

The Discrete Markus-Yamabe Problem for Symmetric Planar Polynomial Maps

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To the memory of Carlos Gutierrez

Abstract

We probe deeper into the Discrete Markus-Yamabe Question for polynomial planar maps and into the normal form for those maps which answer this question in the affirmative. Furthermore, in a symmetric context, we show that the only nonlinear equivariant polynomial maps providing an affirmative answer to the Discrete Markus-Yamabe Question are those possessing \mathbb{Z}_2 as their group of symmetries. We use this to establish two new tools which give information about the spectrum of a planar polynomial map.

Keywords: Markus-Yamabe Conjecture; polynomial maps; symmetry

1 Introduction

The Discrete Markus-Yamabe Question, DMYQ(n), in dimension n was formulated by Cima *et al.* [2] as follows:

[DMYQ(n)] *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a C^1 map such that $F(0) = 0$ and for any $x \in \mathbb{R}^n$, $JF(x)$ has all its eigenvalues with modulus less than one. Is it true that 0 is a global attractor for the discrete dynamical system generated by F ?*

These authors have found sufficient conditions for planar maps to provide an affirmative answer to this question. We proceed with the study in dimension 2, since this is the only interesting dimension: the answer is negative for higher dimensions, see Cima *et al.* [3] for examples in dimensions higher

than 3 and van den Essen and Hubbers [4] for dimensions higher than 4, and is affirmative in dimension 1.

An attentive look at the proof of Theorem B in Cima *et al.* [2] produces a more explicit description of polynomial maps satisfying sufficient conditions for an affirmative answer to the DMYQ(2), leading to a normal form for such maps, as in Chamberland [1]. In particular this may be used for testing a map for eigenvalues outside the unit disk.

After having established the normal form for maps that answer the DMYQ(2) in the affirmative, we look at the symmetric setting. We formulate the Symmetric Discrete Markus-Yamabe Question, SDMYQ(n), as follows:

[SDMYQ(n)] *Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 map such that $F(0) = 0$ and for any $x \in \mathbb{R}^n$, $JF(x)$ has all its eigenvalues with modulus less than one. Suppose that the symmetries of F form a nontrivial compact subgroup of $O(n)$. Is it true that 0 is a global attractor for the discrete dynamical system generated by F ?*

Note that a counterexample to the SDMYQ(2) is given in [2, theorem D], where F is a rational map, and the symmetries of F constitute the group \mathbb{Z}_4 .

We address this question for $n = 2$ when F is polynomial. We find that only when the group of symmetries of the map is \mathbb{Z}_2 (a group of order two), can a nonlinear polynomial map provide an affirmative answer to the DMYQ(2) and *a fortiori* to the SDMYQ(2). In fact, we show that this is the only symmetry group compatible with the hypotheses of the DMYQ(2). This is then used as a test for the existence of expanding eigenvalues in symmetric polynomial maps.

2 Normal Forms for Planar Polynomial Maps

We look deeper into the admissible form of polynomial maps which provide an affirmative answer to the DMYQ(2).

Theorem 2.1 (Normal Form). *Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a polynomial map such that $F(0) = 0$ and all the eigenvalues of $JF(x, y)$ have modulus smaller than one for all $(x, y) \in \mathbb{R}^2$. Then $F(x, y) = B(x, y)^T + u^2 p(u)(\alpha, \beta)^T$, where B is a real matrix, $\alpha, \beta \in \mathbb{R}$, p is a real polynomial and $u = ax + by$ for $a, b \in \mathbb{R}$.*

Proof. Theorem B in Cima *et al.* [2] proves that the condition on the eigenvalues of JF implies that F is obtained by an affine transformation from a triangular map. The assumption that the origin is a fixed-point allows us to work with linear instead of affine transformations. Furthermore, the triangular map is such that the off-diagonal terms can be described by

a polynomial in one variable alone. Therefore, from the triangular map $G(u, v) = K(u, v)^T + (0, uq(u))^T$ with K a real diagonal matrix and q a real polynomial, we obtain by a linear change of coordinates L ,

$$F(x, y) = L^{-1}G(L(x, y)).$$

Let

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then for $u = ax + by$ we get

$$\begin{aligned} F(x, y) &= L^{-1} \left(KL \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ uq(u) \end{pmatrix} \right) = \\ &= L^{-1}KL \begin{pmatrix} x \\ y \end{pmatrix} + uq(u)L^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

Let $A = L^{-1}KL$, then A is a matrix with real eigenvalues. Consider now

$$(\alpha, \beta)^T = L^{-1}(0, 1)^T = \frac{1}{ad - bc} \begin{pmatrix} -b \\ a \end{pmatrix}, \quad (1)$$

then we can rewrite F as follows:

$$F(x, y) = A \begin{pmatrix} x \\ y \end{pmatrix} + uq(u) \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

The proof follows taking

$$B \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + uq(0) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \text{and} \quad p(u) = \frac{q(u) - q(0)}{u}.$$

□

Theorem 2.1 is better used for identifying which maps do not provide an affirmative answer to the DMYQ(2). In fact, while it may not be easy to recognize an admissible form by looking at a map, it will be straightforward to assert that, for instance, $F(x, y) = (\frac{x}{2} + y^2, \frac{y}{3} + y^3)$ will not provide an affirmative answer to the DMYQ(2). This is because the non-linear polynomials in the first and second coordinate have different degrees.

This provides a criterion for studying the spectrum of a polynomial planar map as stated in the following:

Corollary 2.2. *Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a polynomial map. If the quotient of the nonlinear parts of the coordinates of F is not constant, then there exists a point in \mathbb{R}^2 where the jacobian of F has an eigenvalue outside the unit disk.*

3 Symmetric Planar Polynomial Maps

In the context of symmetric maps some further results may be obtained. As usual, the reference for the symmetric context is the book by Golubitsky *et al.* [5]. Assume for the rest of this section that $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ has a compact Lie group Γ as its group of linear symmetries. That is to say that Γ is the largest group such that for all $(x, y) \in \mathbb{R}^2$ and all $\gamma \in \Gamma$ we have

$$F(\gamma \cdot (x, y)) = \gamma \cdot F(x, y). \quad (2)$$

We always assume nontrivial groups and actions.

We single out two possible group elements of Γ . These are represented by κ and ζ_n and act on elements of the plane as

$$\begin{aligned} \kappa \cdot (x, y)^T &= (x, -y)^T \\ \zeta_n \cdot (x, y)^T &= e^{2\pi i/n} \cdot (x, y)^T = \left(x \cos \frac{2\pi}{n} - y \sin \frac{2\pi}{n}, x \sin \frac{2\pi}{n} + y \cos \frac{2\pi}{n} \right)^T, \end{aligned}$$

where $n \in \mathbb{N}$.

Note that any reflection may be written as κ above in suitable coordinates.

Proposition 3.1. *Let Γ be a compact Lie group acting on \mathbb{R}^2 . Assume Γ is the symmetry group of a polynomial map F .*

(i) *If $\kappa \in \Gamma$ then F does not answer the DMYQ(2) in the affirmative unless F is of the form:*

$$F(x, y) = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + y^2 p(y^2) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(ii) *If $\zeta_n \in \Gamma$ for $n \geq 3$ then F does not answer the DMYQ(2) in the affirmative unless F is linear. Moreover, the linear part of F is either a homothety or a rotation matrix.*

Proof. From Theorem 2.1 we know that, in order to satisfy the hypotheses of the DMYQ(2), we must have $F(x, y) = B(x, y)^T + (\alpha u^2 p(u), \beta u^2 p(u))^T$. Then F is γ -equivariant if and only if both B and the nonlinear part satisfy (2). We then write $N(x, y) = (\alpha r(u), \beta r(u))$ with $r(u) = u^2 p(u)$. The proof proceeds in the following two steps:

(i) If $\kappa \in \Gamma$, then $B \cdot \kappa = \kappa \cdot B$ if and only if B is a diagonal matrix. Furthermore, we must have

$$\begin{aligned} N(\kappa \cdot (x, y)) &= (\alpha r(ax - by), \beta r(ax - by))^T \\ &= (\alpha r(ax + by), -\beta r(ax + by))^T = \kappa \cdot N(x, y), \end{aligned}$$

from which it follows that, if both α and β are nonzero, then

$$\begin{cases} r(ax + by) = r(ax - by) \\ r(ax + by) = -r(ax - by) \end{cases},$$

which is to say that $r(ax - by) = -r(ax - by)$ and thus $r(u) = -r(u)$ for all $u \in \mathbb{R}^2$. Hence, r is identically zero.

If $\alpha = 0$, and $\beta \neq 0$ then, from (1) in the proof of Theorem 2.1, $b = 0$. We then have

$$r(ax - by) = -r(ax + by) \quad \Leftrightarrow \quad r(ax) = -r(ax)$$

meaning that r is identically zero.

If $\beta = 0$, and $\alpha \neq 0$ then, from (1) in the proof of Theorem 2.1, $a = 0$. We then have

$$r(ax - by) = r(ax + by) \quad \Leftrightarrow \quad r(-by) = r(by)$$

meaning that r is an even polynomial in y .

If both α and β are zero, the result holds trivially.

(ii) If $\zeta_n \in \Gamma$ for $n \geq 3$, (2) implies that

$$N(\zeta_n \cdot (x, y)) = \begin{pmatrix} \alpha r \left(a(x \cos \frac{2\pi}{n} - y \sin \frac{2\pi}{n}) + b(x \sin \frac{2\pi}{n} + y \cos \frac{2\pi}{n}) \right) \\ \beta r \left(a(x \cos \frac{2\pi}{n} - y \sin \frac{2\pi}{n}) + b(x \sin \frac{2\pi}{n} + y \cos \frac{2\pi}{n}) \right) \end{pmatrix}$$

must be equal to

$$\zeta_n \cdot N(x, y) = \begin{pmatrix} \alpha r(ax + by) \cos \frac{2\pi}{n} - \beta r(ax + by) \sin \frac{2\pi}{n} \\ \alpha r(ax + by) \sin \frac{2\pi}{n} + \beta r(ax + by) \cos \frac{2\pi}{n} \end{pmatrix}.$$

We therefore must have

$$\begin{cases} \alpha r(a(x \cos \frac{2\pi}{n} - y \sin \frac{2\pi}{n}) + b(x \sin \frac{2\pi}{n} + y \cos \frac{2\pi}{n})) = \\ \quad = \alpha r(ax + by) \cos \frac{2\pi}{n} - \beta r(ax + by) \sin \frac{2\pi}{n} \\ \beta r(a(x \cos \frac{2\pi}{n} - y \sin \frac{2\pi}{n}) + b(x \sin \frac{2\pi}{n} + y \cos \frac{2\pi}{n})) = \\ \quad = \alpha r(ax + by) \sin \frac{2\pi}{n} + \beta r(ax + by) \cos \frac{2\pi}{n} \end{cases}.$$

If one of either α or β is zero, we observe that r is identically zero since $n \geq 3$. Otherwise, after some simplification, we obtain

$$-\beta^2 r(ax + by) = \alpha^2 r(ax + by)$$

and again we see that r must be identically zero. \square

The following result finishes our description of planar polynomial maps that provide an affirmative answer to the DMYQ(2).

Theorem 3.2. *A nonlinear equivariant polynomial map satisfying DMYQ(2) can only have $\Gamma = \mathbb{Z}_2$ as its symmetry group.*

Proof. It is known (see, for instance, Golubitsky *et al.* [5], XII §1 (c)) that every compact Lie group in $GL(2)$ can be identified with a subgroup of the orthogonal group $O(2)$. The compact subgroups of $O(2)$ that do not contain a rotation ζ_n , $n \geq 3$, are, in suitable coordinates, the trivial subgroup generated by the identity; \mathbb{Z}_2 , generated by either κ or minus the identity; and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. This last group contains the two reflections κ and $-\kappa$. Therefore if F has these symmetries, it must satisfy both

$$F(x, y) = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} r(y^2) \\ 0 \end{pmatrix}$$

and

$$F(x, y) = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{r}(x^2) \end{pmatrix}$$

and therefore, $r = \tilde{r} = 0$. Since we are assuming Γ to be nontrivial, the proof is finished. \square

We end this note with the following example: the lowest order (and perhaps simplest) nonlinear polynomial map whose symmetry group is \mathbb{Z}_4 is of the form $F(x, y) = (\alpha x - \beta y^3, \alpha y + \beta x^3)$. By Theorem 3.2 this map cannot answer the DMYQ(2) in the affirmative. Indeed, it is clear either by direct computation or by applying Lemma 1.1 in [2] that the eigenvalues of the jacobian of F are not all inside the unit disk.

In fact Theorem 3.2 together with Theorem B of [2] lead to a second criterion for studying the spectrum of a polynomial planar map as follows:

Corollary 3.3. *Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a polynomial map. If F has a non-trivial symmetry group different from \mathbb{Z}_2 , then there exists a point in \mathbb{R}^2 where the jacobian of F has an eigenvalue outside the unit disk.*

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