) > s^y_1(n) + 1\}, \tag{4.3}$$

Define $M_E(n) = \sup_{y\in[a,b]} M(n, y)$.

Lemma 4.8. *(Notation and assumptions as above.) If $n \in \mathbb{N}$, we have*

1. *$M_E(n) < \infty$.*
2. *If $t > 0$ is n-singular for A_1, then the number of turns of $\phi^2_t(z)$ about Σ_2 is at most $M_E(n)$.*

4.1. **The ℓ_2 measure of E_∞.** We start by estimating $\ell_2(E_n)$.

Lemma 4.9. *There exists $C > 0$ such that for all $n \in \mathbb{N}$,*

$$\ell_2(E_n) \leq C \sum_{m=1}^{\infty} \frac{1}{\lambda_1\beta_m + \lambda_2\alpha_n} e^{\frac{\lambda_2}{\beta_m}(\tau_2 m - \tau_1(n-1))}.$$

Proof. Let $n \in \mathbb{N}$. We have

$$\begin{aligned}
\ell_2(E_n) &\leq \int_a^b \sum_{m=1}^{M_E(n)} (\overline{Z}_m^2(y,n) - \overline{Z}_m^1(y,n))\, dy \\
&= \int_a^b \sum_{m=1}^{M_E(n)} e^{\frac{\lambda_2}{\beta_m}(\tau_2 m - \tau_1(n-1))}(1 - e^{\frac{-\lambda_2}{\beta_m}(\tau_1+\tau_2)}) y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m}} m_1^{\frac{\lambda_2\pi_n}{\lambda_1\theta_m}} m_2^{-\frac{\theta_m}{\beta_m}}\, dy, \\
&= \sum_{m=1}^{M_E(n)} e^{\frac{\lambda_2}{\beta_m}(\tau_2 m - \tau_1(n-1))}(1 - e^{\frac{-\lambda_2}{\beta_m}(\tau_1+\tau_2)}) m_1^{\frac{\lambda_2\pi_n}{\lambda_1\theta_m}} m_2^{-\frac{\theta_m}{\beta_m}} \left(\int_a^b y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m}}\, dy\right).
\end{aligned}$$

By lemma 4.3, there exists $C_1 = C_1(m_2) > 0$ such that $m_2^{-\frac{\theta_m}{\beta_m}} \leq C_1$, for all $m \in \mathbb{N}$. Since $m_1 < 1$, we have $m_1^{\frac{\lambda_2\pi_n}{\lambda_1\theta_m}} \leq 1$, for all $m, n \in \mathbb{N}$. We also have

$$\int_a^b y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m}}\, dy \leq \int_0^1 y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m}}\, dy = \frac{\lambda_1\beta_m}{\lambda_1\beta_m + \lambda_2\alpha_n}.$$

Hence

$$\begin{aligned}
\ell_2(E_n) &\leq C_1 \sum_{m=1}^{M_E(n)} \frac{\lambda_1\beta_m}{\lambda_1\beta_m + \lambda_2\alpha_n} e^{\frac{\lambda_2}{\beta_m}(\tau_2 m - \tau_1(n-1))}(1 - e^{\frac{-\lambda_2}{\beta_m}(\tau_1+\tau_2)}) \\
&\leq C_1 \sum_{m=1}^{M_E(n)} \frac{\lambda_1\lambda_2(\tau_1+\tau_2)}{\lambda_1\beta_m + \lambda_2\alpha_n} e^{\frac{\lambda_2}{\beta_m}(\tau_2 m - \tau_1(n-1))},
\end{aligned}$$

where we have used $(1 - e^{\frac{-\lambda_2}{\beta_m}(\tau_1+\tau_2)}) \leq (\tau_1+\tau_2)\frac{\lambda_2}{\beta_m}$. Hence we have the estimate

$$\ell_2(E_n) \leq C \sum_{m=1}^{\infty} \frac{1}{\lambda_1\beta_m + \lambda_2\alpha_n} e^{\frac{\lambda_2}{\beta_m}(\tau_2 m - \tau_1(n-1))}.$$

where $C = \lambda_1\lambda_2 C_1(\tau_1+\tau_2)$. ∎

Lemma 4.10. $\ell_2(E_\infty) = 0$.

Proof. We start by proving that $\sum_{n=1}^{\infty} \ell_2(E_n) < \infty$. By lemma 4.9, it suffices to prove that $\sum_{n,m=1}^{\infty} \frac{1}{\lambda_1\beta_m+\lambda_2\alpha_n} e^{\frac{\lambda_2}{\beta_m}(\tau_2 m-\tau_1(n-1))}$ converges. By the arithmetic-geometric mean inequality we have the estimate

$$\frac{1}{\lambda_1\beta_m+\lambda_2\alpha_n} \leq 1/(2\sqrt{\lambda_1\lambda_2}\sqrt{\beta_m\alpha_n}),\ m,n \geq 1.$$

It follows easily from the definition of α_n, β_m that there exists $C > 0$ such that

$$\frac{1}{\lambda_1\beta_m+\lambda_2\alpha_n} \leq C\rho_1^{-n/2}\rho_1^{-m/2}, m,n \geq 1. \tag{4.4}$$

Observe that if $\tau_2 m - \tau_1(n-1) \leq 0$, then $e^{\frac{\lambda_2}{\beta_m}(\tau_2 m-\tau_1(n-1))} \leq 1$. On the other hand if $\tau_2 m - \tau_1(n-1) > 0$, then $(\tau_2 m - \tau_1(n-1))/\beta_m \leq \tau_2$, since $\beta_m \geq m$, and so $e^{\frac{\lambda_2}{\beta_m}(\tau_2 m-\tau_1(n-1))} \leq e^{\tau_2\lambda_2}$. Hence we have a uniform bound on $e^{\frac{\lambda_2}{\beta_m}(\tau_2 m-\tau_1(n-1))}$ and the convergence of the double sum follows from (4.4). Since $\sum_{n=1}^{\infty} \ell_2(E_n) < \infty$, it follows by the Borel-Cantelli lemma [14, Theorem 13.1], that $\ell_2(E_\infty) = 0$. □

Theorem 4.11. *The likely limit set of the product of two planar homoclinic attractors (Σ_1, ϕ_t^1) and (Σ_2, ϕ_t^2) is Σ.*

Proof. Our argument for the vanishing of $\ell_2(E_\infty)$ did not depend on our choices of $b, d \in [0,1]$. We now follow the arguments given at the end of section 3: by σ-additivity of ℓ_2, the set of bad points in $(\{1\} \times [0,1]) \times (\{1\} \times [0,1])$ has measure zero. Using the rescaling argument given in section 3, we deduce that the set of bad points in $(\{x\} \times [0,1]) \times (\{x'\} \times [0,1])$ has measure zero for all $x, x' \in (0,1]$. It follows by Fubini's theorem that the set of bad points in $A_1 \times A_2$ has measure zero. Hence $\mathcal{L}(A_1 \times A_2) = \mathcal{L}(N_1 \times N_2) \neq \Sigma_1 \times \Sigma_2$. and the result follows by theorem 2.6. □

Remark 4.12. Neither of the cycles in the network $\Sigma \subset \mathbb{R}^4$ is essentially asymptotically stable: for all points (not lying on the stable manifolds), the associated trajectory makes infinitely many traversals arbitrarily close to both homoclinic cycles, even when $\rho_1 = \rho_2$.

4.2. **Switching.** In the remainder of the section, we briefly indicate why we cannot expect switching in the network Σ.

Following the notation of Homburg *et al.* [12], let $\gamma_1 = \Sigma_1 \times \{(0,0)\}$ and $\gamma_2 = \{(0,0)\} \times \Sigma_2$ denote the homoclinic orbits which define the heteroclinic network Σ. Let U_Σ be a tubular neighbourhood of Σ and $U_{\mathbf{0}} \subset U_\Sigma$ be a small neighbourhood of $\mathbf{0}$. Let S_1 and S_2 be two mutually disjoint cross sections transverse to γ_1 and γ_2 respectively. We suppose $S_1, S_2 \subset U_\Sigma \setminus U_{\mathbf{0}}$.

Let $\kappa = (k_i) \in \{1,2\}^{\mathbb{N}}$ be a symbolic sequence. We say that the trajectory with initial condition $\overline{x}$ is a forward realization of κ if the forward trajectory of $\overline{x}$ is contained in U_Σ and there exists an increasing sequence of times $(t_i)_{i\in\mathbb{Z}^+}$, with $t_0 = 0$, such that:

- $\phi^1_{t_i} \times \phi^2_{t_i}(\overline{x}) \in S_{k_i}, i \in \mathbb{N}$;
- $\phi^1_{t_i} \times \phi^2_{t_i}(\overline{x}) \notin S_1 \cup S_2$, all $t \in (t_i, t_{i+1})$, $i \in \mathbb{N}$;
- for $t \in (t_i, t_{i+1})$, the trajectory visits $U_{\mathbf{0}}$ exactly once, $i \in \mathbb{N}$.

In other words, a realization of κ is a trajectory that, after an initial transient, follows the homoclinic connections γ_{k_i} in the order prescribed by κ. Our next definition is based on that of Aguiar *et al.* [2] adapted to our context.

Definition 4.13. The product of two homoclinic cycles is *switching* if for each symbolic sequence $\kappa \in \{1,2\}^{\mathbb{N}}$, there exists a forward realization of κ in U_Σ.

Remarks 4.14. (1) We say that the product of two homoclinic cycles is *finite switching* if the previous set-up holds for finite sequences κ (instead of infinite sequences).
(2) The notion of switching can be extended to a product of any two heteroclinic attractors (the idea is similar but due to complexity in notation, we restrict to the case of homoclinic cycles).

Proposition 4.15. *The network in $\Sigma \subset \mathbb{R}^4$ is not switching or finite switching.*

Proof. In order to show that $\Sigma \subset \mathbb{R}^4$ is not switching, it suffices to prove that Σ is not finite switching. Without loss of generality, suppose that $U_{\mathbf{0}} \subset A_1 \times A_2$ and $K \subset U_{\mathbf{0}}$ is compact and disjoint from Σ. Given $\overline{x} = (y,z) \in U_{\mathbf{0}}$, suppose that $t > 0$ is n-singular for A_1 (that is, the ϕ^1_t trajectory through y). The number of turns of $\phi^2_t(z)$ about Σ_2 is at most $M_K(n) < \infty$, where we may choose $M_K(n)$ independent of $\overline{x} \in K$, as in lemma 4.8. All we have to do now is choose a finite sequence where the proportion of 2's grows at a rate faster than $M(n)/n$. For example, if we assume $\rho_1 \leq \rho_2$, then it is easy to show that there exists $N \in \mathbb{N}$, such that $M_K(n) < 2n$, for $n \geq N$. It follows that if we define the finite block κ by concatenating (1222) sufficiently many times, then κ has no realization. $\square$.

5. Products of heteroclinic attractors

In this section, we study the product of two heteroclinic attractors, both contained in a compact surface. These attractors may be homoclinic cycles, *figure eight* homoclinic cycles, heteroclinic cycles or heteroclinic networks. We only provide provide detailed computations for the cases where one heteroclinic attractor is a homoclinic loop and the other is either a heteroclinic cycle with two equilibria or a figure eight homoclinic cycle. The general case is proved along very similar lines (though with more notation).

As we did in the previous section, we assume a simplifying condition on the connection maps (which we remove in section 6).

Let M_1, M_2 denote compact surfaces (possibly with boundary). Suppose that $\Sigma_1 \subset M_1$, $\Sigma_2 \subset M_2$ are heteroclinic attractors contained and that Σ_1, Σ_2 connect saddle point sets $\mathcal{E}_1, \mathcal{E}_2$ respectively. Set $\Sigma = (\Sigma_1 \times \mathcal{E}_2) \cup (\mathcal{E}_1 \times \Sigma_2)$.

Theorem 5.1. *(Notation and assumptions as above.) The likely limit set of $\mathcal{B}(\Sigma_1 \times \Sigma_2)$ is the heteroclinic network Σ.*

In order to prove this result it suffices to take the product of any pair of heteroclinic cycles $\Sigma_1^\star \subset \Sigma_1$, $\Sigma_2^\star \subset \Sigma_2$ and show that the likely limit set of $\mathcal{B}(\Sigma_1^\star \times \Sigma_2^\star)$ is the heteroclinic network $\Sigma^\star = (\Sigma_1^\star \times \mathcal{E}_2^\star) \cup (\mathcal{E}_1^\star \times \Sigma_2^\star)$, where $\mathcal{E}_i^\star \subset \Sigma_i$ is the set of saddle points on Σ_i, $i = 1, 2$. Note that we allow for the basin of attraction to be an interior or exterior neighbourhood of the cycle and as well as the cycle being a single homoclinic loop or a figure eight homoclinic cycle. We remark that if $\Sigma^\star$ is a subset of the connected surface M, then $M \setminus \Sigma^\star$ has either two or three connected components. (The complement of the figure eight homoclinic cycle, figure 1(b), has three connected components as do the sub-cycles of the network shown in figure 1(d). If we allow the underlying manifold to have as its boundary the cycle — allowing corners — then the complement can have one component.)

As indicated above, we only give detailed arguments for the case when $\Sigma_1^\star$ is a planar homoclinic loop and $\Sigma_2^\star$ is either a planar heteroclinic cycle with two equilibria or a planar figure eight homoclinic cycle. Our analysis covers all of the issues that arise in the general case.

5.1. **Product of a homoclinic loop and heteroclinic cycle.** Let $(\Sigma_1, \phi_t^1, D_1, N_1)$ be a planar attracting homoclinic loop and follow all the notational conventions used in the previous two sections. In particular, N_1 will be an interior neighbourhood of Σ_1 and Σ_1 will have a hyperbolic saddle point $\{(0,0)\} \in \mathbb{R}^2$. We assume that $(\Sigma_2, \phi_t^2, D_2, N_2)$ is a planar attracting heteroclinic cycle with hyperbolic equilibria $\mathbf{p}_1, \mathbf{p}_2$ – see figure 7. As we have drawn this, N_2 will be an interior neighbourhood of Σ_2. Denote the connections between $\mathbf{p}_1$ and $\mathbf{p}_2$ by Σ_{21}, Σ_{22} so that $\Sigma_2 = \Sigma_{21} \cup \Sigma_{22}$.

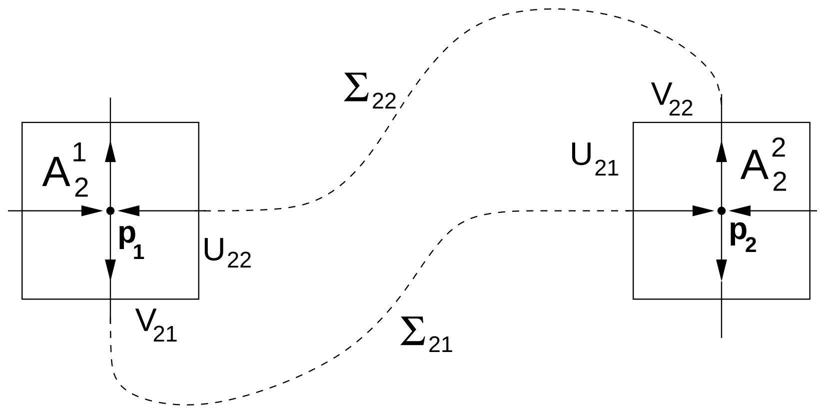

FIGURE 7. Notational conventions for heteroclinic attractor Σ_2

Suppose the hyperbolic saddle at $\mathbf{p_j}$ has eigenvalues $-\mu_2^j < 0 < \lambda_2^j$, and set $\eta_1 = \mu_2^1/\lambda_2^1$, $\eta_2 = \mu_2^2/\lambda_2^2$. We assume that $\rho_2 = \eta_1\eta_2 > 1$ so that Σ_2 is attracting. Since $\mathbf{p}_1, \mathbf{p}_2$ are hyperbolic saddle points for a C^2 planar flow, we may C^1-linearize the flow ϕ_t^2 on box neighbourhoods A_2^1 of $\mathbf{p}_1$ and A_2^2 of $\mathbf{p}_2$. Set $A_2 = A_2^1 \cup A_2^2$ and assume that $A_2 \subset N_2$. We assume coordinates on A_2^1, A_2^2 are chosen so that equilibria correspond to the origin $(0,0) \in \mathbb{R}^2$ and the stable manifold at $(0,0)$ is tangent to the x-axis, the unstable manifold to the y-axis and $A_2^j = [-1,1] \times [-1,1]$, $j = 1,2$. Define

$$U_{21} = \{-1\} \times [0,1]\,, \quad V_{21} = [0,1] \times \{-1\},$$
$$U_{22} = \{1\} \times [-1,0]\,, \quad V_{22} = [-1,0] \times \{1\}.$$

(See figure 7.) We have C^1 time of first return maps $\xi_{21} : V_{21} \to \mathbb{R}$, $\xi_{22} : V_{22} \to \mathbb{R}$ and associated C^1 connection maps $C_{21} : V_{21} \to U_{21}$, $C_{22} : V_{22} \to U_{22}$. For this section we assume there are strictly positive constants $\tau_{21}, \tau_{22}, m_{21}, m_{22}$ such that

$$\xi_{21}(x) = \tau_{21},\ \xi_{22}(x) = \tau_{22},\ C_{21}(x) = m_{21}x,\ C_{22}(x) = m_{22}x. \tag{5.5}$$

Since $A_2^2, A_2^1 \subset N_2$, we may after linearly rescaling one coordinate direction if necessary, always assume that $m_{21}, m_{22} < 1$. In particular, the connection maps C_{21}, C_{22} are defined on all of V_{21}, V_{22}. Let $\mathbf{0} = (0,0,0,0) \in \Sigma$ denote the unique equilibrium for the product flow in $N_1 \times N_2$.

Remark 5.2. If we are given a finite set of planar linear flows ϕ_t^j, each with a hyperbolic saddle point at the origin, box neighbourhoods $A^j \approx [-1,1] \times [-1,1]$, C^1 connection maps $C_j : V_j \to U_j$, and C^1 time of first return maps $\xi_j : V_j \to \mathbb{R}$, $j = 1, \cdots, r$, where each product $V_j \times U_j$ determines a quadrant of A^j, then this data determines a heteroclinic cycle Σ between r equilibria. In general, Σ will be defined on a surface, possibly non-orientable. We have chosen the particular configuration shown in figure 7 because it extends naturally to the exterior of a figure eight attracting homoclinic cycle.

Given $d \in U_{22} \approx [-1, 0]$, let $c > d$ denote the first point of intersection of the forward ϕ_t^2-trajectory through d with U_{22}. Necessarily $[d, c] \subset U_{22}$. Given $z \in [d, c]$, we define strictly monotone increasing sequences $(t_2^{z,2}(n))_{n\geq 0}$, $(s_2^{z,1}(n))_{n\geq 1}$, $(t_2^{z,1}(n))_{n\geq 1}$, $(s_2^{z,2}(n))_{n\geq 1}$ by

(1) $t_2^{z,2}(0) = 0$ and for $n > 0$, $t_2^{z,j}(n)$ is the time to the nth intersection of the forward ϕ_t^2 trajectory through z with U_{2j}, $j = 1, 2$.
(2) $s_2^{z,j}(n)$ is the time to the nth intersection of the forward ϕ_t^2 trajectory through z with V_{2j}, $j = 1, 2$.

We have

$$0 = t_2^{z,2}(0) < s_2^{z,1}(1) < t_2^{z,1}(1) < s_2^{z,2}(1) < t_2^{z,2}(1) < s_2^{z,1}(2) < \cdots$$

Set $z = z^2(0)$ and for $n \geq 1$, let $z^2(n)$ denote the successive points of intersection of the forward trajectory through z with U_{22} ($z^2(n) = \phi_{t_2^{z,2}(n)}(z)$). We similarly let $z^1(n)$ denote the nth point of intersection of the forward trajectory through z with U_{21} ($z^1(n) = \phi_{t_2^{z,1}(n)}(z)$). A straightforward computation shows that

$$\begin{aligned} z^1(n) &= m_{21}^{(1+\cdots+\rho_2^{n-1})} m_{22}^{\eta_1(1+\cdots+\rho_2^{n-2})} z^{\eta_1 \rho_2^{n-1}}, \\ z^2(n) &= m_{21}^{\eta_2(1+\cdots+\rho_2^{n-1})} m_{22}^{(1+\cdots+\rho_2^{n-1})} z^{\rho_2^n}. \end{aligned}$$

As we did in the previous section, it is now straightforward to compute the sequences $(s_2^{z,j}(n))$, $(t_2^{z,j}(n))$, $j = 1, 2$.

Lemma 5.3. *For $n \geq 1$ we have*

$$t_2^{z,1}(n) = s_2^{z,1}(n) + \tau_{21}.$$

$$t_2^{z,2}(n) = s_2^{z,2}(n) + \tau_{22}.$$

$$\begin{aligned} t_2^{z,1}(n) &= n\tau_{21} + (n-1)\tau_{22} - \frac{1}{\lambda_2^1}\left(\sum_{j=0}^{n-1} \log z^2(j)\right) - \frac{1}{\lambda_2^2}\left(\sum_{j=1}^{n-1} \log z^1(j)\right). \\ t_2^{z,2}(n) &= n(\tau_{21} + \tau_{22}) - \frac{1}{\lambda_2^1}\left(\sum_{j=0}^{n-1} \log z^2(j)\right) - \frac{1}{\lambda_2^2}\left(\sum_{j=1}^{n} \log z^1(j)\right). \end{aligned}$$

For $\rho > 1$, $n \geq 0$, define

$$\begin{aligned}\pi_n(\rho) &= \frac{\rho^n - n\rho + n - 1}{(\rho-1)^2},\\ \alpha_n(\rho) &= \frac{\rho^n - 1}{\rho - 1}.\end{aligned}$$

We have the following expressions for the summations in lemma 5.3.

$$\begin{aligned}\sum_{j=0}^{n-1} \log z^2(j) &= \eta_2\pi_n(\rho_2)\log m_{21} + \pi_n(\rho_2)\log m_{22} + \alpha_n(\rho_2)\log z\\ \sum_{j=1}^{n-1} \log z^1(j) &= \pi_n(\rho_2)\log m_{21} + \eta_1\pi_{n-1}(\rho_2)\log m_{22} + \eta_1\alpha_{n-1}(\rho_2)\log z\end{aligned}$$

Definition 5.4. We say $t > 0$ is *m-singular* for Σ_2 if

$$t \in [s_2^{z,1}(m), t_2^{z,1}(m)] \cup [s_2^{z,2}(m), t_2^{z,2}(m)].$$

Note that if t is m-singular for Σ_2, then $\phi_t^2(z) \notin A_2^\circ$.

Set $E = [a,b] \times [d,c] \subset U_1 \times U_{22}$ (the interval $[a,b]$ is as defined in the previous section). For $n, m \in \mathbb{N}$, $j = 1,2$, define

$$\begin{aligned}E^j_{m,n} &= \{(y,z) \in E \mid \exists t \in [s_1^y(n), t_1^y(n)] \cap [s_2^{z,j}(n), t_2^{z,j}(n)\},\\ E^j_n &= \cup_{m=1}^\infty E^j_{m,n},\\ E_n &= E_n^1 \cup E_n^2.\end{aligned}$$

A point $(y,z) \in E_n$ if there exists $t > 0$ such that t is n-singular for Σ_1 and there exists $m \in \mathbb{N}$ such that t is m-singular for Σ_2. We refer to the points of E_n as the n-bad points of E.

Fix $y \in [a,b]$. For $m,n \in \mathbb{N}$, $j = 1,2$, define (possibly empty) closed subintervals $[Z^1_{j,m}(y,n), Z^2_{j,m}(y,n)]$ of $[d,c]$ by

$$[Z^1_{j,m}(y,n), Z^2_{j,m}(y,n)] = \{z \in [d,c] \mid (y,z) \in E^j_{m,n}\}.$$

We refer to $[Z^1_{j,m}(y,n), Z^2_{j,m}(y,n)]$ as a bad subinterval.

Proposition 5.5. *Let $y \in [a,b]$. Given $m,n \in \mathbb{N}$ and $j = 1,2$, we have $[Z^1_{j,m}(y,n), Z^2_{j,m}(y,n)] \subset [\overline{Z}^1_{j,m}(y,n), \overline{Z}^2_{j,m}(y,n)]$, where*

$$\begin{aligned}\overline{Z}^2_{1,m}(y,n) &= e^{\frac{1}{u_m}(-(n-1)\tau_1 + (m-1)(\tau_{21}+\tau_{22})+\tau_{21})} y^{\frac{\alpha_n}{u_m\lambda_1}} m_1^{\frac{\pi_n}{u_m\lambda_1}} m_{21}^{-\frac{a_m}{u_m}} m_{22}^{-\frac{b_m}{u_m}},\\ \overline{Z}^2_{2,m}(y,n) &= e^{\frac{1}{v_m}(-(n-1)\tau_1 + m(\tau_{21}+\tau_{22}))} y^{\frac{\alpha_n}{v_m\lambda_1}} m_1^{\frac{\pi_n}{v_m\lambda_1}} m_{21}^{-\frac{c_m}{v_m}} m_{22}^{-\frac{d_m}{v_m}},\\ \overline{Z}^1_{1,m}(y,n) &= e^{-\frac{1}{u_m}(\tau_1+\tau_{21})}\overline{Z}^2_{1,m}(y,n),\\ \overline{Z}^1_{2,m}(y,n) &= e^{-\frac{1}{v_m}(\tau_1+\tau_{22})}\overline{Z}^2_{2,m}(y,n),\end{aligned}$$

and

$$\begin{aligned} u_m &= \frac{\alpha_m(\rho_2)}{\lambda_2^1} + \frac{\eta_1\alpha_{m-1}(\rho_2)}{\lambda_2^2}, & v_m &= \frac{\alpha_m(\rho_2)}{\lambda_2^1} + \frac{\eta_1\alpha_m(\rho_2)}{\lambda_2^2}, \\ a_m &= \frac{\eta_2\pi_m(\rho_2)}{\lambda_2^1} + \frac{\pi_m(\rho_2)}{\lambda_2^2}, & b_m &= \frac{\pi_m(\rho_2)}{\lambda_2^1} + \frac{\eta_1\pi_{m-1}(\rho_2)}{\lambda_2^2}, \\ c_m &= \frac{\eta_2\pi_m(\rho_2)}{\lambda_2^1} + \frac{\pi_{m+1}(\rho_2)}{\lambda_2^2}, & d_m &= \frac{\pi_m(\rho_2)}{\lambda_2^1} + \frac{\eta_1\pi_m(\rho_2)}{\lambda_2^2}. \end{aligned}$$

Proof. For $j = 1, 2$, the interval $[\overline{Z}^1_{j,m}(y,n), \overline{Z}^2_{j,m}(y,n)]$ is contained in an interval with end points given by the solution of $s_2^{z,j}(m) = s_1^y(n) + \tau_1, s_1^y(n) - \tau_{2j}$. Solving for z, using lemma 5.3, gives the result. □

Lemma 5.6. *If we define*

$$\begin{aligned} S_1 &= \sum_{m=1}^{\infty} \frac{1}{u_m\lambda_1 + \alpha_n} e^{\frac{1}{u_m}(-(n-1)\tau_1 + (m-1)(\tau_{21}+\tau_{22}) + \tau_{21})}, \\ S_2 &= \sum_{m=1}^{\infty} \frac{1}{v_m\lambda_1 + \alpha_n} e^{\frac{1}{v_m}(-(n-1)\tau_1 + m(\tau_{21}+\tau_{22}))}, \end{aligned}$$

then there exists $C > 0$ such that for all $n \in \mathbb{N}$,

$$\ell_2(E_n) \le C\left(S_1 + S_2\right).$$

Proof. As in lemma 4.9, we have

$$\ell_2(E_n) = \ell_1(E_n^1) + \ell_2(E_n^2) \le \sum_{j=1}^{2} \int_a^b \sum_{m=1}^{\infty} \left(\overline{Z}^2_{j,m}(y,n) - \overline{Z}^1_{j,m}(y,n)\right).$$

By lemma 4.3, we have $0 \le \frac{a_m}{u_m}, \frac{b_m}{u_m} \le K_1\frac{\pi_m(\rho_2)}{\alpha_m(\rho_2)} \le K_1\frac{1}{\rho_2 - 1}$, $0 \le \frac{c_m}{v_m}, \frac{d_m}{v_m} \le K_2\frac{\pi_m(\rho_2)}{\alpha_m(\rho_2)} \le K_2\frac{1}{\rho_2-1}$, for some constants $K_1, K_2 > 0$. Therefore, there exists $C_1 = C_1(m_1, m_{21}, m_{22}) > 0$ such that $m_1^{\frac{\pi_n}{u_m\lambda_1}} m_{21}^{-\frac{a_m}{u_m}} m_{22}^{-\frac{b_m}{u_m}} \le C_1$, $m_1^{\frac{\pi_n}{v_m\lambda_1}} m_{21}^{-\frac{c_m}{v_m}} m_{22}^{-\frac{d_m}{v_m}} \le C_1$, for all $m, n \in \mathbb{N}$. The remainder of the proof follows that of lemma 4.9. □

Lemma 5.7. $\ell_2(E_\infty) = 0$.

Proof. It is easy to see that $u_m, v_m \ge \frac{\alpha_m(\rho_2)}{\lambda_2^1}$. Hence for some constant $C > 0$

$$\frac{1}{u_m\lambda_1 + \alpha_n}, \frac{1}{v_m\lambda_1 + \alpha_n} \le \frac{\lambda_2^1}{\alpha_m(\rho_2)\lambda_1 + \alpha_n\lambda_2^1} \le C\rho_1^{-n/2}\rho_2^{-m/2}$$

Along similar lines to the proof of lemma 4.1, we show that $\sum_{n=1}^{\infty} \ell_2(E_n) < \infty$. It follows by the Borel-Cantelli lemma that $\ell_2(E_\infty) = 0$. □

We have the following special case of theorem 5.1.

Proposition 5.8. *The likely limit set of the product of the planar homoclinic cycle* (Σ_1, ϕ_t^1) *and the attracting planar heteroclinic cycle* (Σ_2, ϕ_t^2) *is* Σ.

Proof. Similar to that of theorem 4.11. □

5.2. **Product of a homoclinic loop and a figure eight cycle.** We now consider the case where $(\Sigma_2, \phi_t^2, D_2, N_2)$ is an attracting figure eight homoclinic cycle. Write $N_2 = N_2^- \cup N_2^+$ where $N_2^- = N_{21}^- \cup N_{22}^-$ is an interior neighbourhood of Σ_2 in $\mathbb{R}^2$ and N_2^+ is an exterior neighbourhood of Σ_2 in $\mathbb{R}^2$ (see figure 8). We may write Σ_2 as the union of two attracting homoclinic loops Σ_{21}, Σ_{22}.

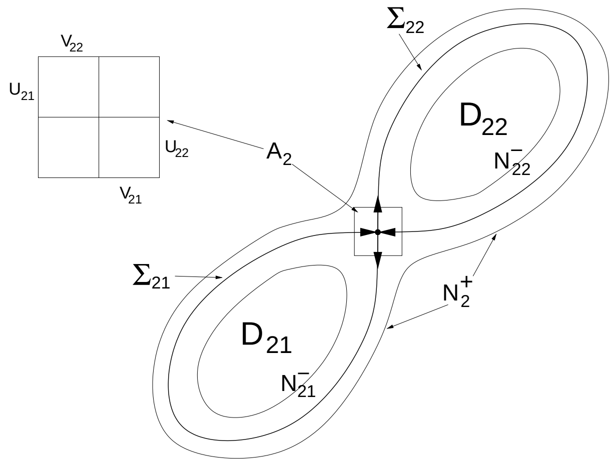

FIGURE 8. Notational conventions for figure eight homoclinic attractor Σ_2

It follows from the results of section 3 applied to the products $\Sigma_1 \times \Sigma_{21}$ and $\Sigma_1 \times \Sigma_{22}$, that $\mathcal{L}(N_1 \times N_2^-) = \Sigma$. It remains to show that $\mathcal{L}(N_1 \times N_2^+) = \Sigma$.

We suppose that the hyperbolic saddle at $(0,0)$ has associated eigenvalues $-\mu_2 < 0 < \lambda_2$. Set $\rho_2 = \mu_2/\lambda_2$ and assume $\rho_2 > 1$ so that Σ_2 is an attracting homoclinic cycle. Since $(0,0)$ is a hyperbolic saddle point for a C^2 planar flow, we may C^1-linearize the flow ϕ_t^2 on a box neighbourhood $A_2 \subset N_2$ of $(0,0)$. We assume coordinates are chosen so that the stable manifold at $(0,0)$ is tangent to the x-axis, the unstable manifold to the y-axis and $A_2 = [-1,1] \times [-1,1]$. Define

$$U_{21} = \{-1\} \times [0,1], \quad V_{21} = [0,1] \times \{-1\},$$

$$U_{22} = \{1\} \times [-1,0], \quad V_{22} = [-1,0] \times \{1\}.$$

(See inset to figure 8.)

But now we are exactly in the situation described by figure 7. Our previous results extend immediately (with $\lambda_2 = \lambda_2^1 = \lambda_2^2$, $\mu_2 = \mu_2^1 = \mu_2^2$ and the terms η_1, η_2 replaced by $\rho_2 = \mu_2/\lambda_2$, and ρ_2 by ρ_2^2, $m_2 = m_{21} = m_{22}$, $\tau_2 = \tau_{21} = \tau_{22}$).

Summarizing, we have shown

Proposition 5.9. *The likely limit set of the product of a planar homoclinic attractor* (Σ_1, ϕ_t^1) *and an attracting planar figure eight cycle* (Σ_2, ϕ_t^2)*, is* Σ.

6. General Global maps

In the previous sections, we assumed that the connection maps $C : V \to U$ were linear and the associated time maps $\xi : V \to \mathbb{R}$ were constant. In this section, we remove this restriction and show that our results continue to hold. We give the details only for product of two attracting homoclinic loops (details for the general case of products of heteroclinic attractors are similar and use the same methods). As far as possible we follow the notational conventions of sections 3, 4. The reader should note that a particular concern is obtain results where we can use the rescaling strategy described at the end of section 3.

Since we are assuming flows are at least C^2, the maps $C_i : V_i \to U_i$, $\xi_i : V_i \to \mathbb{R}$ defined in section 3 are C^2 (see Wiggins [24, Section 10.3]). Hence for $i = 1, 2$ we may write

$$C_i(x) = \gamma^i(x)x \tag{6.6}$$

$$\xi_i(x) = \tau_i + O(|x|), \tag{6.7}$$

where γ^i is C^1, $m_i = \gamma^i(0) \in (0, 1)$ and $\tau_i = \xi_i(0)$.

Variation in the time maps ξ_i causes only minor problems. However, the case of general connection maps is delicate — this is already evident in the analysis given in [4, §3] for the product of a homoclinic loop and limit cycle. We start with an analysis of the case when $\rho_1 \neq \rho_2$ and conclude with the harder case $\rho_1 = \rho_2$. Note that the results in the first part of the section hold with no restriction on ρ_1, ρ_2 (> 1). We indicate in the text results which do not hold with $\rho_1 = \rho_2$.

Rescaling. We start with an analysis of how the connection map changes when we restrict to a smaller rectangular $\hat{A} \subset A$ (we omit subscripts in what follows). Let $A = [0, 1] \times [0, 1]$ and $\hat{A} = [0, \alpha] \times [0, \beta] \subset A$. Let $\alpha = e^{-\mu S}$, $\beta = e^{-\lambda T}$. Linearly rescale coordinates on A so that, in the new coordinates $(\hat{x}, \hat{y})$, $\hat{A} = [0, 1] \times [0, 1]$. If we let $\hat{C} : V^\star \to U^\star$ denote the connection map associated to $\hat{A}$, we have

$$\hat{C}(\hat{x}) = e^{(T+S)\lambda} C(e^{-\mu(T+S)}\hat{x})$$

(See remarks 4.1(2).) Writing $C(x) = \gamma(x)x$, gives

$$\hat{\gamma}(\hat{x}) = e^{(T+S)(\lambda-\mu)}\gamma(e^{-(T+S)\mu}\hat{x}) \tag{6.8}$$

$$\frac{\hat{\gamma}'(\hat{x})}{\hat{\gamma}(\hat{x})} = e^{-(T+S)\mu}\frac{\gamma'(e^{-(T+S)\mu}\hat{x})}{\gamma(e^{-(T+S)\mu}\hat{x})} \tag{6.9}$$

Let $c_1 = \sup_{x\in[0,1]} |\gamma'(x)|$.

Lemma 6.1. *Let $T, S \geq 0$ and define $\hat{m} = \hat{\gamma}(0)$, $\hat{m}^+ = \sup_{\hat{x}\in[0,1]} \hat{\gamma}(\hat{x})$, $\hat{m}^- = \inf_{\hat{x}\in[0,1]} \hat{\gamma}(\hat{x})$. We have*

1. *$\hat{m} = e^{(T+S)(\lambda-\mu)}m$.*
2. *$\hat{m}(1 - \frac{c_1}{m}e^{-\mu(T+S)}) \leq \hat{m}^- \leq \hat{m}^+ \leq \hat{m}(1 + \frac{c_1}{m}e^{-\mu(T+S)})$.*

These estimates continue to hold if we increase either T or S.

Proof. The estimates follow straightforwardly from (6.8,6.9). □

Define $\varepsilon_0 > 0$ by

$$\varepsilon_0 = \min\{1/4, (\rho_2 - 1)/4\}.$$

It follows from lemma 6.1 and (6.9) that we can choose A_2 so that

$$m_2^+ \leq \min\{\rho_2^{-2/\rho_2}, 2^{-3}\}. \tag{6.10}$$

$$\left|\frac{\gamma'(x)}{\gamma(x)}\right| \leq \frac{\varepsilon_0}{2}. \tag{6.11}$$

Shrinking A_i if necessary, we can assume that there exist $\tau_i^-, \tau_i^+ > 0$ so that $\tau_i^- \leq \xi_i(x) \leq \tau_i^+$, for all $x \in V_i$.

Remarks 6.2. (1) If we shrink A_2 further then (6.10,6.11) continue to hold — m_2^+ is a decreasing function of $T + S$.
(2) If we shrink A_i, $\tau_i^\pm$ increase, $i = 1, 2$.

Defining the sequences. Exactly as in section 3, choose a rectangle $E = [a, b]\times[c, d] \subset U_1\times U_2$. Given $y = y_0 \in U_1$, $z = z_0 \in U_2$ let (y_n) and (z_n) denote the successive points of intersection of the forward trajectories through y_0 with U_1 and through z_0 with U_2. For $n \geq 1$, we have

$$y_n = \gamma^1(y_{n-1}^{\rho_1})y_{n-1}^{\rho_1}, \qquad z_n = \gamma^2(z_{n-1}^{\rho_2})z_{n-1}^{\rho_2}.$$

For $n \geq 0$ define $\gamma_n^1(y) = \gamma^1(y_n)$, $\gamma_n^2(y) = \gamma^2(z_n)$.

We have a straightforward generalization of lemma 4.2.

Lemma 6.3. *Let $(y, z) \in E$, $n \in \mathbb{N}$. We have*

1. *$t_1^y(n) = s_1^y(n) + \xi_1(y_n^{\rho_1})$ and $t_2^z(n) = s_2^z(n) + \xi_2(z_n^{\rho_2})$.*
2. *$t_1^y(n) = \sum_{i=0}^{n-1}\xi_1(y_i^{\rho_1}) - \frac{1}{\lambda_1}(\alpha_n \log y + \log\prod_{j=0}^{n-2}\left(\gamma_j^1(y)\right)^{\alpha_{n-1-j}})$.*

(3) $t_2^z(n) = \sum_{i=0}^{n-1} \xi_2(z_i^{\rho_2}) - \frac{1}{\lambda_2}(\beta_n \log z + \log \prod_{j=0}^{n-2} \left(\gamma_j^2(z)\right)^{\beta_{n-1-j}})$.

Remark 6.4. If we assume the connection maps are linear, $\gamma^i = m_i$, $i = 1, 2$, we find that

$$\prod_{j=0}^{n-2} \left(\gamma_j^1(y)\right)^{\alpha_{n-1-j}} = m_1^{\sum_{j=0}^{n-2} \alpha_{n-1-j}} = m_1^{\pi_n},$$

with a similar expression for $\prod_{j=0}^{n-2} \left(\gamma_j^2(z)\right)^{\beta_{n-1-j}}$. These are the terms that appear in lemma 4.2.

The main estimate. In this section we derive the main estimate on $|(\gamma_n^2)'(z)/\gamma_n^2(z)|$ that we use for our proof of convergence of $\sum \ell_2(E_n)$. So as to simplify the notation, we generally drop the identifying subscript 2 from γ_j^2, γ_2 and ρ_2. Given $n \in \mathbb{N}$, define $\overline{\gamma}_n : [c, d] \to \mathbb{R}$ by

$$z_n = \overline{\gamma}_n(z) z^{\rho^n},\ z \in [c, d].$$

Since we have $\gamma(x)x \leq m^+ x$, for all $x \in (0, 1]$, we have the easy estimate

$$z_n \leq (m^+)^{\beta_n} z^{\rho^n},\ n \geq 1, \tag{6.12}$$

where $\beta_n = \sum_{j=0}^{n-1} \rho_2^j$.

Lemma 6.5. *For $n \geq 1$ we have*

(1) $\left|\frac{\overline{\gamma}_n'(z)}{\overline{\gamma}_n(z)}\right| \leq \varepsilon_0 \rho^n$.

(2) $\left|\frac{\gamma_n'(z)}{\gamma_n(z)}\right| \leq \varepsilon_0$.

If $S \geq 0$ and we linearly rescale coordinates in the x-direction so that $e^{-\mu S}$ is rescaled to 1, then estimate (2) changes to

$$\left|\frac{\gamma_n'(z)}{\gamma_n(z)}\right| \leq e^{-\mu S} \varepsilon_0.$$

Proof. The proof of (1) is very similar to that of lemma 3.4 in [4] and we only indicate the main points. The proof goes by induction on n with the hypothesis at step n being

$$\left|\frac{\overline{\gamma}_n'(z)}{\overline{\gamma}_n(z)}\right| \leq \varepsilon_0 (1 - 2^{-n}) \rho^n.$$

When $n = 1$, we have $\overline{\gamma}_1(z) = \gamma(z^\rho)$ and differentiating gives

$$\frac{\overline{\gamma}_1'(z)}{\overline{\gamma}_1(z)} = \rho z^{\rho-1} \frac{\gamma'(z^\rho)}{\gamma(z^\rho)}.$$

Hence by (6.11), we have $|\rho z^{\rho-1} \frac{\gamma'(z^\rho)}{\gamma(z^\rho)}| \leq \varepsilon_0 \frac{\rho}{2} z^{\rho-1} \leq \varepsilon_0 \frac{\rho}{2}$, since $z \in (0, 1]$. This verifies the result when $n = 1$. Given the truth of the statement for $n-1$, the proof of step n proceeds by estimating the derivative $\overline{\gamma}'_n$ and uses the bound $m_2^+ \leq 2^{-3}$ (see (6.10)) and (6.12).

Turning to (2), we have $z_n = \overline{\gamma}_n(z) z^{\rho^n}$ and differentiating with respect to z gives

$$z'_n = \overline{\gamma}'_n(z) z^{\rho^n} + \overline{\gamma}_n(z) \rho^n z^{\rho^n - 1}.$$

We have $\gamma_n(z) = \gamma(z_n^\rho)$. Differentiating and dividing by $\gamma_n(z)$,

$$\begin{aligned} \frac{\gamma'_n(z)}{\gamma_n(z)} &= \frac{\gamma'(z_n^\rho)}{\gamma(z_n^\rho)} \rho z_n^{\rho-1} z'_n, \\ &= \frac{\gamma'(z_n^\rho)}{\gamma(z_n^\rho)} \rho z_n^{\rho-1} \left(\overline{\gamma}'_n(z) z^{\rho^n} + \overline{\gamma}_n(z) \rho^n z^{\rho^n - 1} \right). \end{aligned}$$

By (6.11), we have

$$\left| \frac{\gamma'_n(z)}{\gamma_n(z)} \right| \leq \varepsilon_0 \frac{\rho z_n^{\rho-1}}{2} \left(\left| \overline{\gamma}'_n(z) z^{\rho^n} \right| + \left| \overline{\gamma}_n(z) \rho^n z^{\rho^n - 1} \right| \right).$$

By (1) and choice of ε_0, we have $|\overline{\gamma}'_n(z)| \leq \varepsilon_0 \rho^n \overline{\gamma}_n(z) \leq \rho^n \overline{\gamma}_n(z)$ and so

$$\begin{aligned} \frac{\rho z_n^{\rho-1}}{2} \left| \overline{\gamma}'_n(z) z^{\rho^n} \right| &\leq \frac{\rho z_n^{\rho-1}}{2} \rho^n \overline{\gamma}_n(z) z^{\rho^n}, \\ &= \frac{\rho^{n+1} z_n^\rho}{2}, \\ &\leq \frac{\rho^{n+1} (m^+)^{\rho\beta_n}}{2}. \end{aligned}$$

Turning to the second term, we have

$$\begin{aligned} \frac{\rho z_n^{\rho-1}}{2} \left| \overline{\gamma}_n(z) \rho^n z^{\rho^n - 1} \right| &= \frac{z_n^\rho \rho^{n+1}}{2z}, \\ &\leq \frac{(m^+)^{\rho\beta_n} z^{\rho^{n+1}} \rho^{n+1}}{2z}, \\ &\leq \frac{\rho^{n+1} (m^+)^{\rho\beta_n}}{2}. \end{aligned}$$

Using the bound $m_2^+ \leq \rho_2^{-2/\rho_2}$ (see (6.10)), we get $\rho^{n+1} (m^+)^{\rho\beta_n} \leq 1$.

The final statement is immediate from (6.9). □

Remarks 6.6. (1) Lemma 6.5 continues to hold, with the smaller constants, if we shrink A_2.
(2) Estimate (2) of lemma 6.10 will more than suffice to handle the situation when $\rho_1 \neq \rho_2$. The final statement will be crucial for the analysis when $\rho_1 = \rho_2$.

Estimating $\ell_2(E_n)$ when $\rho_2 > \rho_1$. For this section we assume $\rho_2 \neq \rho_1$. Without loss of generality we suppose $\rho_2 > \rho_1$. Under these conditions, we prove that there exists $N \in \mathbb{N}$, $C > 0$ such that for all $n \geq N$,

$$\ell_2(E_n) \leq C \frac{n}{\alpha_n} \tag{6.13}$$

Since $\sum_{n\geq 1} \frac{n}{\alpha_n} < \infty$, this implies that $\sum_{n\geq 1} \ell_2(E_n) < \infty$ and so $\ell_2(E_\infty) = 0$.

We continue to assume $(y,z) \in E = [a,b] \times [c,d]$. Define

$$M_1 = M_1(n,y) = \prod_{j=0}^{n-2} (\gamma_j^1(y))^{\alpha_{n-1-j}}, \quad M_2(n,z) = \prod_{j=0}^{n-2} (\gamma_j^2(z))^{\beta_{n-1-j}}$$

$$N_1 = N_1(n,y) = \sum_{i=0}^{n-2} \xi_1(y_i^{\rho_1}), \quad N_2(n,z) = \sum_{i=0}^{n-2} \xi_2(z_i^{\rho_2})$$

The bad intervals are contained in intervals $[Z_m^1(y,n), Z_m^2(y,n)]$ which have nonempty intersection with $[c,d]$. The points of $Z_m^1(y,n), Z_m^2(y,n)$ are obtained by solving the following equations for z

$$s_1^y(n) = s_2^z(m) + \xi_2(z_m^{\rho_2}), \quad s_1^y(n) = s_2^z(m) - \xi_1(y_n^{\rho_1}). \tag{6.14}$$

For $i = 1,2$ define

$$N_{2i} = N_2(m, Z_m^i(y,n)), \quad M_{2i} = M_2(m, Z_m^i(y,n)) \tag{6.15}$$

Using (6.14), we find that

$$\begin{aligned} Z_m^1(y,n) &= e^{\frac{\lambda_2}{\beta_m}(N_{21} - N_1 + \xi_2(z_m^{\rho_2}))} y^{\frac{\lambda_2 \alpha_n}{\lambda_1 \beta_m}} M_1^{\frac{\lambda_2}{\lambda_1 \beta_m}} M_{21}^{-\frac{1}{\beta_m}} \\ Z_m^2(y,n) &= e^{\frac{\lambda_2}{\beta_m}(N_{22} - N_1 - \xi_1(y_n^{\rho_1}))} y^{\frac{\lambda_2 \alpha_n}{\lambda_1 \beta_m}} M_1^{\frac{\lambda_2}{\lambda_1 \beta_m}} M_{22}^{-\frac{1}{\beta_m}} \end{aligned}$$

The maximum number of turns around Σ_2 in time $s_1^y(n) + \tau_1^+$ is taken when $z = d$, and is estimated by

$$M_E(n,y) = \min\{m \mid s_2^d(m+1) > s_1^y(n) + \tau_1^+\} \tag{6.16}$$

Define $M_E(n) = \sup_{y\in[a,b]} M_E(n,y)$. Choose $r > 0$ such that $\rho_2^r < \rho_1$.

Lemma 6.7. *There exists $N_1 \in \mathbb{N}$ such that $M_E(n) \leq rn$ for all $n \geq N_1$.*

Proof. By definition, $s_2^d(M_E(n)+1) > s_1^y(n) + \tau_1^+$. The result now follows by a straightforward computation using the expressions for $s_2^d(M_E(n)+1)$ and $s_1^y(n)$ given by lemma 6.3. Indeed, the condition $\rho_2 > \rho_1$ implies that $\lim_{n\to\infty} M_E(n)/n = 0$. ∎

Let $J(n) = \{m \in \mathbb{N} \mid [Z^1_m(y,n), Z^2_m(y,n)] \cap [c,d] \neq \emptyset\}$. For $n \in \mathbb{N}$, define

$$K_n = \sup_{m\in J(n)} \{e^{-\frac{\lambda_2}{\beta_m}\left((n-1)\tau_1^- - m\tau_2^+\right)}(m_1^+)^{\frac{\pi_n\lambda_2}{\lambda_1\beta_m}}(m_2^-)^{-\frac{\theta_m}{\beta_m}}\}$$

Lemma 6.8. *There exists $N \geq N_1$ such that*

$$K_n \leq 2,\ n \geq N. \tag{6.17}$$

Proof. If $n \geq N_1$, then for all $m \in J(n)$, $m \leq rn$. It follows easily from the condition $\rho_2^r < \rho_1$ and lemma 4.3 that $\lim_{n\to\infty} K_n = 0$. □

Remark 6.9. Lemma 6.8 does not extend to the case when $\rho_2 = \rho_1$. The difficulty lies with the term $(m_2^-)^{-\frac{\theta_m}{\beta_m}}$ which we can only bound by $(m_2^-)^{-\frac{1}{\rho_2-1}}$ (lemma 4.3). We handle this problem at the end of the section.

For $n \in \mathbb{N}$, $m \in J(n)$ set $D^y_{m,n} = Z^2_m(n,y) - Z^1_m(n,y)$, $m \in J(n)$. We have

$$\ell_2(E_n) \leq \int_a^b \sum_{m=1}^{M_E(n)} D^y_{m,n}\, dy$$

Lemma 6.10. *For $n \in \mathbb{N}, m \in J(n)$*

$$D^y_{m,n} \leq (I_1 + I_2 + I_3)y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m}} \tag{6.18}$$

where

$$I_1 = K_n\frac{\lambda_2}{\beta_m}(m-1)(\tau_2^- - \tau_2^+),\ I_2 = K_n\frac{\lambda_2}{\beta_m}(\tau_1^+ + \tau_2^+)\ \ I_3 = K_n\left|\log\left(\frac{M_{22}}{M_{21}}\right)\right|.$$

Proof. We have

$$\begin{aligned}
D^y_{m,n} &\leq e^{-\frac{\lambda_2}{\beta_m}N_1}y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m}}M_1^{\frac{\lambda_2}{\lambda_1\beta_m}}e^{\frac{\lambda_2}{\beta_m}((m-1)\tau_2^+)}e^{\frac{\lambda_2}{\beta_m}\tau_2^+}(M_{21})^{-\frac{1}{\beta_m}} \\
&\quad -e^{-\frac{\lambda_2}{\beta_m}N_1}y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m}}M_1^{\frac{\lambda_2}{\lambda_1\beta_m}}e^{\frac{\lambda_2}{\beta_m}((m-1)\tau_2^-)}e^{-\frac{\lambda_2}{\beta_m}\tau_1^+}(M_{22})^{-\frac{1}{\beta_m}} \\
&\leq e^{-\frac{\lambda_2}{\beta_m}N_1}y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m}}(m_1^+)^{\frac{\pi_n\lambda_2}{\lambda_1\beta_m}}e^{\frac{\lambda_2}{\beta_m}((m-1)\tau_2^+)}e^{\frac{\lambda_2}{\beta_m}\tau_2^+}(m_2^-)^{-\frac{\theta_m}{\beta_m}} \\
&\quad \times\left(1 - e^{\frac{\lambda_2}{\beta_m}\left((m-1)(\tau_2^- - \tau_2^+) - (\tau_1^+ + \tau_2^+)\right)}\left(\frac{M_{22}}{M_{21}}\right)^{-\frac{1}{\beta_m}}\right) \\
&\leq K_n\left(\frac{\lambda_2}{\beta_m}\left((m-1)(\tau_2^+ - \tau_2^-) + (\tau_1^+ + \tau_2^+)\right) + \frac{1}{\beta_m}\log\left(\frac{M_{22}}{M_{21}}\right)\right)y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m}}
\end{aligned}$$

By separating the three terms, we obtain the result. □

Remark 6.11. In the case when $\tau_2^+ = \tau_2^- = \tau_2$, $M_{21} = M_{22} = m_2^{\theta_m}$, the terms I_1 and I_3 vanish, and we are only left with I_2 which is precisely the term in lemma 4.9. The crucial term we have to estimate is $\left|\log\left(\frac{M_{22}}{M_{21}}\right)\right|$ in I_3.

Lemma 6.12. *We have* $\left|\log\left(\frac{M_{22}}{M_{21}}\right)\right| \leq \varepsilon_0 \frac{\beta_m}{\rho_2 - 1} D^y_{m,n}$.

Proof. Using (6.15), we get

$$\begin{aligned}
\left|\log\left(\frac{M_{22}}{M_{21}}\right)\right| &= \left|\log\prod_{j=0}^{m-2}\left(\frac{\gamma_j(Z_m^2(n,y))}{\gamma_j(Z_m^1(n,y))}\right)^{\beta_{m-j-1}}\right|, \\
&\leq \sum_{j=0}^{m-2}\beta_{m-j-1}\left|\log\frac{\gamma_j(Z_m^2(n,y))}{\gamma_j(Z_m^1(n,y))}\right|, \\
&\leq \sum_{j=0}^{m-2}\beta_{m-j-1}\left|\frac{\gamma_j'(w)}{\gamma_j(w)}\right| D^y_{m,n}, \\
&\leq \varepsilon_0\sum_{j=0}^{m-2}\beta_{m-j-1}\, D^y_{m,n}, \quad \text{(lemma 6.5)}, \\
&= \varepsilon_0\frac{1}{\rho_2-1}\sum_{j=0}^{m-2}\left(\rho_2^{m-j-1}-1\right) D^y_{m,n} \\
&\leq \varepsilon_0\frac{1}{\rho_2-1}\sum_{j=0}^{m-2}\rho_2^{m-j-1}\, D^y_{m,n}, \\
&\leq \varepsilon_0\frac{\beta_m}{\rho_2-1}\, D^y_{m,n}.
\end{aligned}$$

The second inequality is obtained by using mean value theorem on the interval $[Z_m^1(n,y), Z_m^2(n,y)]$, $w \in (Z_m^1(n,y), Z_m^2(n,y))$. ☐

Lemma 6.13. *For* $n \geq N$*, there exists* $C > 0$ *such that*

$$\ell_2(E_n) \leq C\frac{n}{\alpha_n}$$

Proof. We have

$$I_1 \leq C_1\,\frac{m}{\beta_m}\,y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m}},\ I_2 = C_2\,\frac{1}{\beta_m}\,y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m}},\ I_3 \leq C_3\,y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m}}\,D^y_{m,n},$$

where $C_1 = K_n\lambda_2(\tau_2^+ - \tau_2^-)$, $C_2 = K_n\lambda_2(\tau_1^+ + \tau_2^+)$, $C_3 = \varepsilon_0 K_n/(\rho_2 - 1)$.

Proof. Since $D^y_{m,n} \leq I_1 + I_2 + I_3$ we have

$$\begin{aligned} D^y_{m,n} &\leq (mC_1 + C_2)\frac{1}{\beta_m} y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m}} + C_3 y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m}} D^y_{m,n} \\ &\leq (C_1 + C_2)\frac{m}{\beta_m} y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m}} + C_3 y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m}} D^y_{m,n} \\ &\leq (C_1 + C_2)\frac{m}{\beta_m} y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m}} + \frac{1}{2} y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m}} D^y_{m,n}, \end{aligned}$$

where the last inequality holds by lemma 6.8 for $n \geq N$ and our choice of $\varepsilon_0 = \min\{1/4, (\rho_2 - 1)/4\}$. Hence for $n \geq N$, we have the estimate

$$\begin{aligned} D^y_{m,n} &\leq (C_1 + C_2)\frac{m}{\beta_m} \frac{y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m}}}{1 - y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m}}/2} \\ &\leq \left(\frac{2m\lambda_1(C_1 + C_2)}{\lambda_2\alpha_n}\right) \frac{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m} y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m} - 1}/2}{1 - y^{\frac{\lambda_2\alpha_n}{\lambda_1\beta_m}}/2} \end{aligned}$$

Integrating, we get

$$\begin{aligned} \int_{y=a}^{b} D^y_{m,n}\, dy &\leq \int_0^1 D^y_{m,n}\, dy \\ &= \left(\frac{2m\lambda_1(C_1 + C_2)}{\lambda_2\alpha_n}\right) \log 2 \\ &\leq C\frac{n}{\alpha_n}. \end{aligned}$$

where $C = \frac{2r\lambda_1(C_1+C_2)}{\lambda_2}$. □

Remark 6.14. Our arguments extend immediately to the case of rectangles $E \subset (\{x\} \times [0, 1]) \times (\{x'\} \times [0, 1])$, where $x \in V_1, x' \in V_2$. The key observation is that lemma 6.7 continues to hold and so our arguments all continue to apply with the *same* ε_0.

Estimating $\ell_2(E_n)$ when $\rho_2 = \rho_1$. Assume that $\rho_1 = \rho_2 = \rho$. We need replacements for lemmas 6.7, 6.8. Following our earlier notation, we may choose $N_1 = N(\rho, \lambda_2, \tau_1^-, \tau^+) \in \mathbb{N}$ such that

$$\sup_{n \geq N_1} K_n \leq 2(m_2^-)^{-\frac{1}{\rho-1}}.$$

Set $k = m_2^-$. Linearly rescale A_2 in the x-direction by $e^{-\mu_2 S}$, where S is chosen so that

$$e^{S(\lambda_2 - \mu_2)} k^{-\frac{1}{\rho-1}} \leq 1. \tag{6.19}$$

If we let $\hat{A}_2 = [0, e^{-\mu_2 S}] \times [0,1]$, then a lower bound for $\hat{m}_2^-$ on $\hat{A}_2$ is $e^{S(\lambda_2-\mu_2)}k$ and the constant ε_0 in lemma 6.5 rescales to $\hat{\varepsilon}_0 = \varepsilon_0 e^{-\mu S}$.

Replace A_2 by $\hat{A}_2$. On the rescaled A_2 we have

$$\begin{aligned}(m_2^-)^{-\frac{1}{\rho-1}} &\leq (ke^{S(\lambda_2-\mu_2)})^{-\frac{1}{\rho-1}} = k^{-\frac{1}{\rho-1}}e^{S\lambda_2},\\ \varepsilon_0 &= e^{-\mu S}\min\{\frac{1}{4}, \frac{\rho-1}{4}\},\\ \sup_{n\geq N_1} K_n &\leq 2k^{-\frac{1}{\rho-1}}e^{S\lambda_2}.\end{aligned}$$

Lemma 6.12 holds without any assumptions on ρ_1, ρ_2 and so we still have $\left|\log\left(\frac{M_{22}}{M_{21}}\right)\right| \leq \varepsilon_0 \frac{\beta_m}{\rho-1} D^y_{m,n}$. We have the following straightforward variation on lemma 6.7.

Lemma 6.15. *If $\rho_1 = \rho_2$, there exists $N \geq N_1$ such that*

$$M_E(n) \leq 2n,\ n \geq N.$$

Finally, for lemma 6.13 to hold, it is enough that $\varepsilon_0 K_n/(\rho-1) \leq \frac{1}{2}$, $n \geq N$. Computing, we find that

$$\begin{aligned}\varepsilon_0 K_n/(\rho-1) &\leq 2\min\{\frac{1}{4}, \frac{\rho-1}{4}\}e^{-\mu_2 S}k^{-\frac{1}{\rho-1}}e^{S\lambda_2}/(\rho_2-1)\\ &\leq \frac{2}{\rho-1}\min\{\frac{1}{4}, \frac{\rho-1}{4}\},\ \text{by (6.19)},\\ &\leq \frac{1}{2}.\end{aligned}$$

Remark 6.16. Of course, we could use our argument for the resonant case $\rho_1 = \rho_2$ in the general case. However, we prefer to present the arguments separately as we feel both arguments have intrinsic interest and may be relevant in problems where product structure is broken.

7. Simulations

In this section, we present numerical simulations which illustrate our results on the likely limit set for the product flow of two attracting homoclinic orbits. We consider the product of identical dynamical systems given by

$$x' = y - x + (x-a)^2, \tag{7.20}$$

$$y' = y + 3x - x^3, \tag{7.21}$$

where $a = 1.7611050$. The two-dimensional system has an attracting homoclinic cycle Σ_1 associated to the saddle point $p = (0.452, -1.263)$ (see figure 9). Since the attractors are identical for the product system, we see that if the initial conditions are equal then the ω-limit set will be the diagonal $\{(u,u), u \in \Sigma_1\} \subset \Sigma_1 \times \Sigma_1$. We look at the ω-limit set of the trajectory with initial condition $((x_1,y_1),(x_2,y_2)) = ((2,2.1),(1.7,1))$. Ignoring the initial transient, we show the projections of this trajectory on the (x_1,x_2)- and (y_1,y_2)-planes in figures 10 and 11.

The projections of the product flow on the (x_1,x_2)- and (y_1,y_2)-planes are illustrated in figures 10 and 11. It is clear from figures that when the (x_1,y_1) components of the trajectory are in the neighbourhood of p then the (x_2,y_2) components of the trajectory visits each point of the homoclinic orbit, and vice-a-versa. The results of the simulation are consistent with the ω-limit being equal to Σ and not $\Sigma_1 \times \Sigma_1$. (Simulations were performed using XPPAUT software [9].)

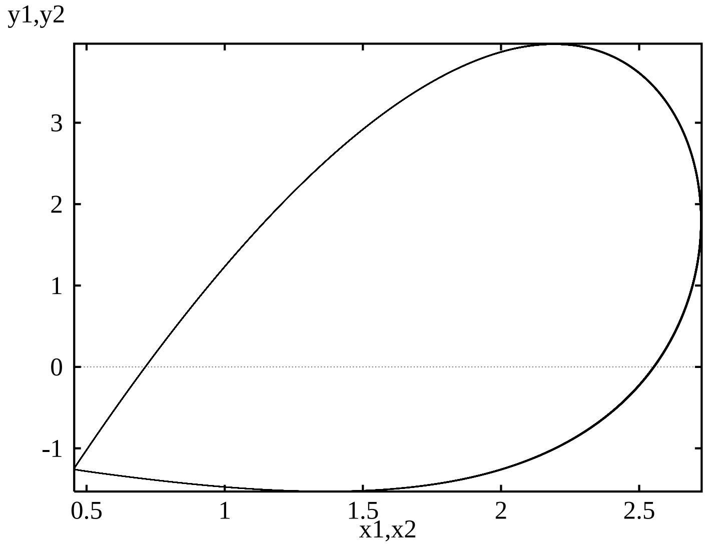

Figure 9. Projection of ω-limit set onto x_1, y_1 plane.

8. Bifurcations near the Product of Attractors

Consider the product dynamics for two planar attracting homoclinic loops $\Sigma_1, \Sigma_2 \subset \mathbb{R}^2$ (for example, the loops associated to the differential equations (7.20,7.21) described in the previous section). We investigate bifurcations that occur when we break either or both of the homoclinic connections but preserve the product structure. We will make use of the Andronov-Leontovich theorem and assume the presence of a splitting parameter (for background and more details we refer to Kuznetsov [17, §6.2], Wiggins [23] or Andronov *et al.* [3]).

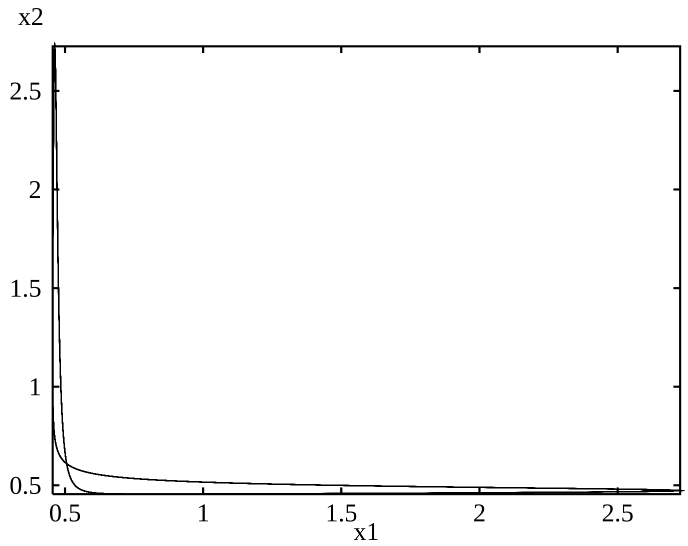

FIGURE 10. Projection of ω-limit set onto x_1, x_2 plane.

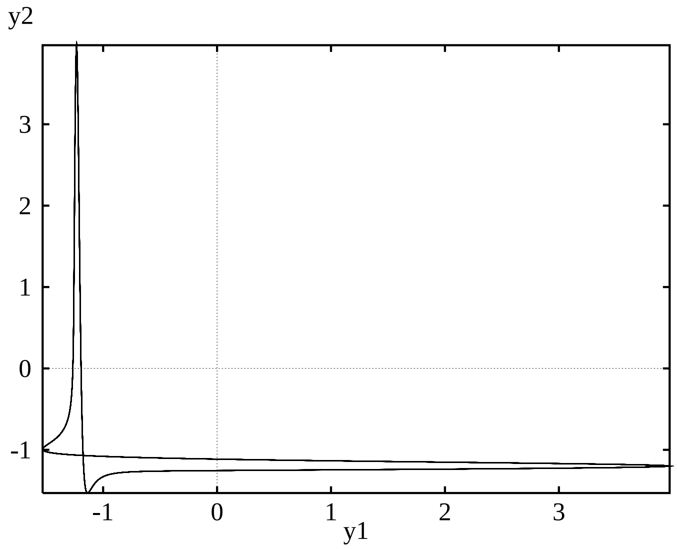

FIGURE 11. Projection of ω-limit set onto y_1, y_2 plane.

Specifically, for $i = 1, 2$, we assume that ξ_i is the splitting parameter governing the homoclinic cycle Σ_i, so that (see figure 12)

(a) for $\xi_i < 0$, the stable manifold lies inside the unstable manifold;
(b) for $\xi_i = 0$, the stable manifold coincides with the unstable manifold giving the homoclinic cycle Σ_i;

(c) for $\xi_i > 0$, the stable manifold lies outside the unstable manifold.

Since the cycles $\Sigma_i \subset \mathbb{R}^2$ are attracting, it follows by the Andronov-Leontovich theorem [3, 17, 23] that for sufficiently small $\xi_i > 0$, there exists a unique stable (hyperbolic) limit cycle $C_i(\xi_i) \subset N_i$ such that as $\xi_i \to 0^+$, the limit cycle $C_i(\xi_i)$ approaches the locus of Σ_i and its period $P_i(\xi_i)$ tends to $+\infty$.

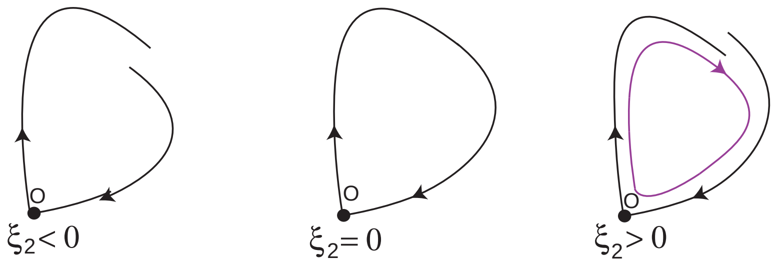

FIGURE 12. Phase plane near the homoclinicity.

Keeping our earlier notation for the eigenvalues of the linearization at the equilibrium $\mathbf{p}_i$ of Σ_i, we assume that

(NR) $\frac{\mu_2}{\lambda_2} \notin \left\{\frac{3}{2}, 2, 3, 4\right\}$ and

(DI) ϕ_t^2 is at least C^7,

(Conditions (NR,DI) imply that the flow ϕ_t^2 is C^3-linearizable at $\mathbf{p}_2$.) Keeping $\xi_2 = 0$ (so the cycle Σ_2 persists), we break the cycle Σ_1 by varying ξ_1. By Andronov-Leontovich theorem, there exists $\delta_1 > 0$ such that for all $\xi_1 \in (0, \delta)$, there exists a unique stable limit cycle $C_1(\xi_1) \subset N_1$. Applying theorem 1.2 of Ashwin and Field [4], it follows that the minimal Milnor attractor for the product system $\Phi_t^{\xi_1,0} = (\phi_t^{1,\xi_1}, \phi_t^{2,0})$ is the topological torus $C_1(\xi_1) \times \Sigma_2$.

Remark 8.1. Theorem 1.2 of Ashwin and Field is stated for the case when the flow ψ_t on the limit cycle is linear. That is, $\psi_t(\theta) = \theta + \omega t$. However, the result extends immediately to general C^1-flows on a limit cycle. Return times to a section transverse to the homoclinic loop, will be equidistributed, modulo the period. However, points of intersection with the section will generally not be equidistributed.

The torus $C_1(\xi_1) \times \Sigma_2$ is the minimal Milnor attractor that appears along the horizontal axis of the bifurcation diagram depicted in figure 13(a). We have a similar argument when we fix the first system at $\xi_1 = 0$ and perturb the second one.

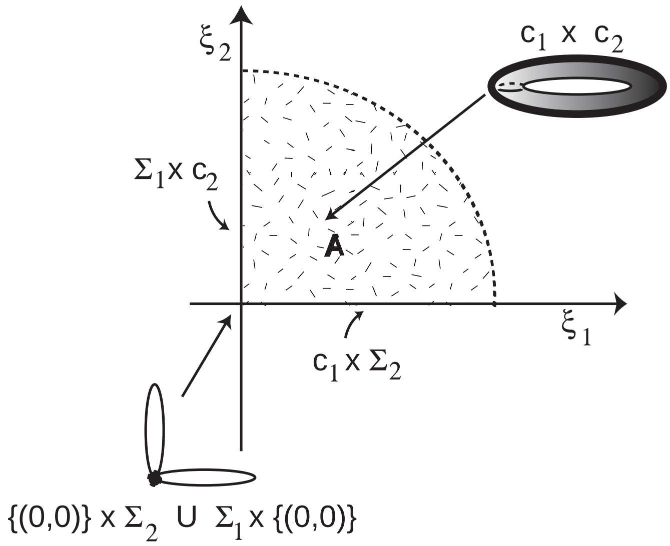

FIGURE 13. Bifurcation diagram for the likely limit set of $N_1 \times N_2$ for a product of homoclinic attractors.

Moreover, if we set $\mathcal{A} = (0, \delta_1) \times (0, \delta_2)$, then for all $(\xi_1, \xi_2) \in \mathcal{A}$, the likely limit set of $N_1 \times N_2$ is the attracting normally hyperbolic two dimensional torus $\mathbb{T}(\boldsymbol{\xi}) = C_1(\xi_1) \times C_2(\xi_2)$. Note that $\mathbb{T}(\boldsymbol{\xi})$ will be a minimal Milnor attractor if $P_1(\xi_1)/P_2(\xi_1)$ is irrational — this will happen on a full measure subset of $\mathcal{A}$. If the ratio is rational, then the induced flow on $\mathbb{T}(\boldsymbol{\xi})$ is a rational torus flow. In this case, since $\mathbb{T}(\boldsymbol{\xi})$ is normally hyperbolic, the stable manifold of each periodic orbit on $\mathbb{T}(\boldsymbol{\xi})$ will be three dimensional and so $\mathbb{T}(\boldsymbol{\xi})$ cannot be a minimal Milnor attractor.

If the vector field associated to the flow ϕ_t^2 is equivariant under the group $\mathbb{Z}_2(-I)$ generated by $-I(x, y) = (-x, -y)$, then the set $\gamma(\Sigma_2) \neq \Sigma_2$ is also a homoclinic orbit associated to the origin and $\widetilde{\Sigma}_2 = \Sigma_2 \cup \gamma(\Sigma_2)$ defines a figure of eight homoclinic cycle (see section 5). Homoclinic cycles of this type appear often in the literature on the bifurcation theory of planar systems (for example, Dangelmayr & Guckenheimer [7]).

Just as above, we can break the connections for either of the homoclinic loops contained in $\widetilde{\Sigma}_2$. Furthermore, if we analyse the first return map to a cross section in the external part of an attracting figure of eight, we can show the existence of a stable fixed point for $\xi_2 < 0$. That is, a periodic orbit surrounding the figure of eight bifurcates for $\xi_2 < 0$, as shown in figure 14 (see Guckenheimer & Holmes [11], Wiggins [23]).

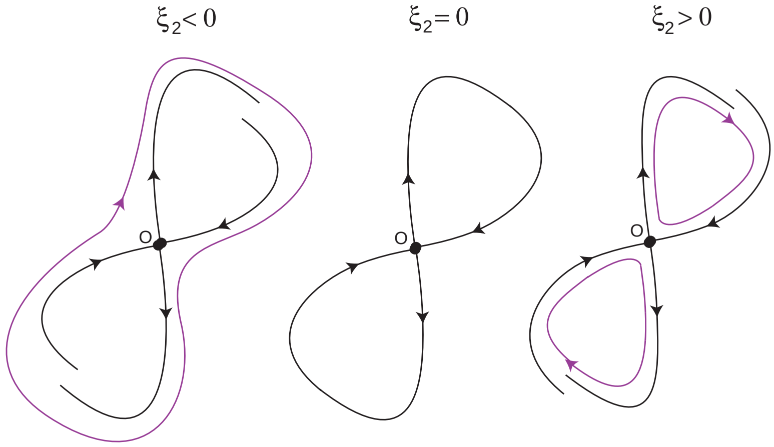

FIGURE 14. Bifurcations to periodic solutions near an attracting figure of eight cycle.

Next we consider the unfolding of the product of an attracting homoclinic cycle and an attracting figure of eight (see figure 15).

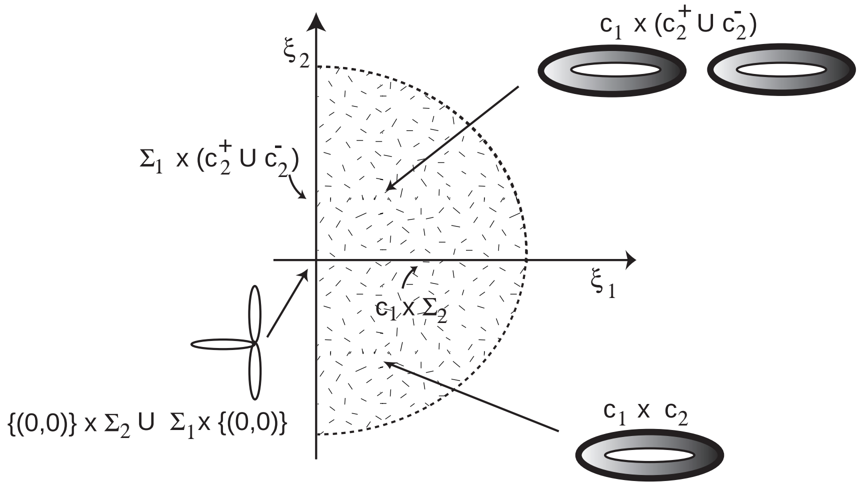

FIGURE 15. Bifurcation diagram for the likely limit set of $N_1 \times N_2$ for the product of a homoclinic attractor and an attracting figure of eight cycle.

Along the ξ_1 axis (where we break the homoclinic cycle Σ_1), the likely limit set consists of two two-dimensional topological tori $C_1(\xi_1) \times \widetilde{\Sigma}_2$, which intersect in a topological circle. This will be a Milnor attractor, but not minimal. Along the positive ξ_2-axis (where we break the figure of eight $\widetilde{\Sigma}_2$), the Milnor attractor has two connected components $(\Sigma_1 \times C_{12}(\xi_2)) \cup (\Sigma_1 \times C_{22}(\xi_2))$, each of which is a minimal Milnor attractor. Along the negative ξ_2-axis, the minimal Milnor attractor is the product of Σ_1 and the attracting limit cycle that appears outside $\widetilde{\Sigma}_2$. In the first quadrant of the bifurcation diagram there will be two tori which are the likely limit sets for $N_1 \times N_2$. These will be minimal Milnor attractors if the induced flows are irrational torus flows.

Finally, we briefly consider the product of a single homoclinic orbit and a heteroclinic attractor consisting of two equilibria and two connections. The analysis is similar to what we did for the product of two single homoclinic attractors because at least one attracting limit cycle bifurcates from the heteroclinic cycle if the interior connection is broken (see figure 16). Given the heteroclinic network shown in figure 16, a characteristic situation is that there is a unique unstable equilibrium enclosed by each cycle (when $\xi = 0$). When we break the interior heteroclinic connection from A to B, an attracting limit cycle is created. We show a typical scenario in figure 16.

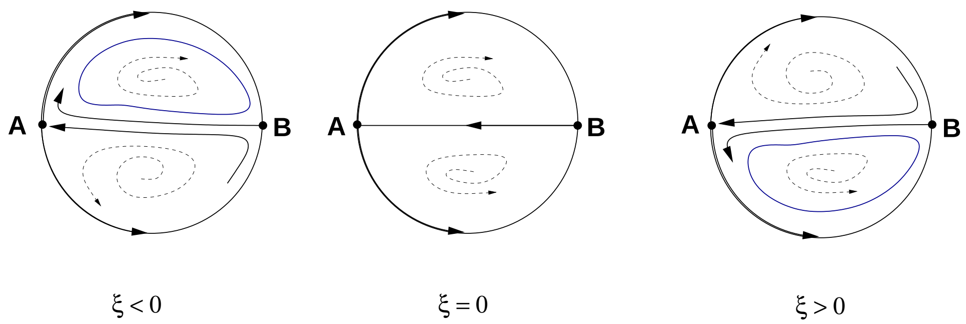

FIGURE 16. An unfolding of a heteroclinic cycle.

9. Discussion and Conclusion

Motivated by the partial results in Ashwin and Field [4] and supported by numerical simulations, we have proved that the likely limit set of the product of two planar heteroclinic attractors is the one-dimensional heteroclinic network which covers the attracting networks in the factors. The result implies that generically two independent trajectories around the heteroclinic connections are forward asynchronous. Following the analysis presented here, it is not difficult to prove that the likely limit set of the product of a finite number of heteroclinic attractors is also the 1-dimensional heteroclinic network which covers the attracting networks in the factors.

For the proof of our main result we needed to assume that the flow was at least C^2. The situation when the flow is only C^1 is far from clear, especially in the resonant case when $\rho_1 = \rho_2$. Since theorem 2.6 holds for C^1-flows, a counterexample in the C^1-case would give the likely limit set as the product of heteroclinic attractors and would likely depend on large fluctuations in the derivative of the connection maps. Similar issues of differentiability arise in the case of the product of a homoclinic attractor and a limit cycle (the result in Ashin and Field [4] required the homoclinic flow to be C^7) and it is not clear whether or not the product of a homoclinic attractor and a limit cycle is a minimal Milnor attractor if the flow (for the homoclinic factor) is less regular, for example C^2.

In the context of game theory and the replicator equation, Sato *et al.* [22] studied numerical examples of the product of heteroclinic networks on a simplex. In this case, the product dynamics are dependent on the payoff matrices. Under some conditions, numerical evidence was found for the existence of complicated behaviour near the product network. This is one reason why we believe our result may have interesting applications outside of equivariant dynamics and network dynamics [1, §5] and why it would be worthwhile generalizing to products of two heteroclinic cycles in higher dimensions. In dimensions greater than 2, issues such as the orientability of the homoclinic connection and the existence of degenerate cases of homoclinic cycles, such as orbit flip and inclination flip, complicate the study. There is also the question of considering the product of a homoclinic cycle with a *butterfly* or *bellows*. We refer to the recent survey by Homburg and Sandstede [13] for more details on these types of cycle as well as issues connected to bifurcation theory and the breakdown of hyperbolicity at the saddle point.

The outstanding question is undoubtedly to obtain quantitative results about the effects of perturbations breaking the product structure to, for example, a skew product structure but keeping the heteroclinic cycles. Does the loss of the product structure lead to phenomena related to essential asymptotic stability or do the likely limit set results persist for small enough perturbations? Put another way, is it possible to find cycle preserving perturbations that make a pre-specified subnetwork essentially asymptotically stable?

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