GENERATING FUNCTIONS FOR HOPF BIFURCATION WITH S_n-SYMMETRY

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Abstract. Hopf bifurcation in the presence of the symmetric group $\mathbf{S}_n$ (acting naturally by permutation of coordinates) is a problem with relevance to coupled oscillatory systems. To study this bifurcation it is important to construct the Taylor expansion of the equivariant vector field in normal form. We derive generating functions for the numbers of linearly independent invariants and equivariants of any degree, and obtain recurrence relations for these functions. This enables us to determine the number of invariants and equivariants for all n, and show that this number is independent of n for sufficiently large n. We also explicitly construct the equivariants of degree three and degree five, which are valid for arbitrary n.

1. **Introduction.** One of the few classic problems in equivariant bifurcation theory that has not been completely investigated is Hopf bifurcation with $\mathbf{S}_n$-symmetry. This problem is relevant to, for example, the behaviour of all-to-all coupled nonlinear oscillators [9]. Consider the symmetric group $\mathbf{S}_n$ consisting of all bijections from $\{1, 2, \ldots, n\}$ to itself using the usual composition of functions as the group operation. The *standard* irreducible representation of $\mathbf{S}_n$ can be realized by considering the restriction of the action of $\mathbf{S}_n$ on $\mathbb{R}^n$ by permutation of coordinates to the invariant subspace given by the vectors with coordinates summing zero, denoted by $\mathbb{R}^{n,0}$. It follows then that Hopf bifurcation occurs for the sum of two isomorphic copies of such representation. Moreover, $\mathbb{R}^{n,0} \oplus \mathbb{R}^{n,0}$ is isomorphic to the subspace of $\mathbb{C}^n$ given by the vectors with complex coordinates summing zero, denoted by $\mathbb{C}^{n,0}$ and where $\mathbf{S}_n$ acts by permutation of the coordinates. When studying Hopf bifurcation, it is important to find the truncation of appropriate degree of the Taylor expansion at the bifurcation point of the commuting vector field assumed in Birkhoff normal form. Also, we need to consider the equivariance under the circle group $\mathbf{S}^1$; it can be assumed that the action of $\mathbf{S}^1$ on $\mathbb{C}^{n,0}$ is given by multiplication by $e^{i\theta}$ for $\theta \in \mathbf{S}^1$. See Golubitsky *et al.* [7, Section XVI 3].

When Hopf bifurcation occurs for problems posed on $\mathbb{C}^{n,0}$, the Equivariant Hopf Theorem [7, Theorem XVI 4.1] guarantees the generic existence of branches of periodic solutions for each isotropy subgroup of $\mathbf{S}_n \times \mathbf{S}^1$ with two-dimensional fixed-point subspace. These isotropy subgroups are called $\mathbb{C}$-*axial*, have been calculated by Stewart [14] and are formed by the spatio-temporal symmetries of the periodic solutions. In order to investigate the stability of these periodic solutions, we need to construct the polynomial invariant functions

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and the equivariant mappings with polynomial components under $\mathbf{S}_n$. Previous work has done this for the specific cases $n = 3$ [3] and $n = 4$ [2].

In this paper we ask how many invariants and equivariants for $\mathbf{S}_n \times \mathbf{S}^1$ there are, degree by degree. We give the answer by constructing generating functions for finite n and using these to find recursive relations. With these relations we obtain results for general n. Rodrigues [11] proves that when $n \geq 5$, degree 5 terms of the vector field in Birkhoff normal form are necessary to determine the stability of the periodic solutions guaranteed by the Equivariant Hopf Theorem. We show explicit generators for equivariant mappings with cubic and quintic homogeneous polynomial components under $\mathbf{S}_n \times \mathbf{S}^1$.

The paper is organized in the following way. In Section 2 we review a few facts about generating functions, Hilbert-Poincaré series, Cohen-Macaulay rings and the standard irreducible representation of $\mathbf{S}_n$. In Section 3 we present Hilbert-Poincaré series for the ring of invariant polynomial functions and the module of equivariant mappings with polynomial components under the standard irreducible representation of $\mathbf{S}_n$. Section 4 reviews recently obtained formulas for invariants and equivariants under $\mathbf{S}_n \times \mathbf{S}^1$ on $\mathbb{C}^{n,0}$. In particular, generating functions for $\mathbf{S}_n \times \mathbf{S}^1$ can be obtained by averaging over $\mathbf{S}^1$ the generating functions for the ring of invariants and module of equivariants for $\mathbf{S}_n$ acting on the real vector space $\mathbb{C}^n$ by permutation of complex coordinates.

Section 5 contains our main results. We obtain recursive formulas for deriving the generating functions for $\mathbf{S}_n$ from the generating functions for $\mathbf{S}_k$, $k < n$ (Theorem 5.1, Corollary 5.4 and Proposition 5.8). Using these we relate in Theorem 5.5 and Corollaries 5.6, 5.9 the generating functions for $\mathbf{S}_n$ and $\mathbf{S}_{n-1}$, leading to general results for arbitrary n. In Section 6 we use our results of Section 5 to derive the numbers of polynomial invariant functions and equivariant mappings with polynomial components of degree less than 10 for Hopf bifurcation with $\mathbf{S}_n$-symmetry. We finish with Section 7 where we give explicit generators for the vector spaces of equivariant mappings with polynomial components of degree 3 and 5 under the action of the group $\mathbf{S}_n \times \mathbf{S}^1$ on $\mathbb{C}^{n,0}$.

2. Background.

The aim of this section is to set up notation and review a few facts about generating functions, Hilbert-Poincaré series and the standard irreducible representation of $\mathbf{S}_n$.

2.1. Generating Functions.

We start by introducing generating functions. See for example Sagan [12, Chapter 4] and references therein.

Definition 2.1. Given a sequence $(a_n)_{n\geq 0} = a_0, a_1, a_2, \ldots$ of complex numbers, the corresponding *generating function* is the power series

$$f(t) = \sum_{n\geq 0} a_n t^n.$$

If the a_n enumerate some set of combinatorial objects, then $f(t)$ is said to be the generating function for those objects. $\diamond$

To obtain information about a sequence it is often easier to manipulate its generating function. Moreover, sometimes there is no known simple expression for a_n and yet $f(t)$ is easy to compute.

Example 2.2. A *partition* of n is a sequence $\lambda = (\lambda_1, \ldots, \lambda_l)$ where the λ_i are weakly decreasing and $\sum_{i=1}^{l} \lambda_i = n$. The generating function $\sum_{n\geq 0} p(n)t^n$, where $p(n)$ is the number of partitions of n is

$$\frac{1}{1-t}\,\frac{1}{1-t^2}\,\frac{1}{1-t^3}\cdots$$

This result is a famous theorem of Euler [4]. There is no known closed-form formula for $p(n)$ itself. In the context of the symmetric group $\mathbf{S}_n$, $p(n)$ is the number of conjugacy classes (and hence also the number of irreducible representations). ◇

2.2. **Hilbert-Poincaré Series.** Let G be a compact Lie group *acting* linearly on a finite-dimensional real or complex vector space V. In what follows, $\mathbf{K} = \mathbb{R}$ or $\mathbf{K} = \mathbb{C}$ and to simplify notation we denote the linear action of $g \in G$ on a vector $v \in V$ by gv.

A polynomial function $f : V \to \mathbf{K}$ is *invariant* under G if $f(gv) = f(v)$ for all $g \in G, v \in V$. A polynomial mapping $F : V \to V$ is *equivariant* under G if $F(gv) = gF(v)$ for all $g \in G$, $v \in V$. The vector space $\mathcal{P}_V(G)$ of G-invariant polynomials is a sub-algebra of the algebra of all polynomial functions $\mathcal{P}_V$ on V and $\mathcal{P}_V^k(G) = \mathcal{P}_V(G) \cap \mathcal{P}_V^k$ is the vector space of homogeneous G-invariant polynomials of degree k.

The space of G-equivariant polynomial mappings from V to V is a module over the ring $\mathcal{P}_V(G)$, and we denote it by $\vec{\mathcal{P}}_V(G)$. Similarly, the space of homogeneous G-equivariant polynomial maps from V to V of degree k is $\vec{\mathcal{P}}_V^k(G) = \vec{\mathcal{P}}_V(G) \cap \vec{\mathcal{P}}_V^k$.

We are interested in calculating the number of linearly independent homogeneous G-invariants or G-equivariants of a certain degree. Generating functions for these dimensions are generally known as “Molien functions” or “Hilbert-Poincaré series”.

The original definition of Hilbert-Poincaré series is for complex representations. In this paper we are interested in real representations. As we explain (see Remark 2.3 below) the ‘real’ and ‘complex’ Hilbert-Poincaré series are the same.

Let G be a compact Lie group acting on $V = \mathbb{R}^m$. Without loss of generality, we can assume that G acts orthogonally and linearly on V, so that any $g \in G$ acts as an orthogonal matrix M_g with real entries. Moreover, we can view it as a matrix acting on $V^{\mathbb{C}} = \mathbb{C}^m$. If $(x_1, \ldots, x_m)$ denote real coordinates on $\mathbb{R}^m$, $x_j \in \mathbb{R}$, then we obtain complex coordinates on $\mathbb{C}^m$ by permitting the x_j to be complex. Note that there is a natural inclusion

$$\mathbb{R}[x_1, \ldots, x_m] \;\subseteq\; \mathbb{C}[x_1, \ldots, x_m]$$

where these are the rings of polynomials in the x_j with coefficients in $\mathbb{R}$, $\mathbb{C}$ respectively.

Remark 2.3. Every real-valued G-invariant in $\mathbb{R}[x_1, \ldots, x_m]$ is also a complex-valued G-invariant in $\mathbb{C}[x_1, \ldots, x_m]$. Conversely, the real and imaginary parts of a complex valued invariant are real invariants (because the matrices M_g have real entries). Therefore a basis over $\mathbb{R}$ for the real vector space of degree k real-valued invariants is also a basis over $\mathbb{C}$ for the complex vector space of degree k $\mathbb{C}$-valued invariants. Similar remarks apply to the equivariants. $\diamond$

We suppose now that V is a m-dimensional vector space over $\mathbb{C}$, where $x_1, \ldots, x_m$ denote coordinates relative to a basis for V, and $G \subseteq \mathrm{GL}(V)$ is a compact Lie group acting on V. Let $\mathcal{P}_V(G)$ denote the sub-algebra of $\mathbb{C}[x_1, \ldots, x_m]$ formed by the invariant polynomials under G (over $\mathbb{C}$). Note that $\mathbb{C}[x_1, \ldots, x_m]$ is graded:

$$\mathbb{C}[x_1, \ldots, x_m] \;=\; R_0 \oplus R_1 \oplus R_2 \oplus \cdots$$

where R_k consists of all homogeneous polynomials of degree k. Now observe that if $f(x) \in R_k$ for some k then $f(gx) \in R_k$ for all $g \in G$. Therefore the space $\mathcal{P}_V(G)$ has the structure

$$\mathcal{P}_V(G) \;=\; \mathcal{P}_V^0(G) \oplus \mathcal{P}_V^1(G) \oplus \mathcal{P}_V^2(G) \oplus \cdots$$

of a graded $\mathbb{C}$-algebra given by $\mathcal{P}_V^k(G) = R_k \cap \mathcal{P}_V(G)$.

The *Hilbert-Poincaré series* of the graded algebra $\mathcal{P}_V(G)$ is a generating function for the dimension of the vector space of invariants at each degree defined by

$$\Phi_G(t) \;=\; \sum_{d=0}^{\infty} (\dim \mathcal{P}_V^d(G)) t^d.$$

Consider the normalised Haar measure μ_G defined on G and denote by $\int_G f$ the integral with respect to μ_G of a continuous function f defined on G. Molien's Theorem gives an explicit formula for Φ_G:

$$\Phi_G(t) \;=\; \int_G \frac{1}{\det(1 - gt)}\, \mathrm{d}\mu_G(g)\,.$$

See Molien [10] for the original proof of the finite case, and Sattinger [13] for the extension to a compact group.

If G is finite, the Molien formula for the Hilbert-Poincaré series of $\mathcal{P}_V(G)$ is

$$\Phi_G(t) \;=\; \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - gt)}\,. \tag{2.1}$$

The *Hilbert series* for the graded module $\vec{\mathcal{P}}_V(G)$ over the ring $\mathcal{P}_V(G)$ is the generating function

$$\Psi_G(t) \;=\; \sum_{d=0}^{\infty} \dim(\vec{\mathcal{P}}_V^d(G)) t^d$$

and an explicit formula for Ψ_G is given by:

$$\Psi_G(t) \;=\; \int_G \frac{\chi(g^{-1})}{\det(1 - gt)}\, \mathrm{d}\mu_G(g) \tag{2.2}$$

where χ is the character for the G action on V [13]. Observe that if the action of G on V is orthogonal then $g^{-1} = g^t$ and $\chi(g^{-1}) = \chi(g)$.

2.3. **The Cohen-Macaulay Property.** Invariant rings admit a nice decomposition: they are Cohen-Macaulay. We review a few concepts and results related with this. See for example Sturmfels [15, Section 2.3] and references therein.

Let $p_1, p_2, \ldots, p_k$ be algebraically independent elements of $\mathbb{C}[x_1, \ldots, x_m]$ which are homogeneous of degrees $d_1, d_2, \ldots, d_k$, respectively. Then the Hilbert series of the graded ring subring $\mathbb{C}[p_1, \ldots, p_k]$ is

$$\frac{1}{(1-t^{d_1})(1-t^{d_2})\cdots(1-t^{d_k})} \tag{2.3}$$

(see for example [15, Lemma 2.2.3]).

Let $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ be a graded $\mathbb{C}$-algebra of dimension n. Thus $R_0 = \mathbb{C}$, $R_i \cdot R_j \subseteq R_{i+j}$ and n is the maximal number of elements of R which are algebraically independent over $\mathbb{C}$. The number n is called the *Krull dimension* of R. A set $\{\theta_1, \ldots, \theta_n\}$ of homogeneous elements of positive degree in R is said to be a *homogeneous system of parameters* (h.s.o.p.) provided R is finitely generated as a module over its subring $\mathbb{C}[\theta_1, \ldots, \theta_n]$. In particular this implies that $\theta_1, \ldots, \theta_n$ are algebraically independent. A basic result of commutative algebra, the Noether Normalization Lemma, implies that an h.s.o.p. for R always exists. Moreover, the following result from commutative algebra [15, Theorem 2.3.1] holds:
If $\theta_1, \ldots, \theta_n$ is an h.s.o.p. for R, then the following conditions are equivalent:
(a) R is a finitely generated *free* module over $\mathbb{C}[\theta_1, \ldots, \theta_n]$. That is, there exist $\eta_1, \ldots, \eta_t \in R$ (which may be chosen to be homogeneous) such that

$$R = \bigoplus_{i=1}^{t} \eta_i \mathbb{C}[\theta_1, \ldots, \theta_n]. \tag{2.4}$$

(b) For every h.s.o.p. $\phi_1, \ldots, \phi_n$ of R, the ring R is a finitely-generated free $\mathbb{C}[\phi_1, \ldots, \phi_n]$-module.

A graded $\mathbb{C}$-algebra R satisfying the conditions (a) and (b) above is said to be *Cohen-Macaulay*. The decomposition (2.4) is called a *Hironaka decomposition* of the Cohen-Macaulay algebra R. If we know the explicit decomposition (2.4) then the Hilbert series of R is

$$\frac{\displaystyle\sum_{i=1}^{t} t^{\deg \eta_i}}{\displaystyle\prod_{j=1}^{n} (1-t^{\deg \theta_j})}. \tag{2.5}$$

If G is compact then the invariant ring $\mathcal{P}_V(G)$ is Cohen-Macaulay. See Hochster and Roberts [8], or Sturmfels [15, Theorem 2.3.5] for the case of finite groups. If

$$\mathcal{P}_V(G) = \bigoplus_{i=1}^{t} \eta_i \mathbb{C}[\theta_1, \ldots, \theta_n]$$

then every invariant $I(x)$ can be written uniquely as

$$I(x) = \bigoplus_{i=1}^{t} \eta_i(x) p_i(\theta_1(x), \ldots, \theta_n(x))$$

where $p_1, \ldots, p_t$ are suitable n-variate polynomials. Moreover, $\{\theta_1, \ldots, \theta_n, \eta_1, \ldots, \eta_t\}$ is a set of *fundamental invariants* for G, also called a *Hilbert basis* of the ring $\mathcal{P}_V(G)$. The polynomials θ_i in the h.s.o.p. are called *primary invariants* and the η_j are called *secondary invariants*. Note that for a given group G there are many different Hironaka decompositions and also the degrees of the primary and secondary invariants are not unique.

2.4. **Standard Irreducible Representation of $\mathbf{S}_n$.** Here we review some classic results about the standard action of $\mathbf{S}_n$ [12].

Consider $\mathbf{S}_n$ acting on $\mathbb{R}^n$ by permutation of coordinates:

$$\sigma\left(x_1, x_2, \ldots, x_n\right) = \left(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(n)}\right) \tag{2.6}$$

for $\sigma \in \mathbf{S}_n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. The action of $\mathbf{S}_n$ is reducible, and decomposes $\mathbb{R}^n$ into the direct sum of two distinct irreducible $\mathbf{S}_n$-invariant spaces:

$$\mathbb{R}^n = \mathbb{R}^{n,0} \oplus U \tag{2.7}$$

where

$$\mathbb{R}^{n,0} = \{x \in \mathbb{R}^n : \ x_1 + x_2 + \cdots + x_n = 0\}, \quad U = \{(x, x, \ldots, x) : \ x \in \mathbb{R}\}.$$

The action of $\mathbf{S}_n$ is trivial on U and absolutely irreducible on $\mathbb{R}^{n,0}$. A representation is said to be absolutely irreducible if the only equivariant linear maps are scalar multiples of the identity. Also, absolute irreducibility implies irreducibility. To prove that the action on $\mathbb{R}^{n,0}$ is absolutely irreducible, let $L : \ \mathbb{R}^{n,0} \to \mathbb{R}^{n,0}$ be linear and $\mathbf{S}_n$-equivariant. Write it as $L(x) = (L_1(x), \ldots, L_n(x))$ and so $L_1(x) + \cdots + L_n(x) = 0$ when $\sum_{i=1}^n x_i = 0$. Now observe that $\mathbf{S}_n$ is generated by the transpositions $(12), (13), \ldots, (1n)$. From the equivariance conditions

$$L((1j)x) = (1j)L(x)$$

for all $x \in \mathbb{R}^{n,0}$ and $j = 2, \ldots, n$, it follows in particular that

$$L_j(x) = L_1((1j)x)$$

for $j = 1, \ldots, n$. Thus in order L to be $\mathbf{S}_n$-equivariant it is necessary that L has the following form:

$$L(x) = (L_1(x), L_1((12)x), \ldots, L_1((1n)x)) \,. \tag{2.8}$$

We claim that L in the above form is $\mathbf{S}_n$-equivariant if and only if L_1 is $\mathbf{S}_{n-1}$-invariant in the $n-1$ coordinates $x_2, \ldots, x_n$. Therefore $L_1(x) = ax_1 + b(x_2 + \cdots + x_n)$ for some real constants $a, b \in \mathbb{R}$ and so

$$L(x) = (a-b)x + b\sum_{i=1}^{n} x_i(1, \ldots, 1)\,.$$

As $x \in \mathbb{R}^{n,0}$ and so $\sum_{i=1}^n x_i = 0$ we have that

$$L(x) = (a-b)x,$$

that is, L is a scalar multiple of identity on $\mathbb{R}^{n,0}$. Thus $\mathbb{R}^{n,0}$ is $\mathbf{S}_n$-absolutely irreducible. To prove the claim, use for example the equivariance of (2.8) under the transposition (12): from $(12)L(x) = L((12)x)$ for all $x \in \mathbb{R}^{n,0}$ it follows that $L_1((1q)x) = L_1((1q)(12)x)$ for any $q \geq 3$ and so $L_1(y) = L_1((1q)(12)(1q)y) = L_1((2q)y)$. Thus L_1 is $\mathbf{S}_{n-1}$-invariant in the last $n-1$ coordinates. Obviously if we take L as in (2.8) where L_1 satisfies this $\mathbf{S}_{n-1}$-invariance condition then L is $\mathbf{S}_n$-equivariant.

Denoting by χ_S the character of the representation of $\mathbf{S}_n$ on the $\mathbf{S}_n$-invariant space S, we have then that

$$\chi_{\mathbb{R}^n}(\sigma) = \chi_U(\sigma) + \chi_{\mathbb{R}^{n,0}}(\sigma) = 1 + \chi_{\mathbb{R}^{n,0}}(\sigma) \tag{2.9}$$

for $\sigma \in \mathbf{S}_n$.

3. **Hilbert-Poincaré Series for $\mathbf{S}_n$- Equivariant Steady-State Theory.** Consider $\mathbf{S}_n$ acting on $\mathbb{R}^n$ as in (2.6). It is known that $\mathcal{P}_{\mathbb{R}^n}(\mathbf{S}_n)$ is a polynomial ring generated for example by the algebraically independent polynomials $x_1^i + x_2^i + \cdots + x_n^i$ for $i = 1, \ldots, n$. Moreover, $\vec{\mathcal{P}}_{\mathbb{R}^n}(\mathbf{S}_n)$ is a free module over the ring $\mathcal{P}_{\mathbb{R}^n}(\mathbf{S}_n)$ generated by the equivariant mappings $E_i : \mathbb{R}^n \to \mathbb{R}^n$ defined by $E_i(x) = (x_1^i, x_2^i, \ldots, x_n^i)^t$ for $i = 0, \ldots, n-1$. See for example Golubitsky and Stewart [6, Proposition 2.27]. That is,

$$\begin{aligned}
\mathcal{P}_{\mathbb{R}^n}(\mathbf{S}_n) &= \mathbb{R}[x_1 + x_2 + \cdots + x_n, \ \ldots, \ x_1^n + x_2^n + \cdots + x_n^n], \\
\vec{\mathcal{P}}_{\mathbb{R}^n}(\mathbf{S}_n) &= \bigoplus_{i=0}^{n-1} E_i \mathbb{R}[x_1 + x_2 + \cdots + x_n, \ \ldots, \ x_1^n + x_2^n + \cdots + x_n^n].
\end{aligned}$$

Denoting by f_n, g_n the Hilbert-Poincaré series of $\mathcal{P}_{\mathbb{R}^n}(\mathbf{S}_n)$ and $\vec{\mathcal{P}}_{\mathbb{R}^n}(\mathbf{S}_n)$, respectively, it follows then by (2.3) and (2.5) that

$$f_n(t) = \frac{1}{(1-t)(1-t^2)\cdots(1-t^n)} \tag{3.10}$$

and

$$g_n(t) = \frac{1 + t + t^2 + \cdots + t^{n-1}}{(1-t)(1-t^2)\cdots(1-t^n)}.$$

Remark 3.1. By an analogous argument to the one used in Section 2.4, we have that a polynomial function $f = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ is $\mathbf{S}_n$-equivariant if and only if $f_1 : \mathbb{R}^n \to \mathbb{R}$ is invariant under the permutation group $\mathbf{S}_{n-1}$ in the last $n-1$-variables and $f_i(x) = f_1((1i)x)$ for $i = 2, \ldots, n$. Thus, the number of linearly independent $\mathbf{S}_n$-equivariant mappings with polynomial homogeneous components of a given degree d is equal to the number of linearly independent $\mathbf{S}_{n-1}$-invariant polynomial functions from $\mathbb{R}^n$ to $\mathbb{R}$ where $\mathbf{S}_{n-1}$ acts trivially on

x_1 and permutes the last coordinates $x_2, \ldots, x_n$. Thus an alternative way to describe the generating function g_n for $\vec{\mathcal{P}}_{\mathbb{R}^n}(\mathbf{S}_n)$ is given by

$$g_n(t) = \frac{1}{1-t} f_{n-1}(t) = \frac{1}{1-t}\, \frac{1}{(1-t)\,(1-t^2)\ldots(1-t^{n-1})}$$

which is the generating function for the polynomial ring $\mathbb{R}[x_1,\ x_2 + \cdots + x_n,\ \ldots,\ x_2^{n-1} + \cdots + x_n^{n-1}]$. ◇

We are now interested in the restriction of the action (2.6) of $\mathbf{S}_n$ on $\mathbb{R}^n$ to the absolutely irreducible space $\mathbb{R}^{n,0}$. We make the following observations:

Remarks 3.2. (a) The restriction to $\mathbb{R}^{n,0}$ of a $\mathbf{S}_n$-invariant polynomial function on $\mathbb{R}^n$ is a $\mathbf{S}_n$-invariant on $\mathbb{R}^{n,0}$. Moreover, any $\mathbf{S}_n$-invariant polynomial function on $\mathbb{R}^{n,0}$ is the restriction to $\mathbb{R}^{n,0}$ of a $\mathbf{S}_n$-invariant polynomial function on $\mathbb{R}^n$. Equivalently, if we denote by $v = (v_1, u)$ coordinates on $\mathbb{R}^n$ according to the decomposition (2.7), then the function $F : \mathcal{P}_{\mathbb{R}^n}(\mathbf{S}_n) \to \mathcal{P}_{\mathbb{R}^{n,0}}(\mathbf{S}_n)$ defined by

$$F(f)(v_1) = f(v_1, 0) \qquad \big(f \in \mathcal{P}_{\mathbb{R}^n}(\mathbf{S}_n)\big)$$

is well defined and it is a surjection.

(b) The restriction to $\mathbb{R}^{n,0}$ of a $\mathbf{S}_n$-equivariant polynomial function on $\mathbb{R}^n$ followed by projection according to the decomposition (2.7) onto $\mathbb{R}^{n,0}$ is a $\mathbf{S}_n$-equivariant polynomial function on $\mathbb{R}^{n,0}$. Conversely, any $\mathbf{S}_n$-equivariant polynomial function on $\mathbb{R}^{n,0}$ is the restriction to $\mathbb{R}^{n,0}$ of a $\mathbf{S}_n$-equivariant polynomial function on $\mathbb{R}^n$ followed by projection according to the decomposition (2.7) onto $\mathbb{R}^{n,0}$. Equivalently, the function $H :\ \vec{\mathcal{P}}_{\mathbb{R}^n}(\mathbf{S}_n) \to \vec{\mathcal{P}}_{\mathbb{R}^{n,0}}(\mathbf{S}_n)$ defined by

$$H(f)(v_1) = f_1(v_1, 0)$$

for $f = (f_1, f_2) :\ \mathbb{R}^n \to \mathbb{R}^n$, is well-defined and it is a surjection.

◇

Denote by F_n, G_n the generating functions for $\mathcal{P}_{\mathbb{R}^{n,0}}(\mathbf{S}_n), \vec{\mathcal{P}}_{\mathbb{R}^{n,0}}(\mathbf{S}_n)$, respectively. We have then the following result:

Theorem 3.3. *(a) The ring $\mathcal{P}_{\mathbb{R}^{n,0}}(\mathbf{S}_n)$ is a polynomial ring generated by $p_i :\ \mathbb{R}^{n,0} \to \mathbb{R}$ for $i = 2, \ldots, n$ defined by $p_i(x_1, \ldots, x_n) = x_1^i + x_2^i + \cdots + x_n^i$. Its generating function is*

$$F_n(t) = \frac{1}{(1-t^2)\cdots(1-t^n)}. \tag{3.11}$$

(b) The module $\vec{\mathcal{P}}_{\mathbb{R}^{n,0}}(\mathbf{S}_n)$ over the ring $\mathcal{P}_{\mathbb{R}^{n,0}}(\mathbf{S}_n)$ is free and it is generated by the functions $H_i :\ \mathbb{R}^{n,0} \to \mathbb{R}^{n,0}$ for $i = 1, \ldots, n-1$, defined by

$$H_1(x) = (x_1, x_2, \ldots, x_n)^t, \quad H_i(x) = (x_1^i, x_2^i, \ldots, x_n^i)^t - \frac{1}{n}(x_1^i + x_2^i + \cdots + x_n^i)(1, 1, \ldots, 1)^t$$

for $i \geq 2$. Its generating function G_n is

$$G_n(t) = \frac{t + t^2 + \cdots + t^{n-1}}{(1-t^2)\cdots(1-t^n)} = \frac{t - t^n}{(1-t)\cdots(1-t^n)}\,. \tag{3.12}$$

Proof. It follows directly from Remarks 3.2 and from the facts that $\mathcal{P}_{\mathbb{R}^n}(\mathbf{S}_n)$ is a polynomial ring generated by $x_1^i+x_2^i+\cdots+x_n^i$ for $i=1,\ldots,n$ and $\vec{\mathcal{P}}_{\mathbb{R}^n}(\mathbf{S}_n)$ is a free module over the ring $\mathcal{P}_{\mathbb{R}^n}(\mathbf{S}_n)$ generated by the equivariant functions $E_i(x)=(x_1^i,x_2^i,\ldots,x_n^i)^t$ for $i=0,\ldots,n-1$.

An alternative way to prove this theorem is by using the Molien formulas (2.1) and (2.2) and the decomposition of the reducible representation described in Section 2.4. Since $\det(1-\sigma t)_{\mathbb{R}^n}=(1-t)\det(1-\sigma t)_{\mathbb{R}^{n,0}}$,

$$f_n(t)=\frac{1}{(1-t)}F_n(t)$$

which is (3.11). By (2.9) we have that $\chi_{\mathbb{R}^n}(\sigma^{-1})=1+\chi_{\mathbb{R}^{n,0}}(\sigma^{-1})$ and thus

$$g_n(t)=\frac{1}{(1-t)}\left(F_n(t)+G_n(t)\right),$$

which can be rearranged to give (3.12). □

4. **Hilbert-Poincaré Series for $\mathbf{S}_n$-Equivariant Hopf Theory.** Consider now the action of $\mathbf{S}_n$ on the $2n$-dimensional real vector $\mathbb{C}^n$ given by

$$\sigma\left(z_1,z_2,\ldots,z_n\right)=\left(z_{\sigma^{-1}(1)},z_{\sigma^{-1}(2)},\ldots,z_{\sigma^{-1}(n)}\right) \tag{4.13}$$

for $\sigma\in\mathbf{S}_n$ and $(z_1,z_2,\ldots,z_n)\in\mathbb{C}^n$. When studying Hopf bifurcation, we consider the action of $\mathbf{S}_n\times\mathbf{S}^1$, where the circle group $\mathbf{S}^1$ acts by

$$\theta\left(z_1,z_2,\ldots,z_n\right)=\left(e^{i\theta}z_1,e^{i\theta}z_2,\ldots,e^{i\theta}z_n\right) \tag{4.14}$$

for $\theta\in\mathbf{S}^1$. Observe that if we write $\overline{z}=(\overline{z}_1,\ldots,\overline{z}_n)$ and use multi-indices, any polynomial function $p:\ \mathbb{C}^n\to\mathbb{R}$ can be written as

$$p(z,\overline{z})\ =\ \sum_{\alpha,\beta}a_{\alpha\beta}z^{\alpha}\overline{z}^{\beta} \tag{4.15}$$

where $\alpha,\beta\in(\mathbb{Z}_0^+)^n$, $z^\alpha=z_1^{\alpha_1}z_2^{\alpha_2}\ldots z_n^{\alpha_n}$ and the coefficients $a_{\alpha\beta}$ may be required to be complex. Moreover, p is $\mathbf{S}^1$-invariant if and only if for each α,β such that $a_{\alpha\beta}\neq 0$ we have $|\alpha|=|\beta|$. In particular, it follows that p has even degree in $z,\overline{z}$. Similarly, if $g:\ \mathbb{C}^n\to\mathbb{C}^n$ has components

$$g_j(z,\overline{z})\ =\ \sum_{\alpha,\beta}b_{\alpha\beta}z^{\alpha}\overline{z}^{\beta}$$

then the $\mathbf{S}^1$-equivariance is equivalent to having $|\alpha|=|\beta|+1$ if $b_{\alpha\beta}\neq 0$. This is [7, Lemma XVI 9.3]. Therefore g has odd degree components in $z,\overline{z}$.

Applying the formulas obtained by Antoneli *et al.* [1] we have that for fixed n and degree $2k$, the number of $\mathbf{S}_n\times\mathbf{S}^1$-invariant homogeneous polynomials of degree $2k$ is given by

$$\dim_{\mathbb{R}}\mathcal{P}^{2k}_{\mathbb{C}^n}(\mathbf{S}_n\times\mathbf{S}^1)\ =\ \frac{1}{n!}\sum_{\sigma\in\mathbf{S}_n}\chi_{(k)}(\sigma)^2.$$

Also, the number of $\mathbf{S}_n \times \mathbf{S}^1$-equivariant polynomial functions with homogeneous components of degree $2k+1$ is

$$\dim_{\mathbb{C}} \vec{\mathcal{P}}^{2k+1}_{\mathbb{C}^n}(\mathbf{S}_n \times \mathbf{S}^1) \; = \; \frac{1}{n!} \sum_{\sigma \in \mathbf{S}_n} \chi_{(k+1)}(\sigma)\, \chi_{(k)}(\sigma)\, \chi(\sigma) \; .$$

Here, $\chi_{(k)}$ denotes the character of the induced action of $\mathbf{S}_n$ on the k-th symmetric tensor power $S^k\mathbb{R}^n$ of $\mathbb{R}^n$. Observe that if we have a representation of a group G on a vector space V, then there is a natural representation of G on the tensor product $V \otimes V$ given by $g(v \otimes w) = gv \otimes gw$. By iteration of this construction one obtains an action of G on the k-th tensor powers $V^{\otimes k}$. By restriction, one obtains a representation of G on the k-th symmetric tensor power S^kV, since it is an invariant subspace of $V^{\otimes k}$ under the action of G.

We use now the concept of bigraded Hilbert-Poincaré series introduced by Forger [5]. Denoting by $c_{q,r}$ the dimension of the space of real $\mathbf{S}_n$-invariant functions from $\mathbb{C}^n \to \mathbb{R}$ of bidegree (q,r), the generating function of two variables

$$f_n(t,s) \; = \; \sum_{q,r=0}^{\infty} c_{q,r} t^q s^r$$

is the *bigraded Hilbert-Poincaré series* for $\mathcal{P}_{\mathbb{C}^n}(\mathbf{S}_n)$. The two variables t and s correspond to z and $\overline{z}$ in (4.15). Moreover, Forger [5] obtains the following integral form:

$$f_n(t,s) \; = \; \frac{1}{n!} \sum_{\sigma \in \mathbf{S}_n} \frac{1}{\det(1-\sigma t)\det(1-\sigma s)} \; . \tag{4.16}$$

Here, σ denotes the matrix representing the action of σ on $\mathbb{R}^n$.

Antoneli *et al.* [1] generalize this concept to the equivariants. Denoting by $e_{q,r}$ the *complex* dimension of the space of $\mathbf{S}_n$-equivariant mappings from $\mathbb{C}^n \to \mathbb{C}^n$ with homogeneous polynomial components of bidegree (q,r), the generating function of two variables

$$g_n(t,s) \; = \; \sum_{q,r=0}^{\infty} e_{q,r} t^q s^r$$

is the *bigraded Hilbert-Poincaré series* of $\vec{\mathcal{P}}_{\mathbb{C}^n}(\mathbf{S}_n)$ and the integral form for that is:

$$g_n(t,s) = \frac{1}{n!} \sum_{\sigma \in \mathbf{S}_n} \frac{\chi\big(\sigma^{-1}\big)}{\det(1-\sigma t)\det\big(1-\sigma s\big)} \tag{4.17}$$

where χ is the character of the representation of $\mathbf{S}_n$ on $\mathbb{R}^n$.

Another result of [1] is that the bigraded Hilbert-Poincaré series for $\mathcal{P}_{\mathbb{C}^n}(\mathbf{S}_n \times \mathbf{S}^1)$ and for $\vec{\mathcal{P}}_{\mathbb{C}^n}(\mathbf{S}_n \times \mathbf{S}^1)$ are given by

$$\Phi_{\mathbf{S}_n \times \mathbf{S}^1}(t,s) \; = \; \frac{1}{2\pi} \int_0^{2\pi} f_n\left(e^{i\theta}t, e^{-i\theta}s\right) \mathrm{d}\theta \tag{4.18}$$

and

$$\Psi_{\mathbf{S}_n \times \mathbf{S}^1}(t,s) \; = \; \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta} g_n\left(e^{i\theta}t, e^{-i\theta}s\right) \mathrm{d}\theta, \tag{4.19}$$

where f_n and g_n are as in (4.16) and (4.17), respectively. Note that (4.18) extracts from f_n those terms which have the same degree in t and s, that is, those that satisfy the $\mathbf{S}^1$-invariance condition $|\alpha| = |\beta|$; hence $\Phi_{\mathbf{S}_n\times\mathbf{S}^1}(t,s)$ in fact depends only on the single variable ts. Similarly, (4.19) selects the terms that satisfy the $\mathbf{S}^1$-equivariance condition $|\alpha| = |\beta|+1$.

These results can also be used when taking $\mathbf{S}_n$ acting on the $\mathbf{S}_n$-invariant space $\mathbb{C}^{n,0} = \{z \in \mathbb{C}^n : \ z_1 + z_2 + \cdots + z_n = 0\}$. In (4.18) and (4.19), instead of using the generating functions f_n, g_n, we use the generating functions for the ring $\mathcal{P}_{\mathbb{C}^{n,0}}(\mathbf{S}_n)$ and the module $\vec{\mathcal{P}}_{\mathbb{C}^{n,0}}(\mathbf{S}_n)$ that we will denote by F_n, G_n, respectively.

These formulas (4.18) and (4.19) show that the generating functions for the invariants and equivariants for $\mathbf{S}_n \times \mathbf{S}^1$ are obtained by integrating over the group $\mathbf{S}^1$ the two-variable generating functions for $\mathbf{S}_n$. In the next section we obtain recursive formulas for f_n, g_n, F_n, G_n (for the $\mathbf{S}_n$ reducible and irreducible cases). In bifurcation theory we are interested in the irreducible case; but it is easier to start from the reducible case.

5. **Generating Functions for $\mathbf{S}_n$ Hopf Bifurcation.** Throughout we consider $\mathbf{S}_n$ acting on the real vector space $\mathbb{C}^n$ by permutation of the coordinates as in (4.13). As before, we denote by $f_n(t,s)$ the bigraded generating function for the ring of $\mathbf{S}_n$-invariant polynomial functions $\mathbb{C}^n \to \mathbb{R}$ and $g_n(t,s)$ the bigraded generating function for the module of the $\mathbf{S}_n$-equivariant mappings with polynomial components $\mathbb{C}^n \to \mathbb{C}^n$.

There is no simple criterion for writing down the generating functions for the case of Hopf bifurcation as in the steady-state case. The ring of invariants is not a polynomial ring. Moreover, the module of the equivariants is not a free module over the ring of the invariants.

Using the Molien formula (4.16) we can derive explicit formulas for specific values of n. For $n = 2$ we have

$$\begin{aligned} f_2(t,s) &= \frac{1}{2}\left[\frac{1}{(1-t)^2(1-s)^2} + \frac{1}{(1-t^2)(1-s^2)}\right] \\ &= \frac{1+ts}{(1-t)(1-s)(1-t^2)(1-s^2)} \end{aligned}$$

indicating that there are four primary invariants: two of degree 1, one in the variables z_1, z_2 and the other in the variables $\overline{z}_1, \overline{z}_2$; and two of degree 2, again, one in the variables z_1, z_2 and the other in the variables $\overline{z}_1, \overline{z}_2$. Moreover, there is a secondary invariant of degree 2. We can take the degree 1 primary invariant generators: z_1+z_2, $\overline{z}_1+\overline{z}_2$; the degree 2 primary invariant generators: $z_1^2+z_2^2$, $\overline{z}_1^2+\overline{z}_2^2$; it can be easily checked that the ring of polynomial $\mathbf{S}_2$-invariant functions $\mathbb{C}^2 \to \mathbb{R}$ has the Hironaka decomposition:

$$\mathcal{P}_{\mathbb{C}^2}(\mathbf{S}_2) = \mathbb{C}[z_1+z_2,\, \overline{z}_1+\overline{z}_2,\, z_1^2+z_2^2,\, \overline{z}_1^2+\overline{z}_2^2] \oplus (z_1\overline{z}_1+z_2\overline{z}_2)\,\mathbb{C}[z_1+z_2,\, \overline{z}_1+\overline{z}_2,\, z_1^2+z_2^2,\, \overline{z}_1^2+\overline{z}_2^2]\,.$$

The integral (4.18) can be evaluated using standard contour integral techniques, substituting $z = e^{i\theta}$ and using the Cauchy residue theorem, and the result is

$$\Phi_{\mathbf{S}_2\times\mathbf{S}^1}(t,s) = \frac{1+t^2s^2}{(1-ts)^2(1-t^2s^2)}.$$

For $n = 3$, using (4.16) we get

$$\begin{aligned} f_3(t,s) &= \frac{1}{6}\left[\frac{1}{(1-t)^3(1-s)^3} + \frac{3}{(1-t)(1-s)(1-t^2)(1-s^2)} + \frac{2}{(1-t^3)(1-s^3)}\right] \\ &= \frac{1+ts+t^2s+ts^2+t^2s^2+t^3s^3}{(1-t)(1-s)(1-t^2)(1-s^2)(1-t^3)(1-s^3)}. \end{aligned}$$

As we increase n we have more complicated formulas where the number of primary and secondary invariants increase. We present here a useful recursive formula that permits the derivation of f_n in terms of f_i for $i < n$.

Theorem 5.1. *Let $n \geq 1$. We have the following recursive formula for the two variable generating function f_n of the ring of the $\mathbf{S}_n$-invariant polynomial functions from $\mathbb{C}^n$ to $\mathbb{R}$:*

$$n f_n(t,s) = \sum_{k=1}^{n} \frac{1}{1-t^k}\frac{1}{1-s^k} f_{n-k}(t,s). \tag{5.20}$$

Proof. By (4.16) we have that

$$f_n(t,s) = \frac{1}{n!}\sum_{\sigma\in\mathbf{S}_n} \frac{1}{\det(1-\sigma t)\det(1-\sigma s)}.$$

For $k = 1, 2, \ldots, n$, let P_k denote the set of permutations of $\mathbf{S}_n$ that have the integer 1 in a k-cycle. We have then that $\mathbf{S}_n$ decomposes into the disjoint union

$$\mathbf{S}_n = P_1 \,\dot\cup\, P_2 \,\dot\cup\, \cdots \,\dot\cup\, P_n$$

and

$$n\, f_n(t,s) = \frac{1}{(n-1)!}\sum_{k=1}^{n}\sum_{\sigma\in P_k} \frac{1}{\det(1-\sigma t)\det(1-\sigma s)}. \tag{5.21}$$

Given k between 1 and n, we calculate now

$$\sum_{\sigma\in P_k} \frac{1}{\det(1-\sigma t)\det(1-\sigma s)}.$$

Observe that the number of k-cycles that have a 1 from n symbols is $(n-1)(n-2)\cdots(n-k+1) = (n-1)!/(n-k)!$. As the order of $\mathbf{S}_{n-k}$ is $(n-k)!$, we have that the cardinality of the set P_k is

$$\frac{(n-1)!}{(n-k)!}\,(n-k)! = (n-1)!\,.$$

Choose the subset of P_k formed by the permutations of P_k that include in their cycle decomposition the k-cycle $(123\ldots k)$. Denote that subset by $(123\ldots k)\mathbf{S}_{\{k+1,\ldots,n\}}$. Then

$$\sum_{\sigma\in(123\ldots k)\mathbf{S}_{\{k+1,\ldots,n\}}} \frac{1}{\det(1-\sigma t)\det(1-\sigma s)}$$

$$= \frac{1}{1-t^k}\,\frac{1}{1-s^k} \sum_{\sigma\in\mathbf{S}_{\{k+1,\ldots,n\}}} \frac{1}{\det(1-\sigma t)\det(1-\sigma s)}$$

$$= \frac{(n-k)!}{(1-t^k)(1-s^k)}\, f_{n-k}(t,s).$$

Observe that if we choose any other subset of P_k given by a specific k-cycle including 1 we will get the same answer. That is, if $i_2, i_3, \ldots, i_n$ are any distinct integers of the set $\{2,3,\ldots,n\}$ then

$$\sum_{\sigma\in(1i_2i_3\ldots i_k)\mathbf{S}_{\{1,2,\ldots,n\}\setminus\{1,i_2,\ldots,i_k\}}} \frac{1}{\det(1-\sigma t)\det(1-\sigma s)} = \frac{(n-k)!}{(1-t^k)(1-s^k)}\, f_{n-k}(t,s).$$

Since the number of k-cycles that have a 1 from n symbols is $(n-1)!/(n-k)!$, we have that

$$\sum_{\sigma\in P_k} \frac{1}{\det(1-\sigma t)\det(1-\sigma s)}$$

$$= \frac{(n-1)!}{(n-k)!}\,\frac{(n-k)!}{(1-t^k)(1-s^k)}\, f_{n-k}(t,s)$$

$$= (n-1)!\,\frac{1}{(1-t^k)(1-s^k)}\, f_{n-k}(t,s).$$

Finally, using (5.21),

$$\begin{aligned} nf_n(t,s) &= \frac{1}{(n-1)!}\sum_{k=1}^{n}(n-1)!\frac{1}{(1-t^k)(1-s^k)}\, f_{n-k}(t,s) \\ &= \sum_{k=1}^{n}\frac{1}{(1-t^k)(1-s^k)}\, f_{n-k}(t,s). \end{aligned}$$

□

Remark 5.2. In one variable, when $\mathbf{S}_n$ acts on $\mathbb{R}^n$ by permutation of coordinates, if $f_n(t)$ denotes the one-variable generating function for the ring of $\mathbf{S}_n$-invariant polynomial functions $\mathbb{R}^n \to \mathbb{R}$, we have

$$nf_n(t) = \sum_{k=1}^{n}\frac{1}{1-t^k} f_{n-k}(t).$$

Now recall that there is an explicit formula for $f_n(t)$ given by (3.10) and so $f_n(t) = \frac{1}{1-t^n} f_{n-1}(t)$.

$\diamond$

Lemma 5.3. *Let $n \geq 1$. The two variable generating function g_n of the module of the $\mathbf{S}_n$-equivariant polynomial functions $\mathbb{C}^n \to \mathbb{C}^n$ is related to f_n in the following way:*

$$g_n(t,s) = \frac{1}{1-t}\,\frac{1}{1-s} f_{n-1}(t,s). \tag{5.22}$$

Proof. Observe that the number of linearly independent polynomial mappings $\mathbb{C}^n \to \mathbb{C}^n$ which are $\mathbf{S}_n$-equivariant and have homogeneous components of degree d is equal to the number of linearly independent polynomial functions $\mathbb{C}^n \to \mathbb{C}$ of degree d that have the form

$$p(z_1, \overline{z}_1)\, q(z_2, \ldots, z_n, \overline{z}_2, \ldots, \overline{z}_n)$$

where $p(z_1, \overline{z}_1)$ is a polynomial in the variables $z_1, \overline{z}_1$ and $q(z_2, \ldots, z_n, \overline{z}_2, \ldots, \overline{z}_n)$ is a polynomial $\mathbf{S}_{n-1}$-invariant. $\square$

Corollary 5.4. *Let $n \geq 1$. The recursive formula for g_n is given by*

$$(n-1)\, g_n(t,s) = \sum_{k=1}^{n-1} \frac{1}{1-t^k}\,\frac{1}{1-s^k}\, g_{n-k}(t,s).$$

Proof. Direct application of Lemma 5.3 and Theorem 5.1 leads to the above recursive formula. $\square$

In the next theorem, $O(n)$ means a polynomial where every term is greater or equal to n in total degree.

Theorem 5.5. *For $n \geq 1$ we have:*

$$f_n(t,s) - f_{n-1}(t,s) = O(n). \tag{5.23}$$

Proof. We prove by induction that $f_n(t,s) - f_{n-1}(t,s) = O(n)$ for all $n \geq 1$. For $n = 1$ we have

$$f_1(t,s) - f_0(t,s) = \frac{1}{(1-t)(1-s)} - 1 = \frac{t+s-ts}{(1-t)(1-s)} = O(1)$$

and so (5.23) is true for $n = 1$. Assume (5.23) is true for $1 \leq n \leq k$. We prove now that it is true for $n = k+1$. By Theorem 5.1 we have that

$$\begin{aligned} (k+1) f_{k+1}(t,s) &= \sum_{i=1}^{k+1} \frac{1}{(1-t^i)(1-s^i)} f_{k+1-i}(t,s), \\ k f_k(t,s) &= \sum_{i=1}^{k} \frac{1}{(1-t^i)(1-s^i)} f_{k-i}(t,s). \end{aligned}$$

Subtracting the above expressions, we obtain

$$\begin{aligned} & k\left(f_{k+1}(t,s)-f_k(t,s)\right)+f_{k+1}(t,s) \\ &=\sum_{i=1}^{k}\frac{1}{(1-t^i)(1-s^i)}\left(f_{k+1-i}(t,s)-f_{k-i}(t,s)\right)+\frac{1}{(1-t^{k+1})(1-s^{k+1})}\,f_0(t,s) \\ &=\sum_{i=1}^{k}\left(1+O(i)\right)\left(f_{k+1-i}(t,s)-f_{k-i}(t,s)\right)+\left(1+O(k+1)\right)f_0(t,s)\ . \end{aligned}$$

As $k+1-i \leq k$ for $i=1,\ldots,k$, by the induction hypothesis, we have $f_{k+1-i}(t,s)-f_{k-i}(t,s)=O(k+1-i)$. Thus

$$\begin{aligned} & k\left(f_{k+1}(t,s)-f_k(t,s)\right)+f_{k+1}(t,s) \\ &=\sum_{i=1}^{k}\left(f_{k+1-i}(t,s)-f_{k-i}(t,s)\right)+\sum_{i=1}^{k}O(i)O(k+1-i)+\left(1+O(k+1)\right)f_0(t,s) \\ &=f_k(t,s)-f_0(t,s)+O(k+1)+f_0(t,s)+O(k+1)\,. \end{aligned}$$

We have then that

$$(k+1)\left(f_{k+1}(t,s)-f_k(t,s)\right)=O(k+1)$$

and so $f_{k+1}(t,s)-f_k(t,s)=O(k+1)$. □

Corollary 5.6. *For $n \geq 1$ we have*

$$g_n(t,s)-g_{n-1}(t,s)=O(n-1).$$

Proof. From Lemma 5.3 and the above theorem we obtain the formula. □

Remarks 5.7. (a) From Theorem 5.5, we have that $\mathbf{S}_n$ and $\mathbf{S}_{n-1}$ have the same number of invariants for degree $d<n$. Fixing the degree d, for all $n \geq d$, the number of $\mathbf{S}_n$-invariants of degree d is the same. Therefore we have results for $\mathbf{S}_n$ for arbitrary n.

(b) From Corollary 5.6, we have that $\mathbf{S}_n$ and $\mathbf{S}_{n-1}$ have the same number of equivariants for degree $d<n-1$. Thus, for all n such that $n-1 \geq d$, the number of $\mathbf{S}_n$-equivariants of degree d is the same. ◇

See Section 6 for the numbers of polynomial invariant functions and equivariant mappings with polynomial components of several degrees for $\mathbf{S}_n$ Hopf bifurcation.

All of the above results are for the reducible representation; we now obtain the analogous results for the irreducible representation.

Proposition 5.8. *Let $n \geq 1$. Consider the restriction of the action (4.13) of $\mathbf{S}_n$ on $\mathbb{C}^n$ to the subspace $\mathbb{C}^{n,0}$ formed by the vectors satisfying $z_1+\cdots+z_n=0$. The generating function for the ring of $\mathbf{S}_n$-invariant polynomial functions $\mathbb{C}^{n,0}\to\mathbb{R}$ is*

$$F_n(t,s)=(1-t)(1-s)f_n(t,s) \qquad (5.24)$$

and the generating function for the module of $\mathbf{S}_n$-equivariant mappings $\mathbb{C}^{n,0}\to\mathbb{C}^{n,0}$ with polynomial components is

$$G_n(t,s)=f_{n-1}(t,s)-(1-t)(1-s)f_n(t,s)\,. \qquad (5.25)$$

Proof. Recall from Section 2.4 that the reducible representation of $\mathbf{S}_n$ on $\mathbb{R}^n$ decomposes into a trivial one-dimensional representation and the irreducible on $\mathbb{R}^{n,0}$ (2.7). So the action of any σ splits into its trivial action on U and its action on $\mathbb{R}^{n,0}$. Thus $\det(1-\sigma t)_{\mathbb{R}^n} = (1-t)\det(1-\sigma t)_{\mathbb{R}^{n,0}}$, for all $\sigma \in \mathbf{S}_n$. Hence by applying (4.16) we obtain

$$f_n(t,s) = F_n(t,s)/(1-t)(1-s).$$

For the generating function for the equivariants, we apply (4.17) to the reducible representation, and make use of (2.9). This gives

$$g_n(t,s) = \frac{1}{n!}\sum_{\sigma\in G} \frac{1+\chi_{\mathbb{R}^{n,0}}(\sigma^{-1})}{(1-t)\det(1-\sigma t)_{\mathbb{R}^{n,0}}(1-s)\det(1-\sigma s)_{\mathbb{R}^{n,0}}}$$

which can be written as

$$g_n(t,s) = \frac{1}{(1-t)(1-s)}(F_n(t,s)+G_n(t,s)).$$

Rearranging this formula and making use of (5.22) and (5.24) gives the result (5.25). □

Using Proposition 5.8 and Theorem 5.5 we obtain:

Corollary 5.9. *For $n \geq 1$ we have the following formulas:*

$$\begin{aligned} F_n(t,s) - F_{n-1}(t,s) &= O(n), \\ G_n(t,s) - G_{n-1}(t,s) &= O(n-1). \end{aligned}$$

Example 5.10. Consider the case $n = 3$. Note that it is the generating function $G_n(t,s)$ for the number of equivariants in the irreducible representation that is of most interest for bifurcation theory. Using (5.25) and the explicit formulas for $f_2(t,s)$ and $f_3(t,s)$ given at the beginning of section 5, we obtain

$$G_3(t,s) = \frac{t+s-st+ts^2+st^2}{(1-t)(1-s)(1-t^3)(1-s^3)}.$$

Carrying out the integral analogous to (4.19) using contour integration gives the result

$$\Psi_{\mathbf{S}_n\times\mathbf{S}^1}(t,s) = \frac{t(1+t^2s^2)}{(1-ts)^2(1-t^3s^3)} = t + 2t^2s + 4t^3s^2 + 7t^4s^3 + \ldots,$$

so we deduce that for the Hopf bifurcation with $\mathbf{S}_3$ symmetry there are two linearly independent equivariants of degree three, four of degree five, and seven of degree seven. These results are consistent with the work of Dias and Paiva [3] who studied this bifurcation problem in detail.

6. **Numbers of Invariants and Equivariants for $\mathbf{S}_n$ Hopf Bifurcation.** In this section we present the numbers of polynomial invariant functions and equivariant mappings with polynomial components of all degrees $d < 10$ for Hopf bifurcation with $\mathbf{S}_n$-symmetry, for all n. We do this both for the reducible representation of $\mathbf{S}_n \times \mathbf{S}^1$ on $\mathbb{C}^n$ and for the standard irreducible representation of $\mathbf{S}_n \times \mathbf{S}^1$ on $\mathbb{C}^{n,0}$. Recall that the action of $\mathbf{S}_n$ on $\mathbb{C}^{n,0}$ is $\mathbf{S}_n$-simple, that is, $\mathbb{C}^{n,0}$ is the sum of two isomorphic absolutely irreducible representations of $\mathbf{S}_n$. In what follows, we refer to the first and the second actions as reducible and irreducible, respectively, for Hopf bifurcation with $\mathbf{S}_n$-symmetry.

Table 1 gives the numbers of invariants in the reducible representation. These are found by using the recurrence relation (5.20) to construct the generating function $f_n(t, s)$, and then taking the term in $t^{d/2}s^{d/2}$ in its Taylor expansion (using Maple). Recall Theorem 5.1, Remark 5.7 (a) and the discussion in Section 4. Alternatively, we may first evaluate the integral (4.18), after which only a one-variable Taylor expansion is required.

For the equivariants in Table 2, we use the generating function (5.22) obtained in Lemma 5.3 and then pick terms in $x^{(d+1)/2}y^{(d-1)/2}$ in its Taylor expansion.

For the irreducible representation we use the formulas obtained in Proposition 5.8: (5.24) for the number of invariants in Table 3, and (5.25) for the number equivariants in Table 4.

Remarks 6.1. (a) From Theorem 5.5, we have that for a fixed degree d, the number of $\mathbf{S}_n \times \mathbf{S}^1$-invariant polynomial functions on the reducible space $\mathbb{C}^n$ is constant for $n \geq d$. Also from Corollary 5.6, the number of $\mathbf{S}_n \times \mathbf{S}^1$-equivariant mappings with polynomial components on $\mathbb{C}^n$ is constant for $n - 1 \geq d$. In particular, using Table 2, we conclude that for $n \geq 4$ there are 11 linearly independent $\mathbf{S}_n \times \mathbf{S}^1$-equivariants of degree three (this result was proved using a different method in [1]). Moreover, for $n \geq 6$, there are 52 linearly independent $\mathbf{S}_n \times \mathbf{S}^1$-equivariants of degree five.

(b) By Corollary 5.9, we have that for a fixed degree d, the number of $\mathbf{S}_n \times \mathbf{S}^1$-invariant polynomial functions on the irreducible space $\mathbb{C}^{n,0}$ is constant for $n \geq d$. Also the number of $\mathbf{S}_n \times \mathbf{S}^1$-equivariant mappings with polynomial components on $\mathbb{C}^{n,0}$ is constant for $n-1 \geq d$. Using Table 4, we conclude that for $n \geq 4$ there are 3 linearly independent $\mathbf{S}_n \times \mathbf{S}^1$-equivariants of degree three, and for $n \geq 6$, there are 12 linearly independent $\mathbf{S}_n \times \mathbf{S}^1$-equivariants of degree five.

◇

$d \backslash n$	2	3	4	5	6	7	8	9	≥10
2	2	2	2	2	2	2	2	2	2
4	5	8	9	9	9	9	9	9	9
6	8	19	27	30	31	31	31	31	31
8	13	42	74	95	105	108	109	109	109

TABLE 1. Number of reducible invariants of degrees $d = 2, 4, 6, 8$, for $\mathbf{S}_n$ Hopf bifurcation.

$d\backslash n$	2	3	4	5	6	7	8	9	≥10
3	6	10	11	11	11	11	11	11	11
5	12	32	46	51	52	52	52	52	52
7	20	78	145	188	206	211	212	212	212
9	30	162	382	581	703	758	777	782	783

TABLE 2. Number of reducible equivariants of degrees $d = 3, 5, 7, 9$, for $\mathbf{S}_n$ Hopf bifurcation.

$d\backslash n$	2	3	4	5	6	7	8	9	≥10
2	1	1	1	1	1	1	1	1	1
4	1	2	3	3	3	3	3	3	3
6	1	3	6	7	8	8	8	8	8
8	1	5	13	19	24	25	26	26	26

TABLE 3. Number of irreducible invariants of degrees $d = 2, 4, 6, 8$, for $\mathbf{S}_n$ Hopf bifurcation.

$d\backslash n$	2	3	4	5	6	7	8	9	≥10
3	1	2	3	3	3	3	3	3	3
5	1	4	9	11	12	12	12	12	12
7	1	7	21	33	41	43	44	44	44
9	1	10	43	84	119	137	146	148	149

TABLE 4. Number of irreducible equivariants of degrees $d = 3, 5, 7, 9$, for $\mathbf{S}_n$ Hopf bifurcation.

7. **Cubic and Quintic Equivariants for $\mathbf{S}_n$ Hopf Bifurcation.** In this section we obtain bases for the complex vector spaces of $\mathbf{S}_n \times \mathbf{S}^1$-equivariant mappings from $\mathbb{C}^{n,0}$ to $\mathbb{C}^{n,0}$ with cubic and quintic homogeneous polynomial components. This is done by explicit construction, which is greatly facilitated by the fact that we know the correct number of independent terms from Table 4.

Theorem 7.1. *Suppose $n \geq 4$. Consider the action of $\mathbf{S}_n \times \mathbf{S}^1$ on $\mathbb{C}^{n,0}$ defined by the restrictions to $\mathbb{C}^{n,0}$ of the actions (4.13) and (4.14) on $\mathbb{C}^n$. Then the following functions $H_i : \mathbb{C}^{n,0} \to \mathbb{C}^{n,0}$ for $i = 1, 2, 3$ constitute a basis of the complex vector space of the $\mathbf{S}_n \times \mathbf{S}^1$-equivariant functions with homogeneous polynomial components of degree 3:*

$$H_i(z) = (h_i(z), h_i((12)z), \ldots, h_i((1n)z))$$

where $z = (z_1, \ldots, z_n) \in \mathbb{C}^{n,0}$ *and*

$$\begin{aligned} h_1(z) &= |z_1|^2 z_1 - \frac{1}{n}\sum_{j=1}^{n} |z_j|^2 z_j, \\ h_2(z) &= \overline{z}_1 \sum_{j=1}^{n} z_j^2, \\ h_3(z) &= z_1 \sum_{j=1}^{n} |z_j|^2 . \end{aligned}$$

Proof. The $\mathbf{S}_n \times \mathbf{S}^1$-equivariant functions with homogeneous polynomial components of degree 3 are obtained by restriction to $\mathbb{C}^{n,0}$ and projection onto $\mathbb{C}^{n,0}$ of the $\mathbf{S}_n \times \mathbf{S}^1$-equivariant functions from $\mathbb{C}^n$ to $\mathbb{C}^n$ with homogeneous polynomial components of degree 3.

Observe that with respect to the direct sum decomposition of $\mathbb{C}^n$ into $\mathbf{S}_n$-invariant spaces,

$$\mathbb{C}^n = \{(z,\, z, \ldots, z) :\ z \in \mathbb{R}\} \oplus \mathbb{C}^{n,0},$$

the projection vector of $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ onto $\mathbb{C}^{n,0}$ is:

$$z - \frac{1}{n}\left(z_1 + \cdots + z_n\right)(1, \ldots, 1)\,.$$

Thus given a $\mathbf{S}_n \times \mathbf{S}^1$-equivariant function $f:\ \mathbb{C}^n \to \mathbb{C}^n$ where $f = (f_1, \ldots, f_n)$ for $f_i : \mathbb{C}^n \to \mathbb{C}$, the restriction of f to $\mathbb{C}^{n,0}$ and projection onto $\mathbb{C}^{n,0}$ is given by

$$f|_{\mathbb{C}^{n,0}} - \frac{1}{n}\sum_{i=1}^{n} f_i|_{\mathbb{C}^{n,0}}\,(1, \ldots, 1)\ .$$

By Remark 6.1 (a), we have that there are 11 linearly independent functions from $\mathbb{C}^n$ to $\mathbb{C}^n$ with homogeneous polynomial components of degree 3 that are $\mathbf{S}_n \times \mathbf{S}^1$-equivariant. Let $f:\ \mathbb{C}^n \to \mathbb{C}^n$ be $\mathbf{S}_n \times \mathbf{S}^1$-equivariant with homogeneous polynomial components of degree 3. The equivariance of f under $\mathbf{S}_n$ is equivalent to the invariance say of the first component f_1 under $\mathbf{S}_{n-1}$ in the last $n-1$ coordinates $z_2, \ldots, z_n$, and then

$$f(z) \ = \ (f_1(z_1, z_2, \ldots, z_{n-1}, z_n), f_1(z_2, z_1, \ldots, z_{n-1}, z_n), \ldots, f_1(z_n, z_2, \ldots, z_{n-1}, z_1))\,.$$

This follows from

$$f\left((1i)(z_1, z_2, \ldots, z_n)\right) \ = \ (1i) f\left(z_1, z_2, \ldots, z_n\right)$$

for $i = 2, 3, \ldots, n$. Now using the $\mathbf{S}^1$-equivariance, for $z = (z_1, \ldots, z_n)$, taking $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_n)$ and using multi-indices, we have that f_1 can be written as

$$f_1(z) \ = \ \sum a_{\alpha\beta} z^{\alpha} \overline{z}^{\beta}$$

with $|\alpha| = |\beta| + 1$, as discussed in section 4, so for terms of degree 3, $|\alpha| = 2$ and $|\beta| = 1$.

The rest of the proof consists in characterizing the first component f_1. That is, we describe the homogeneous polynomials of degree 3 that are $\mathbf{S}_{n-1}$-invariant in the last $n-1$-coordinates $z_2, \ldots, z_n$ and are $\mathbf{S}^1$-equivariant. We consider the $\mathbf{S}_n \times \mathbf{S}^1$-equivariants where the first component is an homogeneous polynomial of degree 3 which can be written as

$$z_1^a \overline{z_1}^b p(z_2, \ldots, z_n)$$

where $a, b \in \mathbb{Z}_0^+$, $a+b \geq 0$ and p is $\mathbf{S}_{n-1}$-invariant. See [11, Theorems 4.2, 4.6] for details. □

Remark 7.2. Table 4 shows that for $n = 3$ there are only two linearly independent equivariants, indicating that $h_1(z)$, $h_2(z)$, $h_3(z)$, are not linearly independent for $n = 3$. In fact it can easily be shown that

$$6H_1(z) - H_2(z) - 2H_3(z) = 0 \quad \text{for } n = 3,\ z \in \mathbb{C}^{3,0}.$$

◇

As shown by Rodrigues [11, Section 4], when studying Hopf bifurcation with $\mathbf{S}_n$-symmetry posed on the standard $\mathbf{S}_n$-simple space $\mathbb{C}^{n,0}$, the quintic terms of the Taylor expansion at the bifurcation point of a general $\mathbf{S}_n \times \mathbf{S}^1$-equivariant Birkhoff normal form are necessary to determine the stability of the branches of periodic solutions guaranteed by the Equivariant Hopf Theorem [7, Theorem XVI 4.1]. That is, the cubic terms of the normal form are too degenerate to determine the stability in certain directions. Moreover, the quintic terms determine completely the stability for most of those branches. See [11, Theorem 4.13] for details. By Remark 6.1 (b) we have that for $n \geq 6$ there are 12 linearly independent $\mathbf{S}_n \times \mathbf{S}^1$-equivariants with quintic polynomial components.

Theorem 7.3. *Suppose $n \geq 6$. Consider the action of $\mathbf{S}_n \times \mathbf{S}^1$ on $\mathbb{C}^{n,0}$ defined by the restrictions to $\mathbb{C}^{n,0}$ of the actions (4.13) and (4.14) on $\mathbb{C}^n$. Then the following functions $H_i : \mathbb{C}^{n,0} \to \mathbb{C}^{n,0}$ for $i = 1, \ldots, 12$ constitute a basis of the complex vector space of the $\mathbf{S}_n \times \mathbf{S}^1$-equivariant functions with homogeneous polynomial components of degree 5:*

$$H_i(z) = (h_i(z), h_i((12)z), \ldots, h_i((1n)z))$$

where $z = (z_1, \ldots, z_n) \in \mathbb{C}^{n,0}$ and

$$\begin{array}{llllll}
h_1(z) & = & |z_1|^4 z_1 - \frac{1}{n}\sum_{i=1}^n |z_i|^4 z_i, & h_2(z) & = & \sum_{i=1}^n |z_i|^4 z_1, \\
h_3(z) & = & \sum_{i=1}^n z_i^2 \sum_{j=1}^n \overline{z}_j^2\, z_1, & h_4(z) & = & \left(\sum_{i=1}^n |z_i|^2\right)^2 z_1, \\
h_5(z) & = & \sum_{j=1}^n |z_j|^2 \overline{z}_j\, z_1^2 - \frac{1}{n}\sum_{j=1}^n |z_j|^2 \overline{z}_j \sum_{i=1}^n z_i^2, & h_6(z) & = & \sum_{j=1}^n \overline{z}_j^2 z_1^3 - \frac{1}{n}\sum_{j=1}^n \overline{z}_j^2 \sum_{i=1}^n z_i^3, \\
h_7(z) & = & \sum_{i=1}^n |z_i|^2 \sum_{j=1}^n z_j^2\, \overline{z}_1, & h_8(z) & = & \sum_{i=1}^n |z_i|^2 z_i^2\, \overline{z}_1, \\
h_9(z) & = & \sum_{j=1}^n z_j^3 \overline{z}_1^2 - \frac{1}{n}\sum_{j=1}^n z_j^3 \sum_{i=1}^n \overline{z}_i^2, & h_{10}(z) & = & \sum_{j=1}^n |z_j|^2 z_j |z_1|^2 - \frac{1}{n}\sum_{j=1}^n |z_j|^2 z_j \sum_{i=1}^n |z_i|^2, \\
h_{11}(z) & = & \sum_{j=1}^n |z_j|^2 |z_1|^2 z_1 - \frac{1}{n}\sum_{j=1}^n |z_j|^2 \sum_{i=1}^n |z_i|^2 z_i, & h_{12}(z) & = & \sum_{j=1}^n z_j^2 |z_1|^2 \overline{z}_1 - \frac{1}{n}\sum_{j=1}^n z_j^2 \sum_{i=1}^n |z_i|^2 \overline{z}_i\,.
\end{array}$$

Proof. The proof follows the lines of the proof of Theorem 7.1. Now recall Remark 6.1 (b) where we conclude that there are 12 linearly independent $\mathbf{S}_n \times \mathbf{S}^1$-equivariant functions on

$\mathbb{C}^{n,0}$. By Remark 6.1 (a), we know that 52 is the number of linearly independent functions from $\mathbb{C}^n$ to $\mathbb{C}^n$ with homogeneous polynomial components of degree 5 and $\mathbf{S}_n \times \mathbf{S}^1$-equivariant. See [11, Theorems 4.5, 4.10] for details. □

Remark 7.4. With the notation of Theorem 7.3, for $n = 5$ we have

$$H_9(z) = 30H_1(z) - \frac{9}{2}H_2(z) + \frac{3}{4}H_3(z) + \frac{3}{2}H_4(z) - 3H_5(z) - \frac{3}{2}H_6(z) + \frac{3}{2}H_7(z) - 3H_8(z) - 6H_{10}(z) - 9H_{11}(z) - \frac{9}{2}H_{12}(z)$$

where $z \in \mathbb{C}^{5,0}$ and so we obtain (over the complex field) only eleven linearly independent $\mathbf{S}_5 \times \mathbf{S}^1$-equivariants on $\mathbb{C}^{5,0}$ with homogeneous polynomial components of degree five. This is consistent with Table 4. ◇

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