QUADRATIC DECOMPOSITION OF LAGUERRE POLYNOMIALS VIA LOWERING OPERATORS

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ABSTRACT. A Laguerre polynomial sequence of parameter $\varepsilon/2$ was previously characterised in a recent work [27] as an orthogonal $\mathscr{F}_\varepsilon$-Appell sequence, where $\mathscr{F}_\varepsilon$ represents a lowering operator depending on the complex parameter $\varepsilon \neq -2n$ for any integer $n \geqslant 0$. Here, we proceed to the quadratic decomposition of an $\mathscr{F}_\varepsilon$-Appell sequence, and we conclude that the four sequences obtained by this approach are also of Appell but with respect to another lowering operator consisting of a fourth order linear differential operator $\mathscr{G}_{\varepsilon,\mu}$, where μ is either 1 or -1. Therefore, we introduce and develop the concept of the $\mathscr{G}_{\varepsilon,\mu}$-Appell sequences and we realize they cannot be orthogonal. At last, we completely describe how to quadratically decompose a Laguerre sequence of parameter $\varepsilon/2$.

Keywords: orthogonal polynomials; Laguerre polynomials; Appell polynomials; lowering operator; Genocchi numbers.

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1. INTRODUCTION AND PRELIMINARY RESULTS

We denote by $\mathscr{P}$ the vector space of the polynomials with coefficients in $\mathbb{C}$ (the field of complex numbers) and by $\mathscr{P}'$ its dual space, whose elements are *forms*. The action of $u \in \mathscr{P}'$ on $f \in \mathscr{P}$ is denoted as $\langle u, f\rangle$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$, $n \geqslant 0$ the moments of u. A linear operator $T : \mathscr{P} \to \mathscr{P}$ has a transpose ${}^tT : \mathscr{P}' \to \mathscr{P}'$ defined by

$$\langle {}^tT(u), f\rangle = \langle u, T(f)\rangle, \quad u \in \mathscr{P}', f \in \mathscr{P}. \tag{1.1}$$

For example, for any form u, any polynomial g, let $Du = u'$ and gu be the forms defined as usually

$$\langle u', f\rangle := -\langle u, f'\rangle \quad , \quad \langle gu, f\rangle := \langle u, gf\rangle,$$

where D is the differential operator. Thus, the differentiation operator D on forms is minus the transpose of the differentiation operator D on polynomials.

Whenever a sequence of polynomials $\{B_n\}_{n\geqslant 0}$ is such that $\deg B_n = n$, for $n \geqslant 0$, we will systematically call it as PS and, in the case where its elements are monic (that is, $B_n(x) = x^n + b_n(x)$, with $\deg b_n \leqslant n-1$ for $n \geqslant 1$) we will refer to it as a *monic polynomial sequence* (MPS). The dual sequence $\{u_n\}_{n\geqslant 0} \subset \mathscr{P}'$ of a MPS $\{B_n\}_{n\geqslant 0}$ is defined by $\langle u_n, B_k\rangle = \delta_{n,k}$, $n,k \geqslant 0$, where $\delta_{n,k}$ denotes the *Kronecker symbol*

[30, 31]. We will denote by $\{B_n^{[1]}\}_{n\geqslant 0}$ the MPS obtained from a given MPS by a single differentiation $B_n^{[1]}(x) := \frac{1}{n+1}B'_{n+1}(x)$, $n \geqslant 0$.

The form u is called *regular* if we can associate with it a PS $\{B_n\}_{n\geqslant 0}$ such that $\langle u, B_nB_m\rangle = k_n\delta_{n,m}$ with $k_n \neq 0$, for all the integers $n,m \geqslant 0$, [13, 30, 31]. The PS $\{B_n\}_{n\geqslant 0}$ is then said to be orthogonal with respect to u. If u is a regular form we can assume that the system (of orthogonal polynomials) is monic. Then, there exists a dual sequence $\{u_n\}_{n\geqslant 0}$ and the original form u is proportional to u_0. Furthermore, we have

$$u_n = \left(\langle u_0, B_n^2\rangle\right)^{-1} B_n u_0,\ n \geqslant 0. \tag{1.2}$$

The sequence $\{B_n\}_{n\geqslant 0}$ is then called a *monic orthogonal polynomial sequence* (MOPS) and it fulfils the second order recurrence relation given by

$$B_0(x) = 1 \qquad B_1(x) = x - \beta_0 \tag{1.3}$$

$$B_{n+2}(x) = (x-\beta_{n+1})B_{n+1}(x) - \gamma_{n+1}B_n(x),\ n \geqslant 0. \tag{1.4}$$

with $\beta_n = \dfrac{\langle u_0, xB_n^2\rangle}{\langle u_0, B_n^2\rangle}$ and $\gamma_{n+1} = \dfrac{\langle u_0, B_{n+1}^2\rangle}{\langle u_0, B_n^2\rangle} \neq 0$, $\quad n \geqslant 0$. A regular form u exists if and only if the *Hankel* determinant $\Delta_n := \det\big[(u)_{i+j}\big]_{0\leqslant i,j\leqslant n}$ is nonzero for any integer $n \geqslant 0$.

When $u \in \mathscr{P}'$ is regular, let Φ be a polynomial such that $\Phi u = 0$, then $\Phi = 0$, [31].

Entailed in the problem of the symmetrysation of sequences of polynomials, comes out the quadratic decomposition (as well as the cubic decomposition) of a PS. Within this context, many authors have dealt with symmetrization problems of orthogonal polynomial sequences either on the real line or in the unit circle. Among them we quote [4, 12, 13, 14, 15, 18, 24, 29, 32]. More specifically, in [13, 15] a symmetric orthogonal polynomial sequence is decomposed into two nonsymmetric sequences. A generalisation of this idea was revealed in [29, 32]: to a given MPS $\{B_n\}_{n\geqslant 0}$, we associate two other MPS, $\{P_n\}_{n\geqslant 0}$ and $\{R_n\}_{n\geqslant 0}$, and two sequences of polynomials, $\{a_n\}_{n\geqslant 0}$ and $\{b_n\}_{n\geqslant 0}$, such that

$$B_{2n}(x) = P_n(x^2) + x\,a_{n-1}(x^2), \quad n \geqslant 0, \tag{1.5}$$

$$B_{2n+1}(x) = b_n(x^2) + x\,R_n(x^2), \quad n \geqslant 0. \tag{1.6}$$

where $0 \leqslant \deg a_n\,, \deg b_n \leqslant n$ for any integer $n \geqslant 0$ and $a_{-1}(\cdot) = 0$, [13, 15, 29]. Under the assumption that $\{B_n\}_{n\geqslant 0}$ is orthogonal, it is not possible to conclude that $\{P_n\}_{n\geqslant 0}$ and $\{R_n\}_{n\geqslant 0}$ are also orthogonal, if some supplementary conditions are not considered. For instance, $a_n = 0 = b_n$, $n \geqslant 0$, if and only if the MPS $\{B_n\}_{n\geqslant 0}$ is symmetric (that is $B_n(-x) = (-1)^nB_n(x), n \geqslant 0$) and its orthogonality supplies the orthogonality of both sequences $\{P_n\}_{n\geqslant 0}$ and $\{R_n\}_{n\geqslant 0}$ [29].

Recently, in [27], the two authors have proceeded to the quadratic decomposition (hereafter QD) of an *Appell polynomial sequence* (that is, a MPS $\{B_n\}_{n\geqslant 0}$ such that $B_n^{[1]}(\cdot) = B_n(\cdot), n \geqslant 0$) [3]. The four associated sequences obtained by this approach are also Appell sequences but with respect to another differential operator

$$\mathscr{F}_\varepsilon := 2D\,x\,D + \varepsilon\,D \tag{1.7}$$

where ε is either 1 or -1, and $D := \frac{d}{dx}$. This operator $\mathscr{F}_\varepsilon$, as well as the differential operator D, decreases in one unit the degree of a polynomial. They are indeed simple examples of the so-called *lowering operators*: a linear mapping $\mathscr{O}$ of $\mathscr{P}$ into itself is called *lowering operator* when $\mathscr{O}(1) = 0$ and $\deg\big(\mathscr{O}(x^n)\big) = n-1$, $n \geqslant 1$. The Appell character of a PS may be generalised in a natural way to other *lowering operators* $\mathscr{O}$ rather than D:

Definition 1.1. *A MPS $\{B_n\}_{n\geqslant 0}$ is called an $\mathscr{O}$-Appell sequence with respect to a lowering operator $\mathscr{O}$ if $B_n(\cdot) = B_n^{[1]}(\cdot,\mathscr{O})$ for any integer $n \geqslant 0$, with*

$$B_n^{[1]}(x;\mathscr{O}) := \rho_n\left(\mathscr{O}\,B_{n+1}\right)(x),\, n \geqslant 0, \tag{1.8}$$

where $\rho_n \in \mathbb{C}-\{0\}$, $n \geqslant 0$, is chosen for making $B_n^{[1]}(x;\mathscr{O})$ monic [7, 8].

This concept is not new. As a matter of fact some authors have considered Appell sequences with respect to other operators like the q-derivative [37], operators reducing or augmenting the degree of a polynomial by k units, with $k \geqslant 1$. Among them we quote [10, 11, 16, 17, 25, 26]). However, such considerations are not useful here.

The primary purpose of this work is to characterise the four sequences associated with the QD of an $\mathscr{F}_\varepsilon$-Appell sequence, in which $\mathscr{F}_\varepsilon$ is the operator given in (1.7) with $\varepsilon \neq -2n$, $n \geqslant 1$. Firstly, in section 2, we show that the four polynomial sequences obtained by this approach are also Appell sequences with respect to a fourth order linear differential operator, denoted by $\mathscr{G}_{\varepsilon,\mu}$, where μ is either 1 or -1. Subsequently, regarding a more accurate information about the arisen $\mathscr{G}_{\varepsilon,\mu}$-Appell sequences, in section 3 we study these MPS through a functional point of view, where the range for the parameter μ was broadened to a dense subset of $\mathbb{C}$ (the set of complex numbers). While ferreting out all the D-Appell and $\mathscr{F}_\varepsilon$-Appell orthogonal sequences, we find the Hermite (a result given by Angelescu [2] and later by other authors [13, 35] but further references may be found in [1]) and the Laguerre polynomial sequences of parameter $\varepsilon/2$, up to a linear change of variable, (achieved in [27]) respectively. However, in section 4 we conclude that a $\mathscr{G}_{\varepsilon,\mu}$-Appell sequence cannot be orthogonal. In spite of this negative result, in the last section we successfully reach the complete description of the QD of the nonsymmetric sequence of Laguerre polynomials, by means of the *Genocchi numbers*.

2. The quadratic decomposition of $\mathscr{F}_\varepsilon$-Appell sequences

Pursuing the idea of the quadratic decomposition of an Appell sequence, we explore the $\mathscr{F}_\varepsilon$-Appell sequences. To accomplish so, it is useful to enlighten some properties of the operator $\mathscr{F}_\varepsilon$; namely for any $f, g \in \mathscr{P}$, we have:

$$\mathscr{F}_\varepsilon\big(f(x)\,g(x)\big) = f(x)\,\mathscr{F}_\varepsilon\big(g(x)\big) + g(x)\,\mathscr{F}_\varepsilon\big(f(x)\big) + 4\,x\,f'(x)\,g'(x),$$

$$\mathscr{F}_\varepsilon\big(f(t^2)\big)(x) = x\left\{8\,x^2 f''(x^2) + 2(4+\varepsilon)\,f'(x^2)\right\}, \tag{2.1}$$

$$\mathscr{F}_\varepsilon\big(t\,f(t^2)\big)(x) = x^2\left\{8x^2 f''(x^2) + 2(8+\varepsilon)\,f'(x^2)\right\} + (2+\varepsilon)\,f(x^2)\,. \tag{2.2}$$

Theorem 2.1. *Consider the QD of a monic sequence $\{B_n\}_{n\geqslant 0}$ as in (1.5)-(1.6). If $\{B_n\}_{n\geqslant 0}$ is an $\mathscr{F}_\varepsilon$-Appell sequence with $\varepsilon \neq -2(n+1)$, $n \geqslant 0$, then the four sequences $\{P_n\}_{n\geqslant 0}$, $\{R_n\}_{n\geqslant 0}$, $\{a_n\}_{n\geqslant 0}$ and $\{b_n\}_{n\geqslant 0}$ are given by*

$$P_n(x) = \frac{1}{\eta_{n+1}(\varepsilon,-1)}\big(\mathscr{G}_{\varepsilon,-1}P_{n+1}\big)(x),\ n \geqslant 0, \tag{2.3}$$

$$R_n(x) = \frac{1}{\eta_{n+1}(\varepsilon,1)}\big(\mathscr{G}_{\varepsilon,1}R_{n+1}\big)(x),\ n \geqslant 0, \tag{2.4}$$

$$a_n(x) = \frac{1}{\eta_{n+2}(\varepsilon,-1)}\big(\mathscr{G}_{\varepsilon,1}a_{n+1}\big)(x),\ n \geqslant 0, \tag{2.5}$$

$$b_n(x) = \frac{1}{\eta_{n+1}(\varepsilon,1)}\big(\mathscr{G}_{\varepsilon,-1}b_{n+1}\big)(x),\ n \geqslant 0, \tag{2.6}$$

where the operators $\mathscr{G}_{\varepsilon,1}$ and $\mathscr{G}_{\varepsilon,-1}$ and the nonzero sequences $\{\eta_{n+1}(\varepsilon,1)\}_{n\geqslant 0}$ and $\{\eta_{n+1}(\varepsilon,-1)\}_{n\geqslant 0}$ are respectively given by

$$\mathscr{G}_{\varepsilon,1} = \big(4DxD + \varepsilon D\big)\,\big(2xD + \mathbb{I}\big)\,\big(4xD + (2+\varepsilon)D\big) \tag{2.7}$$

$$\mathscr{G}_{\varepsilon,-1} = \big(4DxD + \varepsilon D\big)\,\big(2xD - \mathbb{I}\big)\,\big(4xD - (2-\varepsilon)D\big) \tag{2.8}$$

and

$$\eta_{n+1}(\varepsilon,1) = (n+1)\,\big(4(n+1)+\varepsilon\big)\,(2n+3)\,\Big[2\,\big(2\,n+3\big)+\varepsilon\Big]\,,\ n \geqslant 0, \tag{2.9}$$

$$\eta_{n+1}(\varepsilon,-1) = (n+1)\,\big(4(n+1)+\varepsilon\big)\,(2n+1)\,\Big[2\,\big(2\,n+1\big)+\varepsilon\Big]\,,\ n \geqslant 0, \tag{2.10}$$

where $D := \frac{d}{dx}$ and $\mathbb{I}$ represents the identity on $\mathscr{P}$.

Proof. Consider $\rho_{n+1} = (n+1)\big(2(n+1)+\varepsilon\big)$. Operating with $\mathscr{F}_\varepsilon$ on both members of (1.5) and (1.6) with n replaced by $n+1$, then, under the assumption and by virtue of (2.1)-(2.2), we obtain:

$$\begin{aligned}\rho_{2n+2}\{b_n(x^2) + x\,R_n(x^2)\} \;&=\; x\left\{2(4+\varepsilon)\,P'_{n+1}(x^2) + 8x^2\,P''_{n+1}(x^2)\right\}\\ &\quad +(2+\varepsilon)\,a_n(x^2) + 2(8+\varepsilon)\,x^2\,a'_n(x^2)\\ &\quad +8\,x^4\,a''_n(x^2)\,, \quad n \geqslant 0,\end{aligned}$$

$$\begin{aligned}\rho_{2n+1}\{P_n(x^2) + x\,a_{n-1}(x^2)\} \;&=\; x\left\{2(4+\varepsilon)\,b'_n(x^2) + 8x^2\,b''_n(x^2)\right\}\\ &\quad +(2+\varepsilon)\,R_n(x^2) + 2(8+\varepsilon)\,x^2\,R'_n(x^2)\\ &\quad +8\,x^4\,R''_n(x^2)\,, \quad n \geqslant 0,\end{aligned}$$

which consists of polynomials with only even or odd powers. As a result, we necessarily get:

$$\rho_{2n+2}\, R_n(x) = \Big\{2(4+\varepsilon)\, D + 8x\, D^2\Big\} \Big(P_{n+1}(x)\Big),\ n \geqslant 0, \tag{2.11}$$

$$\rho_{2n+1}\, P_n(x) = \Big\{(2+\varepsilon)\,\mathbb{I} + 2(8+\varepsilon)\, x\, D + 8x^2\, D^2\Big\} \Big(R_n(x)\Big),\ n \geqslant 0, \tag{2.12}$$

$$\rho_{2n+2}\, b_n(x) = \Big\{(2+\varepsilon)\,\mathbb{I} + 2(8+\varepsilon)\, x\, D + 8x^2\, D^2\Big\} \Big(a_n(x)\Big),\ n \geqslant 0, \tag{2.13}$$

$$\rho_{2n+1}\, a_{n-1}(x) = \Big\{2(4+\varepsilon)\, D + 8x\, D^2\Big\} \Big(b_n(x)\Big),\ n \geqslant 0. \tag{2.14}$$

Operating with the equalities (2.11) and (2.12), we deduce

$$\rho_{2n+2}\rho_{2n+3}\, R_n(x) = \Big\{2\,\varepsilon\, D + 8\, D\, x\, D\Big\} \cdot \Big\{(2+\varepsilon)\,\mathbb{I} + 2(4+\varepsilon)\, x\, D + 8\, x\, D\, x\, D\Big\} \Big(R_{n+1}(x)\Big),\ n \geqslant 0.$$

and also

$$\rho_{2n+1}\rho_{2n+2}\, P_n(x) = \Big\{(2+\varepsilon)\,\mathbb{I} + 2(8+\varepsilon)\, x\, D + 8x^2\, D^2\Big\} \cdot \Big\{2(4+\varepsilon)\, D + 8\, x\, D^2\Big\} \Big(P_{n+1}(x)\Big),\ n \geqslant 0.$$

Using the identities

$$\left\{\begin{array}{l} x\, D^2 = D\, x\, D - D \\ x^2\, D^2 = x\, D\, x\, D - x\, D \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l} D\, x = x\, D - \mathbb{I} \\ x^2\, D^2 = D\, x\, D\, x - 3\, D\, x + 2\mathbb{I} \end{array}\right. \tag{2.15}$$

in the right-hand side of the first and second previous relations respectively, we deduce

$$\begin{aligned} &\rho_{2n+2}\rho_{2n+3}\, R_n(x) \\ &= \Big\{2\,\varepsilon\, D + 8\, D\, x\, D\Big\} \cdot \Big\{(2+\varepsilon)\,\mathbb{I} + 2(4+\varepsilon)\, x\, D + 8\, x\, D\, x\, D\Big\} \Big(R_{n+1}(x)\Big),\ n \geqslant 0, \end{aligned}$$

yielding (2.4) under the definitions (2.7) and (2.9). We also derive

$$\begin{aligned} &\rho_{2n+1}\rho_{2n+2}\, P_n(x) \\ &= \Big\{(2-\varepsilon)\,\mathbb{I} - 2(4-\varepsilon)\, D\, x + 8\, D\, x\, D\, x\Big\} \cdot \Big\{2\,\varepsilon\, D + 8\, D\, x\, D\Big\} \Big(P_{n+1}(x)\Big),\ n \geqslant 0, \end{aligned}$$

which corresponds to (2.3) under the definitions (2.8) and (2.10).

Likewise, by means of simple manipulations, the system of equalities (2.13) and (2.14) gives rise to another system of two equalities: one involving exclusively elements of the set of polynomials $\{b_n\}_{n\geqslant 0}$ and the other having only elements of the set of polynomials $\{a_n\}_{n\geqslant 0}$, which, on account of the identities (2.15), may be transformed into the following equalities

$$\rho_{2n+2}\rho_{2n+3}\, b_n(x) = \Big\{(2-\varepsilon)\,\mathbb{I} - 2(4-\varepsilon)\, D\, x + 8\, D\, x\, D\, x\Big\} \cdot \Big\{2\,\varepsilon\, D + 8\, D\, x\, D\Big\} \Big(b_{n+1}(x)\Big),\ n \geqslant 0. \tag{2.16}$$

and

$$\rho_{2n+1}\rho_{2n+2}\, a_{n-1}(x) = \Big\{2\,\varepsilon\, D + 8\, D\, x\, D\Big\} \cdot \Big\{(2+\varepsilon)\,\mathbb{I} + 2(4+\varepsilon)\, x\, D + 8\, x\, D\, x\, D\Big\} \Big(a_n(x)\Big),\ n \geqslant 0. \tag{2.17}$$

where $a_{-1}(\cdot) = 0$. The relation (2.16) provides (2.6), whereas the relation (2.17) with n replaced by $n+1$ leads to (2.5), under the definitions (2.7)-(2.10). □

More information about the polynomial sequences is provided in the next result.

Proposition 2.1. *Let $\{B_n\}_{n\geqslant 0}$ be a $\mathscr{F}_\varepsilon$Appell sequence and consider its QD according to (1.5)-(1.6). Then either $\{B_n\}_{n\geqslant 0}$ is symmetric or there exists an integer $p\geqslant 0$ such that $a_p(\cdot)\neq 0$ (respectively, $b_p(\cdot)\neq 0$). In this case, we have*

$$a_n(x)=0,\ b_n(x)=0,\ 0\leqslant n\leqslant p-1,\ \text{when } p\geqslant 1, \tag{2.18}$$

$$a_{p+n}(x)=\binom{n+p+1}{n}\frac{\left(p+\frac{3}{2}\right)_n\left(p+\frac{3}{2}+\frac{\varepsilon}{4}\right)_n\left(p+2+\frac{\varepsilon}{4}\right)_n}{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}+\frac{\varepsilon}{4}\right)_n\left(1+\frac{\varepsilon}{4}\right)_n}\,a_p\,\widehat{a}_n(x), \tag{2.19}$$

$$b_{p+n}(x)=\binom{n+p}{n}\frac{\left(p+\frac{3}{2}\right)_n\left(p+\frac{3}{2}+\frac{\varepsilon}{4}\right)_n\left(p+1+\frac{\varepsilon}{4}\right)_n}{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}+\frac{\varepsilon}{4}\right)_n\left(1+\frac{\varepsilon}{4}\right)_n}\,b_p\,\widehat{b}_n(x),\quad n\geqslant 0 \tag{2.20}$$

where $\widehat{a}_n$ and $\widehat{b}_n$ are two monic polynomials fulfilling $\deg\widehat{a}_n(x)=n$, $\deg\widehat{b}_n(x)=n$, for $n\geqslant 0$, and $(y)_n=y(y+1)\dots(y+n-1)$ represents the Pochhammer symbol.

Proof. If $\{B_n\}_{n\geqslant 0}$ is a symmetric sequence then $a_n(\cdot)=0$, $n\geqslant 0$, and also $b_n(\cdot)=0$, $n\geqslant 0$. Reciprocally, if $a_n(\cdot)=0$, $n\geqslant 0$ (respectively, $b_n(\cdot)=0$, $n\geqslant 0$), then from (2.13) $b_n(\cdot)=0$, $n\geqslant 0$ (respectively $a_n(\cdot)=0$, $n\geqslant 0$, from (2.14)).
When $\{B_n\}_{n\geqslant 0}$ is not a symmetric sequence, let $p\geqslant 0$ be the smallest integer such that $a_p(\cdot)\neq 0$ and $a_n(\cdot)=0$, $0\leqslant n\leqslant p-1$ when $p\geqslant 1$. From (2.14), we have $b_n(\cdot)=\text{constant}=b_n$, $0\leqslant n\leqslant p$ and by virtue of (2.13), $b_n(\cdot)=0$ for $0\leqslant n\leqslant p-1$, $\rho_{2p+2}\,b_p(x)=(2+\varepsilon)\,a_p(x)+2(8+\varepsilon)\,x\,a_p'(x)+8x^2\,a_p''(x)$, which implies $a_p(\cdot)=\text{constant}=a_p\neq 0$. Thus, $(2+\varepsilon)\,a_p=\rho_{2p+2}\,b_p$.
Proceeding by finite induction, then, based on (2.13)-(2.14), we achieve the conclusion $\deg(a_{n+p})=n$ and $\deg(b_{n+p})=n$, $n\geqslant 0$. Therefore, we may consider two nonzero sequences $\{\lambda_n\}_{n\geqslant 0}$ and $\{\mu_n\}_{n\geqslant 0}$ such that

$$a_{n+p}(x)=\lambda_n\,\widehat{a}_n(x)\qquad\text{and}\qquad b_{n+p}(x)=\mu_n\,\widehat{b}_n(x)\,,\quad n\geqslant 0, \tag{2.21}$$

where $\widehat{a}_n(\cdot)$ and $\widehat{b}_n(\cdot)$ represent two monic polynomials of degree $n\geqslant 0$, $\mu_0=b_p$ and $\lambda_0=a_p$. Replacing in (2.13) and (2.14) n by $n+p$ and taking into account (2.21), we obtain

$$\rho_{2n+2p+2}\,\mu_n\,\widehat{b}_n(x)=(2+\varepsilon)\,\lambda_n\,\widehat{a}_n(x)+2(8+\varepsilon)\,x\,\lambda_n\,\widehat{a}_n{}'(x)+8x^2\,\lambda_n\,\widehat{a}_n{}''(x)\,,\quad n\geqslant 0,$$
$$\rho_{2n+2p+1}\,\lambda_{n-1}\,\widehat{a}_{n-1}(x)=2(4+\varepsilon)\,\mu_n\,\widehat{b}_n{}'(x)+8\,x\,\mu_n\,\widehat{b}_n{}''(x),\quad n\geqslant 0.$$

Therefore, the nonzero sequences $\{\lambda_n\}_{n\geqslant 0}$ and $\{\mu_n\}_{n\geqslant 0}$ satisfy the system

$$\rho_{2n+2p+2}\,\mu_n=8\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2}+\frac{\varepsilon}{4}\right)\lambda_n\,,\quad n\geqslant 0,$$
$$\rho_{2n+2p+1}\,\lambda_{n-1}=8\,n\left(n+\frac{\varepsilon}{4}\right)\mu_n,\quad n\geqslant 0.$$

which implies

$$\rho_{2n+2p+2}\,\mu_n = 8\,\left(n+\frac{1}{2}\right)\,\left(n+\frac{1}{2}+\frac{\varepsilon}{4}\right)\,\lambda_n\,,\quad n\geqslant 0,$$

$$\rho_{2n+2p+3}\,\rho_{2n+2p+4}\,\lambda_n = 64\,(n+1)\,\left(n+1+\frac{\varepsilon}{4}\right)\,\left(n+\frac{3}{2}\right)\,\left(n+\frac{3}{2}+\frac{\varepsilon}{4}\right)\,\lambda_{n+1},\quad n\geqslant 0,$$

and, because $\rho_{n+1} = (n+1)\,\big(2(n+1)+\varepsilon\big)$, $n\geqslant 0$, it yields

$$\mu_n = \frac{\left(n+\frac{1}{2}\right)\,\left(n+\frac{1}{2}+\frac{\varepsilon}{4}\right)}{(n+p+1)\big(n+p+1+\frac{\varepsilon}{4}\big)}\,\lambda_n\,,\quad n\geqslant 0,$$

$$\lambda_{n+1} = \binom{n+p+2}{n+1}\frac{(p+\frac{3}{2})_{n+1}\,\left(p+\frac{3}{2}+\frac{\varepsilon}{4}\right)_{n+1}\,\left(p+2+\frac{\varepsilon}{4}\right)_{n+1}}{\left(\frac{3}{2}\right)_{n+1}\,\left(1+\frac{\varepsilon}{4}\right)_{n+1}\,\left(\frac{3}{2}+\frac{\varepsilon}{4}\right)_{n+1}}\,\lambda_0,\quad n\geqslant 0,$$

where $(y)_k$ represents the *Pochhammer* symbol. Finally we achieve,

$$\lambda_n = \binom{n+p+1}{n}\,\frac{\left(p+\frac{3}{2}\right)_n\,\left(p+\frac{3}{2}+\frac{\varepsilon}{4}\right)_n\,\left(p+2+\frac{\varepsilon}{4}\right)_n}{\left(\frac{3}{2}\right)_n\,\left(\frac{3}{2}+\frac{\varepsilon}{4}\right)_n\,\left(1+\frac{\varepsilon}{4}\right)_n}\,\lambda_0,$$

$$\mu_n = \frac{\left(n+\frac{1}{2}\right)\,\left(n+\frac{1}{2}+\frac{\varepsilon}{4}\right)}{(n+p+1)\,\left(n+p+1+\frac{\varepsilon}{4}\right)}\,\lambda_n\,,\quad n\geqslant 0,$$

whence the result. □

The two MPS emerged with the QD of an $\mathscr{F}_\varepsilon$-Appell sequence, are also Appell sequences with respect to the lowering operators $\mathscr{G}_{\varepsilon,1}$ and $\mathscr{G}_{\varepsilon,-1}$, in the light of definition 1.1. Analogously, on account of the relations (2.5)-(2.6) and (2.19)-(2.20) given in Proposition 2.1, we may say that the sequences $\{\widehat{a}_n\}_{n\geqslant 0}$ and $\{\widehat{b}_n\}_{n\geqslant 0}$ are, respectively, $\mathscr{G}_{\varepsilon,1}$ and $\mathscr{G}_{\varepsilon,-1}$-Appell. The study of these arisen Appell sequences will now proceed henceforth as a whole rather than individually, so, under the particular choices of $\mu=-1$ or $\mu=1$, they may be viewed as Appell sequences with respect to the lowering operator

$$\mathscr{G}_{\varepsilon,\mu} := \Big(4DxD+\varepsilon D\Big)\Big(8(xD)^2+2\varepsilon xD+2\mathbb{I}+\mu\,\big(8xD+\varepsilon\mathbb{I}\big)\Big)$$

with the convention: $(xD)^{k+1}=xD\,(xD)^k$ for any integer $k\geqslant 0$. Naturally, it is possible to express:

(2.22)
$$\mathscr{G}_{\varepsilon,\mu} \;:=\; 32\,D(xD)^3+16\varepsilon\,D(xD)^2+2(4+\varepsilon^2)\,DxD+2\varepsilon\,D+\mu\Big\{32\,D(xD)^2+12\varepsilon\,DxD+\varepsilon^2\,D\Big\},$$

The forthcoming developments will be made from a functional point of view, requiring the characterisation of the associated dual sequence, which will be carried out in the next section.

3. The $\mathscr{G}_{\varepsilon,\mu}$-Appell sequences

Let $\{B_n\}_{n\geqslant 0}$ be a MPS with dual sequence $\{u_n\}_{n\geqslant 0}$. Consider the sequence $\{B_n^{[1]}(\cdot;\mathscr{G}_{\varepsilon,\mu})\}_{n\geqslant 0}$ given by

(3.1)
$$B_n^{[1]}(x;\mathscr{G}_{\varepsilon,\mu}) = \frac{1}{\widehat{\rho}_{n+1}}\,\Big(\mathscr{G}_{\varepsilon,\mu}\,B_{n+1}\Big)(x)\,,\; n\geqslant 0$$

where $\mathscr{G}_{\varepsilon,\mu}$ is given by (2.22) and

$$\widehat{\rho}_{n+1} := \widehat{\rho}_{n+1}(\varepsilon,\mu) = (n+1)\,\big(4(n+1)+\varepsilon\big)\,\Big(2+2(n+1)\big(4(n+1)+\varepsilon\big)+(8+8n+\varepsilon)\mu\Big) \tag{3.2}$$

for $n \geqslant 0$. Necessarily the parameters ε and μ must be chosen so that $\widehat{\rho}_{n+1} \neq 0$, for all the integers $n \geqslant 0$, therefore ε and μ are two complex parameters such that

$$\varepsilon \neq -4(n+1) \qquad \text{and} \qquad \mu \neq -\frac{2+2(n+1)(4n+4+x)}{8(n+1)+\varepsilon}\,, \qquad n \geqslant 0.$$

Whenever $\mu \in \{-1,1\}$, then $\widehat{\rho}_{n+1}(\varepsilon,\mu)$ equals $\eta_{n+1}(\varepsilon,\mu)$, given by (2.9)-(2.10), for any integer $n \geqslant 0$.

Before characterising $\mathscr{G}_{\varepsilon,\mu}$-Appell sequences, we must determine the dual sequence of $\{B_n^{[1]}(\cdot;\mathscr{G}_{\varepsilon,\mu})\}_{n\geqslant 0}$, denoted as $\{u_n^{[1]}(\mathscr{G}_{\varepsilon,\mu})\}_{n\geqslant 0}$. For this purpose we need to know the transpose ${}^t\mathscr{G}_{\varepsilon,\mu}$ defined according to (1.1):

$$\begin{aligned}
\langle {}^t\mathscr{G}_{\varepsilon,\mu} u\,,\, f\rangle &= \langle u\,,\, \mathscr{G}_{\varepsilon,\mu} f\rangle \\
&= \Big\langle u\,,\, \big\{32\,D(xD)^3+16(\varepsilon+2\mu)\,D(xD)^2+2(4+\varepsilon^2+6\,\varepsilon\,\mu)\,DxD+\varepsilon(2+\varepsilon\,\mu)\,D\big\}f\Big\rangle
\end{aligned}$$

therefore

$${}^t\mathscr{G}_{\varepsilon,\mu} = 32\,D(xD)^3-16(\varepsilon+2\mu)\,D(xD)^2+2(4+\varepsilon^2+6\,\varepsilon\,\mu)\,DxD-\varepsilon\,(2+\varepsilon\,\mu)\,D\,.$$

However, the convention on D (${}^tD=-D$) permits to write ${}^t\alpha_\nu := (-1)^{\nu+1}D(xD)^\nu$, with $\alpha_\nu := D(xD)^\nu$, leaving out a slight abuse of notation without consequence. Thus ${}^t\mathscr{G}_{\varepsilon,\mu} := \mathscr{G}_{-\varepsilon,-\mu}$ and $\mathscr{G}_{\varepsilon,\mu}$ is defined on $\mathscr{P}$ and $\mathscr{P}'$.

For the sequel, it is worth to express $\mathscr{G}_{\varepsilon,\mu}$ in terms of $x^k\,D^{k+1}$ instead of $D(xD)^k$ (with $k=0,1,2,3$). Based on the identities

$$\begin{aligned}
DxD &= x\,D^2+D \\
D(xD)^2 &= x^2\,D^3+3\,x\,D^2+D \\
D(xD)^3 &= x^3\,D^4+6\,x^2\,D^3+7\,x\,D^2+D\,.
\end{aligned}$$

the operator $\mathscr{G}_{\varepsilon,\mu}$ given by (2.22) may be expressed as follows:

$$\begin{aligned}
\mathscr{G}_{\varepsilon,\mu} &= 32x^3\,D^4+16(12+\varepsilon)x^2\,D^3+2\big(116+\varepsilon(24+\varepsilon)\big)\,x\,D^2+2(4+\varepsilon)(5+\varepsilon)\,D \\
&\quad +\mu\Big\{32\,x^2\,D^3+12(8+\varepsilon)\,x\,D^2+(4+\varepsilon)(8+\varepsilon)\,D\Big\}\,.
\end{aligned} \tag{3.3}$$

and, by means of simple computations, we are able to deduce the $\mathscr{G}_{\varepsilon,\mu}$-derivative of the product of two polynomials:

(3.4)
$$\begin{aligned}
\mathscr{G}_{\varepsilon,\mu}\big(f\,p\big)(x) &= f(x)\,\big(\mathscr{G}_{\varepsilon,\mu}\,p\big)+\big(\mathscr{G}_{\varepsilon,\mu}\,f\big)\;p(x)+128\,x^3\,f'(x)\,p^{(3)}(x) \\
&\quad +48\Big\{(\varepsilon+12+2\mu)f'(x)+4xf''(x)\Big\}x^2\,p''(x)+\Big\{\big(116+\varepsilon^2+48\mu+6\varepsilon(4+\mu)\big)f'(x) \\
&\quad +12(\varepsilon+2(6+\mu))x\,f''(x)+32\,x^2\,f^{(3)}(x)\Big\}4\,x\,p'(x)
\end{aligned}$$

for any $p, f \in \mathscr{P}$. By transposition, we may also compute the $\mathscr{G}_{\varepsilon,\mu}$-derivative of the product of a polynomial by a form:

$$\begin{aligned}\left(\mathscr{G}_{-\varepsilon,-\mu}\, fu\right) = \;& f\left(\mathscr{G}_{-\varepsilon,-\mu} u\right) - \left(\mathscr{G}_{\varepsilon,\mu} f\right) u + f'(x)\, L_3(u) + f''(x)\, L_2(u) \\ & + f^{(3)}(x)\, L_1(u) + 2^6\, x^3\, f^{(4)}(x) u \quad , \qquad f \in \mathscr{P}\,, u \in \mathscr{P}'\,,\end{aligned} \tag{3.5}$$

where

$$\begin{aligned} L_3(u) &= \tau_{3,0}\, u + \tau_{3,1}\, x\, u' + \tau_{3,2}\, x^2\, u'' + 2^7 \cdot x^3\, (u)^{(3)} \\ L_2(u) &= \tau_{2,0}\, x\, u + \tau_{2,1}\, x^2\, u' + 3 \cdot 2^6\, x^3\, u'' \\ L_1(u) &= \tau_{1,0}\, x^2\, u + 2^7\, x^3\, u' \end{aligned} \tag{3.6}$$

with

$$\begin{array}{lll} \tau_{3,0} = 4(20+\varepsilon^2+6\varepsilon\mu); & \tau_{3,1} = 2^2\left(116+\varepsilon^2+6\varepsilon(\mu-4)-48\mu\right); & \tau_{3,2} = -2^4\cdot 3\left(\varepsilon-12+2\mu\right); \\ \tau_{2,0} = 2^2\left(116+\varepsilon^2+6\varepsilon\mu\right); & \tau_{2,1} = 2^4\cdot 3\left(12-\varepsilon-2\mu\right); & \tau_{1,0} = 2^7\cdot 3\,. \end{array}$$

Lemma 3.1. *The dual sequence of $\{B^{[1]}_n(\cdot;\mathscr{G}_{\varepsilon,\mu})\}_{n\geqslant 0}$ denoted as $\{u^{[1]}_n(\mathscr{G}_{\varepsilon,\mu})\}_{n\geqslant 0}$ fulfils*

$$\mathscr{G}_{-\varepsilon,-\mu}\left(u^{[1]}_n(\mathscr{G}_{\varepsilon,\mu})\right) = \widehat{\rho}_{n+1}\, u_{n+1}, \quad n \geqslant 0, \tag{3.7}$$

where $\widehat{\rho}_{n+1}$, $n \geqslant 0$, is given by (3.2).

Proof. Following the definition of a dual sequence, $\langle\, u^{[1]}_n(\mathscr{G}_{\varepsilon,\mu})\,,\, B^{[1]}_m(x;\mathscr{G}_{\varepsilon,\mu})\,\rangle = \delta_{n,m}$ for any integers $n,m \geqslant 0$, which corresponds to $(\widehat{\rho}_{n+1})^{-1}\,\langle\, u^{[1]}_n(\mathscr{G}_{\varepsilon,\mu})\,,\,\mathscr{G}_{\varepsilon,\mu}(B_{m+1})\,\rangle = \delta_{n,m}$ for $n,m \geqslant 0$, that is

$$\langle \mathscr{G}_{-\varepsilon,-\mu}\,\left(u^{[1]}_n(\mathscr{G}_{\varepsilon,\mu})\right), B_{m+1}\rangle = \widehat{\rho}_{n+1}\,\delta_{n,m}\,, \quad n,m \geqslant 0. \tag{3.8}$$

In particular, from the latter we have

$$\langle \mathscr{G}_{-\varepsilon,-\mu}\left(u^{[1]}_n(\mathscr{G}_{\varepsilon\mu})\right), B_{m+1}\rangle = 0, \quad m \geqslant n+1,\, n \geqslant 0\,,$$

which implies [33, 34]

$$\mathscr{G}_{-\varepsilon,-\mu}\left(u^{[1]}_n(\mathscr{G}_{\varepsilon\mu})\right) = \sum_{\nu=0}^{n+1} \lambda_{n,\nu}\, u_\nu, \quad n \geqslant 0,$$

with $\lambda_{n,\nu} = \langle \mathscr{G}_{-\varepsilon,-\mu}\left(u^{[1]}_n(\mathscr{G}_{\varepsilon\mu})\right), B_\nu\rangle$, $0 \leqslant \nu \leqslant n+1$. Consequently, due to (3.8), we obtain (3.7). □

This last result enables us to express all the elements of the dual sequence in terms of the first one:

Proposition 3.1. *The MPS $\{B_n\}_{n\geqslant 0}$ is a $\mathscr{G}_{\varepsilon,\mu}$-Appell sequence if and only if its dual sequence $\{u_n\}_{n\geqslant 0}$ fulfils*

$$u_n = \frac{1}{\alpha_n}\,\mathscr{G}^{\,n}_{-\varepsilon,-\mu}\,(u_0), \quad n \geqslant 0, \tag{3.9}$$

where

$$\alpha_n = 32^n\, n!\, \left(1+\frac{\varepsilon}{4}\right)_n \left(\frac{8+\varepsilon+4\mu-\Delta_{\varepsilon,\mu}}{8}\right)_n \left(\frac{8+\varepsilon+4\mu+\Delta_{\varepsilon,\mu}}{8}\right)_n, \; n \geqslant 0,$$

with $\Delta_{\varepsilon,\mu} = \sqrt{\varepsilon^2+16(\mu^2-1)}$ *, and* $\mathscr{G}^{\,n}_{-\varepsilon,-\mu}$ *represents the* n^{th}*-power of the operator* $\mathscr{G}_{-\varepsilon,-\mu}$.

Proof. The condition is necessary. From (3.7), the sequence $\{u_n\}_{n\geqslant 0}$ satisfies

$$\mathscr{G}_{-\varepsilon,-\mu}(u_n) = \widehat{\rho}_{n+1}(\varepsilon,\mu)\, u_{n+1}, \quad n\geqslant 0, \tag{3.10}$$

with $\widehat{\rho}_{n+1}(\varepsilon,\mu)$ as given in (3.2). In particular, for $n=0$,

$$u_1 = \frac{1}{(4+\varepsilon)(10+8\mu+\varepsilon\,(2+\mu))}\,\mathscr{G}_{-\varepsilon,-\mu}\, u_0\,.$$

Proceeding by finite induction, we easily get (3.9).

The condition is sufficient. From (3.9), it is easy to see that (3.10) is fulfilled. Therefore by comparing it with (3.7), we obtain

$$\mathscr{G}_{-\varepsilon,-\mu}\left(u_n^{[1]}(\mathscr{G}_{\varepsilon,\mu})\right) = \mathscr{G}_{-\varepsilon,-\mu}\, u_n,\; n\geqslant 0\,.$$

The lowering operator $\mathscr{G}_{-\varepsilon,-\mu}$ satisfies $\mathscr{G}_{-\varepsilon,-\mu}(\mathscr{P}) = \mathscr{P}$, and therefore $\mathscr{G}_{-\varepsilon,-\mu}$ is one-to-one on $\mathscr{P}'$. We then get $u_n^{[1]}(\mathscr{G}_{\varepsilon,\mu}) = u_n\,,\; n\geqslant 0$, whence the expected result. □

4. About the orthogonality of a $\mathscr{G}_{\varepsilon,\mu}$-Appell sequence

In this section we seek to find all orthogonal polynomial sequences possessing the $\mathscr{G}_{\varepsilon,\mu}$-Appell character. A somehow unexpected result occurs:

Theorem 4.1. *There is no regularly orthogonal polynomial sequence being* $\mathscr{G}_{\varepsilon,\mu}$*-Appell.*

Proof. Suppose there is a MOPS $\{B_n\}_{n\geqslant 0}$ which is also a $\mathscr{G}_{\varepsilon,\mu}$-Appell sequence and let $\{\beta_n\,,\,\gamma_{n+1}\}_{n\geqslant 0}$ be its recurrence coefficients in accordance with (1.4). From (1.2) and (3.10), we get

$$\mathscr{G}_{-\varepsilon,-\mu}(B_n u_0) = \lambda_n\, B_{n+1} u_0, \quad n\geqslant 0, \tag{4.1}$$

with

$$\lambda_n := \lambda_n(\varepsilon) = \frac{\widehat{\rho}_{n+1}(\varepsilon,\mu)}{\gamma_{n+1}}, \quad n\geqslant 0, \tag{4.2}$$

where $\widehat{\rho}_{n+1}, n\geqslant 0$, is defined in (3.2). We recall that, within the range of ε and μ, $\widehat{\rho}_{n+1}$ is always different from zero for any integer $n\geqslant 0$. The particular choice of $n=0$ in (4.1), provides

$$\mathscr{G}_{-\varepsilon,-\mu}\, u_0 = \lambda_0\, B_1\, u_0\,. \tag{4.3}$$

Consider $n+1$ instead of n in (4.1). Following (3.5)-(3.6), because of the $\mathscr{G}_{\varepsilon,\mu}$-Appell character and on account of (4.3), we derive

$$B'_{n+1}\, L_3(u_0) + B''_{n+1}\, L_2(u_0) + B^{(3)}_{n+1}\, L_1(u_0) = \left\{\lambda_{n+1}\, B_{n+2} - \lambda_0\, B_1\, B_{n+1} + \lambda_n\, \gamma_{n+1}\, B_n - 2^6\, x^3\, B^{(4)}_{n+1}\right\} u_0\,,\; n\geqslant 0, \tag{4.4}$$

In particular, considering $n=0$ in this last relation, u_0 fulfils the equality:

$$L_3(u_0) = U_2(x)\, u_0 \tag{4.5}$$

where $L_3(u_0)$ is given in (3.6) and

$$U_2(x) = \lambda_1 B_2(x) - \lambda_0 B_1^2(x) + \lambda_0\, \gamma_1\, .$$

On account of (4.5), the relation (4.4) becomes like

(4.6)

$$B''_{n+1}\, L_2(u_0) + B^{(3)}_{n+1}\, L_1(u_0) = \Big\{ \lambda_{n+1}\, B_{n+2} - \lambda_0\, B_1\, B_{n+1} + \lambda_n\, \gamma_{n+1}\, B_n - U_2\, B'_{n+1} - 2^6\, x^3\, B^{(4)}_{n+1} \Big\}\, u_0,\ n \geqslant 0,$$

and when $n=1$, this relation becomes like

$$L_2(u_0) = U_3(x)\, u_0 \tag{4.7}$$

where $L_2(u_0)$ is given by (3.6) and

$$U_3(x) = \frac{1}{2}\, \big\{ \lambda_2 B_3(x) - \lambda_0 B_1(x)\, B_2(x) + \lambda_1\, \gamma_2\, B_1(x) - B'_2(x)\, U_2(x) \big\}\, .$$

Therefore, due to (4.7), the relation (4.6) may be transformed into

$$\begin{aligned} B^{(3)}_{n+1}\, L_1(u_0) = \Big\{ & \lambda_{n+1}\, B_{n+2} - \lambda_0\, B_1\, B_{n+1} + \lambda_n\, \gamma_{n+1}\, B_n \\ & - B'_{n+1}\, U_2 - B''_{n+1}\, U_3 - 2^6\, x^3\, B^{(4)}_{n+1} \Big\}\, u_0\, , \qquad n \geqslant 0, \end{aligned} \tag{4.8}$$

and taking $n=2$ we obtain:

$$L_1(u_0) = U_4(x)\, u_0 \tag{4.9}$$

where $L_1(u_0)$ is given in (3.6) and

$$U_4(x) = \frac{1}{6}\, \big\{ \lambda_3 B_4(x) - \lambda_0 B_1(x)\, B_3(x) + \lambda_2\, \gamma_3\, B_2(x) - B'_3(x)\, U_2(x) - B''_3(x)\, U_3(x) \big\}\, .$$

Naturally, $\deg U_k \leqslant k$ for $k = 2,3$ or 4, so there are coefficients $\theta_{k,j}$ with $0 \leqslant j \leqslant k$ such that

$$U_k(x) = \sum_{j=0}^{k} \theta_{k,j}\, x^j, \quad k = 2,3,4. \tag{4.10}$$

A single differentiation on both sides of (4.9) leads to

$$2^7\, x^3\, u''_0 + \big\{ (3 \cdot 2^7 + \tau_{1,0})\, x^2 - U_4(x) \big\}\, u'_0 = \big\{ U'_4(x) - 2\, \tau_{1,0}\, x \big\}\, u_0\, . \tag{4.11}$$

Between (4.11) and (4.7) it is possible to eliminate the term in u''_0, and consequently we have

$$\big\{ (3^2 \cdot 2^8 + 3\, \tau_{1,0} - 2\, \tau_{2,1})\, x^2 - 3\, U_4(x) \big\}\, u'_0 = \big\{ 3\, U'_4(x) - 2\, U_3(x) - 2\, (3\, \tau_{1,0} - \tau_{2,0})\, x \big\}\, u_0 \tag{4.12}$$

The elimination of the term u_0' between the equalities (4.12) and (4.9) and the regularity of u_0 leads to $C_3 \equiv 0$ where

$$C_3(x) = -2^7\, x^3\, \{3\, U_4'(x) - 2\, U_3(x) - 2\, (3\, \tau_{1,0} - \tau_{2,0})\, x\} \\ + \Big((3^2 \cdot 2^7 + 3\, \tau_{1,0} - 2\, \tau_{2,1})\, x^2 - 3\, U_4(x) \Big) \Big(U_4(x) - \tau_{1,0}\, x^2 \Big)$$

Since u_0 is a regular form, necessarily C_3 is identically null, that is, C_3 has all its coefficients in x identically zero. Taking into account the definition of the polynomials U_k with $k = 3,4$ presented in (4.10), we realise that $\deg C_3 \leqslant 8$ and we also achieve:

$$\theta_{4,4} = \theta_{4,0} = \theta_{4,1} = 0 \tag{4.13}$$

As a consequence, $C_3(x) = \sum_{j=3}^{6} c_{3,j}\, x^j$ and the conditions $c_{3,j} = 0$ for $j = 3,4,5,6$ provide

$$\begin{aligned} &\theta_{3,0} = 0 \quad , \quad \theta_{3,3} = \frac{3}{2^8}\, (\theta_{4,3})^2 \quad , \quad \theta_{3,2} = \frac{1}{2^7}\, \theta_{4,3}\, (3\, \theta_{4,2} - 3\, \tau_{1,0} + \tau_{2,1}) \\ &\theta_{3,1} = \frac{1}{2^8} \Big\{ 3\, (\theta_{4,2})^2 + 2^8\, \tau_{2,0} - \theta_{4,2} \big(2^7 \cdot 3 + 6\, \tau_{1,0} - 2\, \tau_{2,1} \big) - \tau_{1,0}(-2^7 \cdot 3 - 3\, \tau_{1,0} + 2\, \tau_{2,1}) \Big\}, \end{aligned} \tag{4.14}$$

whence, $U_4(x) = \big(\theta_{4,3}\, x + \theta_{4,2} \big)\, x^2$ and $U_3(x) = \theta_{3,3}\, x^3 + \theta_{3,2}\, x^2 + \theta_{3,1}\, x$.

Differentiating both sides of (4.7) and then eliminating the term in $u_0^{(3)}$ between the resulting equation and (4.5), we deduce

$$\begin{aligned} &\{2^7 \cdot 3^2 + 2\, \tau_{2,1} - 3\, \tau_{3,2}\}\, x^2\, u_0'' + \{(2\, \tau_{2,0} + 4\, \tau_{2,1} - 3\, \tau_{3,1})\, x - 2\, U_3(x)\}\, u_0' \\ &= \{-2\, \tau_{2,0} + 3\, \tau_{3,0} + 2\, U_3'(x) - 3\, U_2(x)\}\, u_0 \end{aligned} \tag{4.15}$$

We proceed to the elimination of the term in u_0'' between (4.15) and (4.7), and we get:

$$\begin{aligned} &\{ [2^7 \cdot 3\, \tau_{2,0} - 2^6 \cdot 3^2\, \tau_{3,1} + \tau_{2,1}(-2^7 \cdot 3 - 2\, \tau_{2,1} + 3\, \tau_{3,2})]\, x - 2^7 \cdot 3\, U_3(x) \}\, x\, u_0' \\ &= \{ \tau_{2,0} \left(2\, (2^7 \cdot 3 + \tau_{2,1}) - 3\, \tau_{3,2} \right) x - \left(2^7 \cdot 3^2 + 2\, \tau_{2,1} - 3\, \tau_{3,2} \right) U_3(x) \\ &+ 3 \cdot 2^6 \left(3\, \tau_{3,0} - 3\, U_2(x) + 2\, U_3'(x) \right) x \}\, u_0 \end{aligned} \tag{4.16}$$

By eliminating the term in u_0' between (4.16) and (4.9), and by taking into consideration the regularity of u_0, we get the condition: $C_2 \equiv 0$ where

$$\begin{aligned} C_2(x) = \;& -(2^7\, x^3) \Big\{ \tau_{2,0} \left(2\, (2^7 \cdot 3 + \tau_{2,1}) - 3\, \tau_{3,2} \right) x - \left(2^7 \cdot 3^2 + 2\, \tau_{2,1} - 3\, \tau_{3,2} \right) U_3(x) \\ & + 3 \cdot 2^6 \left(3\, \tau_{3,0} - 3\, U_2(x) + 2\, U_3'(x) \right) x \Big\} + \Big\{ \Big[2^7 \cdot 3\, \tau_{2,0} - 2^6 \cdot 3^2\, \tau_{3,1} \\ & + \tau_{2,1}(-2^7 \cdot 3 - 2\, \tau_{2,1} + 3\, \tau_{3,2}) \Big] x^2 - 2^7 \cdot 3\, U_3(x)\, x \Big\} \Big(U_4(x) - \tau_{1,0}\, x^2 \Big) \end{aligned} \tag{4.17}$$

After (4.13), we easily realise that the polynomial C_2 may be expressed as $C_2(x) = \sum_{j=4}^{7} c_{2,j}\, x^j$. Due to (4.13)-(4.14), the condition $c_{2,7} = 0$ implies $\theta_{4,3} = 0$, which, in accordance with (4.14), yields

$$\begin{aligned} &\theta_{3,0} = 0 = \theta_{3,3} = \theta_{3,2} \\ &\theta_{3,1} = \frac{1}{2^8}\left\{3\,\theta_{4,2}^2 + 2^8\,\tau_{2,0} - \theta_{4,2}\big(2^7\cdot 3 + 6\,\tau_{1,0} - 2\,\tau_{2,1}\big) - \tau_{1,0}(-2^7\cdot 3 - 3\,\tau_{1,0} + 2\,\tau_{2,1})\right\}. \end{aligned} \tag{4.18}$$

Consequently, we get $U_3(x) = \theta_{3,1}\, x$ and $U_4(x) = \theta_{4,2}\, x^2$. Since $c_{2,6} = 0 = c_{2,5}$, we deduce $\theta_{2,2} = \theta_{2,1} = 0$. As a result, $U_2(x) = \theta_{2,0}$, $U_3(x) = \theta_{3,1}\, x$ and $U_4(x) = \theta_{4,2}\, x^2$, and, according to (4.9) u_0 fulfils

$$(\tau_{1,0} - \theta_{4,2})\, x^2\, u_0 + 2^7\, x^3\, u_0' = 0,$$

contradicting the regularity of u_0. □

In spite of this negative result, the existence of the $\mathscr{G}_{\varepsilon,\mu}$-Appell being d-orthogonal (for some integer $d \geqslant 2$) ought to be explored in an analogous manner as the one expounded in [20, 9].

5. Applications. The quadratic decomposition of a Laguerre sequence

The quadratic decomposition of a non-symmetric sequence is far from being obvious. Nonetheless, the obtained and some already known results permit to describe the associated polynomial sequences to the QD of a Laguerre sequence with complex parameter.

Proposition 5.1. *A Laguerre sequence $\{B_n\}_{n\geqslant 0}$ of parameter $\frac{\varepsilon}{2}$ (with $\varepsilon \neq -2(n+1), n \geqslant 0$) fulfils (1.5)-(1.6) where $\{R_n\}_{n\geqslant 0}$ and $\{P_n\}_{n\geqslant 0}$ are respectivelly $\mathscr{G}_{\varepsilon,1}$ and $\mathscr{G}_{\varepsilon,-1}$-Appell sequences and $\{a_n\}_{n\geqslant 0}$, $\{b_n\}_{n\geqslant 0}$ are two PS given by*

$$a_n(x) = \sum_{\nu=0}^{n} \lambda_{n,\nu} R_\nu(x), \quad n \geqslant 0 \tag{5.1}$$

$$b_n(x) = \sum_{\nu=0}^{n} \theta_{n,\nu} P_\nu(x), \quad n \geqslant 0\,, \tag{5.2}$$

with

$$\lambda_{n,\nu} = \binom{2n+2}{2\nu} \frac{(-1)^{n-\nu}\, 2^{2n-2\nu+1}}{2\nu+1} \frac{\left(2+\frac{\varepsilon}{2}\right)_{2n+1}}{\left(2+\frac{\varepsilon}{2}\right)_{2\nu}} \mathfrak{G}_{2n-2\nu+2}\,, \quad 0 \leqslant \nu \leqslant n, \quad n \geqslant 0, \tag{5.3}$$

$$\theta_{n,\nu} = \binom{2n+2}{2\nu} \frac{(-1)^{n-\nu}\, 2^{2n-2\nu}}{n+1} \frac{\left(1+\frac{\varepsilon}{2}\right)_{2n+1}}{\left(1+\frac{\varepsilon}{2}\right)_{2\nu}} \mathfrak{G}_{2n-2\nu+2}\,, \quad 0 \leqslant \nu \leqslant n, \quad n \geqslant 0, \tag{5.4}$$

where the symbol $(a)_k = a(a+1)\dots(a+k-1)$, $k \geqslant 0$, denotes the Pochhammer symbol *and $\mathfrak{G}_n$ represent the* unsigned Genocchi numbers.

The *Genocchi numbers* were presumably introduced by Edouard Lucas in [28], but they owe the name to the italian mathematician Angelo Genocchi (1817-1889) [23]. E.T. Bell developed intensive studies on these numbers in the 1920s in [5] and [6]. Such numbers are intimately related to the much more famous *Bernoulli numbers* as it will be presented just after the proof of the precedent result. There are many possibilities for computing the values of the Genocchi numbers (see for example [19], [22] and [38], and also the entry in [36] for further references).

The proof of the latter proposition requires the following known result:

Lemma 5.1. [29] *Given a MPS $\{B_n\}_{n\geqslant 0}$, it is possible to associate two MPS $\{R_n\}_{n\geqslant 0}$ and $\{P_n\}_{n\geqslant 0}$ and two sequences $\{a_n\}_{n\geqslant 0}$ and $\{b_n\}_{n\geqslant 0}$ according to (1.5)-(1.6) and (5.1)-(5.2). If, in addition, $\{B_n\}_{n\geqslant 0}$ is a MOPS fulfilling the second order recurrence relation (1.3)-(1.4), necessarily the coefficients $\lambda_{n,\nu},\theta_{n,\nu}$, $0\leqslant \nu\leqslant n, n\geqslant 0$, satisfy the following system:*

$$\lambda_{n,n} = -\sum_{\nu=1}^{n}\left\{\beta_{2\nu}+\beta_{2\nu+1}\right\},\quad n\geqslant 0, \tag{5.5}$$

$$\theta_{n,n} = -\beta_0-\sum_{\nu=1}^{n}\left\{\beta_{2\nu-1}+\beta_{2\nu}\right\},\quad n\geqslant 0, \tag{5.6}$$

$$\theta_{n+1,\nu}+\gamma_{2n+2}\theta_{n,\nu} = \lambda_{n,\nu-1}+\gamma_{2\nu+1}\lambda_{n,\nu}+\sum_{\mu=\nu}^{n}\lambda_{n,\mu}\,\theta_{\mu,\nu}\,\beta_{2\mu+1} \tag{5.7}$$

$$\lambda_{n+1,\nu}+\gamma_{2n+3}\lambda_{n,\nu} = \theta_{n+1,\nu}+\gamma_{2\nu+2}\theta_{n+1,\nu+1}+\sum_{\mu=\nu}^{n}\theta_{n+1,\mu+1}\,\lambda_{\mu,\nu}\,\beta_{2\mu+2} \tag{5.8}$$

for $0\leqslant \nu\leqslant n$, $n\geqslant 0$, with $\lambda_{n,-1}=0, n\geqslant 0$.

Proof. **(of Proposition 5.1)** Let $\{B_n\}_{n\geqslant 0}$ be a Laguerre sequence of parameter $\frac{\varepsilon}{2}$ with $\varepsilon\neq -2n$, $n\geqslant 1$. The two authors have shown in [27, theorem 6] such sequence to be the unique MOPS being $\mathscr{F}_\varepsilon$-Appell. So, necessarily the second order recurrence relation (1.4) holds and we recall the well known expression for its recurrence coefficients:

$$\beta_n = 2n+1+\frac{\varepsilon}{2}\qquad ;\qquad \gamma_{n+1}=(n+1)\left(n+1+\frac{\varepsilon}{2}\right),\qquad n\geqslant 0. \tag{5.9}$$

Reconsidering the quadratic decomposition of $\{B_n\}_{n\geqslant 0}$ given in (1.5)-(1.6), but this time describing the sequences $\{a_n\}_{n\geqslant 0}$ and $\{b_n\}_{n\geqslant 0}$ by means of the associated MPS $\{P_n\}_{n\geqslant 0}$ and $\{R_n\}_{n\geqslant 0}$, there exist two sets of numbers $\{\lambda_{n,\nu}\}_{0\leqslant\nu\leqslant n}$ and $\{\theta_{n,\nu}\}_{0\leqslant\nu\leqslant n}$ such that (5.1)-(5.2) hold.

By virtue of Theorem 2.1, the MPS $\{R_n\}_{n\geqslant 0}$ and $\{P_n\}_{n\geqslant 0}$ are respectivelly $\mathscr{G}_{\varepsilon,1}$ and $\mathscr{G}_{\varepsilon,-1}$-Appell sequences. Just as it was observed in the proof of Theorem 2.1, the conditions (2.11)-(2.14) hold. In particular from (2.13) and on account of (5.1)-(5.2), we derive

$$2\gamma_{2n+2}\sum_{\nu=0}^{n}\theta_{n,\nu}P_\nu(x) = \sum_{\nu=0}^{n}\lambda_{n,\nu}\left\{(2+\varepsilon)\mathrm{I}+2(8+\varepsilon)xD+8x^2D^2\right\}R_\nu(x),\ n\geqslant 0.$$

Due to (2.12), we have

$$\gamma_{2n+2}\sum_{\nu=0}^{n}\theta_{n,\nu}P_\nu(x)=\sum_{\nu=0}^{n}\lambda_{n,\nu}\gamma_{2\nu+1}P_\nu(x),\ n\geqslant 0,$$

which, because $\{P_n\}_{n\geqslant 0}$ is an independent sequence, provides

$$\theta_{n,\nu}=\frac{\gamma_{2\nu+1}}{\gamma_{2n+2}}\,\lambda_{n,\nu}\ ,\qquad n\geqslant 0,\quad 0\leqslant\nu\leqslant n. \tag{5.10}$$

On the other hand, (5.1)-(5.2) permits to write the relation (2.13) as follows:

$$2\gamma_{2n+1}\sum_{\nu=0}^{n}\lambda_{n-1,\nu}\,R_\nu(x)=\sum_{\nu=0}^{n-1}\theta_{n,\nu+1}\big\{2(4+\varepsilon)D+8xD^2\big\}P_{\nu+1}(x),\ n\geqslant 1.$$

The relation (2.11) allows us to transform the previous into

$$\gamma_{2n+1}\sum_{\nu=0}^{n}\lambda_{n-1,\nu}\,R_\nu(x)=\sum_{\nu=0}^{n-1}\theta_{n,\nu+1}\ \gamma_{2\nu+2}\,R_\nu(x),\ n\geqslant 1,$$

yielding

$$\gamma_{2n+1}\,\lambda_{n-1,\nu}=\gamma_{2\nu+2}\,\theta_{n,\nu+1},\qquad n\geqslant 1\,,\quad 0\leqslant\nu\leqslant n, \tag{5.11}$$

since $\{R_n\}_{n\geqslant 0}$ forms an independent sequence. Combining the relations (5.10) with ν replaced by $\nu+1$ and (5.11) with $n+1$ instead of n, we get

$$\lambda_{n+1,\nu+1}=\frac{\gamma_{2n+4}\ \gamma_{2n+3}}{\gamma_{2\nu+3}\ \gamma_{2\nu+2}}\,\lambda_{n,\nu},\quad 0\leqslant\nu\leqslant n. \tag{5.12}$$

Proceeding by finite induction, it is easy to deduce

$$\lambda_{n+1,\nu+1}=\left\{\prod_{\tau=0}^{2\nu+1}\frac{\gamma_{2n-2\nu+\tau+3}}{\gamma_{\tau+2}}\right\}\lambda_{n-\nu,0},\quad 0\leqslant\nu\leqslant n, \tag{5.13}$$

On account of (5.9), we are able to write

$$\lambda_{n,\nu}=\frac{1}{2\nu+1}\binom{2n+2}{2\nu}\frac{\left(2+\frac{\varepsilon}{2}\right)_{2n+1}}{\left(2+\frac{\varepsilon}{2}\right)_{2\nu}\left(2+\frac{\varepsilon}{2}\right)_{2(n-\nu)+1}}\,\lambda_{n-\nu,0}\ ,\quad 1\leqslant\nu\leqslant n.$$

This last equality is identically verified when we consider the pair (n,ν) to take values on the set $\{(0,0),(1,0)\}$, so it is admissible to write:

$$\lambda_{n,\nu}=\frac{1}{2\nu+1}\binom{2n+2}{2\nu}\frac{\left(2+\frac{\varepsilon}{2}\right)_{2n+1}}{\left(2+\frac{\varepsilon}{2}\right)_{2\nu}\left(2+\frac{\varepsilon}{2}\right)_{2(n-\nu)+1}}\,\lambda_{n-\nu,0}\ ,\quad 0\leqslant\nu\leqslant n. \tag{5.14}$$

Based on Lemma 5.1, we will carry out the determination of the coefficients $\lambda_{n-\nu,0}$. The particular choice $n=0$ in (5.5)-(5.6) and on account of (5.9), respectively, provides

$$\lambda_{0,0}=-2\left(2+\frac{\varepsilon}{2}\right)\quad,\quad\theta_{0,0}=-\left(1+\frac{\varepsilon}{2}\right). \tag{5.15}$$

From (5.10)-(5.11), the two following identities $\gamma_{2n+2}\theta_{n,0} = \gamma_1\lambda_{n,0}$ and $\gamma_{2n+3}\lambda_{n,0} = \gamma_2\theta_{n+1,1}$ hold. Thus, when $\nu = 0$, the relations (5.7)-(5.8) given in Lemma 5.1 become

$$\left\{\begin{aligned} &\theta_{n+1,0} = \sum_{\mu=0}^{n} \lambda_{n,\mu}\,\theta_{\mu,0}\,\beta_{2\mu+1}, \\ &\lambda_{n+1,0} = \theta_{n+1,0} + \sum_{\mu=0}^{n} \theta_{n+1,\mu+1}\;\lambda_{\mu,0}\,\beta_{2\mu+2}, \quad n \geqslant 0. \end{aligned}\right. \tag{5.16}$$

On account of (5.10) and (5.11), we may transform (5.16) into

$$\left\{\begin{aligned} &\frac{1}{\gamma_{2n+4}}\,\lambda_{n+1,0} = \sum_{\mu=0}^{n} \frac{\lambda_{n,\mu}\;\lambda_{\mu,0}}{\gamma_{2\mu+2}}\,\beta_{2\mu+1} \\ &\lambda_{n+1,0} = \frac{\gamma_1}{\gamma_{2n+4}}\,\lambda_{n+1,0} + \gamma_{2n+3}\sum_{\mu=0}^{n} \frac{\lambda_{n,\mu}\;\lambda_{\mu,0}}{\gamma_{2\mu+2}}\,\beta_{2\mu+2}, \quad n \geqslant 0. \end{aligned}\right. \tag{5.17}$$

Since, $\beta_{2\mu+2} = \beta_{2\mu+1} + 2$, for $\mu \geqslant 0$, it follows

$$\sum_{\mu=0}^{n} \frac{\lambda_{n,\mu}\;\lambda_{\mu,0}}{\gamma_{2\mu+2}}\,\beta_{2\mu+2} = 2\sum_{\mu=0}^{n}\left(\frac{\lambda_{n,\mu}\;\lambda_{\mu,0}}{\gamma_{2\mu+2}}\right) + \sum_{\mu=0}^{n}\left(\frac{\lambda_{n,\mu}\;\lambda_{\mu,0}}{\gamma_{2\mu+2}}\,\beta_{2\mu+1}\right), \quad n \geqslant 0.$$

Therefore, from (5.17) we derive

$$\lambda_{n+1,0} = \frac{\gamma_1}{\gamma_{2n+4}}\,\lambda_{n+1,0} + \frac{\gamma_{2n+3}}{\gamma_{2n+4}}\lambda_{n+1,0} + 2\gamma_{2n+3}\sum_{\mu=0}^{n} \frac{\lambda_{n,\mu}\;\lambda_{\mu,0}}{\gamma_{2\mu+2}}, \quad n \geqslant 0, \tag{5.18}$$

which, on account of (5.9), may be expressed like

$$\lambda_{n+1,0} = (n+2)\;\left(2n+3+\tfrac{\varepsilon}{2}\right)\;\left(2n+4+\tfrac{\varepsilon}{2}\right)\sum_{\mu=0}^{n} \frac{\lambda_{n,\mu}\;\lambda_{\mu,0}}{(\mu+1)\left(2\mu+2+\tfrac{\varepsilon}{2}\right)}, \quad n \geqslant 0. \tag{5.19}$$

Now, considering (5.14), the relation (5.19) becomes like

$$\lambda_{n+1,0} = (n+2)\;\left(2+\tfrac{\varepsilon}{2}\right)_{2n+3}\sum_{\mu=0}^{n}\left\{\binom{2n+2}{2\mu}\frac{\lambda_{n-\mu,0}\;\lambda_{\mu,0}}{(2\mu+1)(\mu+1)\left(2+\tfrac{\varepsilon}{2}\right)_{2\mu+1}\left(2+\tfrac{\varepsilon}{2}\right)_{2(n-\mu)+1}}\right\},\ n \geqslant 0. \tag{5.20}$$

Proceeding by finite induction, we infer there is a set of positive integers $\{\chi_n\}_{n\geqslant 0}$, not depending on the parameter ε, fulfilling the equality

$$\lambda_{n,0} = (-1)^{n+1}\,2^{2n+1}\,\chi_n\;\left(2+\tfrac{\varepsilon}{2}\right)_{2n+1}\,, \qquad n \geqslant 0. \tag{5.21}$$

Indeed, on account of (5.15), $\chi_0 = 1$, and, under the assumption, from the relation (5.20) we get

$$\lambda_{n+1,0} = (n+2)\,(-1)^n\,2^{2n+2}\;\left(2+\tfrac{\varepsilon}{2}\right)_{2n+3}\sum_{\mu=0}^{n}\left\{\binom{2n+2}{2\mu}\frac{\chi_{n-\mu}\;\chi_\mu}{(2\mu+1)(\mu+1)}\right\},\ n \geqslant 0.$$

Since the integers χ_n, $n \geqslant 0$, do not depend on ε, they are necessarily related by the equality

$$\chi_{n+1} = \frac{n+2}{2}\sum_{\mu=0}^{n}\binom{2n+2}{2\mu}\frac{\chi_{n-\mu}\;\chi_\mu}{(2\mu+1)(\mu+1)}\,, \qquad n \geqslant 0, \tag{5.22}$$

or, equivalently,

$$\frac{\chi_{n+1}}{(2n+4)!} = \frac{1}{2n+3}\sum_{\mu=0}^{n} \frac{\chi_{n-\mu}}{(2n-2\mu+2)!}\,\frac{\chi_{\mu}}{(2\mu+2)!}\,, \qquad n \geqslant 0. \tag{5.23}$$

Suppose there is an analytic function L defined on an open set of $\mathbb{C}$ such that $L(z) = \sum_{n\geqslant 0} \frac{\chi_n}{(2n+2)!}\, z^n$. Based upon the relation (5.23), $L(z)$ is a solution of the differential equation

$$\left(z\,L(z^2)\right)' = \Lambda_0 + \frac{1}{2}\left(z\,L(z^2)\right)^2 .$$

Therefore, because $\chi_0 = 1$, we trivially conclude that $z\,L\left(z^2\right) = \tan\left(\frac{z}{2}\right)$. Following, per example, [21, 39] and denoting by $\mathfrak{G}_{2n}$ the *unsigned Genocchi numbers*, it is possible to write

$$\tan\left(\frac{z}{2}\right) = \sum_{n\geqslant 0} \mathfrak{G}_{2n+2}\, \frac{z^{2n+1}}{(2n+2)!}$$

whence we have $\chi_n = \mathfrak{G}_{2n+2}$ and (5.21) becomes like

$$\lambda_{n,0} = (-1)^{n+1}\, 2^{2n+1}\, \mathfrak{G}_{2n+2}\, \left(2+\tfrac{\varepsilon}{2}\right)_{2n+1}\,, \qquad n \geqslant 0.$$

Inserting in (5.14), this last equality with $n-\mu$ instead of n, we obtain (5.3) and, on account of (5.10), we get (5.4). $\square$

The unsigned Genocchi numbers are directly related to the *Bernoulli numbers* $\mathfrak{B}_n$ via $\mathfrak{G}_{2n} = 2(1-2^{2n})\mathfrak{B}_{2n}$, where $\mathfrak{B}_n$ are defined by [21, 39]

$$\frac{z}{e^z-1} = 1 - \frac{1}{2}z + \sum_{n\geqslant 1} (-1)^{n+1}\, \mathfrak{B}_{2n}\, \frac{z^{2n}}{(2n)!} \ . \tag{5.24}$$

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REFERENCES

[1] Al-Salam,W. A.. "Characterization theorems for orthogonal polynomials", in: P. Nevai Ed., *Orthogonal polynomials: Theory and Practice*, Vol. C294 (Kluwer, Dordrecht, 1990): 1-23.

[2] Angelescu, A.. "Sur les polynômes orthogonaux en rapport avec d'autres polynômes." *Bull. de la Soc. de Sc. de Cluj Roumanie* **1** (1921-23): 44-59.

[3] Appell, P.. "Sur une classe de polynômes." *Ann. Sci. de l'Ecole Norm. Sup.* (2) **9** (1880): 119-144.

[4] Barrucand, P. and Dickinson, D.. "On cubic transformations of orthogonal polynomials." *Proc. Amer. Math. Soc.* **17** (1966): 810-814.

[5] Bell, E. T.. "An algebra of sequences of functions with an application to the Bernoullian functions." Trans. Amer. Math. Soc. 28 (1926): 129-148.

[6] Bell, E. T.. "Certain invariant sequences of polynomials." *Trans. Amer. Math. Soc.* 31 (1929): 405-421.

[7] Ben Cheikh, Y.. "On obtaining dual sequences via quasi-monomiality." *Georgian Math. J.* **9** (2002): 413-422.

[8] Ben Cheikh, Y.. "Some results on quasi-monomiality." *App. Math. and Comput.* **141** (2003): 63-76.

[9] Ben Cheikh, Y. and Gaied, M.. "Dunkl-Appell d-orthogonal polynomials." *Integral Transforms Spec. Funct.* **18** (2007): 581-597.

[10] Ben Cheikh, Y. and Srivastava, H.M.. "Orthogonality of some polynomial sets via quasi-monomiality." *Appl. Math. and Comp.* **141** (2003): 415-425.

[11] Cesarano, C.. "Monomiality principal and Legendre polynomials." *in: G. Datoli, H.M. Srivastava, C. Cesarano (Eds.)*, Advanced Special Functions and Itegration Methods (Melfi, 2000), Aracne Editrice, Rome 2001, pp. 147-164.

[12] Chihara, T. S.. "On kernel polynomials and related systems." *Boll. Un. Mat. Ital.* **19** (1964): 451-459.

[13] Chihara, T. S., *An Introduction to Orthogonal Polynomials*. New York: Gordon and Breach, 1978.

[14] Chihara, T. S.. "Indeterminate symmetric moment problems." *J. Math. Anal. Appl.* **85** (1982): 331-346.

[15] Chihara, L.M. and Chihara, T. S. . "A class of nonsymmetric orthogonal polynomials." *J. Math. Anal. Appl.* **126** (1987): 275-291.

[16] Dattoli, G.. "Hermite-Bessel and Laguerre-Bessel functions: a by-product of the monomiality principle", *in: D. Cocolicchio, G. Dattoli, H.M. Srivastava (Eds.),* Advanced Special Functions and Applications (Melfi, 1999), Aracne Editrice, Rome, 2000, pp. 83-95.

[17] Dattoli, G., Srivastava, H.M. and Cesarano, C.. "The Laguerre and Legendre polynomials from an operational point of view." *Appl. Math. Comput.* **124** (2001): 117-127.

[18] Dickinson, D., Warsi, S.A.. "On generalized Hermite polynomials and a problem of Carlitz." *Boll. Un. Mat. Ital.* **18** (1963): 256-259.

[19] M. Domaratzki, Combinatorial interpretations of a generalization of the Genocchi numbers, *J. Integer Seq.* **7** (2004), 11 Article 04.3.6 (electronic).

[20] Douak, Khalfa. "The relation of the d-orthogonal polynomials to the Appell polynomials." *J. Comput. Appl. Math.* **70** (1996): 279-295.

[21] Dumond, D. and Foata, D.. "Une propriété de symétrie des nombres de Genocchi." *Bull. Soc. Math. France* **104** (1976): 433-451.

[22] Ehrenborg, R. and Steingrímsson, E.. "Yet another triangle for the Genocchi numbers." *European J. Combin.* **21** (2000): 593-600.

[23] Genocchi, A.. "Intorno all'espressione generale de'numeri Bernoulliani." *Ann. Sci. Mat. Fis.*, **3** (1852): 395-405.

[24] Geronimo, J. S. and Van Assche, W.. "Orthogonal polynomials on several intervals via a polynomial mapping." *Trans. Amer. Math. Soc.* **308** (1988) 559-581.

[25] Ghressi, A. and Khérigi, L. "A new characterization of the generalised Hermite linear form." *Bull. Belg. Math. Soc. Simon Stevin* **15** (2008), 1-7.

[26] He, M.X. and Ricci, P.E.. "Differential equations of Appell polynomials via factorization method." *J. Comp. Appl. Math.* **139** (2002) 231-237.

[27] Loureiro, Ana F. and Maroni, P.. "Quadratic decomposition of Appell sequences." *Expo. Math.* **26** (2008): 177-186.

[28] Lucas, E. *Théorie des nombres. Tome premier: Le calcul des nombres entiers, le calcul des nombres rationnels, la divisibilité arithmétique*. Paris: Éditions Jacques Gabay, 1991. Originally published by Gauthier-Villars et Fils, Impremeurs-Libraires (1891).

[29] Maroni, P.. "Sur la décomposition quadratique d'une suite de polynômes orthogonaux, I." *Rivista di Mat. pura ed. Appl.*, **6** (1990): 19-53.

[30] Maroni, P.. "Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques." C. Brezinski et al. (Eds.), Orthogonal Polynomials and their Applications, in: *IMACS Ann. Comput. Appl. Math.* **9** (1991), 95-130.

[31] Maroni, P.. "Variations around classical orthogonal polynomials. Connected problems." *Journal of Comput. Appl. Math.*, **48** (1993), 133-155.

[32] Maroni, P.. "Sur la décomposition quadratique d'une suite de polynômes orthogonaux, II." *Portugaliae Mathematica*, **50** (1993): 305-329.

[33] Maroni, P.. "Fonctions Eulériennes. Polynômes orthogonaux classiques." *Techniques de l'Ingénieur, traité Généralités (Sciences Fondamentales)*, A **154** (1994): 1-30.

[34] Roman, S.M. and Rota, G.. "The umbral calculus." *Adv. Math.* **27** (1978): 95-188.

[35] Shohat, J.. "The relation of the classical orthogonal polynomials to the polynomials of Appell." *Amer. J. Math.* **58** (1936): 453-464.

[36] N.J.A. Sloane, The on-line encyclopedia of integer sequences. http://www.research.att.com/~njas/sequences/A001469, 2007.

[37] Srivastava, H. M. . "Some characterizations of Appell and q-Appell polynomials." *Ann. Mat. Pura Appl.* (Ser. 4) **130** (1982) 321-329.

[38] Terrill, H.M. and Terrill, E.M.. "Tables of numbers related to the tangent coefficients." *J. Franklin Inst.* **239** (1945): 64-67.

[39] Viennot, Gérard. "Interprétations combinatoires des nombres d'Euler et de Genocchi." Seminar on Number Theory, 1981/1982, Exp. No. 11, 94 pp., Univ. Bordeaux I, Talence, 1982.

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