GENERIC PROFIT SINGULARITIES IN TIME AVERAGED OPTIMIZATION-THE CASE OF A CONTROL SPACE WITH A REGULAR BOUNDARY

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Abstract. For one parameter smooth families of pairs of control systems and profit densities on the circle, we consider the problem of maximizing the averaged profit in the infinite horizon from the singularity theory point of view. Namely studying the generic classification of the optimal averaged profit as function of the parameter. This approach to the problem was introduced in 2002 by V.I. Arnold and all generic classifications in related problems obtained since then, assume a compact control space without boundary.

The existence of a boundary in the control space is usual in real problems and so it is worthwhile to be considered. In this work we considered the existence of a regular boundary in the control space and study the one-dimensional parameter's case. We present all generic singularities of the optimal averaged profit as function of the parameter, and conclude that their appear three new singularities.

1. Introduction

Consider the following smooth control system on the circle S^1:

$$\dot{x} = v(x,u)$$

where x is an angle on the circle and u is a control parameter belonging to the control space U, which is a smooth compact manifold with a regular boundary ∂U. That means that, if dim $U = n$, then every point belonging to ∂U has a neighbourhood diffeomorphic to an open subset of $H^n = \{(u_1, ..., u_n) \in \mathbb{R}^n : u_1 \leq 0\}$.

An *admissible motion* of the control system is an absolutely continuous map $x : t \mapsto x(t)$ from a time interval I to the system phase space S^1 for which the velocity of motion (at each moment of differentiability of the map) belongs to the convex hull of the admissible velocities of the system, more precisely $\dot{x}(t) \in [v_{min}(x(t)), v_{max}(x(t))]$ for a.e. $t \in I$, where

$$v_{min}(x) = \min_{u\in U} v(x,u) \quad \text{and} \quad v_{max}(x) = \max_{u\in U} v(x,u)$$

denote the minimum and maximum admissible velocities at x, respectively.

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Remark 1. Because the phase space is compact, any admissible motion can be extended for all $t \in \mathbb{R}$.

Together with a smooth *profit density* $f : S^1 \to \mathbb{R}$ on the circle, the control system gives rise to the following optimal control problem:

> *To maximize the averaged profit on the infinite time horizon*
> $$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(x(t))dt$$
> *over all the system's admissible motions on the positive semi-axis.*

Remark 2. If the last limit does not exist we must take the upper limit.

In this work we look to this problem through singularity theory. When the problem depends on parameters, that is when both the control system and the profit density depend additionally on parameters, then the optimal strategy can vary with the parameters and the optimal averaged profit on the infinite time horizon, as a function of the parameters, can have singularities (points where it is not smooth). We are so led to the problem of classifying such singularities.

This approach was firstly considered in [1] and more recently in [4] and [5], for the time averaged optimization on the circle and control space without boundary. Those works focuses on two kinds of admissible motions that are crucial for determining the optimal averaged profit on the infinite horizon of a controlled dynamical system, namely

- *a level cycle*: motion using the maximum and minimum velocities when the profit density is less or greater, respectively, than a certain constant, or
- *a stationary strategy*: motion corresponding to an equilibrium point of the controlled dynamical system.

It was proved in [5] that a strategy providing the maximal averaged profit always can be found inside these two kinds of motions. But note that for this statement to be true it is essential the larger concept of equilibrium point of a controlled system considered there, namely, such a point is a point where the convex hull of the admissible velocities of the system contains the zero velocity.

So the classification of the singularities of the optimal averaged profit can be reduced to three cases, namely the singularities for stationary strategies, for level cycles and for transitions between stationary strategies and level cycles.

The generic classification for the one dimensional parameter case is already complete ([1], [4], [5]). The case of the control problem stated before (when the control space has a regular boundary) is treated in this paper.

2. Preliminary results

In this section we do not prove anything new. We just present some results that are crucial for understanding the next sections. Some of the results are presented in a reorganized way to facilitate the understanding of the next sections and some of them can not be found elsewhere in the particular form presented here. So, on those ones, we have chosen not to include any reference.

From now on and to simplify language we will call simply CSB to a control system on the circle having as control space a smooth compact manifold with a regular boundary (as introduced in section 1).

2.1. **Optimal motions.** An admissible motion is said to be *optimal* if it provides the greatest averaged profit on the infinite horizon, which we call just *optimal averaged profit* or *best averaged profit.*

We will use the same definition of *equilibrium point* of a control system as the one given in [5], namely such a point is a point where the convex hull of the admissible velocities contains the zero velocity. Such a point is stationary in the sense that for a control system with a one dimensional phase space there exists an admissible motion circulating close to that point, and converging to it as time goes to infinity. It is clear that the averaged profit on the infinite horizon provided by such motion equals the profit density value at this point, that is, the profit value gained through the permanent staying at the point.

A stationary strategy is a choice of an admissible motion converging to an equilibrium point and the *stationary domain* is the union of all such points.

For a value c of the profit density f we define the *c-level motion* as the one using the maximum and minimum admissible velocities at points where the profit density is not greater and greater than c respectively.

A value of the profit density is called *cyclic* if for all nearby values, the respective level motions provide rotation along the circle. For example, for a system with positive velocities only all values of the profit density are cyclic. For a cyclic value c we call its level motion *c-level cycle* or just *level cycle.* The *period* of a level cycle is its smaller period.

Theorem 2.1. [5] *For a continuous CSB and a continuous profit density on the circle, the best averaged profit on the infinite time horizon can always be provided either by a level cycle or by a stationary strategy.*

Remark 3. This theorem is stated in [5] for a continuous control system on S^1 having as control space a smooth compact manifold (or a disjoint union of smooth compact manifolds). But with respect to the control system, the proof uses only the fact that its extremal velocities are Lipschitz. But that is also the case when the control space is a smooth compact manifold with a regular boundary.

Remark 4. Note that there can exist a lot of different optimal motions. For example, the change of an optimal motion on any finite interval of time preserves its optimality.

When we have a family of control systems and profit densities, the optimal strategy can vary depending on the parameters and the best averaged profit, as a function of the parameters, can have singularities. For example, this profit can be discontinuous, even when the families of control systems and densities are smooth [1]. Theorem 2.1 permits us to subdivide these singularities into three groups in order to analyze them, namely, the singularities for stationary strategies, for level cycles and for the transitions between stationary strategies and level cycles.

2.2. **Extremal velocities.** By a *generic object* (a family of control systems or profit densities, a pair (control system, profit density), etc.) we mean a point in an open dense set in the space of objects endowed with a fine smooth or sufficiently smooth topology. A property or assertion is *generic* (or *holds generically*) if it holds for a generic object.

The maximum and minimum admissible velocities play an important role on both level cycles and stationary strategies and consequently on the singularities of the best averaged profit. In fact the velocity of a c-level cycle is given by:

$$v_c(x) = \begin{cases} v_{max}(x) & \text{if } f(x) \leq c \\ v_{min}(x) & \text{if } f(x) > c \end{cases}$$

and the stationary domain is the set $S = \{x \in S^1 : 0 \in [v_{min}(x), v_{max}(x)]\}$. One of the things that prevent the optimal averaged profit of being smooth is the existence of points where these extremal velocities are not smooth. So we present next their generic singularities.

As said before we will consider one parameter smooth families of pairs of control systems and profit densities on the circle. Let p denote the parameter. As the classification is local we may assume that the parameter's space is just $\mathbb{R}$. So $p \in \mathbb{R}$ and $(v(x,p,u), f(x,p))$ is a one parameter smooth family of pairs of control systems and profit densities on the circle. Recall that the control space U is a smooth compact manifold with a regular boundary ∂U. We begin to look at the extremal admissible velocities, namely to the following 1-parameter vectorfields on S^1:

$$v_{min}(x,p) = \min_{u \in U} v(x,p,u) \quad \text{and} \quad v_{max}(x,p) = \max_{u \in U} v(x,p,u)$$

called minimum and maximum velocities respectively.

The product space of the phase space S^1 by the parameter space is naturally fibred over the parameter, that is, with fibres $\mathcal{F}_p = S^1 \times \{p\}$, for every parameter value p. Two objects of the same nature defined on a fibred space are $\mathcal{F}-$*equivalent* if one of them can be carried out to the other by a fibered smooth diffeomorphism, i. e., by a smooth diffeomorphism that sends fibres to fibres. To simplify language we say that two one parameter families of vectorfields are R^+ equivalent if up to the addition with a smooth function, they are $\mathcal{F}-$*equivalent.*

Theorem 2.2. *For a generic smooth* 1*-parameter family of CSB's, the germ of the maximum velocity at any point, is, up to the product with a non vanishing smooth function,* R^+*-equivalent to the germ at the origin of one of the vectorfields from the second column of Table 1, where* α, α_i *are smooth functions that vanish at the origin. Besides these singularities are stable up to small perturbations of the family of CSB's.*

Table 1.

no.	Singularity	Codimension
1	0	0
2	$\max\limits_{u\leq 0}(-u^2+\alpha(x,p)u)$	1
3	$\max\{\alpha_1(x,p),\alpha_2(x,p)\}$	1
4	$\max\limits_{u\leq 0}(u^3+\alpha_1(x,p)u+\alpha_2(x,p)u^2)$	2
5	$\max\{\alpha_1(x,p),\alpha_2(x,p),\alpha_3(x,p)\}$	2
6	$\max\limits_{u}(-u^4+\alpha_1(x,p)u+\alpha_2(x,p)u^2)$	2
7	$\max\{\max\limits_{u\leq 0}(-u^2+\alpha_1(x,p)u),\alpha_2(x,p)\}$	2

Remark 5. For the minimum velocity we obtain the same result but with a change of sign of the functions from the second column of Table 1.

The generic singularities of the maximum of a parametrically depending function are well known [2], [3] and [6]. These singularities are stable up to small perturbations of the family of functions. The normal forms for the vectorfields are obtained from the normal forms for the functions just by multiplication by a non vanishing smooth function. The last column of Table 1 denotes the codimension in $S^1\times\mathbb{R}$ of the stratum of the respective singularity. Singularities no. 1, 3, 5 and 6 appear already in the case of a control space without boundary. For the case of existence of a regular boundary in the control space, there appear three new singularities (2,4 and 7). These correspond to points where the extremal velocity is C^1 but not C^2. It is important to the forthcoming sections to note that singularities 2 and 3 split into two cases each one as shown in the next table:

no.	Singularity	Codimension
2^*	$\max\limits_{u\leq 0}(-u^2+xu)$	1
$2^t_\pm$	$\max\limits_{u\leq 0}(-u^2+(p\pm x^2)u)$	2
3^*	$\|x\|$	1
3^t	$\|p-x^2\|$	2

Singularities 2* and 3* correspond to the cases $\alpha_x(0,0) \neq 0$ and $(\alpha_1 - \alpha_2)_x(0,0) \neq 0$ respectively, while singularities $2^t_\pm$ and 3^t correspond to the vanishing of these derivatives.

The set where the minimum or maximum of a family of vectorfields is not smooth is called *Maxwell set*. A point of the Maxwell set is called *regular* if at that point the vectorfield has a singularity of type 2* or 3*.

3. Profit singularities for optimal stationary strategies

We will consider now the optimal averaged profit for stationary strategies, that is given by

$$A_s(p) = \max_{x \in S(p)} f(x,p), \tag{1}$$

where $S(p)$ is the set of all phase points x such that (x,p) belongs to the stationary domain $S = \{(x,p) : 0 \in [v_{min}(x,p), v_{max}(x,p)]\}$. It is defined for all parameter values p such that $S(p)$ is not empty.

Thus, to classify the generic singularities of A_s we can first examine the generic singularities of the stationary domain and prove their stability under small perturbations of a generic family of systems, and then analyse the generic singularities of A_s.

To simplify language we will from now on call $A_s(p)$ just optimal profit.

The stationary domain S is a closed subset of $S^1 \times \mathbb{R}$. It is clear that the stationary domain around an interior point is locally $\mathcal{F}-$*equivalent* to $\mathbb{R}^2$. It is also easy to see that at a boundary point of it, one of the extremal admissible velocities vanishes.

Theorem 3.1. *For a generic one-parameter family of CSB's, the germ of the stationary domain at any of its boundary points is the germ at the origin of one of the eight sets from Table 2 in an appropriate smooth coordinate system foliated over the parameter. Besides, the stationary domains for a generic family of CSB's and any one sufficiently close to it can be carried one to another by a C^∞-diffeomorphism that is close to the identity and preserves the natural foliation over the parameter.*

Table 2.

1	$2_\pm$	3	4	$5_\pm$	6
$x \leq 0$	$p \geq \pm x^2$	$p \leq \|x\|$	$x \geq -\|p\|$	$\pm(p^2 - x^2) \leq 0$	$x \leq p\|p\|$
c=1	c=2	c=2	c=2	c=2	c=2

We omit the proof of this theorem. It is based on Theorem 2.2, on transversality theorems and on simple calculations. We just point out the fact that singularities 1, $2_\pm$, 3, 4 and $5_\pm$ already appear in the case of a control system with a smooth compact manifold without boundary as control space [5] and for the case of finite dimensional families of polidynamical systems (when the number of different values of the control parameter is finite) [8]. The existence of a regular boundary in the control space leads, in the general case, to one more singularity in the stationary domain, namely Singularity 6 on Table 2. It corresponds to the ocurrence of a Singularity 2 (Table 1) of an extremal velocity at a boundary point of the stationary domain (more specifically, a Singularity 2*).

The stratum corresponding to Singularity 1 has codimension 1 in $S^1 \times \mathbb{R}$. The points of that stratum are called regular points of the stationary domain. All the strata corresponding to the other singularities have codimension 2 in $S^1 \times \mathbb{R}$.

Theorem 3.2. *For a generic smooth one-parameter family of pairs of CSB's and profit densities on the circle and any value of the parameter admitting equilibrium points, the germ of the best averaged profit over the stationary strategies at such a value is the germ at the origin of one of the five functions in the second row of Table 3 up to the equivalence from the third one. Besides, the graphs of the best averaged profits provided by stationary strategies for a generic pair and any one sufficiently close to it can be reduced one to another by a* Γ*-equivalence which is close to the identity.*

Table 3.

Type	*1*	*2*	*3*	*4*	*5*
Singularity	0	$\lvert p\rvert$	$p\lvert p\rvert$	$\sqrt{p}, p \geq 0$	$\max\left\{0, 1+\sqrt{p}\right\}$
Equivalence	R^+	R^+	R^+	R^+	Γ

Remark 6. All singularities from Table 3 appear already generically in the case of a control system with a smooth compact manifold without boundary as control space [5], as well as in the case of polydinamical systems [7] . We recall that two germs of functions are Γ-*equivalent* if their graphs are equivalent, by a smooth diffeomorphism preserving the natural foliation over the function's domain.

Proof. Suppose that for a parameter value p_0 with $S_{p_0} \neq \emptyset$ the maximum averaged profit $A_s(p_0)$ over the stationary strategies is attained at a unique point Q. Suppose that

a) $\frac{\partial f}{\partial x}(Q) = \cdots = \frac{\partial^i f}{\partial x^i}(Q) = 0 \neq \frac{\partial^{i+1} f}{\partial x^{i+1}}(Q) \quad (i \geq 0)$;
b) the stationary domain at point Q has a codimension c singularity (note that if Q is an interior point of the stationary domain then $c = 0$).

It is easy to see that in a generic case $c + i \leq 2$ (for simplicity we denote $\alpha = c + i$ and call it codimension of Q) . In fact this result follows easily from Thom transversality theorem. So in a generic case only the following situations must be considered:

c	0	1	1	2
i	1	0	1	0

If $c = 0$, the point Q is an interior point of the stationary domain and so it must be a critical point for the profit density, corresponding to a maximum. These situations and the ones treated in [5] and [7] coincide up to a single case. Namely the case $c = 2$, corresponding to a Singularity 6 of Table 2, and $i = 0$. So we just have to analyse this case. The detailed proof for the other cases can be found in [5]. By Theorem 3.1 in an appropriate smooth coordinate system foliated over the parameter, $Q = (0, 0)$ and the stationary domain has the form $x \le p|p|$. As Q is a non critical point of $f(\cdot, p_0)$ in the new coordinates we have $f_x(0, 0) \neq 0$. For $p = 0$ the stationary domain is given by $x \le 0$, and as $f(0, 0) = \max_{x \le 0} f(x, 0)$, we conclude immediately that it must be $f_x(0, 0) > 0$. As $f_x(x, p) \neq 0$ for (x, p) in a neighbourhood of $(0, 0)$, we conclude that around $p = 0$ the maximum averaged profit over the stationary strategies is attained at the boundary of the stationary domain and so $A_s(p) = f(p|p|, p)$. Let $\xi(p) = \dfrac{f(p^2, p) - f(-p^2, p)}{2}$ and $\gamma(p) = \dfrac{f(p^2, p) + f(-p^2, p)}{2}$. Obviously ξ and γ are smooth functions and

$$A_s(p) = \begin{cases} -\xi(p) + \gamma(p) & \text{if } p \le 0 \\ \xi(p) + \gamma(p) & \text{if } p \ge 0 \end{cases}$$

But $\xi(0) = \xi'(0) = 0$ and $\xi''(0) = 2f_x(0, 0) > 0$. So $\xi(p) = p^2 B(p)$ with $B(0) > 0$. Considering the new coordinate $\tilde{p} = p\sqrt{B(p)}$, we conclude that $A_s(p)$ is R^+-equivalent to $\tilde{p}|\tilde{p}|$.

So this new case leads to a already known singularity, namely Singularity 3 of Table 3.

Suppose now that the maximum averaged profit $A_s(p_0)$ over the stationary strategies is attained at $N > 1$ distinct points $Q_1, Q_2, ..., Q_N$ of the stationary domain. Let α_i denote the codimension of the point Q_i, as defined above. It is easy to see that in a generic case $\sum_i^N c_i \le 2$. In fact this result follows easily from multijet transversality theorem. So in a generic case only the situation $N = 2, c_1 = c_2 = 1$ must be considered. This situation, that includes two cases, has been already treated in [5] and [7] and leads to Singularity 2 of Table 3. □

4. Profit singularities for cyclic strategies

Let us consider now a c-level cycle, that is, a rotation along the circle that uses the following velocity:

$$v_c(x, p) = \begin{cases} v_{max}(x, p) & \text{if } f(x) \le c \\ v_{min}(x, p) & \text{if } f(x) > c \end{cases}$$

Recall that c is a cyclic value, that is, for all nearby values, the respective level motions provide rotation along the circle.

Let $T(c,p)$ be the period (the smallest) of the c-level cycle and $P(c,p)$ its profit for a complete rotation along the circle. So $P(c,p) = \int_0^{T(c,p)} f(x(t),p)dt$, where $x(t)$ is the c-level motion. Obviously $v_c(\cdot,p)$ has allways the same sign, which we will suppose positive. So we can rewrite both the period and the profit in terms of spatial integrals, namely:

$$T(c,p) = \oint \frac{1}{v_c(x,p)}dx \qquad \text{and} \qquad P(c,p) = \oint \frac{f(x,p)}{v_c(x,p)}dx.$$

Clearly, the averaged profit on the infinite time horizon, for the c-level cycle is given by $A(c,p) = \dfrac{P(c,p)}{T(c,p)}$.

The following results proved in [4] are fundamental for obtaining the classification of generic singularities of the maximum averaged profit for level cycles.

Theorem 4.1 ([4])**.** *For a value of the parameter p_0, when the maximum averaged profit is provided by a c_0-level cycle then the respective cyclic value c_0 is the unique solution of equation*

$$c = A(c,p_0), \tag{2}$$

if the differentiable profit density has a finite number of critical points and the maximum and minimum velocities of the continuous control system are equal at isolated points only.

Theorem 4.2 ([4])**.** *The averaged profit along a level cycle is a differentiable function of the level near the cyclic value providing the maximum averaged profit, if the control system is continuous and the differentiable profit density has a finite number of critical points.*

Remark 7. Theorems 4.1 and 4.2 play an important role in the classification of generic singularities of the maximum averaged profit for level cycles. The derivative $(P/T)_c$ has to be zero at a level providing the maximum averaged profit, and the derivative of the left hand side of equation (2) at this level is equal to 1. By the implicit function theorem that gives us the possibility to recalculate the singularities of the best averaged profit through the ones of the period of the level cycles and the profit along them.

In order to better understand the situations leading to singularities of the period and the profit, we present next their computations. Suppose c_0 is a regular value of the profit density $f(\cdot,p_0)$. And let $x_1(c,p),...,x_N(c,p)$ be the solutions of equation $f(x,p)=c$ around (c_0,p_0). As c_0 is a regular value of f, the functions $x_i(c,p)$ are smooth. Then

$$T(c,p) = \int_{x_1(c,p)}^{x_2(c,p)} R(x,p)dx + \int_{x_2(c,p)}^{x_3(c,p)} r(x,p)dx + \cdots + \int_{x_{N-1}(c,p)}^{x_N(c,p)} R(x,p)dx + \int_{x_N(c,p)}^{x_1(c,p)} r(x,p)dx$$

with $R(x,p)=\dfrac{1}{v_{min}(x,p)}=\max\limits_{u\in U}\dfrac{1}{v(x,p,u)}$ and $r(x,p)=\min\limits_{u\in U}\dfrac{1}{v(x,p,u)}$. For the profit $P(c,p)$ we just have, in the expression for the period, to multiply R and r by the density f. As R and r are both continuous, T is differentiable, and

$$T_c(c,p)=\sum_i (r(x_i(c,p),p)-R(x_i(c,p),p))\frac{1}{|f_x(x_i(c,p),p)|}$$

$$\begin{aligned} T_p(c,p)= & \sum_i (R(x_i(c,p),p)-r(x_i(c,p),p))\frac{f_p(x_i(c,p),p)}{|f_x(x_i(c,p),p)|}+ \\ & \int_{x_1(c,p)}^{x_2(c,p)} R_p(x,p)dx+\cdots+\int_{x_N(c,p)}^{x_1(c,p)} r_p(x,p)dx \end{aligned}$$

For the profit's derivatives we just have, in the previous expressions, to consider Rf and rf instead of R and r. So, if c_0 is a regular value of the profit density $f(\cdot,p_0)$, then at (c_0,p_0) the functions T and P can loose differentiability only if there are points of the Maxwell set of R and r:

(1) inside the domains where they are used,
(2) where it is necessary to switch between R and r.

Note, that in the first situation T_c is smooth and singularities occur only on T_p, while in the second situation singularities occur on both T_c and T_p. The case of c_0 being a critical value of the density leads also to loss of differentiability of T and P [4] but must be treated case by case. So we can organize the situations leading to singularities (loss of differentiability) of both T and P in three distinct types:

1. Existence of points of the Maxwell set of R and r inside the domains where they are used (*passage through a point of the Maxwell set*).
2. Existence of switching points (points of changing between extremal velocities) on the Maxwell set of R and r (*switching at a point of the Maxwell set*).
3. Coincidence of the optimal averaged profit with a critical value of the profit's density.

and conclude that:

Lemma 4.3. *On the circle for a generic smooth one parameter family of pairs of profit densities and CSB's, both the period and the profit of a level cycle are smooth functions of the level c and the parameter p around any point (c_0,p_0) for which:*

(1) *the level c_0 is not critical for the profit density $f(\cdot,p_0)$;*
(2) *the switching points between extremal velocities lie outside the Maxwell set for this level.*
(3) *there are no passages through points of the Maxwell set corresponding to codimension 2 singularities of the extremal velocities (singularities $2^t_\pm$, 3^t, 4, 5, 6 and 7)*

TABLE 4.

N.	$g(p)$	Equivalence	Conditions
1	0	R^+	
2	$p^{3/2}+p^2$	Γ_a	coincidence with a local minimum of the profit density
3	$p^{3/2}-p^2$	Γ_a	coincidence with a local maximum of the profit density
4	$p^{3/2}$	Γ	passage through a singularity 3^t
5	p^2	R^+	passage through a singularity 5 or 7
6	p^3	R^+	switching at a point with singularity 3*
7	$-p^{7/2}$	Γ	passage through a singularity 6
8	$p^{5/2}$	Γ	passage through a singularity $2^t_\pm$
9	p^5	R^+	passage through a singularity 4
10	p^4	R^+	switching at a point with singularity 2*

Remark 8. Note that a singularity type 2* or 3* of an extremal velocity inside the domain where it is used doesn't lead to loss of differentiability of both the period and the profit, as it is easy to see looking to their expressions above. In those situations we get a smooth period $T(c,p)$ and a smooth profit $P(c,p)$.

Theorem 4.4. *On the circle for a generic smooth one parameter family of pairs of profit densities and CSB's with positive velocities only, the germ of the maximum averaged profit at any value of the parameter, is equivalent to the germ at the origin of*

$$A(p) = \begin{cases} 0 & \text{if } p \leq 0 \\ g(p) & \text{if } p \geq 0 \end{cases}$$

where $g(p)$ is one of the functions in the second column of Table 4 and the equivalence is the one pointed out in the third column. Besides, all these singularities are Γ-stable.

Remark 9. The first seven singularities from Table 4 appear already generically in the case of a control system with a smooth compact manifold without boundary as control space [4], [5]. The last three appear as a consequence of the existence of a boundary in the control space. Note that the passage through a singularity 7 leads to the same singularity as a passage through a singularity 5.

Proof. By Lemma 4.3 and transversality conditions, the cases we have to consider are the ones described in the last column of Table 4. From those cases there are four that are new: passage through a singularity 7, 4 or $2^t_\pm$ and switching at a point with singularity 2*. The other cases appear already generically in the case of a control system with a smooth compact manifold without boundary as control space. A detailed proof of the first seven singularities can be found in [4]. We will treat the remaining cases. For the parameter value p_0, let the maximum averaged profit be provided by c_0. We shift (c_0, p_0) to the origin.

Suppose that the minimum velocity has a type 7 (Table 1) singularity inside the domain where it is used. Easily one can see that

$$R(x,p) = \frac{1}{v_{min}(x,p)} = \max_{u \in U} \frac{1}{v(x,p,u)}$$

has the same kind of singularity (corresponding to a maximum). We can choose coordinates such that the point where singularity 7 takes place is the origin and around it and up to the sum with a smooth vectorfield: $R(x,p) = \max\{v_1, v_2\}$, with $v_1 = x|x|\gamma(x,p)$ and v_2 and γ are smooth and vanish at the origin (generically $\frac{\partial \alpha_2}{\partial x}(0,0) \neq 0$ in the normal form of Table 1) . So for ϵ sufficiently small, we can write

$$T(c,p) = \tilde{T}_\epsilon(c,p) + \int_{-\epsilon}^{\epsilon} v_1 dx + \int_{[-\epsilon,\epsilon] \cap \{v_1 \leq v_2\}} (v_2 - v_1) dx$$

where $\tilde{T}_\epsilon$ corresponds to the part of the integral outside $[-\epsilon, \epsilon]$. The first two summands of the righthandside of this expression are smooth (recall that v_1 has a type 2* singularity at the origin). So we will now concentrate in the last one. Looking at equation $v_2 - v_1 = 0$ and using the implicit function theorem and Hadamard's lemma we get after a suitable change of coordinates:

$$(v_2 - v_1)(x,p) = \begin{cases} (x-p)\omega(x,p) & \text{if } p \leq 0 \\ (x - p\varphi(p))\omega(x,p) & \text{if } p \geq 0 \end{cases}$$

with ω and φ smooth, $\omega(0,0) \neq 0$, $\varphi(0) = 1$ and $\varphi'(0) \neq 0$. We will consider $\omega(0,0) > 0$ (the other situation is treated in a similar way and leads to the same singularity).

So in the new coordinates we have $T(c,p) = \bar{T}_\epsilon(c,p) + D_\epsilon(p)$ where

$$D_\epsilon(p) = \begin{cases} \displaystyle\int_p^\epsilon (x-p)\omega(x,p) dx & \text{if } p \leq 0 \\ \displaystyle\int_{p\varphi(p)}^\epsilon (x - p\varphi(p))\omega(x,p) dx & \text{if } p \geq 0 \end{cases}$$

Note that the computation of the expressions of $D_\epsilon(p)$ on both sides of $p = 0$ can be performed for every p around 0. So we consider both equations:

$$c = \frac{\bar{P}_\epsilon(c,p) + \int_p^\epsilon (x-p)\omega(x,p) f(x,p) dx}{\bar{T}_\epsilon(c,p) + \int_p^\epsilon (x-p)\omega(x,p) dx} \tag{3}$$

$$c = \frac{\bar{P}_\epsilon(c,p) + \int_{p\varphi(p)}^\epsilon (x - p\varphi(p))\omega(x,p) f(x,p) dx}{\bar{T}_\epsilon(c,p) + \int_{p\varphi(p)}^\epsilon (x - p\varphi(p))\omega(x,p) dx} \tag{4}$$

and get their solutions $c_1(p)$ and $c_2(p)$ respectively by the implicit function theorem. So we conclude that the optimal averaged profit is $\mathbb{R}^+$- equivalent to:

$$A(p) = \begin{cases} c_1(p) & \text{if } p \leq 0 \\ c_2(p) & \text{if } p \geq 0 \end{cases}$$

Computing derivatives from (3) and (4) we get $c_1(p) = p^2\phi_1(p)$ and $c_2(p) = p^2\phi_2(p)$, with $\phi_2(0) < \phi_1(0) < 0$. So $A(p)$ is $\mathbb{R}^+$- equivalent to $p|p|$, and so we get singularity 5 of Table 4.

Suppose now that the minimum velocity has a type 4 (Table 1) singularity inside the domain where it is used. Easily one can see that $R(x,p)$ has the same kind of singularity (corresponding to a maximum). We can choose coordinates such that the point where singularity 4 takes place is the origin and around it and up to the sum with a smooth vectorfield:

$$R(x,p) = \max_{u \leq 0}\left(\frac{u^3}{3} + \alpha_1(x,p)u + \alpha_2(x,p)u^2\right)\gamma(x,p)$$

where γ is smooth and nonvanishing. The maximum is easy to calculate and considering the new coordinate $\tilde{x} = (\alpha_2^2 - \alpha_1)(x,p)$ we get the following form for R up to a product with a smooth nonvanishing function and the sum with a smooth vectorfield (tilde is omitted for simplifying the notation):

$$R(x,p) = \begin{cases} -\frac{1}{3}\alpha^3 + \alpha x + \frac{2}{3}x^{3/2} & \text{if } x > 0 \text{ and } \alpha(x,p) + x^{1/2} > 0 \\ 0 & \text{otherwise} \end{cases}$$

with $\alpha(0,0) = 0$. Making $z^2 = x$ and looking to equation $\alpha(z^2,p) + z = 0$ it is easy to conclude that it has an unique solution of the form $z = \varphi(p)$ with $\varphi'(0) \neq 0$. So after a suitable change of coordinates on p, we get (up to the sum with a smooth vectorfield) two possible situations:

$$R(x,p) = \begin{cases} \tilde{R}(x,p) & \text{if } x > 0 \text{ and } p + x^{1/2} \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

or

$$R(x,p) = \begin{cases} \tilde{R}(x,p) & \text{if } x > 0 \text{ and } p + x^{1/2} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\tilde{R}(x,p) = \omega_1^3(x,p) + \omega_1(x,p)x + \omega_2(x,p)x^{3/2}$, with ω_1 and ω_2 smooth and $\omega_1(0,0) = 0$ and $\omega_2(0,0) \neq 0$. We will treat only the first case (the other one is similar and leads to the same singularity). As in the previous case we will write the period $T(c,p)$ and the profit $P(c,p)$ as sums of two components, one of which is smooth and the other one corresponding to the nonsmooth part. So $T(c,p) = \tilde{T}(c,p) + \gamma(p)$ and $P(c,p) = \tilde{P}(c,p) + \psi(p)$ where $\gamma(p)$ and $\psi(p)$ are zero if $p \geq 0$ and equal to $\int_0^{p^2} \tilde{R}(x,p)dx$ and $\int_0^{p^2} \tilde{R}(x,p)f(x,p)dx$ respectively if $p \leq 0$. Let $\tilde{c}(p)$ be the solution of equation $c\tilde{T}(c,p) = \tilde{P}(c,p)$. It is obviously smooth. So subtracting $\tilde{c}$ from the profits density (that is equivalent to subtract $\tilde{c}$ from the averaged profit) we conclude that the optimal averaged profit is $\mathbb{R}^+$-equivalent to a function which is zero if $p \geq 0$ and equal to the solution $c_1(p)$ of equation

$$c = \frac{c^2\bar{P}(c,p) + \psi(p)}{\bar{T}(c,p) + \gamma(p)}$$

if $p \leq 0$, where $\bar{P}(c,p)$ and $\bar{T}(c,p)$ are the "new" smooth parts of the profit and the period. Computing now derivatives from this expression, we get $c_1(p) = p^5\xi(p)$, with $\xi(0) \neq 0$. After subtracting $p^5\xi(p)$ to the averaged profit a changing the coordinate p we get singularity 9 of Table 4.

Suppose now that the minimum velocity has a type $2^t_\pm$ (Table 1) singularity inside the domain where it is used. Easily one can see that $R(x,p)$ has the same kind of singularity (corresponding to a maximum). We can choose coordinates such that the point where singularity takes place is the origin and around it and up to the sum with a smooth vectorfield:

$$R(x,p) = \max_{u \leq 0}(-u^2 + (p \pm x^2)u)\gamma(x,p)$$

where γ is smooth and nonvanishing. The maximum is easy to calculate and considering if necessary a change of sign in p we get two possible forms for R up to a product with a smooth nonvanishing function and the sum with a smooth vectorfield:

$$R(x,p) = \begin{cases} (p-x^2)^2 & \text{if } p - x^2 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

or

$$R(x,p) = \begin{cases} (p-x^2)^2 & \text{if } p - x^2 \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

We will treat only the first case (the other one is similar and leads to the same singularity). Proceeding exactly as in the previous case we conclude that the optimal averaged profit is $\mathbb{R}^+$-equivalent to a function which is zero if $p \leq 0$ and equal to the solution $c_1(p)$ of equation

$$c = \frac{c^2\bar{P}(c,p) + \int_{-\sqrt{p}}^{\sqrt{p}}(p-x^2)^2\omega(x,p)f(x,p)dx}{\bar{T}(c,p) + \int_{-\sqrt{p}}^{\sqrt{p}}(p-x^2)^2\omega(x,p)dx}$$

with ω smooth, $\omega(0,0) > 0$ and $f(0,0) > 0$, if $p \geq 0$. As in [4] we write this equation in the form:

$$c = \frac{c^2\bar{P}(c,p) + \int_0^{\sqrt{p}}(p-x^2)^2F(x^2,p)dx}{\bar{T}(c,p) + \int_0^{\sqrt{p}}(p-x^2)^2G(x^2,p)dx}$$

with $F(x^2,p) = f(x,p)\omega(x,p) + f(-x,p)\omega(-x,p)$ and $G(x^2,p) = \omega(x,p) + \omega(-x,p)$.

We can write this equation in the form:

$$c\bar{T}(c,p) - c^2\bar{P}(c,p) = p^{5/2}(a(p) - cb(p))$$

with a and b smooth functions, $a(0) > 0$ and $b(0) > 0$. Making now the change of coordinate $\tilde{c} = \dfrac{c\bar{T}(c,p) - c^2\bar{P}(c,p)}{a(p) - cb(p)}$ we get the result.

Suppose now that the minimum velocity has a type 2* singularity when switching to the maximum velocity. Easily one can see that $R(x,p)$ has the same kind of singularity (corresponding to a maximum). We can choose coordinates such that the point where singularity 2* takes place is the origin and around it and up to the sum with a smooth vectorfield:

$$R(x,p) = \begin{cases} x^2\gamma(x,p) & \text{if } x \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

with γ smooth and $\gamma(0,0) > 0$.

As the singularity occurs when switching we have $f(0,0) = 0$ and generically $f_x(0,0) \neq 0$. So equation $f(x,p) = c$ has a unique smooth solution $x = \varphi(c,p)$ around the origin. Writing $f(x,p) = f(0,p) + x\tilde{f}(x,p)$, it is easy to see that after a suitable change of coordinates we get for the switching point $x = c - p$ and a vectorfield R that is as above but with a new smooth function γ.

So $T(c,p) = \tilde{T}(c,p) - \varphi(p)$ and $P(c,p) = \tilde{P}(c,p) - \psi(p)$ where $\varphi(p)$ and $\psi(p)$ are zero if $c - p \leq 0$ and equal to $\int_0^{c-p} x^2\gamma(x,p)dx$ and $\int_0^{c-p} x^2\gamma(x,p)f(x,p)dx$ respectively if $c - p \geq 0$. Let $\tilde{c}(p)$ be the solution of equation $c\tilde{T}(c,p) = \tilde{P}(c,p)$ (this equation is smooth and can be considered for (c,p) near the origin, it corresponds to taking $R(x,p$ always equal to $x^2\gamma(x,p)$) . $\tilde{c}(p)$ is obviously smooth, and so subtracting it from the profits density (that is equivalent to subtract $\tilde{c}$ from the averaged profit) we conclude that the optimal averaged profit is $\mathbb{R}^+$-equivalent to a function which is zero if $p \geq 0$ and equal to the solution $c_1(p)$ of equation

$$c = \frac{c^2\bar{P}(c,p) - \psi(p)}{\bar{T}(c,p) - \varphi(p)}$$

if $p \leq 0$, where $\bar{P}(c,p)$ and $\bar{T}(c,p)$ are the "new" smooth parts of the profit and the period. Computing now derivatives from this expression, we get $c_1(p) = p^4\xi(p)$, with $\xi(0) \neq 0$. So we get singularity 9 of Table 4. □

5. Transition singularities

A parameter value is called a *transition value* if in any neighborhood of it, the maximum averaged profit can not be provided by one and only one type of strategy, namely, either by level cycles or by equilibrium points of the controlled dynamical system.

It is clear that for a transition value p_0, the set S_{p_0} is not empty and therefore the best averaged profit $A_s(p_0)$ among all stationary strategies is well defined. As proved in [5], generically, the set of cyclic levels of the profit density at p_0 is open, and $A_s(p_0) \geq A_l(p_0)$, where $A_l(p_0)$ is the upper limit of the averaged profit provided by level cycles when $p \to p_0$. Moreover if A_s is continuous at p_0, then $A_s(p_0) = A_l(p_0)$

Due to stability of the Maxwell set of a generic family of CSB one can fix this set and the respective stationary domain and make only perturbations of the profit density family. Using tranversality theorems and small perturbations of the profit density family one can show that in a generic case for any transition parameter value p_0:

1. The fiber $p = p_0$ contains only regular points of the Maxwell set.
2. The profit $A_s(p_0)$ is provided by only one equilibrium point which is either an interior point of the stationary domain with $f_x = 0 > f_{xx}$ or a regular boundary point of the stationary domain (type 1 Singularity in Table2) with $f_x \neq 0$. In particular, the function A_s is smooth near the point p_0.
3. The value $A_s(p_0)$ of this profit is less than the maximum $m(p_0)$ of the density $f(., p_0)$ on the circle.
4. If the profit $A_s(p_0)$ is a critical value of the profit density $f(., p_0)$, then the level $f(., p_0) = A_s(p_0)$ contains only one critical point of the density and this point belongs to the interior of the stationary domain, and else it is exactly the point providing the maximum averaged profit among stationary strategies.

So, generically only two types of transition occur, namely, to a stationary strategy at an equilibrium point Q either inside the stationary domain or at a regular boundary point of the stationary domain. But these are the situations that occur also generically in the case of a control system with a smooth compact manifold without boundary as control space and so we get the same generic singularities:

Theorem 5.1. [5] *For a generic smooth one parameter family of pairs of CSB's and profit densities on the circle, the germ of the maximum averaged profit at a transition parameter value is R^+-equivalent to the germ at the origin of one of the two functions in Table 5. Besides those singularities are stable.*

Table 5.

N	Singularity	Type
1	$\lvert p \rvert$	Stop at an interior point of the stationary domain with $f_x = 0$
2	$\max\left\{0, -\frac{p}{\ln p}(1 + H)\right\}$	Stop at a regular boundary point of the stationary domain with $f_x \neq 0$

Remark 10. In Table 5, $H = h(p, \frac{1}{\ln p}, \frac{\ln \lvert \ln p \rvert}{\ln p})$ where h is a smooth function of its variables with $h(p, 0, 0) \equiv 0$. Actually the function $c(p) = -\frac{p}{\ln p}(1 + H)$ is given implicitly by an equation of the form $c \ln c = F(c, p)$, with F

smooth. The word "stop" means the switch between the optimal level cycle strategy and the stationary strategy of the given type at the point under consideration.

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