SUBRING DEPTH, FROBENIUS EXTENSIONS AND THEIR TOWERS

LARS KADISON

Abstract. The minimum depth $d(B, A)$ of a subring $B \subseteq A$ recently introduced in a paper by Boltje-Danz-Külshammer is studied and compared with tower depth of a Frobenius extension. In their paper it is proven that $d(B, A)$ is finite if B and A are finite groups algebras over a commutative ring. In this paer we note that $d(B, A) < \infty$ if A is a finite dimensional algebra and B^e has finite representation type. If $A \supseteq B$ is a QF extension, minimum left and right even subring depth are shown to coincide. If $A \supseteq B$ is moreover a Frobenius extension with surjective Frobenius homomorphism, its subring depth is shown to coincide with its tower depth. Formulas for the ring, module, Frobenius and Temperley-Lieb structures are noted for the tower over a Frobenius extension in its realization as tensor powers. A depth 3 QF extension is embedded in a depth 2 QF extension; in turn certain depth n extensions embed in depth 3 extensions if they are Frobenius extensions or other special ring extensions with ring structures on their relative Hochschild bar resolution groups.

1. Introduction and Preliminaries

A theorem of Jans states that if a subalgebra B of an finite dimensional algebra A has $\mu : A \otimes_B A \to A,\ a \otimes a' \mapsto aa'$ a split epimorphism of A-A-bimodules, then A has finite representation type if B has. Weakening the condition on μ to a split epimorphism of A-B-bimodules does not place a restriction on $B \subseteq A$, but the dual hypothesis that a split monomorphism exists from $A \otimes_B A$ into a multiple $nA = A \oplus \cdots \oplus A$ captures the notion of normality of a subalgebra in the context of group algebras [16], Hopf algebras [3] and semisimple algebras [7]. If A is a Frobenius extension of B where A_B is a balanced module, the "depth two" condition as the converse hypothesis is known as, implies that A is a Galois extension of B, where the bimodule endomorphism ring of the extension may be given the structure of a Hopf algebroid (which acts naturally on A) [15, 18]. Such theorems first appeared in [27, 23] for certain finite index subfactors of depth two. The left bialgebroid aspect of the definition of Hopf algebroid was influenced by study of Lie groupoids in Poisson geometry [20], cf. [19]. The publication of [5] clarified the role played by Galois theory in depth two theory.

After the focus on depth two, study of how to generalize depth three and more from subfactor theory to algebra fell into three stages. At first the depth two condition was generalized from a subalgebra pair $B \subseteq A$ to a tower of three rings $C \subseteq B \subseteq A$ [17]. This was applied to the tower of iterated right endomorphism rings above a Frobenius extension $B \subseteq A \subseteq A_1 \hookrightarrow A_2 \hookrightarrow \cdots$, so that $B \subseteq A$ has

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(tower) depth n if $B \hookrightarrow A_{n-3} \hookrightarrow A_{n-2}$ has the generalized depth two property (called a depth-3 tower in [17]). This yields a compact matrix inequality condition $M^{[n+1]} \leq qM^{[n-1]}$ (some $q \in \mathbb{Z}_+$) for when a subalgebra pair of semisimple complex algebras has depth n in terms of the inclusion matrix M, equivalently the incidence matrix of the Bratteli diagram of the inclusion $B \hookrightarrow A$ [3, 7]. Since $M^{[2]} = MM^t$, $M^{[3]} = MM^tM, \ldots$, already in this matrix condition the odd and even depth become distinguished from one another in terms of square and rectangular matrices. From [7] the authors [2, Boltje, Danz, Külshammer] have extended the definition to a subring $B \subseteq A$, which has (right) depth $2n$ if the relative Hochschild $n+1$ bar resolution group $C_{n+1}(A, B)$ maps as a split monomorphism into a multiple of a smaller group, $qC_m(A, B)$ as A-B-bimodules; and of depth $2n + 1$ if this condition only holds as natural B-B-bimodules. Since subring $B \subseteq A$ having depth m implies that it has depth $m+1$, the minimum depth $d(B, A)$ is the more interesting positive integer.

The algebraic definition of depth of subring pairs of Artin algebras is closely related to induced and restricted modules, or characters in the case of group algebras. The depth of many subgroups are recently computed, both as induced complex representations [7] and as induced representations over general commutative rings of group algebras [2]. For example, the minimum depth of the permutation groups $S_n \subset S_{n+1}$ is $2n - 1$ over any ground ring k and depends only on a combinatorial depth of a subgroup $H < G$ defined in terms of $G \times H$-sets and diagonal action in the same way as depth is defined for a subring [2] (and reviewed below in this section). The main theorem in [2] is that an extension $k[G] \supseteq k[H]$ of finite group algebras over any ground ring k has finite depth since $d(k[H], k[G]) \leq d_c(H, G) \leq 2[G : N_G(H)]$.

The notion of subring depth $d(B, A)$ in [2] is defined in equivalent terms below in (3). In case B and A are semisimple complex algebras, it is shown in an appendix of [2] how subring depth equals the notion of depth based on induction-restriction table, equivalently inclusion matrix M in [7] and given below in (4). Such a pair $A \supseteq B$ is a special case of a split, separable Frobenius extension; in Theorem 4.2 below we show that subring depth is equal to tower depth of Frobenius extensions [17] satisfying only a generator module condition. The authors of [2] define a left and right even depth and show these are the same on group algebra extensions; Theorem 2.4 below shows this equality holds for all QF extensions.

There are intriguing relations between relative homological algebra and the subring depth definition and theory. The tower of iterated endomorphism rings above a ring extension becomes in the case of Frobenius extensions a tower of rings on the bar resolution groups $C_n(A, B)$ $(n = 0, 1, 2, \ldots)$ with Frobenius and Temperley-Lieb structures explicitly calculated from their more usual iterative definition in Section 3.1. At the same time Frobenius extensions of depth more than 2 are known to have depth 2 further out in the tower: we extend this observation in [17] with new proofs to include other ring extensions satisfying the hypotheses of Proposition 3.4. In Section 1 it is noted that for a finite dimensional algebra A and a subalgebra B, having finite depth $d(B, A)$ follows from the enveloping algebra B^e having finite representation type.

1.1. **H-equivalent modules.** Let A be a ring. Two left A-modules, ${}_AN$ and ${}_AM$, are said to be *h-equivalent*, denoted by ${}_AM \overset{h}{\sim} {}_AN$ if two conditions are met. First, for some positive integer r, N is isomorphic to a direct summand in the direct sum of r copies of M, denoted by

$$ {}_AN \oplus * \cong {}_AM^r \quad \Leftrightarrow \quad N \,|\, rM \;\Leftrightarrow \tag{1} $$

$$ \exists f_i \in \mathrm{Hom}\,({}_AM, {}_AN),\; g_i \in \mathrm{Hom}\,({}_AN, {}_AM), i = 1, \ldots, r \;:\; \sum_{i=1}^{r} f_i \circ g_i = \mathrm{id}_N $$

Second, symmetrically there is $s \in \mathbb{Z}_+$ such that $M \,|\, sN$. It is easy to extend this definition of h-equivalence (sometimes referred to as similarity) to h-equivalence of two objects in an abelian category, and to show that it is an equivalence relation.

If two modules are h-equivalent, ${}_AN \stackrel{h}{\sim} {}_AM$, then they have Morita equivalent endomorphism rings, $\mathcal{E}_N := \mathrm{End}\,{}_AN$ and $\mathcal{E}_M := \mathrm{End}\,{}_AM$. This is quite easy to see since a Morita context of bimodules are given by $H(M,N) := \mathrm{Hom}\,({}_AM, {}_AN)$, which is an $\mathcal{E}_N$-$\mathcal{E}_M$-bimodule via composition, and the bimodule ${}_{\mathcal{E}_M}H(N,M)_{\mathcal{E}_N}$; these are progenerator modules, by applying to (1) or its reverse, $M \,|\, N^s$, any of the four Hom-functors such as $\mathrm{Hom}\,({}_A-, {}_AM)$ from the category of left A-modules into the category of left E_M-modules showing that ${}_{\mathcal{E}_M}H(N,M)$ is finite projective; similarly, generator. Then the explicit conditions on mappings for h-equivalence show that $H(M,N) \otimes_{\mathcal{E}_M} H(N,M) \to \mathcal{E}_N$ and the reverse mapping given by composition are both bimodule isomorphisms as required. Since $\mathcal{E}_M$ and $\mathcal{E}_N$ are Morita equivalent rings, their centers are isomorphic:

$$ \mathrm{End}\,{}_AM_{\mathcal{E}_M} \cong \mathrm{End}\,{}_AN_{\mathcal{E}_N}. $$

The theory of h-equivalent modules applies to bimodules ${}_TM_B \stackrel{h}{\sim} {}_TN_B$ by letting $A = T \otimes_{\mathbb{Z}} B^{\mathrm{op}}$ which sets up an equivalence of abelian categories between T-B-bimodules and left A-modules. Two additive functors $F, G : \mathcal{C} \hookrightarrow \mathcal{D}$ are h-equivalent if there are natural split epis $F(X)^n \to G(X)$ and $G(X)^m \to F(X)$ for all X in $\mathcal{C}$. We leave the proof of the lemma below as an elementary exercise.

Lemma 1.1. *Suppose two A-modules are h-equivalent, $M \stackrel{h}{\sim} N$ and two additive functors from A-modules to an abelian category are h-equivalent, $F \stackrel{h}{\sim} G$. Then $F(M) \stackrel{h}{\sim} G(N)$.*

For example, the following substitution in equations involving the $\stackrel{h}{\sim}$-equivalence relation follows from the lemma:

$$ {}_AP_T \stackrel{h}{\sim} {}_AQ_T \quad {}_TU_B \stackrel{h}{\sim} {}_TV_B \;\Rightarrow\; {}_AP \otimes_T U_B \stackrel{h}{\sim} {}_AQ \otimes_T V_B \tag{2} $$

Example 1.2. Suppose A is a finite dimensional algebra with indecomposable A-modules $\{P_\alpha | \alpha \in I\}$ (representatives from each isomorphism class for some index set I). By Krull-Schmidt finitely generated modules M_A and N_A have a unique factorization into a direct sum of multiples of finitely many indecomposable module components. Denote the indecomposable constituents of M_A by $\mathrm{Indec}\,(M) = \{P_\alpha \,|\, [P_\alpha, M] \neq 0\}$ where $[P_\alpha, M]$ is the number of factors in M isomorphic to P_α. Note that $M \,|\, qN$ for some positive q if and only if $\mathrm{Indec}\,(M) \subseteq \mathrm{Indec}\,(N)$. It follows that $M \stackrel{h}{\sim} N$ iff $\mathrm{Indec}\,(M) = \mathrm{Indec}\,(N)$.

Suppose $A_A = n_1P_1 \oplus \cdots \oplus n_rP_r$ is the decomposition of the regular module into its projective indecomposables. Let $P_A = P_1 \oplus \cdots \oplus P_r$. Then P_A and A_A are h-equivalent, so that A and $\mathrm{End}\,P_A$ are Morita equivalent. The algebra $\mathrm{End}\,P_A$ is the basic algebra of A.

Example 1.3. Via some more category theory, the definition above extends to positive integers n and m being h-equivalent if $n \,|\, m^r$ and $m \,|\, n^s$ for some positive integers r, s; whence there are primes $p_1, \ldots, p_k$ such that n and m lie in the same h-equivalence class $\{p_1^{r_1} \cdots p_k^{r_k} \,|\, r_1, \ldots, r_k \geq 1\}$. This explains the notation $|$ in eq. (1), although we use additive notation in this paper.

1.2. **Depth two.** A subring pair $B \subseteq A$ is said to have left depth 2 (or be a left depth two extension [15]) if $A \otimes_B A \overset{h}{\sim} A$ as natural B-A-bimodules. Right depth 2 is defined similarly in terms of h-equivalence of natural A-B-bimodules. In [15] it was noted that the left condition implies the right and conversely if A is a Frobenius extension of B. Also in [15] a Galois theory of Hopf algebroids was defined on the endomorphism ring $H := \mathrm{End}\,{}_BA_B$ as total ring and the centralizer $R := A^B$ as base ring. The antipode is the natural anti-isomorphism stemming from following the arrows,

$$\mathrm{End}\, A_B \xrightarrow{\cong} A \otimes_B A \xrightarrow{\cong} (\mathrm{End}\,{}_BA)^{\mathrm{op}}$$

restricted to the intersection $\mathrm{End}\,{}_BA_B = \mathrm{End}\, A_B \cap \mathrm{End}\,{}_BA$.

The Galois extension properties of a depth two extension $A \supseteq B$ are as follows. If A_B is faithfully flat, balanced or B equals its double centralizer in A, the natural action of H on A has invariant subalgebra A^H satisfying the Galois property of $A^H = B$. Also the well-known Galois property of the endomorphism ring as a cross product holds: the right endomorphism ring $\mathrm{End}\, A_B \cong A \# H$, where the latter has smash product product ring structure on $A \otimes_R H$ [15]. There is also a duality structure by going a step further along in the tower above $B \subseteq A \hookrightarrow \mathrm{End}\, A_B \hookrightarrow \mathrm{End}\, A \otimes_B A_A$, where the Hopf algebroid $H' := (A \otimes_B A)^B$ is the R-dual of H and acts naturally on $\mathrm{End}\, A_B$ in such a way that $\mathrm{End}\, (A \otimes_B A)_A$ is a smash product [15].

Conversely, Galois extensions have depth 2. For example, an H-comodule algebra A with invariant subalgebra B and finite dimensional Hopf algebra H over a base field k, which has a Galois isomorphism from $A \otimes_B A \xrightarrow{\cong} A \otimes_k H$ given by $a' \otimes a \mapsto a' a_{(0)} \otimes a_{(1)}$ satisfies (strongly) the depth two condition $A \otimes_B A \cong A^{\dim H}$ as A-B-bimodules. The Hopf subalgebras within a finite dimensional Hopf algebra which have depth 2 are precisely the normal Hopf subalgebras; if normal, it has depth 2 by applying the observation about Hopf-Galois extension just made. The converse follows from an argument noted in [3, Boltje-Külshammer], which divides the normality notion into right and left (like the notion of depth 2), where left normal is invariance under the left adjoint action. In the context of an augmented algebra A (such as a quasi-Hopf algebra or a triangular matrix algebra) their argument is given briefly as follows. Let $\varepsilon : A \to k$ be the algebra homomorphism into a base ring k. Let A^+ denote $\ker \varepsilon$, and for a subalgebra $B \subseteq A$, let B^+ denote $\ker \varepsilon \cap B$.

Proposition 1.4. *Suppose $B \subseteq A$ is a subalgebra of an augmented algebra. If $B \subseteq A$ has right depth 2, then $AB^+ \subseteq B^+A$.*

The proof of this proposition is an exercise in tensoring both sides of $A \otimes_B A \oplus * \cong qA$ by the unit A-module k, then passing to the annihilator ideal of a module and a direct summand. The opposite inclusion is of course satisfied by a left depth 2 extension of augmented algebras.

Example 1.5. Let $A = T_n(k)$ the algebra of n by n upper triangular matrices, and $B = D_n(k)$ the subalgebra of diagonal matrices. Note that there are n augmentations $\varepsilon_i : A \to k$ given by $\varepsilon_i(X) = X_{ii}$, and each of the B_i^+ satisfy the inclusions above if left or right depth two. This is a clear contradiction, thus $d(B, A) > 2$. We will see below that $d(B, A) = 3$.

Also subalgebra pairs of semisimple complex algebras have depth 2 exactly when they are normal in a classical sense of Rieffel. The theorem in [7] is given below and one may prove the forward direction in the manner indicated for the previous proposition.

Theorem 1.6. [7, Theorem 4.6] *Suppose $B \subseteq A$ is a subalgebra pair of semisimple complex algebras. Then $B \subseteq A$ has depth 2 if and only if for every maximal ideal I in A, one has $A(I \cap B) = (I \cap B)A$.*

For example, subalgebra pairs of semisimple complex algebras that satisfy this normality condition are then by our sketch above examples of weak Hopf-Galois extensions, since the centralizer R mentioned above is semisimple (see Kaplansky's Fields and Rings for a C^*-theoretic reason), the extension is Frobenius [3], and weak Hopf algebras are equivalently Hopf algebroids over a separable base algebra [15].

1.3. **Subring depth.** Throughout this paper, let A be a unital associative ring and $B \subseteq A$ a subring where $1_B = 1_A$. Note the natural bimodules ${}_BA_B$ obtained by restriction of the natural A-A-bimodule (briefly A-bimodule) A, also to the natural bimodules ${}_BA_A$, ${}_AA_B$ or ${}_BA_B$, which are referred to with no further ado.

Let $C_0(A, B) = B$, and for $n \geq 1$,

$$C_n(A, B) = A \otimes_B \cdots \otimes_B A \quad (n \text{ times } A)$$

For $n \geq 1$, the $C_n(A, B)$ has a natural A-bimodule structure which restricts to B-A-, A-B- and B-bimodule structures occuring in the next definition.

Definition 1.7. *The subring $B \subseteq A$ has depth $2n + 1 \geq 1$ if as B-bimodules $C_n(A, B) \overset{h}{\sim} C_{n+1}(A, B)$. The subring $B \subseteq A$ has left (respectively, right) depth $2n \geq 2$ if $C_n(A, B) \overset{h}{\sim} C_{n+1}(A, B)$ as B-A-bimodules (respectively, A-B-bimodules).*

It is clear that if $B \subseteq A$ has either left or right depth $2n$, it has depth $2n + 1$ by restricting the h-equivalence condition to B-bimodules. If it has depth $2n+1$, it has depth $2n + 2$ by tensoring the h-equivalence by $- \otimes_B A$ or $A \otimes_B -$. The *minimum depth* is denoted by $d(B, A)$; if $B \subseteq A$ has no finite depth, write $d(B, A) = \infty$.

Note that the minimum left and right minimum *even* depths may differ by 2 (in which case $d(B, A)$ is the least of the two). In the next section we provide a general condition, which includes a Hopf subalgebra pair $B \subseteq A$ of symmetric (Frobenius) algebras, where the left and right minimum even depths coincide.

Also note that a subalgebra pair of Artin algebras $B \subseteq A$ have depth $2n + 1$ if and only if the indecomposable module constituents of $C_{n+m}(A, B)$ remain the same for all $m \geq 0$ as those already found in $C_n(A, B)$ (see example 1.2 above). This corresponds well with the classical notion of finite depth in subfactor theory.

Example 1.8. Again let $A = T_n(k)$ and $B = D_n(k) \cong k^n$. Let e_{ij} denote the matrix units, k_i the n simple B-modules, and k_{ij} for $1 \leq i \leq j \leq n$ the $n(n + 1)/2$ simple components of ${}_BA_B$. Note that $A \otimes_B A$ as a B-bimodule has components

$ke_{is} \otimes_B e_{sj} \cong k_{ij}$ where $i \leq s \leq j$, so $A \otimes_B A \,|\, nA$ as B-bimodules. Thus $d(B, A) \leq 3$. But $d(B, A) \neq 2$ by the remark following Proposition 1.4; then $d(B, A) = 3$.

Remark 1.9. Suppose B is a subring of A. The minimum depth of the subring $B \subseteq A$ as defined in [2, Boltje-Danz-Külshammer] coincides with $d(B, A)$. In fact, for $n > 0$, the depth $2n + 1$ condition in [2] is that for some $q \in \mathbb{Z}_+$

$$C_{n+1}(A, B) \,|\, qC_n(A, B) \tag{3}$$

as B-bimodules. The left depth $2n$ condition in [2] is (3) more strongly as natural B-A-bimodules (and as A-B-bimodules for the right depth $2n$ condition). But (using a pair of classical face and degeneracy maps of homological algebra) we always have $C_n(A, B) \,|\, C_{n+1}(A, B)$ as A-B-, B-A- or B-bimodules, so that the depth $2n$ as well as $2n + 1$ conditions coincide in the case of subring having depth $2n$ and $2n + 1$ conditions above.

Note that depth 1 in this paper is slightly stronger than subring depth 1 in for example [2, 4, 15] since if A is h-equivalent to B as B-bimodules, then A is centrally projective over B (i.e., $A \,|\, qB$ as B-bimodules) and additionally A is a split extension of B as B-bimodules since $B \,|\, qA$ implies $B \,|\, A$; the split extension condition is satisfied by all group algebra extensions and subfactor examples of finite depth.

Example 1.10. Boltje, Danz and Külshammer [2] ask about the depth $d(B, A)$ of invariant subrings in classical invariant theory, where K is a field, $A = K[X_1, \ldots, X_n]$, $B = k[X_1, \ldots, X_n]^G$ and G is a finite group in $\mathrm{GL}_n(K)$ acting by linear substitution of the variables. In any case A_B is finitely generated and B is a finitely generated affine K-algebra. We note here that if G is generated by pseudo-reflections (such as $G = S_n$, the symmetric group) and the characteristic of K is coprime to $|G|$, B is itself an n-variable polynomial algebra and A is a free B-module (part of the Shephard-Todd theorem) [1, 26]. Since A is a commutative algebra, it follows that $d(B, A) = 1$.

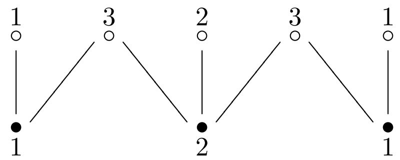
Example 1.11. Let $B \subseteq A$ be a subring pair of semisimple complex algebras. Then the minimum depth $d(B, A)$ may be computed from the inclusion matrix M, alternatively an r-by-s induction-restriction table of r B-simples induced to non-negative integer linear combination of s A-simples along rows, and by Frobenius reciprocity, columns show restriction of A-simples in terms of B-simples). The procedure to obtain $d(B, A)$ given in the paper [7] is the following: let $M^{[2n]} = (MM^t)^n$ and $M^{[2n+1]} = M^{[2n]}M$ (and $M^{[0]} = I_n$), then the matrix M has depth $n \geq 1$ if for some $q \in \mathbb{Z}_+$

$$M^{[n+1]} \leq qM^{[n-1]} \tag{4}$$

The minimum depth of M is equal to $d(B, A)$ by [2, appendix] (or Theorem 4.2 below combined with [6, 7]). One may note that $d(B, A) \leq 2d - 1$ where MM^t has degree d minimal polynomial [7].

In terms of the bipartite graph of the inclusion $B \subseteq A$, $d(B, A)$ is the lesser of the minimum odd depth and the minimum even depth [7]. The matrix M is an incidence matrix of this bipartite graph if all entries greater than 1 are changed to 1, while zero entries are retained as 0: let the B-simples be represented by r black dots in a bottow row of the graph, and A-simples by s white dots in a top row, connected by edges joining black and white dots (or not) according to the 0-1-matrix entries obtained from M. The minimum odd depth of the bipartite graph is 1 plus the diameter in edges of the row of black dots (indeed an odd number), while the minimum even depth is 2 plus the largest of the diameters of the bottom row where a subset of black dots under one white dot is identified with one another.

For example, let $A = \mathbb{C}\, S_4$, the complex group algebra of the permutation group on four letters, and $B = \mathbb{C}\, S_3$. The inclusion diagram pictured below with the degrees of the irreducible representations, is determined from the character tables of S_3 and S_4 or the branching rule (for the Young diagrams labelled by the partitions of n and representing the irreducibles of S_n).



This graph has minimum odd depth 5 and minimum even depth 6, whence $d(B, A) = 5$.

Example 1.12. The induction-restriction table M of the inclusion of permutation groups $S_n \times S_m < S_{n+m}$ via $(\sigma, \tau) \mapsto \begin{pmatrix} 1 & \cdots & n & n+1 & \cdots & n+m \\ \sigma(1) & \cdots & \sigma(n) & n+\tau(1) & \cdots & n+\tau(m) \end{pmatrix}$ may be computed combinatorially from the Littlewood-Richardson coefficients $c^{\gamma}_{\mu\nu} \in \mathbb{N}$, where μ is partition of n, $\nu = (\nu_1, \ldots, \nu_m)$ a partition of m and λ a partition of $n + m$. Briefly, the coefficient number $c^{\gamma}_{\mu\nu}$ is zero if γ does not contain μ, or is the number of Littlewood-Richardson fillings with content ν of γ with μ removed. A Littlewood-Richardson filling of a skew Young tableau is with integers $i = 1, 2, \ldots, m$ occuring ν_i times in rows that are weakly increasing from left to right, columns are strictly increasing from top to bottom, and the entries when listed from right to left in rows, top to bottom row, form a lattice word [10].

For example, computing the matrix M for the subgroup $S_2 \times S_3 < S_5$ with respect to the ordered bases of irreducible characters of the subgroup $\lambda_{(2)} \times \mu_{(1^3)}$, $\lambda_{(1^2)} \times \mu_{(2,1)}$, $\lambda_{(1^2)} \times \mu_{(3)}$, $\lambda_{(2)} \times \mu_{(1^3)}$, $\lambda_{(2)} \times \mu_{(2,1)}$, $\lambda_{(2)} \times \mu_{(3)}$ and of the group $\gamma_{(1^5)}$,$\gamma_{(2,1^3)}$, $\gamma_{(2^2,1)}$, $\gamma_{(3,2)}$, $\gamma_{(3,1^2)}$, $\gamma_{(4,1)}$, $\gamma_{(5)}$ yields

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

The bracketed powers of M satisfy a minimum depth 5 inequality (4) so that $d(S_2 \times S_3, S_5) = 5$. We mentioned before that $d(S_n \times S_1, S_{n+1}) = 2n - 1$ [2, 7]; however, a formula for $d(S_n \times S_m, S_{n+m})$ is not known.

1.4. **Combinatorial depth.** Translating the definition of subring depth with a group algebra-to-group dictionary, one defines a combinatorial depth of a subgroup H in a finite group G following [2]. This makes use of G-sets in an analogous way to how Burnside ring $B(G)$ of a group G is defined with a natural homomorphism into the representation ring $R_k(G)$ of G: the homomorphism $B(G) \to R_k(G)$ is induced by sending a G-set X to its permutation module $k[X]$. One notes that coproduct or disjoint union of G-sets is sent to direct sums of G-modules, direct product of G-sets into tensor products, and something that is useful for depth, monomorphisms of G-sets are functorially associated with split monomorphisms of $k[G]$-modules.

Note that G is a $G \times H$-set via $(g, h) \cdot g' = gg'h^{-1}$, equivalently a G-H-biset via $g \cdot g' \cdot h = gg'h$. Similarly, ${}_H G_H$ denotes the natural H-H-biset G obtained by restriction. Let $G \times_H G$ denote the space of orbits (where an orbit is denoted by a representative $[g_1, g_2]$) in $G \times G$ under the diagonal action of H given by $(g_1, g_2) \cdot h = (g_1 h, h^{-1} g_2)$. Note that $G \times_H G$ is naturally a G-H-biset via $g \cdot [g_1, g_2] \cdot h = [gg_1, g_2 h]$. Define similarly for any right H-set X, $X \times_H G$ as the orbits of $X \times G$ under the diagonal action of H; and a a G-H-biset structure if X is a G-H-biset. Thus $C_n(G, H) := G \times_H \cdots \times_H G$ (n times G) is defined iteratively as a G-H-biset. Denote by qX the disjoint union of q copies of a biset X: $2X = X \coprod X$, and so forth.

Definition 1.13. *A subgroup H in a finite group G has right depth $2n$ (or depth $2n+1$, respectively) if there is a monomorphism of G-H-bisets (respectively, H-H-bisets) $C_{n+1}(G, H) \hookrightarrow qC_n(G, H)$ for some positive integer q. The minimum depth is denoted by $d_c(H, G)$.*

Left depth $2n$ is defined similarly, and it is shown in [2] that H has right depth $2n$ in G iff it has left depth $2n$. Also note that

$$d(k[H], k[G]) \leq d_c(H, G) \tag{5}$$

for any base ring k, which follows from the combinatorial depth m condition of a monomorphism of $C_{m+1}(G, H) \hookrightarrow qC_m(G, H)$ yielding a split monomorphism of permutation modules $k[C_{m+1}(G, H)] = C_{m+1}(k[G], k[H]) \hookrightarrow qC_m(k[G], k[H])$, which is equivalent to the depth m condition on group algebras $B = k[H] \subseteq k[G] = A$ (compare with eq. (3)). To illustrate the methods and definitions of combinatorial depth let us see a second and direct proof of the result in [2] that $d_c(H, G) \leq 2$ if and only if H is a normal subgroup in G.

Proposition 1.14. *A subgroup H of a finite group G is normal in G if and only if $d_c(H, G) \leq 2$.*

Proof. Suppose $H \trianglelefteq G$, $[G : H] = n$ and $\{t_1, \ldots, t_n\}$ is a transversal. Define a G-H-biset homomorphism $G \times_H G \to \coprod_{i=1}^n G_i$ where each $G_i = G$ by $[g_1, g_2] \mapsto g_1 g_2 \in G_j$ where $g_2 \in Ht_j$. If also $[g_3, g_4] \mapsto g_1 g_2 = g_3 g_4 \in G_j$, it follows that

$$[g_1, g_2] = [g_3(g_4 g_2^{-1}), g_2] = [g_3, (g_4 g_2^{-1}) g_2] = [g_3, g_4].$$

Thus $G \times_H G \hookrightarrow nG$.

Conversely, given a monomorphism of G-H-bisets $\Psi : G \times_H G \hookrightarrow \coprod_{i=1}^m G_i$ where each $G_i = G$, we may assume without loss of generality that each $G_i \cap \operatorname{Im} \Psi \neq \emptyset$. In this case it is easy to see that Ψ maps onto its codomain, which has cardinality $m|G|$. Since $|\{[g_1, g_2] \,|\, g_1, g_2 \in G\}| = |G|^2/|H|$, it follows that $m = [G : H]$. Now define $\phi : G \to \coprod_{i=1}^m G_i$ by $\phi(g) = \Psi([1, g])$ which is seen to be an H-H-biset monomorphism. Let $\{t_1, \ldots, t_m\}$ be right coset representatives such that $\phi(t_j) \in G_j$ for all j. Then $\{\phi^{-1}(G_i) | i = 1, \ldots, m\}$ is a partition of G where $Ht_i H \subseteq \phi^{-1}(G_i)$. Thus the number of double cosets of H in G is equal to $[G : H]$, whence H is normal in G. □

It is shown in [2] that depth of subalgebras over base rings (of varying characteristic denoted by a subscript) for $R = k[G]$ and $S = k[H]$ and combinatorial depth fall into a string of inequalities given in [2] as follows:

$$d_0(H,G) \leq d_p(H,G) \leq d_{\mathbb{Z}}(H,G) \leq d_c(H,G) \leq 2[G : N_G(H)]. \tag{6}$$

Unlike the case of permutation groups $G = S_{n+1}$ and $H = S_n$, the inequalities may diverge radically in other cases. For example, a Frobenius complement H in a Frobenius group G has minimal depth three but there is a well-known series $H_p < G_p$ of semi-direct products of cyclic groups where $[G_p : N_{G_p}(H_p)] \to \infty$. Külshammer and students have computed in $\mathrm{PSL}(2,q)$ that a certain subgroup H_q has $d_0(H_q, \mathrm{PSL}(2,q)) = 3$ (see [9]) while $d_c(H_q, \mathrm{PSL}(2,q)) \to \infty$ as $q \to \infty$.

1.5. **Finite depth and finite representation type.** For the next proposition we adopt the notation B^e for the (enveloping) algebra $B \otimes_k B^{\mathrm{op}}$ in homological algebra. Recall that a finite dimensional algebra has finite representation type if it only has finitely many isomorphism classes of indecomposable modules.

For example, a group algebra over a base field of characteristic p has finite representation type if and only if its Sylow p-subgroup is cyclic. (The proof of this may use twice Jans' theorem mentioned in the introduction above, invariance of finite representation type under Morita equivalence between subalgebra and right endomorphism algebra as well as representation type of p-groups.) Thus B having finite representation type does not imply that B^e has finite representation type.

Proposition 1.15. *Suppose $B \subseteq A$ is a subalgebra pair of finite dimensional algebras where B^e has in all r indecomposable B^e-module isomorphism classes. Then $d(B,A) \leq 2r+1$.*

Proof. This follows from the observation in Example 1.2 above that since $C_n(A,B)$ is the image of $C_{n+1}(A,B)$ under an obvious split epimorphism of B^e-modules (equivalently, B-bimodules), there is an increasing chain of subset inclusions

$$\mathrm{Indec}\,(A) \subseteq \mathrm{Indec}\,(A \otimes_B A) \subseteq \mathrm{Indec}\,(A \otimes_B A \otimes_B A) \subseteq \cdots$$

which stops strictly increasing in at most r steps. When $\mathrm{Indec}\,(C_n(A,B)) = \mathrm{Indec}\,(C_{n+1}(A,B))$, then $C_n(A,B) \overset{h}{\sim} C_{n+1}(A,B)$ as B^e-modules, whence $A \supseteq B$ has depth $2n+1 \leq 2r+1$. □

Remarkably, the result in [2] is that all finite group algebra pairs have finite depth. The proposition says something about finite depth of interesting classes of finite dimensional Hopf algebra pairs $B \subseteq A$, where research on which Hopf algebras have finite representation type is a current topic. (Note that B^e is a Hopf algebra and semisimple if B is so.) For example,

Corollary 1.16. *Suppose B is a semisimple Hopf subalgebra in a finite dimensional Hopf algebra A. Suppose that B has n nonisomorphic simple modules. Then $d(B,A) \leq 2n^2+1$.*

2. Even depth of QF extensions

A (proper) ring extension $A \supseteq B$ is a subring or more generally a monomorphism $\iota : B \hookrightarrow A$, which is equivalent to a subring $\iota(B) \subseteq A$. Restricted modules such as $A_{\iota(B)}$ and pullback modules A_B are identified, and these are the type of modules we refer to below unless otherwise stated. (Almost all that we have to say holds for a ring homomorphism $B \to A$ and its pullback modules such as A_B; however, certain conditions needed below such as A_B is a generator, imply that $B \to A$ is monic.)

A ring extension $A \supseteq B$ is a *left QF extension* if the module ${}_BA$ is finitely generated projective and the natural A-B-bimodules satisfy $A \mid q\mathrm{Hom}\,({}_BA, {}_BB)$ for some positive integer q. A right QF extension is oppositely defined. A *QF extension* $A \supseteq B$ is both a left and right QF extension and may be characterized by both A_B and ${}_BA$ being finite projective, and two h-equivalences of bimodules given by ${}_AA_B \overset{h}{\sim} {}_A\mathrm{Hom}\,({}_BA, {}_BB)_B$ and ${}_BA_A \overset{h}{\sim} {}_B\mathrm{Hom}\,(A_B, B_B)_A$ [22]. For example, a Frobenius extension $A \supseteq B$ is a QF extension since it is left and right finite projective and satisfies the stronger conditions that A is *isomorphic* to its right B-dual A^* and its left B-dual ${}^*A$ as natural B-A-bimodules, respectively A-B-bimodules; the more precise definition are given in the next section.

2.1. **β-Frobenius extensions vs. QF extensions.** In Hopf algebras and quantum algebras, examples of Frobenius extensions often occur with a twist foreseen by Nakayama and Tzuzuku, their so-called beta-Frobenius extension. Let β be an automorphism of the ring B and $B \subseteq A$ a subring pair. We next denote the pullback module of a module ${}_BM$ along $\beta : B \to B$ by ${}_\beta M$. A ring extension $A \supseteq B$ is a *β-Frobenius extension* if A_B is finite projective and there is a bimodule isomorphism ${}_BA_A \cong {}_\beta\mathrm{Hom}\,(A_B, B_B)$. One shows that $A \supseteq B$ is a Frobenius extension if and only if β is an inner automorphism. A subring pair of Frobenius algebras $B \subseteq A$ is β-Frobenius extension so long as A_B is finite projective and the Nakayama automorphism η_A of A stabilizes B, in which case $\beta = \eta_B \circ \eta_A^{-1}$ [25]. For instance a finite dimensional Hopf algebra $A = H$ and $B = K$ a Hopf subalgebra of H are a pair of Frobenius algebras satisfying the conditions just given: the formula for β reduces to the following given in terms of the modular functions of H and K and the antipode S [13, 7.8]:

$$\beta(x) = \sum_{(x)} m_H(x_{(1)}) m_K(S(x_{(2)})) x_{(3)} \tag{7}$$

When a β-Frobenius extension is a QF extension is addressed in the next proposition.

Proposition 2.1. *A β-Frobenius extension $A \supseteq B$ is a left QF extension if and only if there are $u_i, v_i \in A$ $(i = 1, \ldots, n)$ such that $su_i = u_i\beta(s)$ and $v_is = \beta(s)v_i$ for all i and $s \in B$, and*

$$\beta^{-1}(s) = \sum_{i=1}^{n} u_i s v_i. \tag{8}$$

Proof. Suppose $A \supseteq B$ is β-Frobenius extension. Then the bimodule isomorphism given above applied to 1_A has value $E : A \to B$, a cyclic generator of ${}_\beta\mathrm{Hom}\,(A_B, B_B)_A$ satisfying $E(b_1ab_2) = \beta(b_1)E(a)b_2$ for all $b_1, b_2 \in B, a \in A$. If $x_1, \ldots, x_m \in A$ and $\phi_1, \ldots, \phi_m \in \mathrm{Hom}\,(A_B, B_B)$ are projective bases of A_B, and $E(y_j-) = \phi_j$ the equations $\sum_{j=1}^m x_jE(y_ja) = a$ and $\sum_{j=1}^m \beta^{-1}(E(ax_j))y_j = a$ hold for all $a \in A$. (Call (E, x_j, y_j) a *β-Frobenius coordinate system* of $A \supseteq B$. Note that also ${}_BA$ is finite projective.)

Given the elements $u_i, v_i \in A$ satisfying the equations above, let $E_i = E(u_i-)$ which defines n mappings in (the untwisted) $\mathrm{Hom}\,({}_BA_B, {}_BB_B)$. Also define n mappings $\psi_i \in \mathrm{Hom}\,({}_A({}^*A)_B, {}_AA_B)$ by $\psi_i(g) = \sum_{j=1}^m x_jg(v_iy_j)$ where it is not hard to show using the β-Frobenius coordinate equations that $\sum_j x_j \otimes_B v_i y_j \in (A \otimes_B A)^A$ for each i (a Casimir element). It follows that $\sum_{i=1}^n \psi_i(E_i) = 1_A$ and that $A \mid {}^*A^n$ as natural A-B-bimodules, whence A is a left QF extension of B.

Conversely, assume the left QF condition ${}_B A^*{}_A \mid A^n$, equivalent to ${}_A A_B \mid {}^*A^n$ by applying the right B-dual functor and noting $({}^*A)^* \cong A$ as well ${}^*(A^*) \cong A$. Also assume the slightly rewritten β-Frobenius condition ${}_{\beta^{-1}} A_A \cong {}_B(A^*)_A$, which then implies ${}_{\beta^{-1}} A_A \mid A^n$. So there are n mappings $g_i \in \mathrm{Hom}\,({}_{\beta^{-1}} A_A, {}_B A_A)$ and n mappings $f_i \in \mathrm{Hom}\,({}_B A_A, {}_{\beta^{-1}} A_A)$ such that $\sum_{i=1}^n f_i \circ g_i = \mathrm{id}_A$. Equivalently, with $u_i := f(1_A)$ and $v_i := g(1_A)$, $\sum_{i=1}^n u_i v_i = 1_A$, and the equations in the proposition are satisfied. □

The following corollary weakens one of the equivalent conditions in [10]. It implies that a finite dimensional Hopf algebra that is QF over a Hopf subalgebra is necessarily Frobenius over it. (Nontrivial examples of QF extensions occur for weak Hopf algebras over their separable base algebra [12].)

Corollary 2.2. *Let H be a finite dimensional Hopf algebra and K a Hopf subalgebra. In the notation of (7) the following are equivalent:*

1. *The automorphism $\beta = \mathrm{id}_K$ and $H \supseteq K$ is a Frobenius extension.*
2. *The algebra extension $H \supseteq K$ is a QF extension.*
3. *The modular functions $m_H(x) = m_K(x)$ for all $x \in K$.*

Proof. $(1 \Rightarrow 2)$ A Frobenius extension is a QF extension. $(2 \Rightarrow 3)$ Set $s = 1$ in (8), apply the counit ε to see that $\varepsilon(\sum_i u_i v_i) = 1$. Re-apply ε to (8) to obtain $\varepsilon \circ \beta = \varepsilon$. Apply ε to (7) and use uniqueness of inverse in convolution algebra $\mathrm{Hom}\,(K, k)$, where $m_K \circ S = m_K^{-1}$ and $1 = \varepsilon$, to show that $m_H = m_K$ on K. $(3 \Rightarrow 1)$ This follows from (7). □

It is well-known that for a Frobenius extension $A \supseteq B$, coinduction of a module, $M_B \mapsto \mathrm{Hom}\,(A_B, M_B))$ is naturally isomorphic as functors to induction of $M_B \mapsto M \otimes_B A)$ (from the category of B-modules into the category of A-modules). Similarly, a QF extension has h-equivalent coinduction and induction functors, which is seen from the naturality of the mappings in the next proof. Let T be an arbitrary third ring.

Proposition 2.3. *Suppose ${}_T M_B$ is a bimodule and $A \supseteq B$ is a QF extension. Then there is an h-equivalence of bimodules,*

$$ {}_T M \otimes_B A_A \overset{h}{\sim} {}_T \mathrm{Hom}\,(A_B, M_B)_A. \tag{9} $$

Proof. Since A_B is f.g. projective, it follows that there is an T-A-bimodule isomorphism

$$ M \otimes_B \mathrm{Hom}\,(A_B, B_B) \cong \mathrm{Hom}\,(A_B, M_B), \tag{10} $$

given by $m \otimes_B \phi \mapsto m\phi(-)$ with inverse constructed from projective bases for A_B. But the right B-dual of A is h-equivalent to ${}_B A_A$, so (9) holds by Lemma 1.1. □

The next theorem shows that minimum right and left even depth of a QF extension are equal (see the Definition 1.7 where as before $C_n(A, B) = A \otimes_B \cdots \otimes_B A$, n times A).

Theorem 2.4. *If $A \supseteq B$ is QF extension, then $A \supseteq B$ has left depth $2n$ if and only if $A \supseteq B$ has right depth $2n$.*

Proof. The left depth $2n$ condition on $A \supseteq B$ recall is $C_{n+1}(A,B) \overset{h}{\sim} C_n(A,B)$ as B-A-bimodules. To this apply the additive functor $\mathrm{Hom}\,(-_A, A_A)$ (into the category of A-B-bimodules), noting that $\mathrm{Hom}\,(C_n(A,B)_A, A_A) \cong \mathrm{Hom}\,(C_{n-1}(A,B)_B, A_B)$ via $f \mapsto f(- \otimes_B \cdots - \otimes_B 1_A)$ for each integer $n \geq 1$. It follows (from Lemma 1.1) that there is an A-B-bimodule h-equivalence,

$$\mathrm{Hom}\,(C_n(A,B)_B, A_B) \overset{h}{\sim} \mathrm{Hom}\,(C_{n-1}(A,B)_B, A_B) \tag{11}$$

(Then in the depth two case, the left depth two condition is equivalent to $\mathrm{End}\, A_B \overset{h}{\sim} A$ as natural A-B-bimodules.)

Given bimodule ${}_AM_B$, we have ${}_AM \otimes_B A_A \overset{h}{\sim} {}_A\mathrm{Hom}\,(A_B, M_B)_A$ by the previous lemma: apply this to $C_{n+1}(A,B) = C_n(A,B) \otimes_B A$ using the hom-tensor adjoint relation: there are h-equivalences and isomorphisms of A-bimodules,

$$\begin{aligned} C_{n+1}(A,B) &\overset{h}{\sim} \mathrm{Hom}\,(A_B, C_n(A,B)_B) \\ &\overset{h}{\sim} \mathrm{Hom}\,(A_B, \mathrm{Hom}\,(A_B, C_{n-1}(A,B)_B)_B) \\ &\cong \mathrm{Hom}\,(A \otimes_B A_B, C_{n-1}(A,B)_B) \\ \cdots &\overset{h}{\sim} \mathrm{Hom}\,(C_p(A,B)_B, C_{n-p+1}(A,B)_B) \end{aligned} \tag{12}$$

for each $p = 1, 2, \ldots, n$ and $n = 1, 2, \ldots$. Compare (11) and (12) with $p = n$ to get ${}_AC_{n+1}(A,B)_B \overset{h}{\sim} {}_AC_n(A,B)_B$ which is the right depth $2n$ condition.

The converse is proven similarly from the symmetric conditions of the QF hypothesis. □

3. Frobenius extensions

As noted above a Frobenius extension $A \supseteq B$ is characterized by any of the following four conditions [13]. First, that A_B is finite projective and ${}_BA_A \cong \mathrm{Hom}\,(A_B, B_B)$. Secondly, that ${}_BA$ is finite projective and ${}_AA_B \cong \mathrm{Hom}\,({}_BA, {}_BB)$. Thirdly, that coinduction and induction of right (or left) B-modules is naturally equivalent. Fourth, there is a Frobenius coordinate system ($E : A \to B; x_1, \ldots, x_m, y_1, \ldots, y_m \in A$), which satisfies

$$E \in \mathrm{Hom}\,({}_BA_B, {}_BB_B), \quad \sum_{i=1}^m E(ax_i)y_i = a = \sum_{i=1}^m x_iE(y_ia) \quad (\forall a \in A). \tag{13}$$

These (dual bases) equations may be used to show the useful fact that $\sum_i x_i \otimes y_i \in (A \otimes_B A)^A$.

We continue this notation in the next lemma. Example 2.7 in [13] provides an example of a matrix algebra Frobenius extension with a non-surjective Frobenius homomorphism.

Lemma 3.1. *The natural module A_B is a generator $\Leftrightarrow$ ${}_BA$ is a generator $\Leftrightarrow$ there are elements $\{a_j\}_{j=1}^n$ and $\{c_j\}_{j=1}^n$ such that $\sum_{j=1}^n E(a_jc_j) = 1_B$ $\Leftrightarrow$ E is surjective.*

Proof. The bimodule isomorphism ${}_BA_A \xrightarrow{\cong} {}_B\mathrm{Hom}\,(A_B, B_B)_A$ is realized by $a \mapsto E(a-)$ (with inverse $\phi \mapsto \sum_{i=1}^m \phi(x_i)y_i$). If A_B is a generator, then there are elements $\{c_j\}_{j=1}^n$ of A and mappings $\{\phi_j\}_{j=1}^n$ of A^* such that $\sum_{j=1}^n \phi_j(c_j) = 1_B$. Let $Ea_j = \phi_j$. Then $\sum_{j=1}^n E(a_jc_j) = 1_B$.

Another bimodule isomorphism ${}_AA_B \xrightarrow{\cong} {}_A\mathrm{Hom}\,({}_BA, {}_BB)_B$ is realized by $a \mapsto E(-a) := aE$. Then writing the last equation as $\sum_j c_j E(a_j) = 1_B$ exhibits ${}_BA$ as a generator.

The last equivalent condition is clear. Note too that any other Frobenius homomorphism is given by Ed for some invertible $d \in A^B$. □

A Frobenius (or QF) extension $A \supseteq B$ enjoys an *endomorphism ring theorem* [22, 21], which shows that $\mathcal{E} := \mathrm{End}\, A_B \supseteq A$ is a Frobenius (respectively, QF) extension, where the default ring homomorphism $A \to \mathcal{E}$ is understood to be the left multiplication mapping $\lambda : a \mapsto \lambda_a$ where $\lambda_a(x) = ax$. It is worth noting that λ is a left split A-monomorphism (by evaluation at 1_A) so ${}_A\mathcal{E}$ is a generator.

The *tower* of a Frobenius (resp. QF) extension is obtained by iteration of the endomorphism ring and λ, obtaining a tower of Frobenius (resp. QF) extensions where occasionally we need the notation $B := \mathcal{E}_{-1}, A = \mathcal{E}_0$ and $\mathcal{E} = \mathcal{E}_1$

$$B \to A \hookrightarrow \mathcal{E}_1 \hookrightarrow \mathcal{E}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_n \hookrightarrow \cdots \tag{14}$$

so $\mathcal{E}_2 = \mathrm{End}\, \mathcal{E}_A$, etc. By transitivity of Frobenius extension or QF extension [25], [22], all sub-extensions $\mathcal{E}_m \hookrightarrow \mathcal{E}_{m+n}$ in the tower are also Frobenius (resp. QF) extensions.

The rings $\mathcal{E}_n$ are h-equivalent to $C_{n+1}(A,B) = A \otimes_B \cdots \otimes_B A$ as A-bimodules in case $A \supseteq B$ is a QF extension. This follows from noting the

$$\mathrm{End}\, A_B \cong A \otimes_B \mathrm{Hom}\,(A_B, B_B) \overset{h}{\sim} A \otimes_B A$$

also holding as natural $\mathcal{E}$-A-bimodules, obtained by substitution of $A^* \overset{h}{\sim} A$. This observation is then iterated followed by cancellations of the type $A \otimes_A M \cong M$.

3.1. **Tower above Frobenius extension.** Specialize now to $A \supseteq B$ a Frobenius extension with Frobenius coordinate system E and $\{x_i\}_{i=1}^m, \{y_i\}_{i=1}^m$. Then the h-equivalences above are replaced by isomorphisms, and $\mathcal{E}_n \cong C_{n+1}(A,B)$ for each $n \geq -1$ as ring isomorphisms with respect to a certain induced "E-multiplication." The E-multiplication on $A \otimes_B A$ is induced from the endomorphism ring $\mathrm{End}\, A_B \xrightarrow{\cong} A \otimes_B A$ given by $f \mapsto \sum_i f(x_i) \otimes_B y_i$ with inverse $a \otimes a' \mapsto \lambda_a \circ E \circ \lambda_{a'}$. The outcome of E-multiplication on $C_2(A,B)$ is given by

$$(a_1 \otimes_B a_2)(a_3 \otimes_B a_4) = a_1 E(a_2 a_3) \otimes_B a_4 \tag{15}$$

with unity element $1_1 = \sum_{i=1}^m x_i \otimes_B y_i$. Note that the A-bimodule structure on $\mathcal{E}_1$ induced by $\lambda : A \hookrightarrow \mathcal{E}$ corresponds to the natural A-bimodule $A \otimes_B A$.

The E-multiplication is defined inductively on

$$\mathcal{E}_n \cong \mathcal{E}_{n-1} \otimes_{\mathcal{E}_{n-2}} \mathcal{E}_{n-1} \tag{16}$$

using the Frobenius homomorphism $E_{n-1} : \mathcal{E}_{n-1} \to \mathcal{E}_{n-2}$ obtained by iterating the following natural Frobenius coordinate system on $\mathcal{E}_1 \cong A \otimes_B A$, given by $E_1(a \otimes_B a') = aa'$ and $\{x_i \otimes_B 1_A\}_{i=1}^m$, $\{1_A \otimes_B y_i\}_{i=1}^m$ [24] as one checks.

The iterative E-multiplication on $C_n(A,B)$ clearly exists as an associative algebra, but it seems worthwhile (and not available in the literature) to compute it explicitly. The multiplication on $C_{2n}(A,B)$ is given by $(\otimes = \otimes_B, n \geq 1)$

$$(a_1 \otimes \cdots \otimes a_{2n})(c_1 \otimes \cdots \otimes c_{2n}) = \tag{17}$$

$$a_1 \otimes \cdots \otimes a_n E(a_{n+1} E(\cdots E(a_{2n-1} E(a_{2n} c_1) c_2) \cdots) c_{n-1}) c_n) \otimes c_{n+1} \otimes \cdots \otimes c_{2n}.$$

The identity on $C_{2n}(A,B)$ is in terms of the dual bases,

$$1_{2n-1} = \sum_{i_1,\ldots,i_n=1}^{m} x_{i_1} \otimes \cdots \otimes x_{i_n} \otimes y_{i_n} \otimes \cdots \otimes y_{i_1}. \tag{18}$$

The multiplication on $C_{2n+1}(A,B)$ is given by

$$(a_1 \otimes \cdots \otimes a_{2n+1})(c_1 \otimes \cdots \otimes c_{2n+1}) = \tag{19}$$

$$a_1 \otimes \cdots \otimes a_{n+1}E(a_{n+2}E(\cdots E(a_{2n}E(a_{2n+1}c_1)c_2)\cdots)c_n)c_{n+1} \otimes \cdots \otimes c_{2n+1}$$

with identity

$$1_{2n} = \sum_{i_1,\ldots,i_n=1}^{m} x_{i_1} \otimes \cdots \otimes x_{i_n} \otimes 1_A \otimes y_{i_n} \otimes \cdots \otimes y_{i_1}. \tag{20}$$

Denote in brief notation the rings $C_n(A,B) := A_n$ and distinguish them from the isomorphic rings $\mathcal{E}_{n-1}$ $(n = 0,1,\ldots)$.

The inclusions $A_n \hookrightarrow A_{n+1}$ are given by $a_{[n]} \mapsto a_{[n]}1_n$, which works out in the odd and even cases to:

$$A_{2n-1} \hookrightarrow A_{2n},$$

$$a_1 \otimes \cdots \otimes a_{2n-1} \longmapsto \sum_i a_1 \otimes \cdots \otimes a_n x_i \otimes y_i \otimes a_{n+1} \otimes \cdots \otimes a_{2n-1} \tag{21}$$

$$A_{2n} \hookrightarrow A_{2n+1},$$

$$a_1 \otimes \cdots \otimes a_{2n} \longmapsto a_1 \otimes \cdots \otimes a_n \otimes 1_A \otimes a_{n+1} \otimes \cdots \otimes a_{2n} \tag{22}$$

The bimodule structure on A_n over a subalgebra A_m (with $m < n$ via composition of left multiplication mappings λ) is just given in terms of the multiplication in A_m as follows:

$$(r_1 \otimes \cdots \otimes r_m)(a_1 \otimes \cdots \otimes a_n) = \tag{23}$$

$$[(r_1 \otimes \cdots \otimes r_m)(a_1 \otimes \cdots \otimes a_m)] \otimes a_{m+1} \otimes \cdots \otimes a_n$$

with a similar formula for the right module structure.

The formulas for the successive Frobenius homomorphisms $E_m : A_{m+1} \to A_m$ are given in even degrees by

$$E_{2n}(a_1 \otimes \cdots \otimes a_{2n+1}) = a_1 \otimes \cdots \otimes a_n E(a_{n+1}) \otimes a_{n+2} \otimes \cdots \otimes a_{2n+1}. \tag{24}$$

for $n \geq 0$. The formulas in the odd case is

$$E_{2n+1}(a_1 \otimes \cdots \otimes a_{2n+2}) = a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}a_{n+2} \otimes a_{n+3} \otimes \cdots \otimes a_{2n+2} \tag{25}$$

for $n \geq 0$.

The dual bases of E_n denoted by x_i^n and y_i^n are given by all-in-one formulas

$$x_i^n = x_i \otimes 1_{n-1} \tag{26}$$

$$y_i^n = 1_{n-1} \otimes y_i \tag{27}$$

for $n \geq 0$ (where $1_0 = 1_A$). Note that $\sum_i x_i^n \otimes_{A_n} y_i^n = 1_{n+1}$.

With another choice of Frobenius coordinate system (F, z_j, w_j) for $A \supseteq B$ there is in fact an invertible element d in the centralizer subring A^B of A such that $F = E(d-)$ and $\sum_i x_i \otimes_B y_i = \sum_j z_j \otimes_B d^{-1}w_j$ [13, 24]; whence an isomorphism of the E-multiplication onto the F-multiplication, both on $A \otimes_B A$, given by $r_1 \otimes r_2 \mapsto$

$r_1 \otimes d^{-1} r_2$. If the tower with E-multiplication is denoted by A_n^E and the tower with F-multiplication by A_n^F, there is a sequence of ring isomorphisms

$$A_{2n}^E \xrightarrow{\cong} A_{2n}^F,$$

$$a_1 \otimes \cdots \otimes a_{2n} \longmapsto a_1 \otimes \cdots \otimes a_n \otimes d^{-1} a_{n+1} \otimes \cdots \otimes d^{-1} a_{2n} \quad (28)$$

$$A_{2n+1}^E \xrightarrow{\cong} A_{2n+1}^F,$$

$$a_1 \otimes \cdots a_{2n+1} \longmapsto a_1 \otimes \cdots \otimes a_{n+1} \otimes d^{-1} a_{n+2} \otimes \cdots \otimes d^{-1} a_{2n+1} \quad (29)$$

which commute with the inclusions $A_n^{E,F} \hookrightarrow A_{n+1}^{E,F}$.

Theorem 3.2. *The multiplication, module and Frobenius structures for the tower $A_n = A \otimes_B \cdots \otimes_B A$ (n times A) above a Frobenius extension $A \supseteq B$ are given by the formulas (15) to (29).*

Proof. First define Temperley-Lieb generators iteratively by $e_n = 1_{n-1} \otimes_{A_{n-2}} 1_{n-1} \in A_{n+1}$ for $n = 1, 2, \ldots$, which results in the explicit formulas,

$$e_{2n} = \sum_{i_1,\ldots,i_{n+1}} x_{i_1} \otimes \cdots \otimes x_{i_n} \otimes y_{i_n} x_{i_{n+1}} \otimes y_{i_{n+1}} \otimes y_{i_{n-1}} \otimes \cdots \otimes y_{i_1} \quad (30)$$

$$e_{2n+1} = \sum_{i_1,\ldots,i_n} x_{i_1} \otimes \cdots \otimes x_{i_n} \otimes 1_A \otimes 1_A \otimes y_{i_n} \otimes \cdots \otimes y_{i_1}$$

These satisfy braid-like relations [15, p. 106]; namely,

$$e_i e_j = e_j e_i, \quad |i - j| \geq 2, \quad e_{i+1} e_i e_{i+1} = e_{i+1}, \quad e_i e_{i+1} e_i = e_i 1_{i+1}. \quad (31)$$

(The generators above fail to be idempotents to the extent that $E(1)$ differs from 1.) The proof that the formulas above are the correct outcomes of the inductive definitions may be given in terms of Temperley-Lieb generators, braid-like relations and important relations

$$e_n x e_n = e_n E_{n-1}(x), \quad \forall x \in A_n \quad (32)$$

$$y e_n = E_n(y e_n) e_n, \quad \forall y \in A_{n+1}, \quad E_n(e_n) = 1_{n-1} \quad (33)$$

$$x e_n = e_n x, \quad \forall x \in A_{n-1} \quad (34)$$

[15, p. 106] (for background see [11]) as well as the symmetric left-right relations. These relations and the Frobenius equations (13) may be checked to hold in terms of the equations above in a series of exercises left to the reader.

The formulas for the Frobenius bases follow from the iteratively apparent $x_i^n = x_i e_1 e_2 \cdots e_n$ and $y_i^n = e_n \cdots e_2 e_1 y_i$ and uniqueness of bases w.r.t. same Frobenius homomorphism. In fact $e_n \cdots e_2 e_1 a = 1_{n-1} \otimes a$ for any $a \in A, n = 1, 2, \ldots$ (a symmetrical formula holds as well) and $1_n = \sum_i x_i e_1 \cdots e_{n-1} e_n e_{n-1} \cdots e_1 y_i$.

Since the inductive definitions of the ring and modules structures on the A_n's also satisfy the relations listed above, and agree on and below A_2, the proof is finished with an induction argument based on expressing tensors as words in Temperley-Lieb generators and elements of A.

We note that

$$\begin{aligned} a_1 \otimes \cdots \otimes a_{n+1} &= (a_1 \otimes \cdots \otimes a_n)(1_{n-1} \otimes a_{n+1}) \quad (35) \\ &= (a_1 \otimes \cdots \otimes a_{n-1})(1_{n-2} \otimes a_n)(e_n \cdots e_1 a_{n+1}) \end{aligned}$$

$$= \cdots = a_1(e_1a_2)(e_2e_1a_3)\cdots(e_{n-1}\cdots e_1a_n)(e_n\cdots e_1a_{n+1})$$

The formulas for multiplication (19), (17) and (23) follow from induction and applying the relations (31) through (34). □

Example 3.3. Let $A = k[\epsilon]$ where $\epsilon^2 = 0$, the ring of dual numbers, a Frobenius algebra with Frobenius homomorphism $E : A \to k$ given by $E(\lambda_1 1 + \lambda_2\epsilon) = \lambda_2$ and dual bases $x_1 = 1$, $x_2 = \epsilon$, $y_1 = \epsilon$, $y_2 = 1$. The Frobenius extension A over $B = k1$ has second tower algebra

$$A_2 = \{ \sum_{i_1,i_2=0}^{1} \lambda_{i_1i_2}\epsilon_1^{i_1}\epsilon_2^{i_2} \mid i_1, i_2 \in \{0,1\}, \lambda_{i_1i_2} \in k\}$$

where $\epsilon_1^{i_1}\epsilon_2^{i_2} = \epsilon^{i_1} \otimes_k \epsilon^{i_2}$, the identity is $1 = \epsilon_1\epsilon_2^0 + \epsilon_1^0\epsilon_2$, and the E-multiplication is given by

$$(\epsilon_1^{i_1}\epsilon_2^{i_2}) \cdot (\epsilon_1^{j_1}\epsilon_2^{j_2}) = \begin{cases} \epsilon_1^{i_1}\epsilon_2^{j_2} & \text{if } i_2 + j_1 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Note that $A_2 \cong M_2(k)$ via $\epsilon_1\epsilon_2^0 \mapsto e_{11}$, $\epsilon_1^0\epsilon_2 \mapsto e_{22}$, $\epsilon_1^0\epsilon_2^0 \mapsto e_{21}$ and $\epsilon_1\epsilon_2 \mapsto e_{12}$. (Note that $e_1 = 1 \otimes 1 = \epsilon_1^0\epsilon_2^0$ satisfies $e_1^2 = 0$.) The inclusion $A \hookrightarrow A_2$ is then given by

$$\lambda_1 1 + \lambda_2\epsilon \mapsto \begin{pmatrix} \lambda_1 & \lambda_2 \\ 0 & \lambda_1 \end{pmatrix}$$

A_3 is isomorphic to the subalgebra of $M_4(k)$ given by $\{\lambda_1(e_{11} + e_{22}) + \lambda_2e_{12} + \lambda_3(e_{13}+e_{24})+\lambda_4e_{14}+\lambda_5(e_{31}+e_{42})+\lambda_6e_{32}+\lambda_7(e_{33}+e_{44})+\lambda_8e_{34} \mid \lambda_i \in k, i = 1, \ldots, 8\} \cong M_2(k[\epsilon])$. The inclusion $A_2 \hookrightarrow A_3$ is given by

$$X \longmapsto \begin{pmatrix} X_{11} & 0 & X_{12} & 0 \\ 0 & X_{11} & 0 & X_{12} \\ X_{21} & 0 & X_{22} & 0 \\ 0 & X_{21} & 0 & X_{22} \end{pmatrix}$$

For the next proposition the main point is that given a Frobenius extension there is a ring structure on the $C_n(A, B)$'s satisfying the hypotheses below (for one compares with (23)). This is true as well if A is a ring with B in its center, since the ordinary tensor algebra on $A \otimes_B A$ may be extended to an n-fold tensor product algebra $A \otimes_B \cdots \otimes_B A$.

Proposition 3.4. *Let $A \supseteq B$ be a ring extension. Suppose that there is a ring structure on each $A_n := C_n(A, B)$ for each $n \geq 0$, a ring homomorphism $A_{n-1} \to A_n$ for each $n \geq 1$, and that the composite $B \to A_n$ induces the natural bimodule given by $b \cdot (a_1 \otimes \cdots a_n) \cdot b' = ba_1 \otimes a_2 \otimes \cdots \otimes a_nb'$. Then $A \supseteq B$ has depth $2n + 1$ if and only if $A_n \mid B$ has depth 3.*

Proof. If $A \supseteq B$ has depth $2n+1$, then $A_n \overset{h}{\sim} A_{n+1}$ as B-bimodules. By tensoring repeatedly by ${}_BA \otimes_B -$, also $A_n \overset{h}{\sim} A_{2n}$ as B-bimodules. But $A_{2n} \cong A_n \otimes_B A_n$. Then $A_n \supseteq B$ has depth three.

Conversely, if $A_n \mid B$ has depth 3, then $A_{2n} \overset{h}{\sim} A_n$ as B-bimodules. But $A_{n+1} \mid A_{2n}$ via the split B-bimodule epi $a_1 \otimes \cdots \otimes a_{2n} \mapsto a_1 \cdots a_n \otimes a_{n+1} \otimes \cdots \otimes a_{2n}$. Then $A_{n+1} \mid qA_n$ for some $q \in \mathbb{Z}_+$. It follows that $A \supseteq B$ has depth $2n + 1$. □

One may in turn embed a depth three extension into a ring extension having depth two. The proof requires the QF condition. Retain the notation for the endomorphism ring introduced earlier in this section.

Theorem 3.5. *Suppose $A \supseteq B$ is a QF extension. If $A \supseteq B$ has depth 3, then $\mathcal{E} \supseteq B$ has depth 2. Conversely, if $\mathcal{E} \supseteq B$ has depth 2, and A_B is a generator, then $A \supseteq B$ has depth 3.*

Proof. Since A is a QF extension of B, we have $\mathcal{E} \stackrel{h}{\sim} A \otimes_B A$ as $\mathcal{E}$-A-bimodules. Then $\mathcal{E} \otimes_B \mathcal{E} \stackrel{h}{\sim} A \otimes_B A \otimes_B A \otimes_B A$ as $\mathcal{E}$-B-bimodules. Given the depth 3 condition, $A \otimes_B A \stackrel{h}{\sim} A$ as B-bimodules, it follows by two substitutions that $\mathcal{E} \otimes_B \mathcal{E} \stackrel{h}{\sim} A \otimes_B A$ as $\mathcal{E}$-B-bimodules. Consequently, $\mathcal{E} \otimes_B \mathcal{E} \stackrel{h}{\sim} \mathcal{E}$ as $\mathcal{E}$-B-bimodules. Hence, $\mathcal{E} \supseteq B$ has right depth 2, and since it is a QF extension by the endomorphism ring theorem and transitivity, $\mathcal{E} \supseteq B$ also has left depth 2.

Conversely, we are given A_B a progenerator, so that $\mathcal{E}$ and B are Morita equivalent rings, where ${}_B\mathrm{Hom}\,(A_B, B_B)_{\mathcal{E}}$ and ${}_{\mathcal{E}}A_B$ are the context bimodules. If $\mathcal{E} \supseteq B$ has depth two, then $\mathcal{E} \otimes_B \mathcal{E} \stackrel{h}{\sim} \mathcal{E}$ as $\mathcal{E}$-B-bimodules. Then $A \otimes_B A \otimes_B A \otimes_B A \stackrel{h}{\sim} A \otimes_B A$ as $\mathcal{E}$-B-bimodules. Since $\mathrm{Hom}\,(A_B, B_B) \otimes_{\mathcal{E}} A \cong B$ as B-bimodules, a cancellation of the bimodules ${}_{\mathcal{E}}A_B$ follows, so $A \otimes_B A \otimes_B A \stackrel{h}{\sim} A$ as B-bimodules. Since $A \otimes_B A \mid A \otimes_B A \otimes_B A$, it follows that $A \otimes_B A \mid qA$ for some $q \in \mathbb{Z}_+$. Then $A \supseteq B$ has depth 3. □

Example 3.6. To illustrate that the theorem does not extend to when $A \supseteq B$ is not a QF extension, consider $A = T_n(k)$ (a hereditary algebra) and $B = D_n(k)$ (a semisimple algebra), and k be an algebraically closed field of characteristic zero. (Since B, A is, is not a QF-algebra it follows by transitivity that $A \supseteq B$ is not a QF extension.) It was computed that $d(B, A) = 3$ in Example 1.8. Thinking of the columns of A as Ae_{ii}, it is quite easy to see that $\mathrm{End}\, A_B \cong M_1(k) \times M_2(k) \times \cdots \times M_n(k)$ and that the inclusion of $A \hookrightarrow \mathrm{End}\, A_B$ is given by

$$X \mapsto (X_{11}, \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix}, \ldots, X)$$

Its restriction to B is given by

$$\mathrm{Diag}(\mu_1, \ldots, \mu_n) \mapsto (\mu_1, \mathrm{Diag}(\mu_1, \mu_2), \ldots, \mathrm{Diag}(\mu_1, \ldots, \mu_n))$$

with inclusion matrix $M = \sum_{i \leq j} e_{ij}$. Then $MM^t > 0$, and from (4) we see that $d(B, \mathcal{E}) = 3$.

4. When tower depth equals subring depth

In this section we review tower depth from [17] and find a general case when it is the same as subring depth defined in (3) and in [2]. We first require a generalization of left and right depth 2 to a tower of three rings. We say that a tower $A \supseteq B \supseteq T$ where $A \supseteq B$ and $B \supseteq T$ are ring extensions, has *generalized right depth* 2 if $A \otimes_B A \stackrel{h}{\sim} A$ as natural A-T-bimodules. (Note that if $T = B$, this is the definition of the ring extension $A \supseteq B$ having right depth 2.)

Throughout the section below we suppose $A \supseteq B$ a Frobenius extension and $\mathcal{E}_i \hookrightarrow \mathcal{E}_{i+1}$ its tower above it, as defined in (14) and the ensuing discussion in Section 3. Following [17] (with a small change in vocabulary), we say that $A \supseteq B$ has *right* *tower depth* $n \geq 2$ if the sub-tower of composite ring extensions $B \to \mathcal{E}_{n-3} \hookrightarrow \mathcal{E}_{n-2}$ has generalized right depth 2; equivalently, as natural $\mathcal{E}_{n-2}$-B-bimodules,

$$\mathcal{E}_{n-2} \otimes_{\mathcal{E}_{n-3}} \mathcal{E}_{n-2} \oplus * \cong q\mathcal{E}_{n-2} \tag{36}$$

for some positive integer q, since the reverse condition is always satisfied. Since $\mathcal{E}_{-1} = B$ and $\mathcal{E}_0 = A$, this recovers the right depth two condition on a subring B of A. To this definition we add that a Frobenius extension $A \supseteq B$ has depth 1 if it is a centrally projective ring extension; i.e., ${}_BA_B \,|\, qB$ for some $q \in \mathbb{Z}_+$. Left tower depth n is just defined using (36) but as natural B-$\mathcal{E}_{n-2}$-bimodules. By [17, Theorem 2.7] the left and right tower depth n conditions are equivalent on Frobenius extensions.

From the definition of tower depth and a comparison of (16) and (1.7) we note that if A is a Frobenius extension of B of tower depth $n > 1$, then $B \subseteq A$ has subring depth $2n - 2$; from (36) we obtain $A_n \,|\, qA_{n-1}$ as A-B-bimodules, since $A_n \cong \mathcal{E}_{n-1} \cong \mathcal{E}_{n-2} \otimes_{\mathcal{E}_{n-3}} \mathcal{E}_{n-2}$.

From [17, Lemma 8.3], it follows that if $A \supseteq B$ has tower depth n, it has tower depth $n + 1$. Define $d_F(A, B)$ to be the minimum tower depth if $A \supseteq B$ has tower depth n for some integer n, $d_F(A, B) = \infty$ if the condition (36) is not satisfied for any $n \geq 2$ nor is it depth 1. Notice that $d_F(A, B) = d(B, A)$ if one of $d(B, A) \leq 2$ or $d_F(A, B) \leq 2$. This is extended to $d_F(A, B) = d(B, A)$ if one of $d(B, A), d_F(A, B) \leq 3$ in the next lemma.

Notice that tower depth n makes sense for a QF extension $A \supseteq B$: by elementary considerations, it has right tower depth 3 if $B \to A \hookrightarrow \mathcal{E}$ satisfies $\mathcal{E} \otimes_A \mathcal{E} \overset{h}{\sim} \mathcal{E}$ as $\mathcal{E}$-B-bimodules. It has been noted elsewhere that a QF extension has right tower depth 3 if and only if it has left tower depth 3 by an argument essentially identical to that in [17, Th. 2.8] but replacing Frobenius isomorphisms with quasi-Frobenius h-equivalences.

Lemma 4.1. *A QF extension $A \supseteq B$ such that A_B is a generator has tower depth 3 if and only if B has depth 3 as a subring in A.*

Proof. ($\Rightarrow$) By the QF property, $\mathcal{E} \overset{h}{\sim} A \otimes_B A$ as $\mathcal{E}$-B-bimodules. By the tower depth 3 condition, $\mathcal{E} \otimes_A \mathcal{E} \overset{h}{\sim} \mathcal{E}$ as $\mathcal{E}$-B-bimodules. Then $A \otimes_B A \otimes_B A \overset{h}{\sim} A \otimes_B A$ as $\mathcal{E}$-B-bimodules. Since A_B is a progenerator, we cancel bimodules ${}_{\mathcal{E}}A_B$ as in the proof of Theorem 3.5 to obtain $A \otimes_B A \overset{h}{\sim} A$ as B-bimodules. Hence $B \subseteq A$ has depth 3.

($\Leftarrow$) Given ${}_BA_B \overset{h}{\sim} {}_BA \otimes_B A_B$, by tensoring with ${}_{\mathcal{E}}A \otimes_B -$ we get $A \otimes_B A \overset{h}{\sim} A \otimes_B A \otimes_B A$ as $\mathcal{E}$-B-bimodules. By the QF property, $\mathcal{E} \otimes_A \mathcal{E} \overset{h}{\sim} \mathcal{E}$ as $\mathcal{E}$-B-bimodules follows, whence $A \supseteq B$ has tower depth 3. □

The theorem below proves that subring depth and tower depth coincide on Frobenius generator extensions. At a certain point in the proof, we use the following fundamental fact about the tower A_n above a Frobenius extension $A \supseteq B$: since the compositions of the Frobenius extensions remain Frobenius, the iterative constructions of E-multiplication on tensor-squares isomorphic to endomorphism rings applies, but gives isomorphic ring structures to those on the A_n. For example, the composite extension $B \to A_n$ is Frobenius with $\mathrm{End}\,(A_n)_B \cong A_n \otimes_B A_n \cong A_{2n}$, isomorphic in its $E \circ E_1 \circ \cdots \circ E_{n-1}$-multiplication or its E-multiplication given in (17) [14].

Theorem 4.2. *Suppose A is a Frobenius extension of B and A_B is a generator. Then $A \supseteq B$ has tower depth m for $m = 1, 2, \ldots$ if and only if the subring $B \subseteq A$ has depth m. Consequently, $d_F(A,B) = d(B,A)$.*

Proof. The cases $m = 1, 2, 3$ have been dealt with above. We divide the rest of the proof into odd m and even m. The proof for odd $m = 2n+1$: ($\Rightarrow$) If $A \supseteq B$ has tower depth $2n+1$, then $A_{2n} \otimes_{A_{2n-1}} A_{2n} \,|\, qA_{2n}$ as A_{2n}-B-bimodules. Continuing with $A_{2n} \cong A_{2n-1} \otimes_{A_{2n-2}} A_{2n-1}$, iterating and performing standard cancellations, we obtain

$$A_{2n+1} \,|\, qA_{2n} \tag{37}$$

as $\mathrm{End}\,(A_n)_B$-B-bimodules. But the module $(A_n)_B$ is a generator for all n by Lemma 3.1, the endomorphism ring theorem for Frobenius generator extensions and transitivity of generator property for modules (if M_A and A_B are generators, then restricted module M_B is clearly a generator). It follows that $(A_n)_B$ is a progenerator and cancellable as an $\mathrm{End}\,(A_n)_B$-B-bimodule (applying the Morita theorem as in the proof of Theorem 3.5). Then ${}_B(A_{n+1})_B \,|\, {}_B(A_n)_B$ after cancellation of A_n from (37), which is the depth $2n+1$ condition in (3).

($\Leftarrow$) Suppose $A_{n+1} \oplus * \cong A_n$ as B-bimodules. Apply to this the additive functor $A_n \otimes_B -$ from category of B-bimodules into the category of $\mathrm{End}\,(A_n)_B$-B-bimodules. We obtain (37) which is equivalent to the tower depth $2n+1$ condition of $A \supseteq B$.

The proof in the even case, $m = 2n$ does not need the generator condition (since even non-generator Frobenius extensions have endomorphism ring extensions that are generators):

($\Rightarrow$) Given the tower depth $2n$ condition $A_{2n-1} \otimes_{A_{2n-2}} A_{2n-1} \cong A_{2n}$ is isomorphic as A_{2n-1}-B-bimodules to a direct summand in qA_{2n-1} for some positive integer q. Introduce a cancellable extra term in $A_{2n} \cong A_n \otimes_A A_{n+1}$ and in $A_{2n-1} \cong A_n \otimes_A A_n$. Now note that $A_{2n-1} \cong \mathrm{End}\,(A_n)_A$ which is Morita equivalent to A. After cancellation of the $\mathrm{End}\,(A_n)_A$-A-bimodule A_n, we obtain $A_{n+1} \,|\, A_n$ as A-B-bimodules as required by (3).

($\Leftarrow$) Given ${}_A(A_{n+1})_B \,|\, {}_A(A_n)_B$, we apply ${}_{\mathrm{End}\,(A_n)_A} A_n \otimes_A -$ obtaining $A_{2n} \,|\, A_{2n-1}$ as A_{2n-1}-B-bimodules, which is equivalent to the tower depth $2n$ condition. □

A depth 2 extension $A \supseteq B$ may have easier equivalent conditions, e.g., a normality condition, to fulfill than the B-A-bimodule condition $A \otimes_B A \,|\, qA$ [3]. Thus the next corollary (or one like it stated more generally for Frobenius extensions) presents a possible simplification in determining whether a special type of ring extension has finite depth. The corollary follows from the theorem above as well as [17, 8.6], Corollary 2.2, Proposition 3.4 and Theorem 3.5.

Corollary 4.3. *Let $K \subseteq H$ be a Hopf subalgebra pair of finite dimensional unimodular Hopf algebras. Then K has finite depth in H if and only if there is a tower algebra H_m such that $K \subseteq H_m$ has depth 2.*

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Departamento de Matematica, Faculdade de Ciências da Universidade do Porto, Rua Campo Alegre, 687, 4169-007 Porto
E-mail address: lkadison@fc.up.pt