On profinite uniform structures
defined by varieties of finite monoids

Jean-Éric Pin[*] and Pedro V. Silva[†]

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This paper arose during the preparation of [11], a paper on pro-p topologies extending the results of [12]. After recollecting various assertions on profinite topologies, we tried to improve existing statements and to arrange our notes into a readable form. We soon realize that many results were spread out in the literature, some of them not being stated in the way we wished and some other ones simply missing. At the end, we decided there was place for a self-contained presentation of this material. This is the way this paper was born.

Let us describe the framework of this paper. We consider pro-**V** uniformities associated with a variety of finite monoids, but in contrast with the standard literature [2, 3, 4], we work with arbitrary monoids and not only with free or free profinite monoids. A more general framework could be achieved by associating uniformities with varieties of finite ordered monoids, like in [13], or even with lattices of recognizable sets, like in [8], but the present level of generality is sufficient for the applications we have in mind in [11].

Thus we fix an arbitrary monoid M and a variety of finite monoids **V**. The intuitive idea underlying the definition of the pro-**V** uniformity on M is that any morphism φ from M onto a monoid of **V** defines an entourage, which consists of the pairs of elements of M that cannot be distinguished by φ. The aim of this paper is to address two general questions on these uniform structures and a few more specialized ones that were motivated by our paper [11].

A first question is whether pro-**V** uniformities can be defined by a metric or a pseudometric. This is a nontrivial question, since the pseudometric naturally associated a variety of finite monoids **V** does not necessarily define the pro-**V** uniformity, even if this uniformity is metrizable... We clarify this question in Section 3 (see in particular Propositions 3.1 and 3.2).

[*]LIAFA, Université Paris-Diderot and CNRS, Case 7014, 75205 Paris Cedex 13, France. `Jean-Eric.Pin@liafa.jussieu.fr`

[†]Centro de Matemática, Faculdade de Ciências, Universidade do Porto, R. Campo Alegre 687, 4169-007 Porto, Portugal. `pvsilva@fc.up.pt`

The second question of general interest is the description of continous and uniformly continuous functions. We first give a characterization of these functions in term of recognizable sets (Theorem 4.1) and use it to extend a result of Reutenauer and Schützenberger on continuous functions for the pro-group topology (Corollary 4.3). Next we introduce the notion of *hereditary continuity* and discuss the behaviour of our three main properties (continuity, uniform continuity, hereditary continuity) under composition, product or exponential.

The last section is more specialized and is mainly motivated by the applications to [11]. We analyse the properties of **V**-uniform continuity when **V** is the intersection — or the join — of a family of varieties and we discuss in some detail the case where **V** is commutative.

1   Topology and uniform structures

This section surveys the basic definitions and results on uniform spaces which will be needed in the sequel. For more details, the reader is referred to [6, 7].

1.1   Uniform spaces

Let X be a set. The subsets of $X \times X$ can be viewed as relations on X. In particular, if U and V are subsets of $X \times X$, we use the notation UV to denote the composition of the two relations, that is, the set

$$UV = \big\{ (x,y) \in X \times X \mid \text{there exists } z \in X, \ (x,z) \in U \text{ and } (z,y) \in V \big\}.$$

Given a relation U, the *transposed relation* of U is the relation

$$^t U = \big\{ (x,y) \in X \times X \mid (y,x) \in U \big\}$$

A relation U is *symmetrical* if $^t U = U$. Finally, if $x \in X$ and $U \subseteq X \times X$, we write $U(x)$ for the set $\{ y \in X \mid (x,y) \in U \}$.

A *uniformity* on a set X is a non empty set $\mathcal{U}$ of subsets of $X \times X$ satisfying the following properties:

(1) if a subset U of $X \times X$ contains an element of $\mathcal{U}$, then $U \in \mathcal{U}$,

(2) the intersection of any two elements of $\mathcal{U}$ contains an element of $\mathcal{U}$,

(3) each element of $\mathcal{U}$ contains the diagonal of $X \times X$,

(4) for each $U \in \mathcal{U}$, $^t U \in \mathcal{U}$,

(5) for each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $VV \subseteq U$.

If $\mathcal{U}$ is a uniformity on the set X, the elements of $\mathcal{U}$ are called *entourages*. Note that $X \times X$ is always an entourage. The pair $(X, \mathcal{U})$ (or the set X if $\mathcal{U}$ is understood) is called a *uniform space*.

For each $x \in X$, let $\mathcal{U}(x) = \{U(x) \mid U \in \mathcal{U}\}$. There exists a unique topology on X, called the *topology induced* by $\mathcal{U}$, for which $\mathcal{U}(x)$ is the filter of neighborhoods of x for each $x \in X$. A uniform space $(X, \mathcal{U})$ is *Hausdorff* if the induced topology is Hausdorff. This is equivalent to requiring that the intersection of all the entourages of $\mathcal{U}$ is equal to the diagonal of $X \times X$.

A *basis* of a uniformity $\mathcal{U}$ is a subset $\mathcal{B}$ of $\mathcal{U}$ such that each element of $\mathcal{U}$ contains an element of $\mathcal{B}$. In particular, $\mathcal{U}$ consists of all the relations on X containing an element of $\mathcal{B}$. We say that $\mathcal{U}$ is *generated* by $\mathcal{B}$. A set $\mathcal{B}$ of subsets of $X \times X$ is a basis of some uniformity if and only if it satisfies properties (2) to (5).

An entourage U is *transitive* if $UU \subseteq U$. A uniformity is said to be *transitive* if it has a basis consisting of transitive entourages. It is said to be *totally bounded* if, for each entourage U, there exist finitely many subsets $B_1, \ldots, B_n$ of X such that $X = \bigcup_i B_i$ and $\bigcup_i (B_i \times B_i) \subseteq U$. The interest of totally bounded uniformities lies in the following result, which is a consequence of [7, TG.II.29, Thm. 3].

Proposition 1.1 *Let $(X, \mathcal{U})$ be an Hausdorff uniform space. Then the completion of X is compact if and only if $\mathcal{U}$ is totally bounded.*

Example 1.1 Let A be a set. Given a finite partition $X = \{X_1, \ldots, X_n\}$ of A, let
$$U_X = \bigcup_{1 \leqslant i \leqslant n} X_i \times X_i$$

The sets of the form U_X, where X runs over the class of finite partitions of A, form the basis of a transitive, totally bounded uniformity. By Proposition 1.1, the profinite completion of A is compact.

If $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are uniform spaces, a function $\varphi \colon X \to Y$ is said to be *uniformly continuous* if, for each entourage V of $\mathcal{V}$, $(\varphi \times \varphi)^{-1}(V)$ is an entourage of $\mathcal{U}$, or, equivalenty, there exists an entourage $U \in \mathcal{U}$ such that $\varphi(U) \subseteq V$. Naturally, (simple) *continuity* of φ refers to the induced topologies.

1.2 Pseudometrics

Recall that a *metric* on a set X is a mapping $d \colon X \times X \to \mathbb{R}^+$ satisfying the following conditions:

(1) for all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.

(2) for all $x, y \in X$, $d(x, y) = d(y, x)$,

(3) if $x, y, z \in X$, then $d(x, z) \leqslant d(x, y) + d(y, z)$.

A *pseudometric* satisfies conditions (2), (3) and the following weaker version of condition (1):

(1') for every $x \in X$, $d(x, x) = 0$.

Finally, a pseudometric is *ultrametric* if it satisfies the following stronger version of condition (3):

(3′) For each $x, y, z \in X$, $d(x, z) \leqslant \max(d(x, y), d(y, z))$.

In this paper, we are mostly interested in *pseudo-ultrametrics*, that we simply call *pu-metrics* to keep a short name.

A pseudometric d on a set X naturally defines a uniformity $\mathcal{U}$ on X. A basis of $\mathcal{U}$ is given by the subsets of $X \times X$ of the form

$$U_\varepsilon = \{(x, y) \in X \times X \mid d(x, y) < \varepsilon\} \qquad (\varepsilon > 0)$$

If d is a pu-metric, the uniformity is transitive. Two pu-metrics on the set X are said to be *uniformly equivalent* if they define the same uniformity.

It is easily verified that if d and d' are pseudometrics, respectively on X and on Y, and if $\mathcal{U}$ and $\mathcal{U}'$ are the uniformities defined by these pseudometrics, then a function $\varphi \colon (X, \mathcal{U}) \to (X', \mathcal{U}')$ is uniformly continuous if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in X$, $d(x, y) < \delta$ implies $d'(\varphi(x), \varphi(y)) < \varepsilon$.

The uniformities that can be defined by a pu-metric are characterized by the following property (see [6, TG.IX.2, Theorem 1]).

Proposition 1.2

(1) *A transitive uniformity can be defined by a pu-metric if and only if it has a countable basis.*

(2) *A transitive Hausdorff uniformity can be defined by an ultrametric if and only if it has a countable basis.*

2 Varieties

2.1 Definitions

A *Birkhoff variety of monoids* is a class of monoids closed under taking submonoids, quotients and direct products.

A *variety of finite monoids* is a class of finite monoids closed under taking submonoids, quotients and finite direct products. In the sequel, we shall use freely the term *variety* instead of *variety of finite monoids*. Examples include the variety $\mathbf{A}$ of finite aperiodic monoids and the variety $\mathbf{G}$ of finite groups.

The join of a family of varieties $(\mathbf{V}_i)_{i \in I}$ is the smallest variety containing all the varieties $\mathbf{V}_i$, for $i \in I$.

Let M be a monoid. Recall that a subset L of M is *recognizable* if there exists a finite monoid F and a surjective morphism $\varphi : M \to F$ such that $L = \varphi^{-1}(\varphi(L))$. More generally, if $\mathbf{V}$ is a variety of finite monoids, we say that L is $\mathbf{V}$-*recognizable* if there exists a finite monoid $F \in \mathbf{V}$ and a surjective morphism $\varphi : M \to F$ such that $L = \varphi^{-1}(\varphi(L))$. It is a well-known fact that the $\mathbf{V}$-recognizable subsets of a monoid form a Boolean algebra.

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2.2 Uniform structure associated with a variety

In the next subsections, $\mathbf{V}$ denotes a fixed variety of finite monoids. Given a morphism φ from a monoid M onto a finite monoid F, let

$$U_\varphi = \{(u, v) \in M \times M \mid \varphi(u) = \varphi(v)\}$$

Since F is finite, U_φ can be written as a finite union

$$U_\varphi = \bigcup_{s \in F} \left(\varphi^{-1}(s) \times \varphi^{-1}(s)\right) \tag{2.1}$$

The sets of the form U_φ, where φ runs over the class of all morphisms from M onto a monoid of $\mathbf{V}$, form the basis of a uniformity on M, called the pro-$\mathbf{V}$ uniformity (see [13] for more details). The topology defined by the pro-$\mathbf{V}$ uniformity is called the pro-$\mathbf{V}$ topology.

The following result is a consequence of the results of [13], but we give a direct proof for the convenience of the reader.

Proposition 2.1 *Let M be a monoid. Then the pro-$\mathbf{V}$ uniformity is transitive and totally bounded.*

Proof. The pro-$\mathbf{V}$ uniformity is transitive, since the relations U_φ are transitive. Let $\varphi : M \to F$ a morphism from M onto a monoid of $\mathbf{V}$. The decomposition (2.1), together with the equality $M = \bigcup_{s \in F} \varphi^{-1}(s)$, show that the pro-$\mathbf{V}$ uniformity is totally bounded. $\square$

We now turn to a more precise description of the pro-$\mathbf{V}$ topology.

Proposition 2.2 *Let M be a monoid. Then the $\mathbf{V}$-recognizable sets of M form a basis of clopen subsets of the pro-$\mathbf{V}$ topology.*

Proof. It follows from [13, Proposition 3.6] that the sets of the form $\varphi^{-1}(s)$, where $\varphi : M \to F$ a morphism from M onto a monoid of $\mathbf{V}$ and $s \in F$ form a basis of clopen sets. These sets are by construction $\mathbf{V}$-recognizable and every $\mathbf{V}$-recognizable set is a finite union of such sets. Therefore, the $\mathbf{V}$-recognizable sets also form a basis of clopen subsets of the pro-$\mathbf{V}$ topology. $\square$

Note that the recognizable pro-$\mathbf{V}$ clopen subsets of M are not necessarily $\mathbf{V}$-recognizable, even if M is a free monoid. For instance, if $A = \{a, b\}$, $M = A^*$ and $\mathbf{V}$ is the variety of monoids with central idempotents, defined by the profinite identity $x^\omega y = yx^\omega$, then the pro-$\mathbf{V}$ topology on A^* is trivial. Indeed, the syntactic monoid of any finite language belongs to $\mathbf{V}$ and hence, every finite language is pro-$\mathbf{V}$ clopen. It follows immediately that every language is pro-$\mathbf{V}$ clopen and thus the pro-$\mathbf{V}$ topology is trivial. Now the language aA^* is recognizable and pro-$\mathbf{V}$ clopen, but it is not a $\mathbf{V}$-language since its syntactic monoid does not belong to $\mathbf{V}$.

The variety $\mathbf{G}$ is a singular exception. The following result was proved in [10, Corollary 3.6].

Proposition 2.3 *Let A be a finite alphabet. A recognizable language of A^* is clopen in the pro-group topology if and only if it is a group language.*

2.3 Pseudometric associated with a variety

Let M be a monoid and let $u, v \in M$. We say that a monoid N *separates* u and v if there exists a monoid morphism $\varphi : M \to N$ such that $\varphi(u) \neq \varphi(v)$.

To each variety $\mathbf{V}$ of finite monoids, one can attach a pu-metric $d_{\mathbf{V}}$ on M defined as follows. Set, for all $u, v \in M$,

$$r_{\mathbf{V}}(u, v) = \min \left\{ |N| \,\middle|\, N \text{ is in } \mathbf{V} \text{ and separates } u \text{ and } v \right\}$$

with the convention $\min \emptyset = \infty$. This *valuation* satisfies the following properties, for all u, v in M:

(1) $r_{\mathbf{V}}(u, v) = r_{\mathbf{V}}(v, u)$

(2) $r_{\mathbf{V}}(u, w) \geqslant \min \{ r_{\mathbf{V}}(u, v), r_{\mathbf{V}}(v, w) \}$

Finally, we put

$$d_{\mathbf{V}}(u, v) = 2^{-r_{\mathbf{V}}(u,v)}$$

with the convention $2^{-\infty} = 0$. Then $d_{\mathbf{V}}$ is a pu-metric.

2.4 Hausdorff property

A monoid M is *residually* $\mathbf{V}$ if any two distinct elements of M are separated by a monoid in $\mathbf{V}$. The next result follows immediately from the definitions.

Proposition 2.4 *The following conditions are equivalent*

(1) *M is residually $\mathbf{V}$,*

(2) *the pro-$\mathbf{V}$ topology is Hausdorff,*

(3) *$d_{\mathbf{V}}$ is an ultrametric.*

Propositions 1.1 and 2.1 now give the following result.

Proposition 2.5 *If a monoid is residually $\mathbf{V}$, its completion for the pro-$\mathbf{V}$ uniformity is compact.*

3 Metrizability and related questions

The question arises to know when the pro-$\mathbf{V}$ uniformity can be defined by a pu-metric, or even by an ultrametric. The answer is given in the next proposition.

Proposition 3.1 *Let M be a monoid. The following conditions are equivalent:*

(1) *the pro-$\mathbf{V}$ uniformity on M has a countable basis,*

(2) *there are at most countably many $\mathbf{V}$-recognizable sets in M,*

(3) *for each monoid $F \in \mathbf{V}$, there are only countably many morphisms from M onto F,*

(4) *the pro-$\mathbf{V}$ uniformity on M can be defined by a pu-metric.*

If these conditions are satisfied, the pro-$\mathbf{V}$ uniformity can be defined by a metric if and only if M is residually $\mathbf{V}$.

Proof. The equivalence of (1) and (4) follows from Proposition 1.2.

(2) implies (1). The entourages U_φ form a basis of the pro-$\mathbf{V}$ uniformity and (2.1) shows that there are countably many such sets.

(3) implies (2). First, $\mathbf{V}$ contains at most a countable number of nonisomorphic monoids. The result follows, since a countable union of countable sets is still countable.

(1) implies (3). Let $\varphi : M \to F$, $\varphi' : M \to F'$ be morphisms from M onto monoids in $\mathbf{V}$. We have $U_\varphi \subseteq U_{\varphi'}$ if and only if there exists some morphism $\theta : F \to F'$ such that $\varphi' = \theta \circ \varphi$. It follows that each U_φ is contained in only finitely many $U_{\varphi'}$. Let now $\mathcal{B}$ be a countable basis of the pro-$\mathbf{V}$ uniformity. For each $B \in \mathcal{B}$, there exists a morphism φ_B such that U_{φ_B} is contained in B. Furthermore, for every morphism φ, U_φ contains some $B \in \mathcal{B}$ and hence U_{φ_B}. Therefore

$$\mathcal{U} = \bigcup_{B \in \mathcal{B}} \{ U_\varphi \mid U_{\varphi_B} \subseteq U_\varphi \}$$

and thus the basis of the pro-$\mathbf{V}$ uniformity formed by the U_φ is itself countable, as a countable union of finite sets. Hence (3) holds.

The final part of the statement follows from Proposition 2.4. □

If the conditions of Proposition 3.1 are satisfied, does it imply that $d_\mathbf{V}$ defines the pro-$\mathbf{V}$ uniformity? The full answer to this question is given by the next proposition.

Proposition 3.2 *Let M be a monoid. The following conditions are equivalent:*

(1) *for each monoid $F \in \mathbf{V}$, there are only finitely many morphisms from M onto F,*

(2) *the pro-$\mathbf{V}$ uniformity can be defined by $d_\mathbf{V}$.*

Proof. Let $\mathcal{U}'$ be the uniformity on M defined by $d_\mathbf{V}$. A basis of $\mathcal{U}'$ is given by the subsets of $M \times M$

$$U'_\varepsilon = \{ (x, y) \in M \times M \mid d_\mathbf{V}(x, y) < \varepsilon \} \qquad (\varepsilon > 0)$$

We first prove the inclusion $\mathcal{U} \subseteq \mathcal{U}'$ by showing that the relation

$$U'_{2^{-|F|}} \subseteq U_\varphi \tag{3.2}$$

holds for every morphism φ from M onto a monoid $F \in \mathbf{V}$.

Indeed, if $(u, v) \in U'_{2^{-|F|}}$, then $d_{\mathbf{V}}(u, v) < 2^{-|F|}$ and so $r_{\mathbf{V}}(u, v) > |F|$. Hence $\psi(u) = \psi(v)$ for every morphism ψ from M onto a monoid of $\mathbf{V}$ of size at most $|F|$. In particular, $\varphi(u) = \varphi(v)$ and so $(u, v) \in U_\varphi$. Thus (3.2) holds.

(1) implies (2). Let $\varepsilon > 0$. Take $n \in \mathbb{N}$ such that $2^{-n} < \varepsilon$. Let $\varphi_i : M \to F_i$ $(i = 1, \ldots, k)$ be an enumeration of all morphisms from M onto (nonisomorphic) monoids of $\mathbf{V}$ of size at most 2^{-n}. Let φ be the morphism from M into $F_1 \times \ldots \times F_k$ defined by $\varphi(u) = (\varphi_1(u), \ldots, \varphi_k(u))$ and let $F = \varphi(M)$. Then F belongs to $\mathbf{V}$ and φ induces a surjective morphism from M onto F. We claim that

$$U_\varphi \subseteq U'_\varepsilon. \tag{3.3}$$

Indeed, $\varphi(u) = \varphi(v)$ implies $\varphi_i(u) = \varphi_i(v)$ for $i = 1, \ldots, k$ and thus u and v cannot be separated by a morphism onto a monoid of $\mathbf{V}$ of size at most n. Hence $r_{\mathbf{V}}(u, v) > n$ and so $d_{\mathbf{V}}(u, v) < 2^{-n} < \varepsilon$, which proves the claim.

It follows from (3.2) and (3.3) that $\mathcal{U}'$ coincides with the pro-$\mathbf{V}$ uniformity, which proves (2).

(2) implies (1). Let $n \in \mathbb{N}$ and let F be a monoid of $\mathbf{V}$ of size n. For every morphism $\varphi : M \to F$, one has $U'_{2^{-n}} \subseteq U_\varphi$ by (3.2). Since $\mathcal{U}'$ is the pro-$\mathbf{V}$ uniformity on M, there exists some morphism $\psi : M \to N$ onto a monoid of $\mathbf{V}$ such that $U_\psi \subseteq U'_{2^{-n}}$. Thus $U_\psi \subseteq U_\varphi$ and there exists some morphism $\theta : N \to F$ such that $\varphi = \theta \circ \psi$. Since there are only finitely many morphisms between the finite monoids N and F, (1) holds. $\square$

If M is a finitely generated monoid and F is a finite monoid, there are only finitely many morphisms from M onto F. Therefore, we get the following corollary.

Corollary 3.3 *On a finitely generated monoid, the pro-$\mathbf{V}$ uniformity can be defined by $d_{\mathbf{V}}$.*

It is important to note that the conclusion of Corollary 3.3 may fail if M is not finitely generated. We give two instructive counterexamples. In both cases, the uniform structure is associated with the variety $\mathbf{M}$ of all finite monoids and is simply called the profinite uniformity. Similarly, we simplify the notation $d_{\mathbf{M}}$ to d.

Proposition 3.4 *On $(\mathbb{N}, \max)$, the metric d is discrete, but the profinite uniformity is not.*

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Proof. It is easy to see that any two distinct elements of $\mathbb{N}$ can be separated by a morphism onto the two-element semilattice. Then d can only take the values 0 or $1/4$ and hence d is discrete.

However, the diagonal of $\mathbb{N}$ does not belong to the profinite uniformity since any morphism from $\mathbb{N}$ onto a finite monoid is necessarily noninjective. $\square$

We now consider the case of a free monoid.

Proposition 3.5 *Let A be an alphabet. Then the following properties hold:*
  (1) The completion of A^* for the profinite uniformity is compact.
  (2) The completion of A^* for d is compact if and only if A is finite.
In particular, the profinite uniformity is defined by d if and only if A is finite.

Proof. Property (1) follows from Proposition 2.5, since A^* is residually finite. Further Corollary 3.3 shows that if A is finite, the profinite uniformity is defined by d and hence its completion is compact. Now, if a and b are letters of A, the value of $d(a, b)$ is 0 or $1/4$. It follows that if A is infinite, any infinite sequence of pairwise distinct letters of A has no Cauchy subsequence and cannot converge in the completion of (A^*, d). $\square$

4 Continuous functions

In this section, we study continuous and uniformly continuous functions.

4.1 Continuity and recognizable sets

Let $\mathbf{V}$ and $\mathbf{W}$ be two varieties of finite monoids and let M and N be two monoids. We say that a function f from M to N is $(\mathbf{V}, \mathbf{W})$-*continuous* if it is continuous with respect to the pro-$\mathbf{V}$ topology on M and the pro-$\mathbf{W}$ topology on N. Similarly, a function f from M to N is $(\mathbf{V}, \mathbf{W})$-*uniformly continuous* if it is uniformly continuous with respect to the pro-$\mathbf{V}$ uniformity on M and the pro-$\mathbf{W}$ uniformity on N. When $\mathbf{V} = \mathbf{W}$, we simply refer to $\mathbf{V}$-*(uniform) continuity.*

If M is compact (not necessarily Hausdorff) for the pro-$\mathbf{V}$ uniformity, then $\mathbf{V}$-continuity is equivalent to $\mathbf{V}$-uniform continuity, but in general $\mathbf{V}$-uniform continuity is a stronger property than $\mathbf{V}$-continuity. For instance, consider the monoid $\mathbb{N}$ of nonnegative integers under addition. It is shown in [11] that all functions from $\mathbb{N}$ to $\mathbb{N}$ are $\mathbf{A}$-continuous. However, a function from $\mathbb{N}$ to $\mathbb{N}$ is $\mathbf{A}$-uniformly continuous if and only if it is eventually constant or tends to infinity when n tends to infinity.

We now give a simple characterization of these topological notions in terms of recognizable sets.

Theorem 4.1 *Let M and N be two monoids and let $f : M \to N$ be a mapping. Then*

   (1) *f is $(\mathbf{V}, \mathbf{W})$-uniformly continuous if and only if for any $\mathbf{W}$-recognizable subset L of N, $f^{-1}(L)$ is a $\mathbf{V}$-recognizable subset of M.*

   (2) *f is $(\mathbf{V}, \mathbf{W})$-continuous if and only if for any $\mathbf{W}$-recognizable subset L of N, $f^{-1}(L)$ is a (possibly infinite) union of $\mathbf{V}$-recognizable subsets of M.*

Proof. (1) Let L be a $\mathbf{W}$-recognizable subset of N. Then there exists a monoid F of $\mathbf{W}$, a morphism $\varphi : N \to F$ such that $L = \varphi^{-1}(\varphi(L))$. If f is $(\mathbf{V}, \mathbf{W})$-uniformly continuous, there is a morphism ψ from M onto a monoid S of $\mathbf{V}$ such that the condition $(u, v) \in U_\psi$ implies $(f(u), f(v)) \in U_\varphi$. In other words, $\psi(u) = \psi(v)$ implies $\varphi(f(u)) = \varphi(f(v))$. We claim that

$$\psi^{-1}(\psi(f^{-1}(L))) = f^{-1}(L)$$

First, $f^{-1}(L)$ is clearly a subset of $\psi^{-1}(\psi(f^{-1}(L)))$. To prove the opposite inclusion, let $u \in \psi^{-1}(\psi(f^{-1}(L)))$. Then $\psi(u) \in \psi(f^{-1}(L))$, that is, $\psi(u) = \psi(v)$ for some $v \in f^{-1}(L)$. Thus $\varphi(f(u)) = \varphi(f(v))$ and, since $f(v) \in L$, one gets $f(u) \in \varphi^{-1}(\varphi(L))$ and finally $f(u) \in L$ since $L = \varphi^{-1}(\varphi(L))$. This proves the claim and shows that $f^{-1}(L)$ is $\mathbf{V}$-recognizable.

Suppose now that, for any $\mathbf{W}$-recognizable subset L of N, $f^{-1}(L)$ is a $\mathbf{V}$-recognizable subset of M. Let φ be a morphism from N onto a monoid F of $\mathbf{W}$. For each $r \in F$, $\varphi^{-1}(r)$ is a $\mathbf{W}$-recognizable subset of N and hence $f^{-1}(\varphi^{-1}(r))$ is a $\mathbf{V}$-recognizable subset of M, recognized by some morphism ψ_r from M onto a monoid F_r of $\mathbf{V}$. Let $\psi : M \to \prod_{r \in F} F_r$ be the morphism defined by $\psi(x) = (\psi_r(x))_{r \in F}$ and let $T = \psi(M)$. As a submonoid of a finite product of monoids of $\mathbf{V}$, T also belongs to $\mathbf{V}$. Now, if $(u, v) \in U_\psi$, then $\psi(u) = \psi(v)$ and hence $\psi_r(u) = \psi_r(v)$ for all $r \in F$. Since ψ_r recognizes $f^{-1}(\varphi^{-1}(r))$, it follows that $u \in f^{-1}(\varphi^{-1}(r))$ if and only if $v \in f^{-1}(\varphi^{-1}(r))$, that is, $f(u) \in \varphi^{-1}(r)$ if and only if $f(v) \in \varphi^{-1}(r)$. Therefore $(f(u), f(v)) \in U_\varphi$, which shows that f is $(\mathbf{V}, \mathbf{W})$-uniformly continuous.

(2) By Proposition 2.2, the $\mathbf{V}$-recognizable sets form a basis $\mathcal{B}$ of clopen sets for the pro-$\mathbf{V}$ topology. It follows that a set is open if and only if it is a union of $\mathbf{V}$-recognizable sets. Now f is $(\mathbf{V}, \mathbf{W})$-continuous if and only if, for every $B \in \mathcal{B}$, $f^{-1}(B)$ is open, which is another way to state (2). $\square$

An interesting case arises when M is a free monoid and $\mathbf{V} = \mathbf{G}$.

Theorem 4.2 *Let A be a finite alphabet and let M be a monoid. Let $f : A^* \to M$ be a mapping such that, for every $\mathbf{W}$-recognizable subset of M, $f^{-1}(L)$ is recognizable. Then the following conditions are equivalent:*

   (1) *f is $(\mathbf{G}, \mathbf{W})$-continuous,*

   (2) *f is $(\mathbf{G}, \mathbf{W})$-uniformly continuous,*

(3) *for any* **W***-recognizable subset L of M, $f^{-1}(L)$ is a group language.*

Proof. The equivalence of (2) and (3) follows from Theorem 4.1. Since (2) implies (1), it remains to prove that (1) implies (3). Suppose that f is $(\mathbf{G}, \mathbf{W})$-continuous and let L be a **W**-recognizable subset of M. Then L is clopen for the pro-**W** topology and since f is $(\mathbf{G}, \mathbf{W})$-continuous, $f^{-1}(L)$ is clopen for the pro-group topology and it is also a recognizable language by the assumption on f. It follows by Proposition 2.3 that $f^{-1}(L)$ is a group language, which proves (3). □

If $\mathbf{W} = \mathbf{G}$ and $M = B^*$, we get the following corollary, which extends a result of Reutenauer and Schützenberger [14].

Corollary 4.3 *Let A and B be two finite alphabets and let $f : A^* \to B^*$ be a mapping such that, for every group language of B^*, $f^{-1}(L)$ is recognizable. Then the following conditions are equivalent:*

(1) *f is continuous for the pro-group topology,*

(2) *f is uniformly continuous for the pro-group uniformity,*

(3) *for any group language L of B^*, $f^{-1}(L)$ is a group language.*

The original statement of Reutenauer and Schützenberger required f to be a rational function, but this condition is not mandatory. For instance, the function from A^* into itself defined by $f(u) = u^2$ is not rational, but it is continuous for the pro-group topology.

4.2 Hereditary continuity

Let **V** be a variety of finite monoids. A function is **V***-hereditarily continuous* if it is **W**-uniformly continuous for each subvariety **W** of **V**.

Note that **V**-hereditary continuity is in general a stronger property than **V**-uniform continuity. For instance, it is shown in [11] that a function f from $\mathbb{Z}$ to $\mathbb{Z}$ is **G**-uniformly continuous if and only if, for all $r \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that for all $u, v \in \mathbb{Z}$,

$$u \equiv v \pmod{s} \quad \text{implies} \quad f(u) \equiv f(v) \pmod{r}$$

On the other hand, f is **G**-hereditarily continuous if and only if, for all $u, v \in \mathbb{Z}$, $u - v$ divides $f(u) - f(v)$. It follows that the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

is **G**-uniformly continuous but not **G**-hereditarily continuous.

Morphisms provide a first general example of hereditarily continuous functions.

Proposition 4.4 *Every monoid morphism is* **V***-hereditarily continuous.*

Proof. Let $\theta : M \to N$ be a morphism of monoids. Let φ be a morphism from N onto a monoid F of **V**. Then $\psi = \varphi \circ \theta$ is a morphism from M onto a monoid of **V**. Since $\psi(u) = \psi(v)$ implies $\varphi(\theta(u)) = \varphi(\theta(v))$ for all $u, v \in M$ trivially, it follows that θ is **V**-uniformly continuous. Since **V** is arbitrary, it is **V**-hereditarily continuous as well. $\square$

The following result is quite useful to deal with unions of chains of varieties.

Proposition 4.5 *Let* **V** *be a non finitely generated variety. Then a function is* **V***-hereditarily continuous if and only if it is* **W***-uniformly continuous for every proper subvariety* **W** *of* **V***.*

Proof. By definition of hereditary continuity, it suffices to prove that if a function $f : M \to N$ is **W**-uniformly continuous for every proper subvariety **W** of **V**, then it is **V**-uniformly continuous.

Let φ be a morphism from N onto a monoid F of **V**. Since **V** is non finitely generated, F belongs to some proper subvariety **W** of **V**. Now since f is **W**-uniformly continuous, there exists a morphism ψ from M onto some monoid of **W** such that, for all $x, y \in M$, $(x, y) \in U_\psi$ implies $(f(x), f(y)) \in U_\varphi$. Since **W** $\subseteq$ **V**, it follows that f is **V**-uniformly continuous. $\square$

4.3   Closure properties

We now show that composition, product and exponential behave well with respect to continuity, uniform continuity and hereditarily continuity.

In this section **V**, **W** and **X** denote three varieties.

Proposition 4.6 *Let M, N and R be monoids and let $f : M \to N$ and $g : N \to R$ be mappings.*

(1) *If f is* $(\mathbf{V}, \mathbf{W})$*-[uniformly] continuous and g is* $(\mathbf{W}, \mathbf{X})$*-[uniformly] continuous, then $g \circ f$ is* $(\mathbf{V}, \mathbf{X})$*-[uniformly] continuous.*

(2) *If f and g are* **V***-hereditarily continuous, then $g \circ f$ is* **V***-hereditarily continuous.*

Proof. Condition (1) follows from the general fact that the composition of two [uniformly] continuous functions is [uniformly] continuous.

Condition (2) follows from (1). $\square$

Given two functions f, g from a monoid M into a monoid N, the *product* of f and g is the function fg from M to N defined by $(fg)(x) = f(x)g(x)$.

Theorem 4.7 *Let f, g be functions from a monoid M into a monoid N.*

(1) *If f and g are both* $(\mathbf{V}, \mathbf{W})$*-[uniformly] continuous, so is* fg.

(2) *If f and g are both* $\mathbf{V}$*-hereditarily continuous, so is* fg.

Proof. Let L be a $\mathbf{W}$-recognizable subset of N and let φ be a monoid morphism from N onto some monoid F of $\mathbf{W}$ recognizing L. We claim that

$$(fg)^{-1}(L) = \bigcup_{\{(r,s) \in F \times F \mid rs \in \varphi(L)\}} f^{-1}(\varphi^{-1}(r)) \cap g^{-1}(\varphi^{-1}(s)) \qquad (4.4)$$

Denote by R the right member of (4.4) and let $u \in (fg)^{-1}(L)$. Setting $r = \varphi(f(u))$ and $s = \varphi(g(u))$, we get

$$rs = \varphi(f(u))\varphi(g(u)) = \varphi(f(u)g(u)) = \varphi((fg)(u))$$

and since $(fg)(u) \in L$, then $rs \in \varphi(L)$. Further, $u \in f^{-1}(\varphi^{-1}(r)) \cap g^{-1}(\varphi^{-1}(s))$ and hence u belongs to R. This proves that $(fg)^{-1}(L)$ is contained in R.

To establish the opposite inclusion, consider an element $u \in R$. Then $u \in f^{-1}(\varphi^{-1}(r)) \cap g^{-1}(\varphi^{-1}(s))$ for some elements r and s of F such that $rs \in \varphi(L)$. One gets $\varphi(f(u)) = r$ and $\varphi(g(u)) = s$, whence $\varphi(f(u)g(u)) = rs \in \varphi(L)$ and thus $f(u)g(u) \in \varphi^{-1}(\varphi(L)) = L$. Thus $u \in (fg)^{-1}(L)$ which proves (4.4).

(1) Suppose now that f and g are $(\mathbf{V}, \mathbf{W})$-continuous. As each of the sets $\varphi^{-1}(r)$ and $\varphi^{-1}(s)$ is $\mathbf{W}$-recognizable by construction, then by Theorem 4.1, the sets $f^{-1}(\varphi^{-1}(r))$ and $f^{-1}(\varphi^{-1}(s))$ are open and by (4.4), $(fg)^{-1}(L)$ is open. Thus fg is $(\mathbf{V}, \mathbf{W})$-continuous.

Suppose now that f and g are $(\mathbf{V}, \mathbf{W})$-uniformly continuous. Then by Theorem 4.1, the sets $f^{-1}(\varphi^{-1}(r))$ and $f^{-1}(\varphi^{-1}(s))$ are $\mathbf{V}$-recognizable and hence $(fg)^{-1}(L)$ is $\mathbf{V}$-recognizable. Thus fg is $(\mathbf{V}, \mathbf{W})$-uniformly continuous.

(2) The case of $\mathbf{V}$-hereditarily continuous functions follows immediately from (1). $\square$

Given a function f from a monoid M into a monoid N and a function g from M to $\mathbb{N}$, the *exponential* of f by g is the function f^g from M to N defined by $(fg)(x) = f(x)^{g(x)}$. If N is a group, the exponential can be actually defined for every function g from M to $\mathbb{Z}$.

Theorem 4.8 *Let f be a function from a monoid M into a monoid N and let g be a function from M to the additive monoid $\mathbb{N}$ of nonnegative integers.*

(1) *If f and g are both* $(\mathbf{V}, \mathbf{W})$*-[uniformly] continuous, so is* f^g.

(2) *If f and g are both* $\mathbf{V}$*-hereditarily continuous, so is* f^g.

Proof. Let L be a $\mathbf{W}$-recognizable subset of N and let φ be a monoid morphism from N onto some monoid F of $\mathbf{W}$ recognizing L. We claim that, for each $r \in F$, the set

$$E_r = \{n \in \mathbb{N} \mid r^n \in \varphi(L)\}$$

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is a $\mathbf{W}$-recognizable subset of $\mathbb{N}$ (considered as an additive monoid) and that

$$(f^g)^{-1}(L) = \bigcup_{r \in F} f^{-1}(\varphi^{-1}(r)) \cap g^{-1}(E_r) \qquad (4.5)$$

Let $r \in F$. Since $F \in \mathbf{W}$, the submonoid $\langle r \rangle$ of F generated by r belongs to $\mathbf{W}$. Further, the map π from $\mathbb{N}$ to $\langle r \rangle$ defined by $\pi(n) = r^n$ is a morphism. Thus E_r is equal to $\pi^{-1}(\varphi(L))$ and hence is recognizable.

Denote by R the right member of (4.5) and let $u \in (f^g)^{-1}(L)$. Setting $r = \varphi(f(u))$ and $n = g(u)$, we get

$$\varphi((f^g)(u)) = \varphi(f(u)^{g(u)}) = \varphi(f(u)^n) = r^n$$

Since $(f^g)(u) \in L$, one has $\varphi((f^g)(u)) \in \varphi(L)$, whence $r^n \in \varphi(L)$ and finally $n \in E_r$. It follows that $u \in f^{-1}(\varphi^{-1}(r)) \cap g^{-1}(E_r)$ and this proves that $(f^g)^{-1}(L)$ is contained in R.

To establish the opposite inclusion, consider an element $u \in R$. Then $u \in f^{-1}(\varphi^{-1}(r)) \cap g^{-1}(n)$ for some $r \in F$ and $n \in E_r$. Therefore $\varphi(f(u)) = r$ and $g(u) = n$, whence $\varphi(f(u)^{g(u)}) = r^n \in \varphi(L)$ since $n \in E_r$. It follows that $f(u)^{g(u)} \in \varphi^{-1}(\varphi(L)) = L$. Thus $u \in (f^g)^{-1}(L)$ which proves (4.5).

The end of the proof is similar to that of Theorem 4.7. □

If N is a group, Theorem 4.8 can be extended as follows.

Theorem 4.9 *Let f be a function from a monoid M into a group G and let g be a function from M to the additive group of integers $\mathbb{Z}$.*

(1) *If f and g are both $(\mathbf{V}, \mathbf{W})$-[uniformly] continuous, so is f^g.*

(2) *If f and g are both $\mathbf{V}$-hereditarily continuous, so is f^g.*

Proof. The proof is almost identical to that of Theorem 4.8, and is therefore omitted. □

5   Intersection and join

In this section, we study $(\mathbf{V}, \mathbf{W})$-uniform (resp. $\mathbf{V}$-hereditary) continuity when $\mathbf{V}$ or $\mathbf{W}$ is the intersection or the union of a family varieties, with special attention to the case where $\mathbf{V}$ is a variety of finite commutative monoids.

5.1   Intersection of varieties

Consider three varieties $\mathbf{V}$, $\mathbf{W}$ and $\mathbf{X}$ and let f be a $(\mathbf{V}, \mathbf{W})$-uniformly continuous function. In this section, we address the following type of questions: when is f $(\mathbf{V}, \mathbf{W} \cap \mathbf{X})$-uniformly continuous (resp. $(\mathbf{V} \cap \mathbf{X}, \mathbf{W})$-uniformly continuous, $(\mathbf{V} \cap \mathbf{X}, \mathbf{W} \cap \mathbf{X})$-uniformly continuous)?

We say that a monoid M has *finite $\mathbf{V}$-quotients* if every finite quotient of M is in $\mathbf{V}$. Note that a monoid with finite $\mathbf{V}$-quotients does not necessarily belongs to the Birkhoff variety generated by $\mathbf{V}$. For instance, the bicyclic monoid has finite commutative quotients, but is not commutative.

Proposition 5.1 *Let $f : M \to N$ be a function.*
1. *If N has finite $\mathbf{X}$-quotients, then f is $(\mathbf{V}, \mathbf{W})$-uniformly continuous if and only if it is $(\mathbf{V}, \mathbf{W} \cap \mathbf{X})$-uniformly continuous.*
2. *If M has finite $\mathbf{X}$-quotients, then f is $(\mathbf{V}, \mathbf{W})$-uniformly continuous if and only if it is $(\mathbf{V} \cap \mathbf{X}, \mathbf{W})$-uniformly continuous.*

Proof. (1) The direct implication is trivial.

Let φ be a morphism from N onto a monoid F of $\mathbf{W}$. Since N has finite $\mathbf{X}$-quotients, we have $F \in \mathbf{W} \cap \mathbf{X}$. Since f is $(\mathbf{V}, \mathbf{W} \cap \mathbf{X})$-uniformly continuous, there exists some morphism ψ from M onto a monoid S of $\mathbf{V}$ such that $\psi(u) = \psi(v)$ implies $\varphi(f(u)) = \varphi(f(v))$ for all $u, v \in M$. Therefore f is $(\mathbf{V}, \mathbf{W})$-uniformly continuous.

(2) The proof is a straighforward adaptation of the proof of (1) and can be omitted. □

Corollary 5.2 *Let M and N have finite $\mathbf{X}$-quotients and let $f : M \to N$ be a function. Then f is $(\mathbf{V}, \mathbf{W})$-uniformly continuous if and only if it is $(\mathbf{V} \cap \mathbf{X}, \mathbf{W} \cap \mathbf{X})$-uniformly continuous.*

Proof. It follows from a double application of Proposition 5.1. □

Corollary 5.3 *Let N be a monoid with finite $\mathbf{X}$-quotients and let $f : M \to N$ be a function. Then f is $\mathbf{V}$-hereditarily continuous if and only if f is $(\mathbf{V} \cap \mathbf{X})$-hereditarily continuous.*

Proof. The direct implication is trivial since any subvariety of $\mathbf{V} \cap \mathbf{X}$ is a subvariety of $\mathbf{V}$.
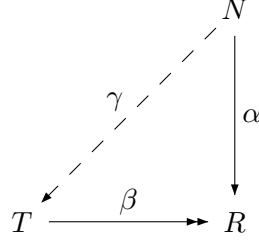
Conversely, let $\mathbf{W}$ be a subvariety of $\mathbf{V}$. Since f is $(\mathbf{V} \cap \mathbf{X})$-hereditarily continuous, then f is $(\mathbf{W} \cap \mathbf{X})$-uniformly continuous and hence $(\mathbf{W} \cap \mathbf{X}, \mathbf{W})$-uniformly continuous by Proposition 5.1 (1). Therefore f is $\mathbf{W}$-uniformly continuous and thus $\mathbf{V}$-hereditarily continuous. □

Applying this result when $\mathbf{X}$ is the variety of finite commutative monoids, we get:

Proposition 5.4 *A function from a monoid into a commutative monoid is $\mathbf{V}$-hereditarily continuous if and only if it is $(\mathbf{V} \cap \mathbf{Com})$-hereditarily continuous.*
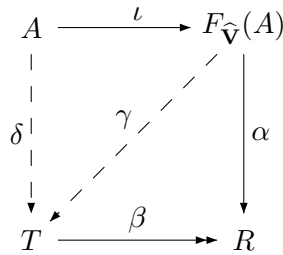
5.2 Join of varieties

To deal with joins of varieties, we introduce the following concept, inherited from ring theory. Let us say that a monoid N is $\mathbf{V}$-*projective* if the following property holds: if $\alpha : N \to R$ is a morphism and if $\beta : T \to R$ is a surjective morphism, with T (and hence R) in $\mathbf{V}$, then there exists a morphism $\gamma : N \to T$ such that $\alpha = \beta \circ \gamma$.

$$\begin{array}{ccc}
 & & N \\
 & \overset{\gamma}{\diagup} & \big\downarrow \alpha \\
T & \underset{\beta}{\longrightarrow} & R
\end{array}$$

For instance, every free monoid is $\mathbf{V}$-projective for any $\mathbf{V}$. Also note that any $\mathbf{V}$-projective monoid is $\mathbf{W}$-projective for any subvariety $\mathbf{W}$ of $\mathbf{V}$. Further examples of $\mathbf{V}$-projective monoids are given by the following result.

Proposition 5.5 *Let $\mathbf{V}$ be a variety of finite monoids and let $\widehat{\mathbf{V}}$ be any Birkhoff variety containing $\mathbf{V}$. Then every $\widehat{\mathbf{V}}$-free monoid is $\mathbf{V}$-projective.*

Proof. Let A be a set and let $F_{\widehat{\mathbf{V}}}(A)$ be the $\widehat{\mathbf{V}}$-free monoid on A. Let also $\iota : A \to F_{\widehat{\mathbf{V}}}(A)$ be the canonical mapping. Let $\alpha : F_{\widehat{\mathbf{V}}}(A) \to R$ be a morphism and let $\beta : T \to R$ be a surjective morphism, with T in $\mathbf{V}$. Since β is surjective, there exists a function $\delta : A \to T$ such that $\beta \circ \delta = \alpha \circ \iota$. Since $T \in \mathbf{V}$, $\mathbf{V}$ is contained in $\widehat{\mathbf{V}}$ and $F_{\widehat{\mathbf{V}}}(A)$ is $\widehat{\mathbf{V}}$-free on A, there exists a morphism $\gamma : F_{\widehat{\mathbf{V}}}(A) \to T$ such that $\gamma \circ \iota = \delta$.

$$\begin{array}{ccc}
A & \overset{\iota}{\longrightarrow} & F_{\widehat{\mathbf{V}}}(A) \\
\big\downarrow{\delta} & \overset{\gamma}{\diagup} & \big\downarrow \alpha \\
T & \underset{\beta}{\longrightarrow} & R
\end{array}$$

Thus $\beta \circ \gamma \circ \iota = \beta \circ \delta = \alpha \circ \iota$, whence $\beta \circ \gamma = \alpha$ since $\iota(A)$ generates $F_{\widehat{\mathbf{V}}}(A)$. Therefore $F_{\widehat{\mathbf{V}}}(A)$ is $\mathbf{V}$-projective. $\square$

The importance of projective monoids in the study of hereditarily continuous functions stems from the property stated in Proposition 5.7 below. We first need a weak form of distributive law for commutative varieties.

Lemma 5.6 *Let* **V** *be the join of a family* $(\mathbf{V}_i)_{i \in I}$ *of varieties of commutative monoids and let* **W** *be a subvariety of a variety* **V**. *Then*

$$\mathbf{W} = \mathbf{V} \cap \mathbf{W} = \left(\bigvee_{i \in I} \mathbf{V}_i \right) \cap \mathbf{W} = \bigvee_{i \in I} (\mathbf{V}_i \cap \mathbf{W})$$

Proof. The corresponding result for varieties in the Birkhoff sense follows from [9]. The lemma now follows from the fact that, if **V** is a Birkhoff variety, and $\mathbf{V}_{fin}$ is the variety of all finite members of **V**, the lattice of all subvarieties of $\mathbf{V}_{fin}$ inherits all lattice identities from the lattice of all subvarieties of **V**. This result follows implicitly from a general result by Ash [5] and is explicitly contained in [1]. □

Proposition 5.7 *Let* **V** *be the join of a family* $(\mathbf{V}_i)_{i \in I}$ *of commutative monoids. A function from a monoid into a* **V**-*projective monoid is* **V**-*hereditarily continuous if and only if it is* $\mathbf{V}_i$-*hereditarily continuous for all* $i \in I$.

Proof. Let f be a function from a monoid M onto a **V**-projective monoid N. If f is **V**-hereditarily continuous, it is also $\mathbf{V}_i$-hereditarily continuous for all $i \in I$, since $\mathbf{V}_i$ is a subvariety of **V**.

Suppose now that f is $\mathbf{V}_i$-hereditarily continuous for all $i \in I$. Let **W** be a subvariety of **V**. Setting, for every $i \in I$, $\mathbf{W}_i = \mathbf{V}_i \cap \mathbf{W}$, one has by Lemma 5.6

$$\mathbf{W} = \mathbf{V} \cap \mathbf{W} = \left(\bigvee_{i \in I} \mathbf{V}_i \right) \cap \mathbf{W} = \bigvee_{i \in I} (\mathbf{V}_i \cap \mathbf{W}) = \bigvee_{i \in I} \mathbf{W}_i$$

Let φ be a morphism from N onto a monoid F of **W**. Then there is a submonoid T of a direct product of the form $R_1 \times \ldots \times R_k$, with $R_j \in \mathbf{W}_{i_j}$ for $1 \leqslant j \leqslant k$, and a surjective morphism from T onto F. Let us denote by $\pi_j : R_1 \times \cdots \times R_k \to R_j$ the natural projection. Since **W** is a subvariety of **V**, T belongs to **V** and since N is **V**-projective, there exists a morphism $\gamma : N \to T$ such that $\beta \circ \gamma = \varphi$. Setting, for $1 \leqslant j \leqslant k$, $\alpha_j = \pi_j \circ \gamma$, we get the following commutative diagram:

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Note that $\alpha_j(N)$ belongs to $\mathbf{W}_{i_j}$. Since f is $\mathbf{V}_{i_j}$-hereditarily continuous, it is also $\mathbf{W}_{i_j}$-uniformly continuous, and there exists a morphism ψ_j from M onto some monoid E_j of $\mathbf{W}_{i_j}$ such that, for all $x, y \in M$,

$$(x, y) \in U_{\psi_j} \text{ implies } (f(x), f(y)) \in U_{\alpha_j}. \tag{5.6}$$

Let $\psi : M \to E_1 \times \ldots \times E_k$ be the diagonal morphism defined by $\psi(x) = (\psi_1(x), \ldots, \psi_k(x))$. We claim that, for all $x, y \in M$,

$$(x, y) \in U_\psi \text{ implies } (f(x), f(y)) \in U_\varphi. \tag{5.7}$$

Indeed, if $\psi(x) = \psi(y)$, then for all $j \in \{1, \ldots, k\}$, we get $\psi_j(x) = \psi_j(y)$ and by (5.6), $\alpha_j(f(x)) = \alpha_j(f(y))$. Since $\alpha_j = \pi_j \circ \gamma$, it follows that $\pi_j(\gamma(f(x)) = \pi_j(\gamma(f(y))$ for all j and finally $\gamma(f(x)) = \gamma(f(y))$. Composing with β, we obtain $\beta(\gamma(f(x))) = \beta(\gamma(f(y)))$, that is, $\varphi(f(x)) = \varphi(f(y))$, which proves (5.7).

Since $\psi(M)$ belongs to $\mathbf{W}$, it follows that f is $\mathbf{W}$-uniformly continuous and therefore $\mathbf{V}$-hereditarily continuous. $\square$

5.3 The commutative case

In this section, we consider the case where $\mathbf{V}$ is a variety of finite commutative monoids.

We denote by $\mathbf{Ab}$ the variety of finite commutative groups, and, for each prime p, by $\mathbf{Ab}_p$ the variety of finite commutative groups whose order is a power of p. The fundamental decomposition theorem for finite commutative groups gives imediately

$$\mathbf{Ab} = \bigvee_{p \text{ prime}} \mathbf{Ab}_p \tag{5.8}$$

We also denote by $\mathbf{CA}$ the variety of finite aperiodic and commutative monoids and, for each $t \geqslant 0$, by $\mathbf{CA}_t$ the variety of finite commutative monoids M such that, for all $x \in M$, $x^t = x^{t+1}$.

Proposition 5.8 *Let f be a function into a $\mathbf{V}$-projective monoid. Then*

(1) *f is $\mathbf{V}$-hereditarily continuous if and only if it is both $(\mathbf{V} \cap \mathbf{Ab})$-hereditarily continuous and $(\mathbf{V} \cap \mathbf{CA})$-hereditarily continuous;*

(2) *f is $(\mathbf{V} \cap \mathbf{Ab})$-hereditarily continuous if and only if it is $(\mathbf{V} \cap \mathbf{Ab}_p)$-hereditarily continuous for every prime p.*

Proof. (1) By [2, Lemma 6.1.9(b)], $\mathbf{Com}$ is generated by the finite monogenic commutative monoids. Since each such monoid embeds in the direct product of a finite cyclic group by a finite monogenic aperiodic monoid, it follows that $\mathbf{Com} = \mathbf{Ab} \vee \mathbf{CA}$. Since $\mathbf{V} \subseteq \mathbf{Com}$, it follows that $\mathbf{V} = \mathbf{V} \cap (\mathbf{Ab} \vee \mathbf{CA})$ and hence by Lemma 5.6, $\mathbf{V} = (\mathbf{V} \cap \mathbf{Ab}) \vee (\mathbf{V} \cap \mathbf{CA})$.

Let N be a **V**-projective monoid and let $f : M \to N$ be a function. It follows by Proposition 5.7 that f is **V**-hereditarily continuous if and only if it is both $(\mathbf{V} \cap \mathbf{Ab})$-hereditarily continuous and $(\mathbf{V} \cap \mathbf{CA})$-hereditarily continuous.

(2) It follows from (5.8) and Lemma 5.6 that

$$\mathbf{V} \cap \mathbf{Ab} = \mathbf{V} \cap \left(\bigvee_{p \text{ prime}} \mathbf{Ab}_p \right) = \bigvee_{p \text{ prime}} (\mathbf{V} \cap \mathbf{Ab}_p)$$

Since N is **V**-projective it is also $(\mathbf{V} \cap \mathbf{Ab})$-projective. Now the result follows from Proposition 5.7. $\square$

The next characterization of $(\mathbf{V} \cap \mathbf{CA})$-hereditarily continuous functions does not require the projectivity of N.

Proposition 5.9 *A function is* $(\mathbf{V} \cap \mathbf{CA})$*-hereditarily continuous if and only if it is* $(\mathbf{V} \cap \mathbf{CA}_t)$*-uniformly continuous for every* $t \geqslant 0$.

Proof. It follows from Proposition 4.5 since **CA** is non finitely generated and its proper subvarieties are precisely the varieties $\mathbf{CA}_t$ [2]. $\square$

These decompositions are applied in [11] to get results such as:

Theorem 5.10 *[11] Let f be a function from $\mathbb{N}$ to $\mathbb{N}$. Then f is* **M**-*hereditarily continuous if and only if f is constant or satisfies the two following conditions:*
  (1) *for all $u, v \in \mathbb{N}$, $u - v$ divides $f(u) - f(v)$,*
  (2) *for all $u \in \mathbb{N}$, $f(u) \geqslant u$.*

Acknowledgements

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