

HEXAGONAL PROJECTED SYMMETRIES

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ABSTRACT. In the study of pattern formation in symmetric physical systems a 3-dimensional structure in thin domains is often modelled as 2-dimensional one. We are concerned with functions in \mathbb{R}^3 that are invariant under the action of a crystallographic group and the symmetries of their projections into a function defined on a plane. We obtain a list of the crystallographic groups for which the projected functions have a hexagonal lattice of periods. The proof is constructive and the result may be used in the study of observed patterns in thin domains, whose symmetries are not expected in 2-dimensional models, like the black-eye pattern.

1. INTRODUCTION

Regular patterns are usually seen directly in nature and experiments. Convection, reaction-diffusion systems and the Faraday waves experiment comprise three commonly studied pattern-forming systems, see for instance [2], [11], [3].

The pattern itself and its observed state can occur in different dimensions. This happens for instance when an experiment is done in a 3-dimensional medium but the patterns are only observed on a surface, a 2-dimensional object. This is the case for reaction-diffusion systems in the Turing instability regime, [11], which have often been described using a 2-dimensional representation [8]. The interpretation of this 2-dimensional outcome is subject to discussion: the black-eye pattern observed by [8] has been explained both as a mode interaction, [5], and as a suitable projection of a 3-dimensional into a 2-dimensional lattice [4]. In her article, Gomes shows how a 2-dimensional hexagonal pattern can be produced by a specific projection of a Body Centre Cubic (bcc) lattice.

In the study of quasicrystals, projection is a mathematical tool for lowering dimension, [10], [6]. In contrast the work of [9] applies it to fully symmetric patterns.

Pinho and Labouriau [9] study projections in order to understand how these affect symmetry. Their necessary and sufficient conditions for identifying projected symmetries are used extensively in our results. In particular, it follows from their results that the lattice of periods of the projected functions is not obtained by deleting the last coordinate of the original.

Motivated by the explanation of [4] we look for all 3-dimensional lattices that exhibit a hexagonal projected lattice. We illustrate our results using the simple cubic lattice.

2. PROJECTED SYMMETRIES

The study of projections is related to patterns. Patterns are level curves of functions $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$. In our work we suppose that these functions are invariant under the action of a particular subgroup of the Euclidean group: a crystallographic group.

The Euclidean group, $E(n+1)$, is the group of all isometries on \mathbb{R}^{n+1} , also described by the semi-direct sum $E(n+1) \cong \mathbb{R}^{n+1} \rtimes O(n+1)$, with elements given as an ordered pair (v, δ) , in which $v \in \mathbb{R}^{n+1}$ and δ is an element of the orthogonal group $O(n+1)$ of dimension $n+1$.

Let Γ be a subgroup of $E(n+1)$. The homomorphism

$$\begin{aligned} \phi : \quad \Gamma &\rightarrow O(n+1) \\ (v, \delta) &\mapsto \delta \end{aligned}$$

has as image a group \mathbf{J} , called the *point group* of Γ , and its kernel forms the *translation subgroup* of Γ .

We say that the translation subgroup of Γ is a $n+1$ -dimensional lattice, \mathcal{L} , if it is generated over the integers by $n+1$ linearly independent elements $l_1, \dots, l_{n+1} \in \mathbb{R}^{n+1}$, which we write:

$$\mathcal{L} = \langle l_1, \dots, l_{n+1} \rangle_{\mathbb{Z}}$$

A *crystallographic group* is a subgroup of $E(n+1)$, such that its translation subgroup is a $n+1$ -dimensional lattice.

This concept is a generalization, given by [9], of the 3-dimensional crystallographic group as defined by [7], page 55.

To get symmetries of objects in \mathbb{R}^{n+1} , consider the group *action* of $E(n+1)$ on \mathbb{R}^{n+1} given by the function:

$$(1) \quad \begin{aligned} E(n+1) \times \mathbb{R}^{n+1} &\rightarrow \mathbb{R}^{n+1} \\ ((v, \delta), (x, y)) &\mapsto (v, \delta) \cdot (x, y) = v + \delta(x, y) \end{aligned}$$

In [1], the reader can see that the action (1) restricted to a point group of a crystallographic group leaves its translation subgroup L invariant. The largest subgroup of $O(n+1)$ that leaves L invariant forms the *holohedry* of L and its denoted by H_L . The holohedry is always a finite group, see [10], subsection 2.4.2.

Crystallographic groups and symmetries of pattern formation are associated by the description of group of symmetries on space of functions. To see this, observe that (1) induces an action of a crystallographic group Γ on the space of functions $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ by:

$$(\gamma \cdot f)(x, y) = f(\gamma^{-1}(x, y)) \text{ for } \gamma \in \Gamma \text{ and } (x, y) \in \mathbb{R}^{n+1}$$

Thus, we can construct a space \mathcal{X}_Γ of Γ -invariant functions, that is

$$\mathcal{X}_\Gamma = \{f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}; \gamma \cdot f = f, \forall \gamma \in \Gamma\}$$

In particular a Γ -invariant function is L -invariant.

A *L-symmetric pattern* or *L pattern* consists of the level curves of a function $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ with periods in the lattice L .

In [4] the black-eye pattern is obtained as a projection of a function, whose level sets form a bcc pattern in \mathbb{R}^3 . In terms of symmetries, the black-eye is a hexagonal pattern, as we can see in [4], it is the level sets of a bidimensional function with periods in a hexagonal plane lattice, that is, a lattice that admits as its holohedry a group isomorphic to the dihedral group of symmetries of the regular hexagon, D_6 .

For $y_0 > 0$, consider the restriction of $f \in \mathcal{X}_\Gamma$ to the region between the hyperplanes $y = 0$ and $y = y_0$. The projection operator Π_{y_0} integrates this restriction of f along the width y_0 , yielding a new function with domain \mathbb{R}^n .

Definition 1. For $f \in \mathcal{X}_\Gamma$ and $y_0 > 0$, the **projection operator** Π_{y_0} is given by:

$$\Pi_{y_0}(f)(x) = \int_0^{y_0} f(x, y) dy$$

The region between $y = 0$ and $y = y_0$ is called the **projection band** and $\Pi_{y_0}(f) : \mathbb{R}^n \rightarrow \mathbb{R}$ is the **projected function**.

The functions $\Pi_{y_0}(f)$ may be invariant under the action of some elements of the group $E(n) \cong \mathbb{R}^n \rtimes O(n)$. The relation between the symmetries of f and those of $\Pi_{y_0}(f)$ was provided by Pinho and Labouriau [9].

To find the group of symmetries of the projected functions $\Pi_{y_0}(\mathcal{X}_\Gamma)$, the authors consider the following data:

- for $\alpha \in O(n)$, the elements of $O(n+1)$:

$$\sigma := \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}, \alpha_+ := \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \alpha_- := \sigma \alpha_+;$$

- the subgroup $\hat{\Gamma}$ of Γ , whose elements are of the form

$$\left((v, y), \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \right); \alpha \in O(n), \beta = \pm 1, (v, y) \in \mathbb{R}^{n+1};$$

- and the projection $h : \hat{\Gamma} \rightarrow E(n) \cong \mathbb{R}^n \rtimes O(n)$ given by:

$$h \left((v, y), \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \right) = (v, \alpha)$$

The group of symmetries of $\Pi_{y_0}(\mathcal{X}_\Gamma)$ is the image by the projection h of the group Γ_{y_0} defined as:

- If $(0, y_0) \in \mathcal{L}$ then $\Gamma_{y_0} = \widehat{\Gamma}$.
- If $(0, y_0) \notin \mathcal{L}$ then Γ_{y_0} contains those elements of $\widehat{\Gamma}$ that are either of the form $((v, 0), \alpha_+)$ or of the form $((v, y_0), \alpha_-)$.

The group Γ_{y_0} depends of how the elements of Γ are transformed by the projection Π_{y_0} . The criterion that clarifies the connection between the symmetries of \mathcal{X}_Γ and $\Pi_{y_0}(\mathcal{X}_\Gamma)$ is provided by the following result:

Theorem 1 (Theorem 1.2 in [9]). *All functions in $\Pi_{y_0}(\mathcal{X}_\Gamma)$ are invariant under the action of $(v, \alpha) \in E(n)$ if and only if one of the following conditions holds:*

- I $((v, 0), \alpha_+) \in \Gamma$;
- II $((v, y_0), \alpha_-) \in \Gamma$;
- III $(0, y_0) \in \mathcal{L}$ and either $((v, y_1), \alpha_+) \in \Gamma$ or $((v, y_1), \alpha_-) \in \Gamma$, for some $y_1 \in \mathbb{R}$.

3. HEXAGONAL PROJECTED SYMMETRIES

As we saw in the last section, there is a connection between a crystallographic group Γ in dimension $n + 1$ and the group of symmetries of the set of projected functions $\Pi_{y_0}(\mathcal{X}_\Gamma)$. In this work we aim to know which crystallographic groups in dimension 3 can yield hexagonal symmetries after projection. In other words, we want to describe how to obtain hexagonal plane patterns by projection.

Given a crystallographic group Γ , with a $n + 1$ -dimensional lattice \mathbb{L} , whose holohedry is $H_\mathcal{L}$, we denote by $\Pi_{y_0}(\mathcal{L})$ the translation subgroup of the crystallographic group of symmetries of $\Pi_{y_0}(\mathcal{X}_\Gamma)$, whose point group is a subset of the holohedry of $\Pi_{y_0}(\mathcal{L})$. From theorem 1 we obtain

Corollary 1. *Let $\tilde{\Gamma}$ be a crystallographic group with lattice $\tilde{\mathcal{L}} \subset \mathbb{R}^n$. Suppose $\tilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$, and let $H_{\tilde{\mathcal{L}}}$ and $H_\mathcal{L}$ be the holohedries of $\tilde{\mathcal{L}}$ and $\mathcal{L} \subset \mathbb{R}^{n+1}$, respectively. If $\alpha \in H_{\tilde{\mathcal{L}}}$ then either $\alpha_+ \in H_\mathcal{L}$ or $\alpha_- \in H_\mathcal{L}$.*

Proof. To prove our claim, it is sufficient to apply theorem 1 to the case $\tilde{\Gamma} = \tilde{\mathcal{L}} \dot{+} H_{\tilde{\mathcal{L}}}$. Since $\alpha \in H_{\tilde{\mathcal{L}}}$, there exists $v \in \mathbb{R}^n$ such that f is (v, α) -invariant for all $f \in \Pi_{y_0}(\mathcal{X}_\Gamma)$. By theorem 1, one of the three conditions holds. Then, depending on whether (I), (II) or (III) is verified, either (w, α_+) or (w, α_-) is in Γ where, $w \in \{(v, 0), (v, y_0), (v, y_1)\}$. By definition of holohedry, we have either $\alpha_+ \in H_\mathcal{L}$ or $\alpha_- \in H_\mathcal{L}$. \square

Remark 1. We note that there is a non-trivial relation between the lattice $\tilde{\mathcal{L}}$ of periods of the projected functions and that of the original one. In fact, consider a $(n + 1)$ -dimensional lattice \mathbb{L} and $\tilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$. If $v \in \tilde{\mathcal{L}}$ then (v, I_n) is a symmetry of $\Pi_{y_0}(\mathcal{X}_\Gamma)$. Applying theorem 1 with $\alpha = I_n$, one of the following holds for each $v \in \tilde{\mathcal{L}}$:

- I $((v, 0), \alpha_+) = ((v, 0), I_{n+1}) \in \Gamma$, or equivalently $(v, 0) \in \mathcal{L}$;
- II $((v, y_0), \alpha_-) = ((v, y_0), \sigma) \in \Gamma$ then $((v, y_0), \sigma)^2 \in \Gamma$ implying that $(2v, 0) \in \mathcal{L}$;
- III $(0, y_0) \in \mathcal{L}$ and either (v, y_1) or $(2v, 0)$ is in \mathbb{L} , for some $y_1 \in \mathbb{R}$.

While condition I implies that $\mathcal{L} \cap \{(x, 0) \in \mathbb{R}^{n+1}\} \subseteq \tilde{\mathcal{L}}$, the other conditions show that this inclusion is often strict. Furthermore, conditions II and III show that we may have no element of the form (v, y_1) in \mathbb{L} and yet $v \in \tilde{\mathcal{L}}$. So, the lattice of periods of the projected functions is not obtained by deleting the last coordinates of the original.

The next proposition provides sufficient conditions for a multiple of the projected lattice to exist as the sublattice of a suspended lattice \mathbb{L} in \mathbb{R}^{n+1} .

Proposition 1. *Consider a crystallographic group Γ with a lattice $\mathcal{L} \subset \mathbb{R}^{n+1}$ and let $\tilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L}) = \langle \tilde{l}_1, \dots, \tilde{l}_n \rangle_{\mathbb{Z}} \subset \mathbb{R}^n$ be the translation subgroup of $\Pi_{y_0}(\mathcal{X}_\Gamma)$. Suppose for each \tilde{l}_j one of the following conditions holds:*

- i $((\tilde{l}_j, 0), I_{n+1}) \in \Gamma$;
- iii $((\tilde{l}_j, y_1), \sigma) \in \Gamma$, for some $y_1 \in \mathbb{R}$;
- iii $(0, y_0)$, and $(\tilde{l}_j, \frac{p}{q}y_0) \in \mathcal{L}$, for some p, q nonzero integers.

Then there exists $r \in \mathbb{Z}$ such that $\mathcal{L}_r = \{r \cdot (v, 0); v \in \tilde{\mathcal{L}} \text{ and } r \in \mathbb{Z}\}$ is a sublattice of \mathbb{L} .

Condition (iii) is a stronger version of condition III in theorem 1. The other condition follow from theorem 1.

Proof. If one of the conditions $i - iii$ is true, for some $j \in \{1, \dots, n\}$, then, using remark 1, either $(\tilde{l}_j, 0)$ or $(2\tilde{l}_j, 0) \in \mathcal{L}$.

If condition iv holds for some $j \in \{1, \dots, n\}$, then $(0, y_0), (\tilde{l}_j, \frac{p_j}{q_j}y_0) \in \mathcal{L}$, where p_j, q_j are nonzero integers. Since \mathbf{L} is a lattice $(q_j\tilde{l}_j, 0) \in \mathcal{L}$.

Therefore, for each j , we can chose $r_j = 2q_j$ such that $(r_j\tilde{l}_j, 0) \in \mathcal{L}$. Let r be the least common multiple of the $\{r_j\}_{j=1}^n$, then $(r\tilde{l}_j, 0) \in \mathcal{L}$ for all j , finishing the proof. \square

For three-dimensional lattices we have the following stronger result. This gives the full description of a 3-dimensional lattice which can be projected onto a prescribed 2-dimensional one.

Theorem 2. *Let $\mathcal{L} \subset \mathbb{R}^3$ be a lattice such that its projection is a plane lattice $\tilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$ generated by two linearly independent vectors*

$$\tilde{l}_1 \text{ and } \tilde{l}_2 = \rho\tilde{l}_1, \text{ for } \rho \in H_{\tilde{\mathcal{L}}}.$$

There exists $r \in \mathbb{Z} \setminus \{0\}$ such that the three-dimensional lattice \mathbf{L} has a sublattice $\mathcal{L}_r = \langle (r\tilde{l}_1, 0), (r\tilde{l}_2, 0) \rangle_{\mathbb{Z}}$ if and only if for each $v \in \{\tilde{l}_1, \tilde{l}_2\}$ one of the following conditions holds:

- a. $((v, 0), I_3) \in \Gamma$;
- b. $((v, y_1), \sigma) \in \Gamma$, for some $y_1 \in \mathbb{R}$;
- c. $(v, y_1) \in \mathcal{L}$, for some $y_1 \in \mathbb{R}$.

Proof. That the conditions are necessary is immediate from remark 1. Let us prove that they are sufficient.

Suppose that for each $v \in \{\tilde{l}_1, \tilde{l}_2\}$ one of the conditions a to c is verified. We will prove that there exists $r \in \mathbb{Z} \setminus \{0\}$, such that \mathcal{L}_r is a sublattice of \mathbf{L} .

Since $\tilde{l}_2 = \rho\tilde{l}_1$, it is sufficient to show that one of $r(\tilde{l}_1, 0)$ or $r(\tilde{l}_2, 0)$ is in \mathbf{L} . To see this, suppose, without loss of generality, that $r(\tilde{l}_1, 0) \in \mathcal{L}$. Then since $\rho \in H_{\tilde{\mathcal{L}}}$, by corollary 1, either $\rho_+ \in H_{\mathcal{L}}$ or $\rho_- \in H_{\mathcal{L}}$. As $\rho_+(r\tilde{l}_1, 0) = \rho_-(r\tilde{l}_1, 0) = (r\tilde{l}_2, 0)$, it implies that $r(\tilde{l}_2, 0) \in \mathcal{L}$ and therefore, \mathcal{L} has a sublattice \mathcal{L}_r .

If for some $v \in \{\tilde{l}_1, \tilde{l}_2\}$ one of the conditions a or b is true then, by remark 1, $(rv, 0) \in \mathcal{L}$, for $r = 1$ or $r = 2$. Hence, all that remains to prove is the case when \tilde{l}_1 and \tilde{l}_2 only satisfy condition c .

By hypothesis,

$$(2) \quad (\tilde{l}_1, y_1) \text{ and } (\tilde{l}_2, y_2) \text{ are in } \mathcal{L}, \text{ for some } y_1, y_2 \in \mathbb{R}$$

this implies that

$$(3) \quad (\tilde{l}_1 + \tilde{l}_2, y_1 + y_2) \in \mathcal{L}$$

Using (2) and corollary 1

$$\text{either } \rho_+(\tilde{l}_1, y_1) = (\tilde{l}_2, y_1) \in \mathcal{L} \text{ or } \rho_-(\tilde{l}_1, y_1) = (\tilde{l}_2, -y_1) \in \mathcal{L}.$$

If $(\tilde{l}_2, y_1) \in \mathcal{L}$ then

$$(\tilde{l}_2, y_1) + (\tilde{l}_2, y_2) = (2\tilde{l}_2, y_1 + y_2) \in \mathcal{L}$$

thus, using (3)

$$(\tilde{l}_1 + \tilde{l}_2, y_1 + y_2) - (2\tilde{l}_2, y_1 + y_2) = (\tilde{l}_1 - \tilde{l}_2, 0) \in \mathcal{L}$$

Since $\{\tilde{l}_1, \tilde{l}_2\}$ is a basis to $\tilde{\mathcal{L}}$ and $\rho \in H_{\tilde{\mathcal{L}}}$ then

$$\rho(\tilde{l}_1 - \tilde{l}_2) = m\tilde{l}_1 + n\tilde{l}_2, \quad m, n \in \mathbb{Z}$$

where m, n are not both equal to zero. Suppose that $n \neq 0$, then

$$(4) \quad n(\tilde{l}_1 - \tilde{l}_2, 0), (m\tilde{l}_1 + n\tilde{l}_2, 0) \in \mathcal{L}$$

implying that the sum of these last two vectors $((n+m)\tilde{l}_1, 0) \in \mathcal{L}$. Therefore, if $n \neq -m$, \mathcal{L}_r is a sublattice of \mathbf{L} , where $r = m + n \in \mathbb{Z}$. If $n = -m$, we subtract the two expressions in (4) to get $(2n\tilde{l}_1, 0) \in \mathcal{L}$.

If $(\tilde{l}_2, -y_1) \in \mathcal{L}$ then

$$-(\tilde{l}_2, -y_1) + (\tilde{l}_2, y_2) = (0, y_1 + y_2) \in \mathcal{L}$$

thus, using (3)

$$(\tilde{l}_1 + \tilde{l}_2, y_1 + y_2) - (0, y_1 + y_2) = (\tilde{l}_1 + \tilde{l}_2, 0) \in \mathcal{L}$$

An analogous argument applied to $\rho(\tilde{l}_1 + \tilde{l}_2, 0)$ finishes the proof. \square

Let $\mathcal{L} \subset \mathbb{R}^3$ be a lattice and $P \subset \mathbb{R}^3$ be a plane such that $P \cap \mathcal{L} \neq \emptyset$. Given $v \in P \cap \mathcal{L}$ there is a rotation $\gamma \in O(3)$ such that $\gamma(P - v)$ is the plane $XOY = \{(x, y, 0); x, y \in \mathbb{R}\}$. Then we define the y_0 -**projection** of \mathcal{L} into P as the lattice $\gamma^{-1}(\tilde{\mathcal{L}}) \subset E(2)$ where $\tilde{\mathcal{L}}$ is the symmetry group of $\Pi_{y_0}(\mathcal{X}_{\gamma(\mathcal{L}-v)})$.

We say that **the y_0 -projection of \mathcal{L} into the plane P is a hexagonal plane lattice** if and only if the lattice $\tilde{\mathcal{L}}$ admits as its holohedry a group isomorphic to D_6 .

Our main result is the following theorem.

Theorem 3. *Let $\mathcal{L} \subset \mathbb{R}^3$ be a lattice. The y_0 -projection of \mathcal{L} into the plane P is a hexagonal plane lattice if and only if:*

- (1) $P \cap \mathcal{L}$ contains at least two elements;
- (2) there exists $\beta \in H_{\mathcal{L}}$ such that:
 - β has order 6;
 - P is β -invariant.

Proof. Suppose first that $(0, 0, 0) \in P \cap \mathcal{L}$. To show that the conditions (1) and (2) are necessary let us consider, without loss of generality, that $P = XOY$. Therefore, the conditions hold by theorem 2.

To prove that the condition (1) and (2) are sufficient consider $\beta \in H_{\mathcal{L}}$, then either β is a rotation or β is a rotation-inversion. In both cases we can write $\beta = (-I_3)^k \gamma$, $k = 0, 1$, where γ is a rotation and I_3 is the identity. Observe that, since the order of β is six, the order of γ should be either three or six.

By [7], theorem 2.1 and the proof of the crystallographic restriction theorem, in the same reference, there exists only one subspace of dimension 2 invariant by β . Such a plane is the plane perpendicular to the rotation axis of γ . So, let P be this plane.

Since $P \cap \mathcal{L} \neq \{0\}$, let v be a nonzero element of minimum length in $P \cap \mathcal{L}$ and the lattice $\mathcal{L}' = \{v, \beta v\}_{\mathbb{Z}} = \{v, \gamma v\}_{\mathbb{Z}}$. If γ has order 6 then \mathcal{L}' is a hexagonal plane lattice. If γ has order 3, notice that $-v, -\gamma v \in \mathcal{L}'$, because \mathcal{L}' is a lattice, hence \mathcal{L}' is also a hexagonal lattice.

If $(0, 0, 0) \notin P \cap \mathcal{L}$, note that the proof can be reduced to the previous case by a translation. \square

Remark 2. Theorem 3 shows that the only possibility to obtain patterns with hexagonal symmetry, by y_0 -projection, is to project the functions $f \in \mathcal{X}_{\mathcal{L}}$ in a plane invariant by the action of some element $\beta \in H_{\mathcal{L}}$ with order six. After finding one of those planes, in order to obtain projections as in definition 1, we only need to change coordinates. The reader can see an example with the bcc lattice in [4].

Moreover, by theorem 2, if \mathcal{L} is a lattice such that the projection, $\tilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$, of \mathcal{L} into the plane XOY is a hexagonal plane sublattice then \mathcal{L} has a sublattice $\mathcal{L}_r = \{r \cdot (v, 0); v \in \tilde{\mathcal{L}} \text{ and } r \in \mathbb{Z}\}$. This implies that $\forall f \in \mathcal{X}_{\mathcal{L}}$, we have \mathcal{L}_r periodicity.

As a consequence of theorem 3, we are able to list all the Bravais lattices that may be projected to produce a 2-dimensional hexagonal pattern.

Theorem 4. *The Bravais lattices that project to a hexagonal plane lattice are:*

1. Simple cubic lattice;
2. Body-centred cubic lattice;
3. Face-centred cubic lattice;
4. Hexagonal lattice; and
5. Rhombohedral lattice.

Moreover, up to change of coordinates, for the first three lattices the plane of projection must be parallel to one of the planes in Table 1. For the hexagonal and rhombohedral lattice the plane of projection must be parallel to the plane XOY .

Proof. It is immediate from theorem 3 that we can exclude the following Bravais lattices: triclinic, monoclinic, orthorhombic and tetragonal, since the holohedries of these lattices do not have elements of order six.

TABLE 1. Two-dimensional spaces perpendicular to the rotation axis of each one of the rotations γ_i . Here we denote by $\langle v \rangle, v \in \mathbb{R}^3$ the subspace generated by v .

Rotation	Rotation Axis	Perpendicular Plane
γ_1	$\langle(1, 1, -1)\rangle$	$P_1 = \{(x, y, z); z = x + y\}$
γ_2	$\langle(1, -1, -1)\rangle$	$P_2 = \{(x, y, z); z = x - y\}$
γ_3	$\langle(1, -1, 1)\rangle$	$P_3 = \{(x, y, z); z = -x + y\}$
γ_4	$\langle(1, 1, 1)\rangle$	$P_4 = \{(x, y, z); z = -(x + y)\}$

To see if the other Bravais lattices have hexagonal projected symmetries, we need to examine the rotations of order three and six in their holohedries and see if the plane perpendicular to their rotation axes intersects the lattice.

The group of rotational symmetries of the cubic lattice (as well as the body centred cubic lattice and the face centred cubic lattice) is isomorphic to S_4 , the group of permutation of four elements. So, in the holohedry of the cubic lattice we only have rotations of order one, two or three. Consider a systems of generators for a representative for the cubic lattice L, in the standard basis of \mathbb{R}^3 , given by:

$$(1, 0, 0), (0, 1, 0), (0, 0, 1)$$

Then, the matrix representation of the rotations of order 3 in $H_{\mathcal{L}}$ are:

$$\gamma_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix},$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Two-dimensional spaces perpendicular to the rotation axis of each one of these rotations are given in Table 1.

This means that for the fist three lattices in the list, the projection of functions $f \in \mathcal{X}_{\mathcal{L}}$ into a plane have hexagonal symmetries only if the plane is parallel to one of the plane subspaces given in table 1.

Consider now a 3-dimensional hexagonal lattice. Its group of rotational symmetries has order twelve and it has a subgroup of order six consisting of the rotational symmetries of the rhombohedral lattice.

Let the representatives for the hexagonal and rhombohedral lattices, be generated by:

$$(1, 0, 0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right), (0, 0, c) \quad c \neq 0, \pm 1.$$

$$(1, 0, 1), \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 1\right), \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 1\right)$$

respectively. Then, the twelve rotations in the holohedry of the hexagonal lattice are generated by:

$$\rho_z = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \gamma_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The generators of the group of rotational symmetries of the rhombohedral lattice are then ρ_z^2 and γ_x .

We conclude that the only rotations of order 6 in the holohedry of the hexagonal lattice are ρ_z and ρ_z^5 , and of order 3 ρ_z^2 and ρ_z^4 .

Therefore, the y_0 -projection of the hexagonal and rhombohedral lattices is a hexagonal plane sublattice if and only if the y_0 -projection is made into a plane parallel to the plane XOY .

□

4. HEXAGONAL PROJECTED SYMMETRIES OF THE SIMPLE CUBIC LATTICE

We conclude the article with an example to illustrate the hexagonal symmetries obtained by z_0 -projection of functions with periods in the simple cubic lattice, for all $z_0 \in \mathbb{R}$.

Consider a three-dimensional crystallographic group, $\Gamma = \mathcal{L} \dot{+} H_{\mathcal{L}}$, where \mathcal{L} is the simple cubic lattice generated by the vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ over \mathbb{Z} , and $H_{\mathcal{L}}$ its holohedry.

Without loss of generality, let us see the projection of Γ on P_1 (see Table 1).

From theorem 3, the cubic lattice has a hexagonal plane sublattice that intersects P_1 . This sublattice is generated by:

$$(5) \quad (0, 1, 1), (1, 0, 1)$$

To make our calculations easier and to set up the hexagonal symmetries in the standard way consider the new basis $\{(0, 1, 1), (1, 0, 1), (0, 0, 1)\}$ for the lattice \mathcal{L} . Now multiply \mathcal{L} by the scalar $\frac{1}{\sqrt{2}}$ in order to normalise the vectors of (5). With these changes the crystallographic group Γ has the new translational subgroup generated by the vectors:

$$v_1 = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), \quad v_2 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}), \quad v_3 = (0, 0, \frac{1}{\sqrt{2}})$$

Projection of Γ on P_1 , as in definition 1, can be done after a change of coordinates that transforms P_1 into XOY . Consider that change given by the orthonormal matrix

$$A = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$$

Then, in the new system of coordinates $X = Ax$, we obtain the base for the simple cubic lattice given by:

$$(6) \quad l_1 = (1, 0, 0), l_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0), l_3 = (\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{-\sqrt{6}}{6})$$

Observe that we changed the position of \mathcal{L} as prescribed by theorem 3.

We proceed to describe the symmetries of the space $\Pi_{z_0}(\mathcal{X}_{\Gamma})$, for each $z_0 \in \mathbb{R}$. For this, we need to write up the subgroups $\hat{\Gamma}$ and Γ_{z_0} of Γ . Denote by $\Sigma_{z_0} = \mathcal{L}_{z_0} \dot{+} J_{z_0}$ the subgroup of $E(2)$ of all symmetries of $\Pi_{z_0}(\mathcal{X}_{\Gamma})$.

It is straightforward to see that the elements of Γ with orthogonal part α_{\pm} are in the group

$$\hat{\Gamma} = \{((v, z), \rho); (v, z) \in \mathcal{L}, \quad \rho \in \hat{J}\}$$

where \hat{J} is the group generated by

$$\gamma = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \kappa = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the group Γ_{z_0} has a subgroup $H = \bar{\mathcal{L}} \dot{+} \bar{J}$, for all $z_0 \in \mathbb{R}$, where $\bar{\mathcal{L}}$ is the translation subgroup $\bar{\mathcal{L}} = \langle l_1, l_2 \rangle_{\mathbb{Z}}$ and \bar{J} is the subgroup generated by $((0, 0, 0), \kappa)$ and $((0, 0, 0), -\gamma)$. Using statement I of theorem 1, for all $z_0 \in \mathbb{R}$ all the functions $f \in \Pi_{z_0}(\mathcal{X}_{\Gamma})$ are $(1, 0)$, and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ periodic and invariant for the action of

$$\kappa' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad -\gamma' = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

In Table 2 we list the group Γ_{z_0} , for each $z_0 \in \mathbb{R}$, and describe the respective projected symmetries.

We assume that all the functions $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ in $\mathcal{X}_{\mathcal{L}}$ admit a unique formal Fourier expansion in terms of the waves

$$w_k(x, y, z) = \exp(2\pi i \langle k, (x, y, z) \rangle)$$

TABLE 2. Projection of $\Gamma = \mathcal{L} \dot{+} H_{\mathcal{L}}$, for each $z_0 \in \mathbb{R}$.

$z_0 \in \mathbb{R}$	Γ_{z_0}	$\Sigma_{z_0} = \mathcal{L}_{z_0} \dot{+} J_{z_0}$
$z_0 = \frac{3n}{\sqrt{6}}, n \in \mathbb{Z} \setminus \{0\}$		
then $(0, 0, \frac{n}{\sqrt{6}}) \in \mathcal{L}$	$\Gamma_{z_0} = \hat{\Gamma}$	$\mathcal{L}_{z_0} = \langle (\frac{1}{2}, \frac{\sqrt{3}}{6}), (\frac{1}{2}, \frac{-\sqrt{3}}{6}) \rangle_{\mathbb{Z}}$ $J_{z_0} = D_6 = \langle \gamma', \kappa' \rangle$
$z_0 = \frac{3n-1}{\sqrt{6}}, n \in \mathbb{Z} \setminus \{0\}$		
then $(\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{3n-1}{\sqrt{6}}) \in \mathcal{L}$	Γ_{z_0} contains $((\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{3n-1}{\sqrt{6}}), \gamma)$	$\mathcal{L}_{z_0} = \langle (1, 0), (\frac{1}{2}, \frac{\sqrt{3}}{2}) \rangle_{\mathbb{Z}}$ $J_{z_0} = \langle ((\frac{1}{2}, \frac{\sqrt{3}}{6}), \gamma'), \kappa' \rangle$
$z_0 = \frac{3n+1}{\sqrt{6}}, n \in \mathbb{Z} \setminus \{0\}$		
then $(\frac{1}{2}, \frac{-\sqrt{3}}{6}, \frac{3n+1}{\sqrt{6}}) \in \mathcal{L}$	Γ_{z_0} contains $((\frac{1}{2}, \frac{-\sqrt{3}}{6}, \frac{3n+1}{\sqrt{6}}), \gamma)$	$\mathcal{L}_{z_0} = \langle (1, 0), (\frac{1}{2}, \frac{\sqrt{3}}{2}) \rangle_{\mathbb{Z}}$ $J_{z_0} = \langle ((\frac{1}{2}, \frac{-\sqrt{3}}{6}), \gamma'), \kappa' \rangle$
For z_0 different		
of the cases before	$\Gamma_{z_0} = H$	$\mathcal{L}_{z_0} = \langle (1, 0), (\frac{1}{2}, \frac{\sqrt{3}}{2}) \rangle_{\mathbb{Z}}$ $J_{z_0} = \langle -\gamma', \kappa' \rangle$

where k is a *wave vector* in the dual lattice, $\mathcal{L}^* = \{k \in \mathbb{R}^3; \langle k, l_i \rangle \in \mathbb{Z}, i = 1, 2, 3\}$, of \mathbb{L} given in (6), with *wave number* $|k|$, $(x, y, z) \in \mathbb{R}^3$ and $\langle \cdot, \cdot \rangle$ is the usual inner product in \mathbb{R}^3 . Thus,

$$f(x, y, z) = \sum_{k \in \mathcal{L}^*} z_k w_k(x, y, z)$$

where z_k is the Fourier coefficient, for each $k \in \mathcal{L}^*$, and with the restriction $z_{-k} = \overline{z_k}$.

Therefore, we can write

$$\mathcal{X}_{\mathcal{L}} = \bigoplus_{k \in \tilde{\mathcal{L}}} V_k$$

for

$$\tilde{\mathcal{L}} = \{k = (k_1, k_2) \in \mathcal{L}; k_1 > 0 \text{ or } k_1 = 0 \text{ and } k_2 > 0\}$$

and

$$V_k = \{Re(zw_k(x, y, z)); z \in \mathbb{C}\} \cong \mathbb{C}$$

Note that \mathcal{X}_{Γ} is a subspace of $\mathcal{X}_{\mathcal{L}}$.

A straightforward calculation shows that the function

$$(7) \quad u(x, y, z) = \sum_{|k|=\sqrt{2}} \exp(2\pi i k \cdot (x, y, z))$$

is Γ -invariant.

The contour plot of the projections of u are shown in the next figures, with respective symmetries given in Table 2.

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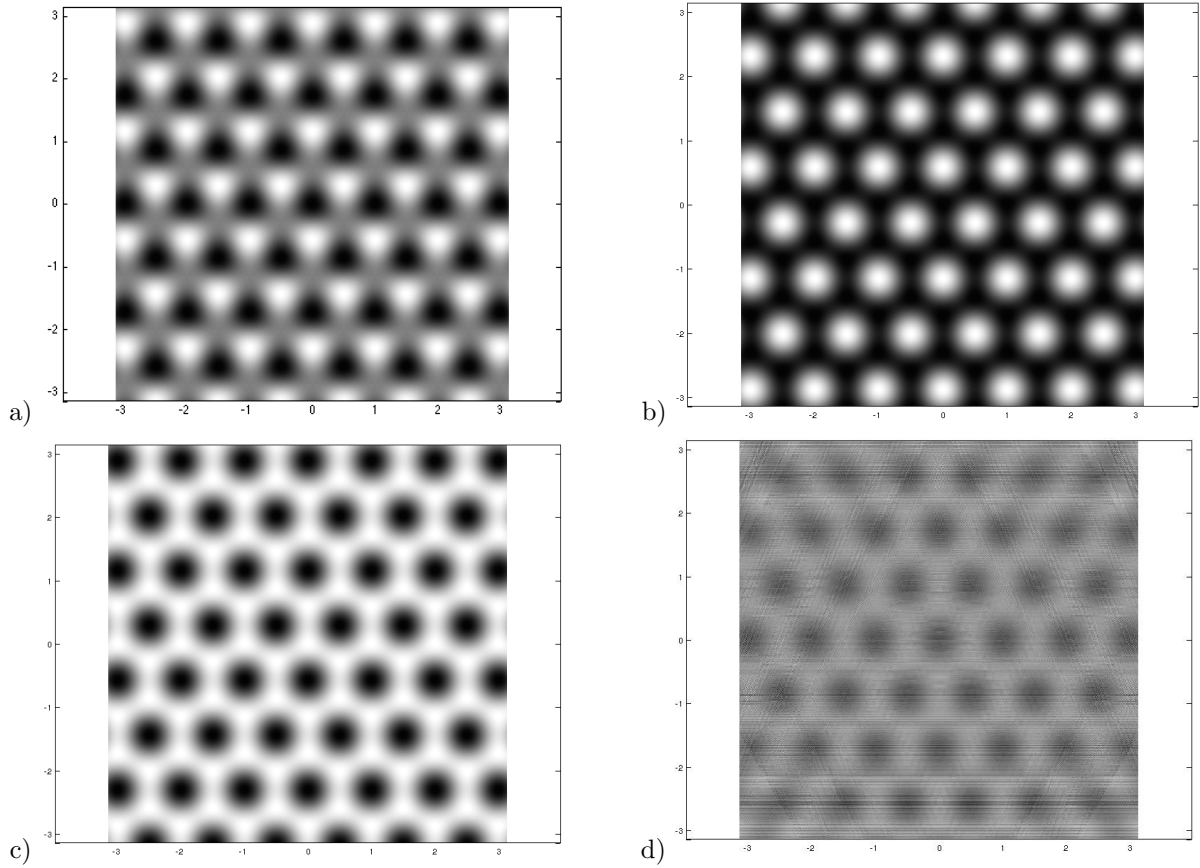


FIGURE 1. The integral over the depth z_0 . (a) $z_0 = \frac{1}{2\sqrt{6}}$. (b) $z_0 = \frac{1}{\sqrt{6}}$. (c) $z_0 = \frac{2}{\sqrt{6}}$. (d) $z_0 = \frac{3}{\sqrt{6}}$

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