REPRESENTATION THEORY OF FINITE SEMIGROUPS, SEMIGROUP RADICALS AND FORMAL LANGUAGE THEORY

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ABSTRACT. In this paper we characterize the congruence associated to the direct sum of all irreducible representations of a finite semigroup over an arbitrary field, generalizing results of Rhodes for the field of complex numbers. Applications are given to obtain many new results, as well as easier proofs of several results in the literature, involving: triangularizability of finite semigroups; which semigroups have (split) basic semigroup algebras, two-sided semidirect product decompositions of finite monoids; unambiguous products of rational languages; products of rational languages with counter; and Černý's conjecture for an important class of automata.

CONTENTS

1. Introduction

For over 100 years, the theory of linear representations has played a fundamental role in studying finite groups, finite dimensional algebras and Lie algebras as well as other parts of algebra. By way of contrast, the theory of semigroup representations, which was intensively developed during the 50s and 60s in classic work such as Clifford [15], Munn [29, 30] and Ponizovsky (see [16, Chapter 5] for an account of this work, as well as [26, 56] for nicer treatments restricting to the case of finite semigroups), has found almost no applications in the theory of finite semigroups. It was pointed out by McAlister in his survey of 1971 [28] that the only paper applying representation theoretic results to finite semigroups was the paper [51] of Rhodes. This paper determined the congruence on a finite semigroup S associated to the direct sum of the irreducible representations of S over the field of complex numbers. Rhodes applied this result to calculate the Krohn-Rhodes complexity [24, 25] of completely regular semigroups. Around the time of McAlister's survey, there also appeared a paper of Zalcstein [72] trying to apply representation theory to finite semigroup theory.

For many years, representation theory of finite semigroups remained dormant until Putcha, in a series of papers (cf. [43, 44, 45, 46] and others), revived the theme. Putcha was primarily interested in relating semigroup theory with modern areas in representation theory such as quasi-hereditary algebras, weights for representations of finite groups of Lie type and with calculating quivers of algebras of finite semigroups. However, his research was not aimed at applying the representation theory of semigroups to the study of finite semigroups in their own right. While to some extent we continue in the vein of relating finite semigroup theory to the rest of modern algebra — for instance we determine over an arbitrary field K which finite semigroups have basic or split basic semigroup algebras over K — we very much focus on using representation theory precisely for the purpose of answering questions from finite semigroup theory. We are particularly interested in varieties of finite semigroups and their connections with formal language theory and other aspects of theoretical computer science, as exposited in the two treatises by Eilenberg [18]. Nonetheless we expect that the first four sections of this paper should be of interest to readers in Algebraic Combinatorics, Representation Theory and Finite Semigroup Theory.

Let us briefly survey the contents of the paper. Following the preliminaries, we define the Rhodes radical of a finite semigroup S with respect to a field K to be the congruence on S induced by the Jacobson radical of its semigroup algebra KS over K. Using classical Wedderburn theory, we give a more conceptual proof of Rhodes's characterization [51] of this radical in characteristic zero and extend it to characteristic p. Further, the radical is shown to be intimately related with the Mal'cev product, which is an integral part of the varietal theory of finite semigroups. We also give an alternative semigroup representation theoretic proof of the description

of the Rhodes radical, along the original lines of Rhodes [51], that allows a more precise and usable characterization of the radical.

Part of our aim is to render things in a form understandable to both specialists and non-specialists. Recent work of Bidigaire *et al.* [10] and Brown [12, 13], for instance, spends quite some time in redeveloping basic aspects of the representation theory of idempotent semigroups (known as bands) that were already in the literature [28, 56, 31, 26], but perhaps not in a form accessible to most mathematicians. Our results handle the general case in a form that both semigroup theorists as well as workers in finite dimensional algebras, group representation theory and other related fields should find useful.

We then proceed to applications. The first application gives abstract, algebraic characterizations of finite semigroups that are triangularizable over a field K; in the language of the theory of finite dimensional algebras, we characterize those finite semigroups whose semigroup algebras are split basic K-algebras. The case of a finite field was handled by three of the authors in [4] without using representation theory, leading to a much more complicated proof. Here we handle all fields K in a uniform manner by simply characterizing those semigroups all of whose irreducible representations over K have degree one. It turns out that the collection of finite semigroups triangularizable over a given field K is a variety of finite semigroups (that of course depends on K) and that those "triangularizable" varieties are in fact some of the most commonly studied varieties in finite semigroup theory.

Our next application is to obtain simpler proofs of some bilateral semidirect product decomposition results of Rhodes, Tilson and Weil [54, 55] using representation theory. The original proofs rely on a case-by-case analysis of Rhodes's classification of maximal proper surjective morphisms.

After purely algebraic applications, we switch to those dealing with important objects of theoretical computer science such as formal languages and finite automata. We use modular representation theory to give simpler proofs of results of Péladeau and Weil [36, 71] on marked products with modular counter and characteristic zero representation theory to obtain simpler proofs of results of Pin, Straubing and Therien [41] on unambiguous marked products. Our final application uses representation theory to confirm the longstanding Černý conjecture on synchronizing automata in the special case that the transition monoid belongs to the much-studied variety **DS**.

Further applications of our results have been obtained by the third author [62, 63]; in particular the results of Bidigaire *et al.* [10] and Brown [12, 13] on random walks on hyperplane arrangements and on bands have been extended to the varieties **DA** and $\mathbf{DO} \cap \overline{\mathbf{Ab}}$, which is as far as these results can be extended.

We have tried to make the representation part (Section 3) of this paper accessible both to readers from semigroup theory and readers familiar with representation theory from other contexts. Having the latter category of readers in mind, in the next section we give a concise overview of standard

notions and terminology of semigroup theory needed for the representation part. The application part (Sections 4–7) requires further background in semigroup theory, formal languages and automata.

2. Preliminaries

Good sources for semigroup theory, in particular finite semigroup theory, are [16, 25, 18, 40, 1]. Here we introduce some standard notions and terminology. The reader is welcome to skip this section, referring back only as needed.

A *congruence* on a semigroup S is an equivalence relation $\equiv$ such that
$$s \equiv s' \implies ts \equiv ts', \ st \equiv s't$$
for all $s, s', t \in S$. Left and right congruences are defined analogously. If $\varphi : S \to T$ is a morphism, then the congruence *associated to* φ is defined by $s \equiv_\varphi t$ if and only if $s\varphi = t\varphi$.

An *idempotent* e of a semigroup is an element such that $e = e^2$. It is well known that in a finite semigroup S some power of each element is an idempotent; namely, for all $s \in S$, one verifies that $s^{|S|!}$ is idempotent. The set of idempotents of a semigroup S is denoted $E(S)$. It is a partially ordered set via the order
$$e \leq f \iff ef = fe = e. \tag{2.1}$$
A *semilattice* E is an idempotent commutative semigroup. In this case, the order (2.1) has all finite meets, the meet being given by the product in E.

A *right ideal* of a semigroup S is a subset R such that $RS \subseteq R$. Left ideals and (two-sided) ideals are defined similarly. If $s \in S$, we use $R(s), L(s), J(s)$ for the respective right, left and two-sided principal ideals generated by s. This leads to the definitions of Green's relations [20, 25, 16], which play an essential role in semigroup theory. We define an equivalence relation $\mathcal{R}$ on S by setting, for $s, t \in S$, $s \mathrel{\mathcal{R}} t$ if and only if $R(s) = R(t)$; in this case one writes R_s for the $\mathcal{R}$-class of s. One similarly defines the equivalence relations $\mathcal{L}$ and $\mathcal{J}$, whose classes of s are denoted L_s and J_s respectively. Define $s \mathrel{\mathcal{H}} t$ if $s \mathrel{\mathcal{R}} t$ and $s \mathrel{\mathcal{L}} t$; the $\mathcal{H}$-class of s is denoted H_s. There are also associated preorders. For instance, $s \leq_\mathcal{R} t$ if and only if $R(s) \subseteq R(t)$. It is easy to see that $\mathcal{R}$ is a left congruence and $\mathcal{L}$ is a right congruence.

In a finite semigroup (or even in an algebraic semigroup [42, 48]), the following *stability* relations hold [25]:
$$s \mathrel{\mathcal{J}} st \iff s \mathrel{\mathcal{R}} st, \quad t \mathrel{\mathcal{J}} st \iff t \mathrel{\mathcal{L}} st.$$
From these relations, it follows that in a finite semigroup, if $s \mathrel{\mathcal{J}} t$ then there exists $u \in S$ such that $s \mathrel{\mathcal{R}} u \mathrel{\mathcal{L}} t$ and $v \in S$ such that $s \mathrel{\mathcal{L}} v \mathrel{\mathcal{R}} t$. In the case that J_s is a subsemigroup one can take u and v to be idempotents as every $\mathcal{H}$-class within J_s contains an idempotent.

An element $s \in S$ is called *(von Neumann) regular* if $s \in sSs$. In a finite semigroup, s is regular if and only if J_s contains an idempotent if and only if R_s contains an idempotent if and only if L_s contains an idempotent.

A $\mathcal{J}$-class (respectively, $\mathcal{R}$-class, $\mathcal{L}$-class) is called *regular* if it contains an idempotent. If e is an idempotent, then H_e is a group, called the *maximal subgroup* at e. It is the group of units of the *local monoid* eSe and so it is the largest subgroup of S with identity e. By a subgroup of a semigroup S, we mean simply a subsemigroup that is a group; it need not have the same identity as S in the case that S is a monoid. The local monoid eSe is the largest subsemigroup of S with identity e. For example, if S is the monoid of $n \times n$ matrices over K and e is an idempotent of rank r, then eSe is isomorphic to the monoid of $r \times r$ matrices over K and the maximal subgroup H_e is isomorphic to the general linear group of degree r over K.

If S is a semigroup, we set S^1 to be S with an adjoined identity if S is not a monoid and S otherwise. We shall frequently use the following fact: suppose that $e, f \in E(S)$; then

$$e \mathcal{L} f \Leftrightarrow ef = e, \ fe = f$$
$$e \mathcal{R} f \Leftrightarrow ef = f, \ fe = e \tag{2.2}$$

For instance, if $e \mathcal{L} f$, then $e = xf$ for some $x \in S^1$. Hence

$$ef = xff = xf = e;$$

the other equalities are handled similarly.

A semigroup is called *simple* if it has no proper (two-sided) ideals. A semigroup S with 0 is called 0-*simple* if $S^2 = \{st \mid s, t \in S\} \neq 0$ and the only ideals of S are $\{0\}$ and S. Simple semigroups and 0-simple semigroups were classified up to isomorphism by Rees and Suschewitsch [16]. We shall need in the sequel only the following properties that are the content of [68, XI. Propositions 1.2–1.4].

Proposition 2.1. *Let S be a finite simple semigroup. Then every element of S belongs to a subgroup of S. For any idempotents $e, f \in S$, there exist $x \in eSf$ and $y \in fSe$ such that $e = xy$ and $f = yx$. Moreover, eSe is the group H_e, fSf is the group H_f and the map $H_e \to H_f$ given by $h \mapsto yhx$ is a group isomorphism.*

This proposition says that the idempotents of a simple semigroup are *conjugate* and that the local monoids are the maximal subgroups; moreover, they are all isomorphic to the same group.

An ideal of a semigroup S is called *minimal* if it contains no other ideal of S; the minimal ideal of a finite semigroup is a simple semigroup and is a regular $\mathcal{J}$-class [16]. An ideal of a semigroup S with 0 is called 0-*minimal* if the only ideal of S properly contained in it is $\{0\}$; a 0-minimal ideal I of a finite semigroup is either 0-simple (and then $I \setminus \{0\}$ is a regular $\mathcal{J}$-class) or it is *null*, meaning $I^2 = 0$.

The following definition, introduced by Eilenberg and Schützenberger [19, 18], is crucial in finite semigroup theory. A class $\mathbf{V}$ of semigroups closed under formation of finite direct products, subsemigroups and homomorphic images is called a *variety of finite semigroups* (or sometimes a *pseudovariety* of semigroups). Varieties of finite monoids and groups are defined

analogously. We remark that in universal algebra, the term variety is used differently, but since we shall not consider such varieties, no confusion should arise.

Some varieties that shall play an important role in this paper are the trivial variety $\mathbf{I}$ (containing only the trivial semigroup) and the variety of finite p-groups (p a prime) $\mathbf{G}_p$. The variety of finite Abelian groups is denoted $\mathbf{Ab}$. The variety of finite semilattices is denoted $\mathbf{Sl}$.

The following notion shall be used throughout this paper. If $\mathbf{V}$ is a variety of finite semigroups, a morphism $\varphi : S \to T$ is called a $\mathbf{V}$-*morphism* if, for each idempotent $e \in T$, its preimage $e\varphi^{-1}$ (which is then a subsemigroup in S) belongs to $\mathbf{V}$. The congruence associated to a $\mathbf{V}$-morphism is called a $\mathbf{V}$-*congruence*. In other words, a congruence on S is a $\mathbf{V}$-congruence if and only if all its congruence classes that are subsemigroups belong to $\mathbf{V}$. For instance, if $\varphi : G \to H$ is a group homomorphism, then φ is a $\mathbf{V}$-morphism if and only if $\ker \varphi \in \mathbf{V}$.

Finally, we recall two fundamental varietal constructions. If $\mathbf{V}$ is a variety of finite monoids, then $\mathbf{LV}$ denotes the class of all finite semigroups S such that, for each idempotent $e \in S$, the local monoid eSe belongs to $\mathbf{V}$. It is easy to see that $\mathbf{LV}$ is a variety of finite semigroups. If $\mathbf{V}$ and $\mathbf{W}$ are varieties of finite semigroups, their *Mal'cev product* $\mathbf{V} \,\text{ⓜ}\, \mathbf{W}$ consists of all finite semigroups S such that there is a finite semigroup T mapping homomorphically onto S such that T admits a $\mathbf{V}$-morphism to a semigroup in $\mathbf{W}$. Again, it is well known and easy to verify that $\mathbf{V} \,\text{ⓜ}\, \mathbf{W}$ is a variety of finite semigroups.

The most important example is when $\mathbf{V}$ is a variety of finite groups. A semigroup is a *local group* if eSe is a group for each idempotent e. For instance, by Proposition 2.1 simple semigroups are local groups. If one considers all $n \times n$ upper triangular matrices over a field K that have a fixed zero/non-zero pattern on the diagonal, we will see in Section 4 that one obtains a local group. Thus the monoid of all upper triangular matrices is a disjoint union of local groups.

Our goal is to state the well known version of Proposition 2.1 for local groups. Unfortunately, even though this is folklore in semigroup theory, we could not pinpoint an exact reference. First we need the following well-known finiteness result, which is a "Pumping Lemma" for finite semigroups [1, Proposition 5.4.1]. Set S^n be the ideal of S consisting of all elements of S that can be expressed as a product of n elements of S.

Lemma 2.2 (Pumping Lemma)**.** *Let S be a semigroup with n elements. Then $S^n = SE(S)S$.*

Now we can state the main property of local groups.

Proposition 2.3. *Let S be a finite semigroup. Then the following are equivalent:*

(1) *S is a local group;*

(2) S^n is a simple semigroup for some $n > 0$ (i.e. S is a nilpotent ideal extension of a simple semigroup);

(3) S^n is the minimal ideal of S for some $n > 0$ (i.e. S is a nilpotent extension of its minimal ideal);

(4) S does not contain a semigroup isomorphic to the two-element semi-lattice $\{0, 1\}$ with multiplication.

Furthermore for any idempotents $e, f \in S$, there exist $x \in eSf$ and $y \in fSe$ such that $e = xy$ and $f = yx$. Moreover, the groups eSe and fSf are isomorphic via the map $eSe \to fSf$ given by $h \mapsto yhx$.

Proof. Suppose first that (1) holds. Then (4) must hold since if $\{e, f\} \subseteq S$ is isomorphic to $\{0, 1\}$ with e as the identity, then $efe = f$ and so $e, f \in eSe$, showing that eSe is not a group.

For (4) implies (3), let I be the minimal ideal of S. We show that $E(S) \subseteq I$. Suppose $e \in E(S) \setminus I$. Let $s \in I$ be any element and set $n = |S|$. Then $f = (ese)^{n!}$ is an idempotent belonging to I (so in particular $f \neq e$) and $ef = fe = f$. Thus $\{e, f\}$ is a subsemigroup isomorphic to $\{0, 1\}$. This contradiction shows that $E(S) \subseteq I$. Now by the Pumping Lemma, if $n = |S|$, then $S^n = SE(S)S \subseteq I$. However, $SE(S)S$ is clearly an ideal, so $I \subseteq SE(S)S$. Hence $S^n = I$.

We noted that the minimal ideal of any finite semigroup is a simple semigroup so the implication (3) implies (2) is trivial. For (2) implies (1), suppose that $T = S^n$ is a simple semigroup. Notice that $E(S) \subseteq T$ and that $eSe \subseteq T$ for any idempotent e since $e \in S^n$ for all n. Thus

$$eSe = e(eSe)e \subseteq eTe \subseteq eSe,$$

so $eSe = eTe$. But eTe is a group by Proposition 2.1. This proves (1).

The proof that (2) implies (1) shows that in a local group S with minimal ideal I, one has that I contains all the idempotents of S and $eSe = eIe$ for each idempotent e of S. Proposition 2.1 then implies the final statement of the proposition. $\square$

3. The Rhodes Radical

3.1. **Background and Motivation.** Let K be a field and S a finite semigroup. The semigroup algebra of S over K is denoted KS. Recall that this is the K-vector space with basis S and the multiplication extending the multiplication in S. If A is a finite dimensional K-algebra (for instance KS), then it has a largest nilpotent ideal $\mathrm{Rad}(A)$, called its (Jacobson) *radical*. Consider the composite mapping

$$S \to KS \to KS/\mathrm{Rad}(KS);$$

this is a morphism of semigroups where the latter two are viewed with respect to their multiplicative structure. We define $\mathrm{Rad}_K(S)$, called the *Rhodes radical* of S, to be the associated congruence on S. Let us briefly discuss the role of the Rhodes radical for the representation theory of finite semigroups.

Let V be a K-vector space of finite dimension n. Then $\mathrm{End}_K(V)$ denotes the monoid of K-endomorphisms of V. We shall identify $\mathrm{End}_K(V)$ with the monoid $M_n(K)$ of $n \times n$ matrices over K whenever it is convenient. A *representation* of a finite semigroup S over K of *degree* n is a morphism $\rho : S \to M_n(K)$ or, equivalently, a morphism $\rho : S \to \mathrm{End}_K(V)$ where V is an n-dimensional vector space over K. It is easy to see that via ρ we can view V as a finite dimensional (right) KS-module and that all finite dimensional (right) KS-modules arise in this way. The *regular representation* of S is the faithful representation on the K-vector space with basis S^1 and where the action is induced by right multiplication of S on the basis elements.

A subsemigroup S of $\mathrm{End}_K(V)$ is called *irreducible* if there is no proper, non-zero subspace of V that is invariant under S. A representation $\rho : S \to \mathrm{End}_K(V)$ of a semigroup S is called *irreducible* if $S\rho$ is an irreducible subsemigroup of $\mathrm{End}_K(V)$. A representation is irreducible if and only if the associated KS-module is simple.

It is well known that the radical $\mathrm{Rad}(A)$ of a finite dimensional K-algebra A is the intersection of the kernels of the irreducible representations of A. Since every irreducible representation $\rho : S \to \mathrm{End}_K(V)$ of a finite semigroup S uniquely extends to an irreducible representation of the semigroup algebra KS, and vice versa, every irreducible representation of KS restricts to an irreducible representation of S, we conclude that the Rhodes radical $\mathrm{Rad}_K(S)$ of S is precisely the intersection of the congruences of the form $\equiv_\rho$ where $\rho : S \to \mathrm{End}_K(V)$ is an irreducible representation of S. Thus, the Rhodes radical in the finite semigroup setting naturally corresponds to the (Jacobson) radical in the setting of finite dimensional algebras. Moreover, in spite of the fact that the irreducible representations of S and KS are basically the same objects, we will see that working with the Rhodes radical $\mathrm{Rad}_K(S)$ has some advantages over considering the radical $\mathrm{Rad}(KS)$ of the corresponding semigroup algebra. The point is that, as we are going to show, the Rhodes radical $\mathrm{Rad}_K(S)$ can be explicitly calculated in terms that are internal with respect to the semigroup S while determining the radical $\mathrm{Rad}(KS)$ requires studying invertibility of certain matrices in the matrix ring over the algebras KH for all maximal subgroups H of S (cf. [22]) which is, generally speaking, a highly non-trivial task.

Rhodes [51] calculated $\mathrm{Rad}_K(S)$ for K the field of complex numbers, but his arguments work for any field of characteristic 0. Extensions of these results in a more general context have been obtained by Okniński [32], but without the varietal viewpoint [18] that we use to tie the results to language theory. Here we furnish two descriptions of the Rhodes radical. The first proceeds via an argument using the theory of finite dimensional algebras. Afterwards we give a description along the lines of Rhodes [51], using semigroup representation theory developed by Clifford, Munn and Ponizovsky [16, 56, 26] and the semi-local theory of Krohn, Rhodes and Tilson [25]. Both proofs are informative, the first being technically easier, the second giving a more concrete description of the congruence.

Given a field K, let

$$\mathbf{G}_K = \begin{cases} \mathbf{I} & char\ K = 0 \\ \mathbf{G}_p & char\ K = p. \end{cases}$$

It is well known that this is the variety of finite groups that are "unipotent" over K (i. e. a finite group G has a faithful unitriangular representation over K if and only if $G \in \mathbf{G}_K$). This notation will allow us to phrase our results in a characteristic-free manner.

We shall also often encounter the variety $\mathbf{LG}_K$. By Proposition 2.3 a finite semigroup S belongs to this variety if and only if there is an integer n such that S^n is a simple semigroup U, all of whose maximal subgroups are in $\mathbf{G}_K$. Equivalently, $S \in \mathbf{LG}_K$ if and only if it does not contain a copy of the two element semilattice $\{e, f \mid ef = fe = e^2 = e,\ f^2 = f\}$ and the maximal subgroups of S belong to $\mathbf{G}_K$.

3.2. Rhodes Radical via Wedderburn Theory.
Our first goal is to relate the notion of a $\mathbf{V}$-morphism to algebra morphisms.

Lemma 3.1. *Let $\varphi : A \to B$ be a morphism of K-algebras with $\ker \varphi$ nilpotent. Let S be a finite subsemigroup of A. Then $\varphi|_S$ is an $\mathbf{LG}_K$-morphism.*

Proof. Without loss of generality, we may assume that S spans A and hence that A is finite dimensional. Let $e_0 \in E(B)$ and $U = e_0 \varphi|_S^{-1}$. First we show that U does not contain a copy of the two element semilattice. Indeed, suppose that $e, f \in E(U)$ and $ef = fe = e$. Then

$$(f - e)^2 = f^2 - ef - fe + e^2 = f - e.$$

Since $f - e \in \ker \varphi$, a nilpotent ideal, we conclude $f - e = 0$, that is $f = e$.

Now let G be a maximal subgroup of U with identity e. Then $g - e \in \ker \varphi$. Since g and e commute, if the characteristic is p, then, for large enough n,

$$0 = (g - e)^{p^n} = g^{p^n} - e$$

and so G is a p-group. If the characteristic is 0, then we observe that $(g - e)^n = 0$ for some n (take n minimal). So by taking the regular representation ρ of G, we see that $g\rho$ is a matrix with minimal polynomial of the form $(g\rho - 1)^n$; that is $g\rho$ is unipotent. A quick consideration of the Jordan canonical form for such $g\rho$ shows that if $g\rho \neq 1$, then it has infinite order. It follows that $g = e$ and so G is trivial. This completes the proof that $U \in \mathbf{LG}_K$. $\square$

Let $\varphi : S \to T$ be a morphism and let $\overline{\varphi} : KS \to KT$ denote the linear extension of φ to the semigroup algebra KS. Our goal is to prove the converse of Lemma 3.1 for $\overline{\varphi}$. Of particular importance is the case where T is the trivial semigroup. In this case $\ker \overline{\varphi}$ is called the *augmentation ideal*, denoted ωKS, and $\overline{\varphi}$ the *augmentation map*. It is worth observing that if U is a subsemigroup of S, then the augmentation map for U is the restriction of the augmentation map of S and hence $\omega KU = \omega KS \cap KU$.

So we begin by giving a varietal characterization of finite semigroups with nilpotent augmentation ideal.

First we prove a classical lemma showing how to find generators for the ideal $\ker \overline{\varphi}$ in terms of $\varphi : S \to T$.

Lemma 3.2. *Let $\varphi : S \to T$ be a morphism and let $\overline{\varphi} : KS \to KT$ denote the linear extension of f to the semigroup algebra KS. Then the set*

$$X = \{s_1 - s_2 \mid s_1\varphi = s_2\varphi\}$$

generates the ideal $\ker \overline{\varphi}$ as a vector space over K.

Proof. Clearly, $X \subseteq \ker \overline{\varphi}$. Now take an arbitrary $u = \sum_{s \in S} c_s s \in \ker \overline{\varphi}$ where $c_s \in K$. Applying the morphism $\overline{\varphi}$ to u, we obtain

$$0 = \sum_{t \in S\varphi} (\sum_{s\varphi = t} c_s)t$$

whence for each $t \in S\varphi$,

$$\sum_{s\varphi = t} c_s = 0 \tag{3.1}$$

as elements of T form a basis of KT. Now picking for each $t \in S\varphi$ a representative $s_t \in S$ with $s_t\varphi = t$ and using (3.1), we can rewrite the element u as follows:

$$u = \sum_{t \in S\varphi} (\sum_{s\varphi = t} c_s(s - s_t)),$$

that is, as a linear combination of elements in X. $\qquad \square$

We recall a standard result from the theory of finite-dimensional algebras due to Wedderburn [70].

Lemma 3.3. *Let A be a finite dimensional associative algebra over a field K. Assume that A is generated as a K-vector space by a set of nilpotent elements. Then A is a nilpotent algebra.*

The following can be proved using representation theory of finite semigroups or extracted from a general result of Ovsyannikov [34]. We give a simple direct proof using the above lemma. A similar proof for groups can be found, for instance, in [35].

Proposition 3.4. *Let S be a finite semigroup. Then the augmentation ideal ωKS is nilpotent if and only if $S \in \mathbf{LG}_K$.*

Proof. Recall that ωKS is the kernel of the morphism $KS \to K$ induced by the trivial morphism $S \to \{1\}$; it consists of all elements $\sum_{s \in S} c_s s \in KS$ such that $\sum_{s \in S} c_s = 0$.

Suppose first that $S \in \mathbf{LG}_K$. By Lemmas 3.2 and 3.3, it suffices to prove that $s - t$ is a nilpotent element of KS for all $s, t \in S$.

We first make some reductions. By Proposition 2.3 that there is an integer n such that every product of at least n elements in S belongs to its minimal ideal U. In particular, for all $s, t \in S$, $(s-t)^n$ belongs to $KU \cap \omega KS = \omega KU$.

Thus it suffices to show ωKU is nilpotent. In other words, we may assume without loss of generality that S is simple with maximal subgroups in $\mathbf{G}_K$ and we change notation accordingly.

Since S is a simple semigroup, it constitutes a single $\mathcal{J}$-class whence, as observed in Section 2, for every two $s, t \in S$, there exists an idempotent $e \in S$ such that $s \mathrel{\mathcal{R}} e$ and $e \mathrel{\mathcal{L}} t$. Then

$$s - t = (s - e) + (e - t).$$

So the augmentation ideal of S is generated as a vector space by differences of elements in either the same $\mathcal{R}$-class or the same $\mathcal{L}$-class, with one of them being an idempotent.

Assume that $s \mathrel{\mathcal{R}} e$ or $s \mathrel{\mathcal{L}} e$ and let $f = f^2$ be the idempotent in the $\mathcal{H}$-class of s. Then

$$s - e = (s - f) + (f - e).$$

Thus we see that the augmentation ideal is generated as a vector space by elements that are either the difference of an element and the idempotent in its $\mathcal{H}$-class or the difference of two idempotents in the same $\mathcal{R}$-class or the same $\mathcal{L}$-class.

Consider an element $s - f$ where $f^2 = f \mathrel{\mathcal{H}} s$. If char $K = 0$, then $s = f$, since $\mathbf{G}_K$ is the trivial variety and there is nothing to prove. If char $K = p$, then there is an n such that $s^{p^n} = f$ and since s commutes with f, we have

$$(s - f)^{p^n} = s^{p^n} - f = 0,$$

so in all cases $s - f$ is a nilpotent element.

Now consider an element $f - e$ where e and f are idempotents and either $f \mathrel{\mathcal{L}} e$ or $f \mathrel{\mathcal{R}} e$. Then

$$(f - e)^2 = f^2 - ef - fe + e^2 = 0$$

by (2.2).

Therefore, the augmentation ideal is generated as a vector space by nilpotent elements and we have shown ωKS is nilpotent.

The converse is a consequence of Lemma 3.1 with $A = KS$, $B = K$ and φ the augmentation map. $\qquad\square$

Theorem 3.5. *Let $\varphi : S \to T$ be a morphism of finite semigroups. Then φ is an $\mathbf{LG}_K$-morphism if and only if $\ker \overline{\varphi}$ is a nilpotent ideal of KS.*

Proof. Sufficiency is immediate from Lemma 3.1. For necessity, suppose φ is an $\mathbf{LG}_K$-morphism. Then by Lemmas 3.2 and 3.3, it suffices to prove that $s_1 - s_2$ is a nilpotent element of KS for each $s_1, s_2 \in S$ with $s_1\varphi = s_2\varphi$.

Let n be an integer such that $(s_1\varphi)^n$ is an idempotent f of T. Since φ is an $\mathbf{LG}_K$-morphism, $U = f\varphi^{-1}$ is in $\mathbf{LG}_K$. Also any product involving n elements of the set $\{s_1, s_2\}$ belongs to U. Therefore,

$$(s_1 - s_2)^n \in KU \cap \omega KS = \omega KU$$

and is hence nilpotent by Proposition 3.4. It follows that $s_1 - s_2$ is nilpotent, as desired. $\qquad\square$

Theorem 3.5 is a semigroup theorist's version of a classical and central result of the theory of finite dimensional algebras and holds in an appropriate sense for all finite dimensional algebras. Indeed, it has been known since the early 1900's that if A is a finite dimensional algebra and N is a nilpotent ideal of A, then every idempotent of A/N lifts to an idempotent of A. Furthermore, if we assume that algebras have identity elements then two lifts of an idempotent in A/N are conjugate by an element of the group of units of A of the form $g = 1 + n, n \in N$ and more generally, a conjugacy class of idempotents of A/N can be lifted to a single conjugacy class of idempotents in A. Lastly, two idempotents e, f of a finite dimensional algebra A are conjugate by an element of the group of units of A if and only if $e \mathcal{J} f$ in the multiplicative monoid of A. Putting this all together, it can be shown that considered as a morphism between multiplicative monoids, the morphism from A to A/N is an **LG**-morphism.

Theorem 3.6. *The Rhodes radical of a finite semigroup S over a field K is the largest $\mathbf{LG}_K$-congruence on S.*

Proof. Since the map $KS \to KS/\mathrm{Rad}(KS)$ has nilpotent kernel, Lemma 3.1 shows that $\mathrm{Rad}_K(S)$ is an $\mathbf{LG}_K$-congruence. If $\varphi : S \twoheadrightarrow T$ is any $\mathbf{LG}_K$-morphism, then $\overline{\varphi} : KS \twoheadrightarrow KT$ has nilpotent kernel by Theorem 3.5, whence $\ker \overline{\varphi} \subseteq \mathrm{Rad}(KS)$. Thus if $s_1\varphi = s_2\varphi$, then $s_1 - s_2 \in \ker \overline{\varphi} \subseteq \mathrm{Rad}(KS)$, showing that $(s_1, s_2) \in \mathrm{Rad}_K(S)$, as desired. $\qquad\square$

As a consequence we now give a simpler proof of some results of Krohn–Rhodes–Tilson [25, 52, 67].

Lemma 3.7. *Let $\varphi : S \twoheadrightarrow T$ be a surjective morphism of finite semigroups. Then φ induces a surjective morphism $\widetilde{\varphi} : S/\mathrm{Rad}_K(S) \twoheadrightarrow T/\mathrm{Rad}_K(T)$.*

Proof. Clearly $\mathrm{Rad}(KS)\overline{\varphi}$ is a nilpotent ideal of KT and hence contained in $\mathrm{Rad}(KT)$. Therefore a morphism $KS/\mathrm{Rad}(KS) \twoheadrightarrow KT/\mathrm{Rad}(KT)$ is well defined. $\qquad\square$

Theorem 3.8. *Let $\mathbf{V}$ be a variety of finite semigroups and S a finite semigroup. Then the following are equivalent:*

1. $S \in \mathbf{LG}_K \textcircled{m} \mathbf{V}$;
2. $S/\mathrm{Rad}_K(S) \in \mathbf{V}$;
3. *There is an $\mathbf{LG}_K$-morphism $\varphi : S \to T$ with $T \in \mathbf{V}$.*

Proof. Since $\mathrm{Rad}_K(S)$ is a $\mathbf{LG}_K$-congruence, (2) implies (3). Clearly (3) implies (1). For (1) implies (2), suppose $\varphi : T \twoheadrightarrow S$ and $\psi : T \twoheadrightarrow U$ are surjective morphisms with $U \in \mathbf{V}$ and ψ is an $\mathbf{LG}_K$-morphism. Then, by Theorem 3.6, the canonical morphism $T \twoheadrightarrow T/\mathrm{Rad}_K(T)$ factors through ψ and so $T/\mathrm{Rad}_K(T)$ is a quotient of U and hence belongs to $\mathbf{V}$. It now follows from Lemma 3.7 that $S/\mathrm{Rad}_K(S) \in \mathbf{V}$. $\qquad\square$

The central question about a variety of finite semigroups is usually the decidability of its membership problem. We say that a variety $\mathbf{V}$ is said to

have *decidable membership* if there exists an algorithm to recognize whether a given finite semigroup S belongs to $\mathbf{V}$. The above results imply that the Mal'cev product $\mathbf{LG}_K \, \text{\textcircled{m}} \, \mathbf{V}$ has decidable membership whenever the variety $\mathbf{V}$ has. Indeed, given a finite semigroup S, one effectively constructs its Rhodes radical $\text{Rad}_K(S)$ as the largest $\mathbf{LG}_K$-congruence on S and then verifies, using decidability of membership in $\mathbf{V}$, the condition (2) of Theorem 3.8. This observation is important because Mal'cev products of decidable pseudovarieties need not be decidable in general. See [53, 9].

3.3. **Rhodes Radical via Semigroup Theory.** We now indicate how to prove Theorem 3.6 using semigroup representation theory. Here we use the characterization of the Rhodes radical as the intersection of the congruences corresponding to all irreducible representations of S over K. This method will give us an explicit description of $\text{Rad}_K(S)$.

Krohn and Rhodes introduced the notion of a generalized group mapping semigroup in [24]. A semigroup S is called *generalized group mapping* [25, 24] (GGM) if it has a (0-)minimal ideal I on which it acts faithfully on both the left and right by left and right multiplication respectively. This ideal I is uniquely determined and is of the form $I = J(\cup\{0\})$ where J is a regular $\mathcal{J}$-class. We shall call I the *apex* of S, written $\text{Apx}(S)$. We aim to show that finite irreducible matrix semigroups are generalized group mapping.

The following result was stated by Rhodes for the case of the field of complex numbers [51, 56] but holds true in general. Our proof for the general case uses the results of Munn and Ponizovsky [16, 56].

Theorem 3.9. *Let K be a field, V be a finite dimensional K-vector space and $S \leq \text{End}_K(V)$ be a finite, irreducible subsemigroup. Then S is generalized group mapping.*

Proof. If S is the trivial semigroup, then it is clearly generalized group mapping. So we may assume S is non-trivial. Let I be a 0-minimal ideal of S; if S has no zero, take I to be the minimal ideal. It is shown in [16, Theorem 5.33] that the identity of $\text{End}_K(V)$ is a linear combination of elements of I. We shall provide a proof of this for the sake of completeness. It will then immediately follow that S acts faithfully on both the left and right of I by left and right multiplication.

The proof proceeds in several steps. Let $I^{\perp} = \{v \in V \mid vI = 0\}$. We first show that $I^{\perp} = 0$. To do this, we begin by showing that $I^{\perp}$ is S-invariant. Indeed, if $s \in S$, $t \in I$ and $v \in I^{\perp}$, then using that $st \in I$, we have

$$(vs)t = v(st) = 0,$$

showing that $vs \in I^{\perp}$. Since $I \neq \{0\}$, we cannot have $I^{\perp} = V$; thus $I^{\perp} = 0$ by irreducibility of S.

Next we show that I itself is irreducible. Let $\{0\} \neq W \leq V$ be an I-invariant subspace. Let $W_0 = \text{Span}\{wt \mid w \in W, \ t \in I\}$. Notice that $W_0 \subseteq W$. If $w \in W$, $t \in I$ and $s \in S$, then $(wt)s = w(ts) \in W$ since $ts \in I$ and W is I-invariant. Hence W_0 is S-invariant and so W_0 is either $\{0\}$ or

V. Since $I^{\perp} = 0$, we cannot have that $W_0 = \{0\}$ and so $W \supseteq W_0 = V$ establishing that $W = V$. We conclude that I is irreducible.

Let A be the K-span of I inside of $\mathrm{End}_K(V)$. Then A is an irreducible algebra acting on V and hence is a simple algebra by a well-known result of Burnside [27, 16]. Thus A has an identity element e by Wedderburn's theorem. But e commutes with the irreducible semigroup I and hence, by Schur's lemma, is non-singular. But the only non-singular idempotent endomorphism of V is the identity map and so the identity map belongs to A, the linear span of I. $\qquad\square$

Corollary 3.10. *A finite irreducible subsemigroup of $M_n(K)$ has a unique 0-minimal ideal, which is regular.*

We recall some notions and results of Krohn and Rhodes. The reader is referred to [25] for details. Fix a finite semigroup S. Choose for each regular $\mathcal{J}$-class J a fixed maximal subgroup G_J.

Proposition 3.11 ([25, Fact 7.2.1]). *Let $\varphi : S \twoheadrightarrow T$ be a surjective morphism. Let J' be a $\mathcal{J}$-class of T and let J be a $\leq_{\mathcal{J}}$-minimal $\mathcal{J}$-class of S with $J\varphi \cap J' \neq \emptyset$. Then $J\varphi = J'$. Moreover if J' is regular, then J is unique and regular, and the images of the maximal subgroups of J are precisely the maximal subgroups of J'.*

If T is GGM and $J' = \mathrm{Apx}(T) \setminus 0$, then we shall call the $\mathcal{J}$-class J of the above proposition the *apex* of φ, denoted $\mathrm{Apx}(\varphi)$. Let K_φ be the group theoretic kernel of $\varphi|_{G_{\mathrm{Apx}(\varphi)}}$. We call K_φ the *kernel* of φ. Krohn and Rhodes showed [25] that φ is completely determined by its apex and kernel.

Let J be a regular $\mathcal{J}$-class of S and $N \lhd G_J$ be a normal subgroup. We denote by R_a, $a \in A$, the $\mathcal{R}$-classes of J and by L_b, $b \in B$, the $\mathcal{L}$-classes of J. Suppose that $G_J = R_1 \cap L_1$. For each $a \in A, b \in B$, choose according to Green's Lemma [16] $r_a \in J$ such that $s \mapsto r_a s$ is a bijection $R_a \to R_1$ and $l_b \in J$ such that $s \mapsto sl_b$ is a bijection $L_b \to L_1$. With this notation if $H_{ab} = R_a \cap L_b$, then $s \mapsto r_a s l_b$ is a bijection $H_{ab} \to G_J$.

We define a congruence by $s \equiv_{(J,G_J,N)} t$ if and only if, for all $x, y \in J$,

$$xsy \in J \iff xty \in J \tag{3.2}$$

and, in the case where $xsy \in J$, if $x \in R_a$ and $y \in L_b$, then

$$r_a x s y l_b N = r_a x t y l_b N. \tag{3.3}$$

The quotient $S/\equiv_{(J,G_J,N)}$ is denoted $\mathsf{GGM}(J, G_J, N)$ [25]. The following result is the content of [25, Proposition 8.3.28, Remark 8.3.29].

Theorem 3.12. *Let S be a finite semigroup and J a regular $\mathcal{J}$-class with maximal subgroup G_J. Suppose $\varphi : S \twoheadrightarrow T$ is a surjective morphism with T a generalized group mapping semigroup. Let $J = \mathrm{Apx}(\varphi)$ and let K_φ be the kernel. Then the congruence associated to φ is $\equiv_{(J,G_J,K_\varphi)}$. In particular, $T \cong \mathsf{GGM}(J, G_J, K_\varphi)$.*

It follows from the above theorem that the definition of $\mathsf{GGM}(J, G_J, K_\varphi)$ doesn't depend on the choices made. The following result from [25] is an immediate consequence of the definition of $\equiv_{(J, G_J, N)}$.

Corollary 3.13. *Let $\varphi_1 : S \twoheadrightarrow T_1$, $\varphi_2 : S \twoheadrightarrow T_2$ be surjective morphisms to generalized group mapping semigroups with $\mathrm{Apx}(\varphi_1) = \mathrm{Apx}(\varphi_2)$. Then φ_2 factors through φ_1 if and only if $K_{\varphi_1} \leq K_{\varphi_2}$.*

We shall need the following fundamental result on semigroup representations, due to Clifford, Munn and Ponizovsky, which is one of the main results of [16, Chapter 5] (see also [56]).

Theorem 3.14. *Let S be a finite semigroup, J a regular $\mathcal{J}$-class of S and G_J a maximal subgroup of J. Then any irreducible representation $\rho : G_J \to GL(V)$ can be extended uniquely to an irreducible representation of S with apex J.*

It is proved in [16, 56] that every irreducible representation of a finite semigroup S is obtained by extending an irreducible representation of a maximal subgroup G_J for some regular $\mathcal{J}$-class J, although we shall not need this result.

We are now ready to prove Theorem 3.6 via representation theoretic means. First we need the following classical result, which is a consequence of Maschke's theorem and Clifford's theorem from finite group representation theory, handling the group case [27, Corollary 8.6][1]. If G is a finite group, we define G_K, called the *unipotent radical* of G, to be the largest normal subgroup of G belonging to $\mathbf{G_K}$.

Theorem 3.15. *Let G be a group and K be a field. Then $\mathrm{Rad}_K(G)$ is the congruence whose classes are the cosets of G_K.*

We remark that Theorem 3.15 also follows from our first proof of Theorem 3.6 since the largest $\mathbf{LG_K}$ congruence on a finite group clearly has kernel G_K.

Theorem 3.16. *Let S be a semigroup and K be a field. Then $\mathrm{Rad}_K(S)$ is the congruence associated to the direct sum over all regular $\mathcal{J}$-classes J of the maps*

$$S \twoheadrightarrow \mathsf{GGM}(J, G_J, (G_J)_K). \tag{3.4}$$

Proof. Let $\sim$ be the congruence associated to the direct sum of the maps (3.4). Let φ be an irreducible representation of S with apex $J = \mathrm{Apx}(\varphi)$. Then, by Theorem 3.15, $(G_J)_K \leq K_\varphi$ and so, by Corollary 3.13, φ factors through $S \twoheadrightarrow \mathsf{GGM}(J, G_J, (G_J)_K)$. Thus $\sim \subseteq \mathrm{Rad}_K(S)$.

For the reverse inclusion, it suffices to show that the congruence associated to each map $S \twoheadrightarrow \mathsf{GGM}(J, G_J, (G_J)_K)$ can be realized by a direct sum of irreducible representations. Fix a regular $\mathcal{J}$-class J and let $\{N_i\}$ be a collection of normal subgroups of G_J. Then the congruence associated to

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[1]We thank John Dixon for pointing this result out to us.

the direct sum of the maps $S \twoheadrightarrow \mathsf{GGM}(J, G_J, N_i)$ is the congruence associated to $S \twoheadrightarrow \mathsf{GGM}(J, G_J, \bigcap N_i)$. In particular, consider the collection $\{K_\varphi\}$ where φ is an irreducible representation of S with apex J. Then, by Theorem 3.14, the K_φ run over all kernels of irreducible representations of G_J, so by Theorem 3.15, we obtain $\bigcap K_\varphi = (G_J)_K$. The theorem now follows. $\quad\square$

Notice that this theorem allows for an explicit determination of $\mathrm{Rad}_K(S)$ via (3.2) and (3.3). The fact that the above congruence is the largest $\mathbf{LG}_K$-congruence is contained in [25, 52, 67].

4. APPLICATIONS TO DIAGONALIZABILITY AND TRIANGULARIZABILITY

Our first application of the Rhodes radical is to the question of diagonalizablity and triangularizability of finite semigroups. In [4], three of the authors characterized the varieties of finite semigroups that can be (uni)triangularized over finite fields. Using our techniques, we give a shorter, more conceptual proof that works over a general field.

Let K be a field. Define $\mathbf{Ab}_K$ to be the variety of finite Abelian groups generated by all finite subgroups of K^*. It is well known that any finite subgroup G of K^* is cyclic and is the set of roots of $x^{|G|} - 1$. Moreover, there is a cyclic subgroup of K^* of order m if and only if $x^m - 1$ splits into distinct linear factors over K. It is not hard to see that if $x^e - 1$ and $x^f - 1$ split into distinct linear factors, then so does $x^{\mathrm{lcm}(e,f)} - 1$. Also if $x^e - 1$ splits into linear factors, then so does $x^d - 1$ for any divisor d of e. Hence $\mathbf{Ab}_K$ can be described as the variety of all finite Abelian groups whose exponent e has the property that $x^e - 1$ splits into e distinct linear factors over K. We remark that if the characteristic of K is $p > 0$, then e and p must be relatively prime for this to happen. If K is algebraically closed of characteristic 0, then $\mathbf{Ab}_K = \mathbf{Ab}$. If K is algebraically closed of characteristic $p > 0$, then $\mathbf{Ab}_K$ consists of all finite Abelian p'-groups, that is of all finite Abelian groups whose orders are relatively prime to p.

If $\mathbf{H}$ is a variety of finite groups, then the elements of the variety $\mathbf{H} \textcircled{m} \mathbf{Sl}$ are referred to as *semilattices of groups from* $\mathbf{H}$. Such semigroups are naturally "graded" by a semilattice in such a way that the homogeneous components (which are the $\mathcal{H}$-classes) are groups from $\mathbf{H}$. It turns out that $\mathbf{H} \textcircled{m} \mathbf{Sl}$ is the varietal join $\mathbf{Sl} \vee \mathbf{H}$. See [16, 1] for more details. The following exercise in Linear Algebra captures diagonalizability.

Theorem 4.1. *Let K be a field and S a finite semigroup. Then the following are equivalent:*

(1) *S is commutative and satisfies an identity $x^{m+1} = x$ where $x^m - 1$ splits into distinct linear factors over K;*
(2) *S is a semilattice of Abelian groups from $\mathbf{Ab}_K$;*
(3) *Every representation of S is diagonalizable;*
(4) *S is isomorphic to a subsemigroup of K^n for $n = |S^1|$.*

Proof. The equivalence of (1) and (2) follows from Clifford's Theorem [16, Theorem 4.11]. For (1) implies (3), suppose $\rho : S \to \mathrm{End}_K(V)$ is a representation. Since S satisfies $x^{m+1} = x$, we must have that $s\rho$ satisfies $x(x^m - 1) = 0$. It follows that the minimal polynomial of $s\rho$ for any $s \in S$ has distinct roots and splits over K. Hence $s\rho$ is diagonalizable for all $s \in S$. To show that ρ is diagonalizable we induct on the degree of the representation. If ρ is of degree one, then clearly it is diagonalizable. If $S\rho$ is contained in the scalar matrices, then we may also deduce that the representation is diagonalizable. Otherwise, there is an element $s \in S$ such that $s\rho$ is not a scalar matrix. Since $s\rho$ is diagonalizable, we can write $V = \bigoplus_{\lambda \in \mathrm{Spec}(s\rho)} E_\lambda$ where E_λ is the eigenspace of λ. We claim that E_λ is S-invariant. Indeed, if $t \in S$ and $u \in E_\lambda$, then

$$ut\rho s\rho = us\rho t\rho = \lambda ut\rho$$

and so $ut\rho \in E_\lambda$. Since $s\rho$ is not a scalar, each E_λ has smaller dimension and so the restriction of ρ can be diagonalized by induction. Thus we have diagonalized ρ.

(3) implies (4) follows immediately from considering the right regular representation of S, that is by having S act on S^1 by right multiplication and extending linearly. To show that (4) implies (1), first observe that S embeds in a direct product of finite subsemigroups of K. A finite subsemigroup of K is commutative and must satisfy an identity of the form $x^i = x^{p+i}$, for some $i \geq 0, p > 0$. Since K is a field, we deduce that S satisfies an identity of the form $x^{m+1} = x$ for some $m \geq 0$. If m is minimum, then $x^m - 1$ splits into distinct linear factors, as discussed above. This completes the proof. $\square$

With a little more work, we can improve condition (4) in the previous theorem a bit. It is easy to prove that any semilattice of groups S acts faithfully by right multiplication on S considered as a set, even if S does not have an identity. Thus we can replace condition (4) by S is isomorphic to a subsemigroup of K^n for $n = |S|$. From this it immediately follows that the finite semigroups described in Theorem 4.1 are precisely the finite semigroups S such that $KS \cong K^n$ where $n = |S|$. Details are left to the reader.

The semigroups satisfying the conditions of the above theorem for diagonalizability form a variety of finite semigroups that we denote by $\mathbf{D}_K$. It is precisely the variety $\mathbf{Ab}_K \textcircled{m} \mathbf{Sl}$ by Clifford's Theorem [16, Theorem 4.11]. It is the varietal join $\mathbf{Sl} \vee \mathbf{Ab}_K$, which can be seen from the above theorem. For example, if $K = \mathbb{C}$, then $\mathbf{D}_K$ consists precisely of semilattices of Abelian groups (i.e. commutative inverse semigroups). If $K = \mathbb{F}_q$, the finite field of q elements, then $\mathbf{D}_K$ consists of semilattices of Abelian groups with exponent dividing $q - 1$. If K is the algebraic closure of $\mathbb{F}_p$, then $\mathbf{D}_K$ consists of semilattices of Abelian p'-groups.

We shall call a matrix *unidiagonal* if it is diagonal and its entries are contained in $\{0, 1\}$. We have the following analogue of Theorem 4.1 whose proof we leave to the reader.

Theorem 4.2. *Let K be a field and S a finite semigroup. Then the following are equivalent:*

(1) *S is a semilattice;*
(2) *Every representation of S is unidiagonalizable;*
(3) *$KS \cong K^n$ for $n = |S|$ and the image of S under this isomorphism is contained in $\{0,1\}^n$;*
(4) *S is isomorphic to a subsemigroup of $\{0,1\}^n$ for some n.*

The above theorem shows that **Sl** is the variety of finite unidiagonalizable semigroups.

Let K be a field. Let $T_n(K)$ denote the semigroup of upper triangular $n \times n$ matrices over K. Recall that a finite dimensional K-algebra A is called *basic* if $A/\mathrm{Rad}(A)$ is commutative. If $A/Rad(A) \cong K^n$ for some n, then A is called a *split basic K-algebra*.

Theorem 4.3. *Let S be a finite semigroup and K a field. Then the following are equivalent.*

(1) *$S \in \mathbf{LG}_K \circledm \mathbf{D}_K$;*
(2) *$S/\mathrm{Rad}_K(S) \in \mathbf{D}_K$;*
(3) *$KS/\mathrm{Rad}(KS) \cong K^m$ for some m;*
(4) *KS is a split basic K-algebra;*
(5) *Every irreducible representation of S over K has degree one;*
(6) *Every representation of S is triangularizable;*
(7) *$S \leq T_n(K)$, $n = |S^1|$;*
(8) *$S \leq T_m(K)$, some m.*

Proof. We have already seen the equivalence of (1) and (2). For (2) implies (3), let $T = S/\mathrm{Rad}_K(S)$. Then $KT \twoheadrightarrow KS/\mathrm{Rad}(KS)$. By Theorem 4.1 $KT = K^{|T|}$. Hence $KS/\mathrm{Rad}(KS)$ is a direct product of copies of K. The equivalence of (3) and (4) is the definition.

The implication (3) implies (5) follows immediately from the Wedderburn theory, since the only irreducible representations of a direct product of fields are the projections. For (5) implies (6), let $\varphi : S \to M_m(K)$ be a representation. Then by choosing a composition series for the right KS-module associated to φ, we can put $S\varphi$ in block upper triangular form where the diagonal blocks are irreducible representations or the zero representation. But since all such are of degree one, we conclude that $S\varphi$ has been brought to triangular form.

One establishes (6) implies (7) by considering the regular representation of S. That (7) implies (8) is trivial. For (8) implies (1), observe that the projection φ from $T_m(K)$ to the diagonal is an algebra homomorphism with nilpotent kernel. Thus $\varphi|_S$ is an $\mathbf{LG}_K$-morphism by Lemma 3.1 and so (1) follows from Theorem 4.1. $\square$

Let $UT_n(K)$ denote the semigroup of upper unitriangular $n \times n$ matrices over K, where by unitriangular we mean triangular with only 1's and 0's on the diagonal. By a *trivial representation* of S, we mean a homomorphism

$\varphi : S \to \{0,1\}$. The following theorem is proved similarly to the above theorem. We omit the proof.

Theorem 4.4. *Let S be a finite semigroup and K a field. Then the following are equivalent.*

(1) *$S \in \mathbf{LG}_K \textcircled{m} \mathbf{Sl}$;*
(2) *$S/\mathrm{Rad}_K(S) \in \mathbf{Sl}$;*
(3) *$KS/\mathrm{Rad}(KS) \cong K^m$ for some m and the image of S is contained in $\{0,1\}^m$;*
(4) *Every irreducible representation of S over K is trivial;*
(5) *Every representation of S is unitriangularizable;*
(6) *$S \leq UT_n(K)$, $n = |S^1|$;*
(7) *$S \leq UT_m(K)$, some m.*

Notice that unitriangularizability depends only on the characteristic and not the field. The proofs of condition (6) in the above theorems show that a [uni]triangularizable monoid can be realized as a submonoid of $S \leq T_n(K)$ $[S \leq UT_n(K)]$ and a [uni]triangular group can be realized as a subgroup of $T_n^*(K)$ $[UT_n^*(K)]$ (where here $*$ denotes the group of units of a monoid). We remark that if a finite semigroup S is triangularizable over the algebraic closure $\overline{K}$ of K, then it is triangularizable over a finite extension of K. Indeed, S can be faithfully represented in $T_n(K)$ $(UT_n(K))$, where $n = |S^1|$. Since only finitely many entries appear amongst the entries of S, we can just take the extension field generated by these entries. The same remarks apply to diagonalization.

We now determine the above varieties. Recall that if $\mathbf{H}$ is a variety of finite groups, then $\overline{\mathbf{H}}$ denotes the variety of finite semigroups all of whose subgroups belong to $\mathbf{H}$. Usually $\overline{\mathbf{I}}$ is denoted $\mathbf{A}$ (for aperiodic). If $\mathbf{V}$ is a variety of finite semigroups, then $\mathbf{DV}$ is the variety of semigroups whose regular $\mathcal{J}$-classes are subsemigroups that belong to $\mathbf{V}$. If $\mathbf{H}$ is a variety of finite groups, then $\mathbf{Sl} \vee \mathbf{H}$ is the variety of semilattices of groups from $\mathbf{H}$ [1]. We denote by $\mathbf{O}$ the variety of finite orthodox simple semigroups. A simple semigroup S is *orthodox* if $E(S)$ is a subsemigroup. If $\mathbf{V}$ is a variety of finite semigroups, $\mathbf{EV}$ is the variety of finite semigroups S such that $E(S)$ generates a subsemigroup in $\mathbf{V}$.

To handle the case of characteristic zero, we need a result that can easily be verified by direct calculations with generalized group mapping congruences. Since a syntactic proof can be found in [2, Corollary 3.3] we skip the proof.

Lemma 4.5. *Let $\mathbf{H}$ be a variety of finite groups, then*

$$\mathbf{LI} \textcircled{m} (\mathbf{Sl} \vee \mathbf{H}) = \mathbf{DO} \cap \overline{\mathbf{H}}.$$

In particular we obtain the following corollary:

Corollary 4.6. *The variety of unitriangularizable semigroups in characteristic zero is $\mathbf{DA}$. The variety of triangularizable semigroups over a field K*

of characteristic zero is $\mathbf{DO} \cap \overline{\mathbf{Ab}_K}$*. In particular, the variety of triangularizable semigroups over an algebraically closed field of characteristic zero is* $\mathbf{DO} \cap \overline{\mathbf{Ab}}$*.*

Of course $\mathbf{DA}$ and $\mathbf{DO} \cap \overline{\mathbf{Ab}}$ are decidable varieties. In general, decidability of $\mathbf{DO} \cap \overline{\mathbf{Ab}_K}$ depends on K. Notice that $\mathbf{DA}$ contains all finite bands, that is, all finite idempotent semigroups. The triangularizability of bands can be found in the work of [12, 13]. Corollary 4.6 is useful for computing spectra of random walks on semigroups in $\mathbf{DA}$ or $\mathbf{DO} \cap \overline{\mathbf{Ab}_K}$ [62]. In particular, some famous Markov chains, such as the Tsetlin library, arise as random walks on bands [10, 12, 13]. Another consequence of Corollary 4.6 is that the semigroup algebra of a finite semigroup S is split basic over the reals if and only if $S \in \mathbf{DO}$ and every subgroup of S has exponent two.

We now turn to the case of characteristic p.

Lemma 4.7. *Let p be a prime and let $\mathbf{H}$ be a variety of finite p'-groups. Then*

$$\mathbf{LG}_p \,\textcircled{m}\, (\mathbf{Sl} \vee \mathbf{H}) = \mathbf{D}(\overline{\mathbf{G}_p \,\textcircled{m}\, \mathbf{H}}) \cap \mathbf{E}\overline{\mathbf{G}_p}. \qquad (4.1)$$

Proof. To see that the left hand side of (4.1) is contained in the right hand side, suppose $S \in \mathbf{LG}_p \,\textcircled{m}\, (\mathbf{Sl} \vee \mathbf{H})$. Let $T = S/\mathrm{Rad}_{\mathbb{F}_p}(S)$ and let $\varphi : S \to T$ be the canonical homomorphism. Then φ is an $\mathbf{LG}_p$-morphism and $T \in \mathbf{Sl} \vee \mathbf{H}$ by Theorem 3.8. Hence if J is a regular $\mathcal{J}$-class of T, then $J\varphi^{-1}$ is a nilpotent extension of a simple semigroup by Proposition 2.3. It easily follows that $S \in \mathbf{DS}$ (since regular $\mathcal{J}$-classes are mapped into regular $\mathcal{J}$-classes). Suppose G is a subgroup of S. Then $G\varphi \in \mathbf{H}$ and $\ker \varphi|_G \in \mathbf{G}_p$ since φ is an $\mathbf{LG}_p$-morphism. We conclude $S \in \mathbf{D}(\overline{\mathbf{G}_p \,\textcircled{m}\, \mathbf{H}})$. Let J be a regular $\mathcal{J}$-class of S. Let $E(J)$ be the idempotents of J. Then $E(J)\varphi$ is the unique idempotent f of the $\mathcal{J}$-class $J\varphi$ of T (since T is a semilattice of groups). Hence $\langle E(J) \rangle \varphi = f$. Since $f\varphi^{-1} \in \mathbf{LG}_p$ it follows that every maximal subgroup of $\langle E(J) \rangle$ belongs to $\mathbf{G}_p$. This shows that $S \in \mathbf{E}\overline{\mathbf{G}_p}$. This establishes the inclusion from left to right in (4.1).

For the reverse inclusion, it suffices to show that if J is a regular $\mathcal{J}$-class of a finite semigroup S in the right hand side of (4.1), then

$$\mathsf{GGM}(J, G_J, (G_J)_{\mathbb{F}_p}) \in \mathbf{Sl} \vee \mathbf{H}.$$

First note that since $\mathbf{H}$ consists of p'-groups, $G_J \in \mathbf{G}_p \,\textcircled{m}\, \mathbf{H}$ means precisely that G_J has a normal p-Sylow subgroup N and that $G_J/N \in \mathbf{H}$. We remark that N is the p-radical $(G_J)_{\mathbb{F}_p}$. By the results of [21], there is a Rees matrix representation $\mathcal{M}^0(G_J, A, B, C)$ of J^0 with the entries of C generating the maximal subgroup K of the idempotent-generated subsemigroup. Since $S \in \mathbf{E}\overline{\mathbf{G}_p}$, K is a p-subgroup of G_J and hence contained in N. According to [25, 8.2.22 Fact (e)]) to obtain the image of J in $\mathsf{GGM}(J, G_J, (G_J)_{\mathbb{F}_p})$, we project to $\mathcal{M}^0(G_J/N, A, B, \overline{C})$, where $\overline{C}$ is obtained from C by first reducing modulo N, and then identifying proportional rows and columns. But since the entries of C belong to N, this results in identifying all rows and columns and so the image of J in $\mathsf{GGM}(J, G_J, (G_J)_{\mathbb{F}_p})$ is simply G_J/N. Since

$\mathsf{GGM}(J, G_J, (G_J)_{\mathbb{F}_p})$ acts faithfully on the right of its apex by partially defined right translations and the only non-zero, partially defined right translations of a group are zero and right translations by elements of the group, we see that

$$\mathsf{GGM}(J, G_J, (G_J)_{\mathbb{F}_p}) = G_J/N \text{ or } (G_J/N) \cup 0$$

(depending on whether J is the minimal ideal, or not). Thus

$$\mathsf{GGM}(J, G_J, (G_J)_{\mathbb{F}_p}) \in \mathbf{Sl} \vee \mathbf{H}$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Observing that extensions of p-groups by Abelian groups are the same thing as extensions of p-groups by Abelian p'-groups, we have the following corollary.

Corollary 4.8. *The variety of unitriangularizable semigroups over any field of characteristic p is $\mathbf{D}\overline{\mathbf{G}}_p$. The variety of triangularizable semigroups over a field K of characteristic p is $\mathbf{D}(\overline{\mathbf{G}_p \, \text{\small{ⓜ}} \, \mathbf{Ab}_K}) \cap \mathbf{E}\overline{\mathbf{G}}_p$. In particular, the variety of semigroups triangularizable over an algebraically closed field of characteristic p is $\mathbf{D}(\overline{\mathbf{G}_p \, \text{\small{ⓜ}} \, \mathbf{Ab}}) \cap \mathbf{E}\overline{\mathbf{G}}_p$.*

In particular, commutative semigroups are triangularizable over any characteristic. More precisely, every finite commutative semigroup is triangularizable over some field of characteristic 0 and for some field of characteristic p for each prime p. In fact, the semigroups triangularizable over any characteristic are precisely those in $\mathbf{DO} \cap \overline{\mathbf{Ab}}$. Pseudoidentities for many of these varieties can be found in [4]. A method of constructing pseudoidentities for $\mathbf{LI} \, \text{\small{ⓜ}} \, \mathbf{V}$ from those of $\mathbf{V}$ can be found in [47] and for $\mathbf{LG}_p \, \text{\small{ⓜ}} \, \mathbf{V}$ from those of $\mathbf{V}$ can be found in [5].

We now turn to characterize those finite semigroups whose semigroup algebras are basic over a field K. The case of split basic K-algebras has already been handled in Theorem 4.3. Recall that a finite dimensional K-algebra A is called *basic* if $A/\mathrm{Rad}(A)$ is commutative, or equivalently, a direct product of fields. Since $KS/\mathrm{Rad}(KS)$ is generated as an algebra by $S/\mathrm{Rad}_K(S)$, to be basic $S/\mathrm{Rad}_K(S)$ must be a semilattice of Abelian groups (embedding in a direct product of fields). Conversely, if $S/\mathrm{Rad}_K(S)$ is a semilattice of Abelian groups, then $KS/\mathrm{Rad}(KS)$ (being generated by $S/\mathrm{Rad}_K(S)$) must be a commutative algebra. Thus we have proved:

Corollary 4.9. *Let S be a finite semigroup and K a field. Then KS is a basic algebra if and only if*

$$S \in \mathbf{LG}_K \, \text{\small{ⓜ}} \, (\mathbf{Sl} \vee \mathbf{Ab}) = \begin{cases} \mathbf{DO} \cap \overline{\mathbf{Ab}} & char\ K = 0 \\ \mathbf{D}(\overline{\mathbf{G}_p \, \text{\small{ⓜ}} \, \mathbf{Ab}}) \cap \mathbf{E}\overline{\mathbf{G}}_{\mathbf{p}} & char\ K = p. \end{cases}$$

5. Applications to Semigroup Decomposition Theory

Our next application of the Rhodes radical is to recover some deep algebraic decomposition results of Rhodes–Tilson–Weil. For the remainder of the paper we will deal with monoids and varieties of finite monoids.

First of all we recall the definition of the two-sided semidirect product of two monoids. Let M and N be monoids and suppose that N has a bi-action on M (that is commuting left and right actions on N). For convenience we write M additively and N multiplicatively although we assume no commutativity. Then the *two-sided semidirect product* $M \bowtie N$ consists of all 2×2 upper triangular matrices $\begin{pmatrix} n & m \\ 0 & n \end{pmatrix}$ with the usual matrix multiplication. There is an obvious projection to N via the diagonal. The variety generated by two-sided semidirect products $M \bowtie N$ with $M \in \mathbf{V}$ and $N \in \mathbf{W}$ is denoted $\mathbf{V} ** \mathbf{W}$.

Rhodes and Tilson introduced in [54] the kernel category as a way to determine membership in $\mathbf{V} ** \mathbf{W}$. We restrict ourselves to considering the kernel category of a morphism and to a special case of the results of [54] to avoid getting technical. Let $\varphi : M \to N$ be a homomorphism. Following [54], we define a category K_φ, called the *kernel category* of φ. The object set is $N \times N$. The arrows are equivalence classes of triples $(n_L, m, n_R) \in N \times M \times N$ where $(n_L, m, n_R) : (n_L, m\varphi n_R) \to (n_L m\varphi, n_R)$ and two coterminal triples (n_L, m, n_R) and (n_L, m', n_R) are identified if and only if $m_L m m_R = m_L m' m_R$ for all $m_L \in n_L \varphi^{-1}, m_R \in n_R \varphi^{-1}$. Composition is given by

$$[(n_L, m, m'\varphi n_R)][(n_L m\varphi, m', n_R)] = [(n_L, mm', n_R)];$$

the identity at (n_L, n_R) is $[(n_L, 1, n_R)]$.

We consider categories as partial algebras whose elements consist of all of its arrows. If C is a category and c is an object of C, then the collection of all arrows $C(c, c)$ from c to itself is a monoid called the *local monoid at* c. It is clear that if we add a new zero element to C, we obtain a semigroup C^0 called the *consolidation* of C. In C^0, the identity element e_c at c is an idempotent and then it is easy to see that $C(c, c)$ is isomorphic to the local monoid (in the sense of our previous usage of that term in semigroup theory) $e_c C(c, c) e_c$.

Let $\mathbf{V}$ be a variety of finite monoids. A category C is said to be *locally in* $\mathbf{V}$ if each of the local monoids $C(c, c)$ belongs to $\mathbf{V}$; we use the notation $C(c)$ as a shorthand for $C(c, c)$. The collection of categories locally in $\mathbf{V}$ is denoted $\ell\mathbf{V}$. The following is an amalgamation of results of [69] and a special case of the results of [54].

Theorem 5.1. *Let M be a finite monoid, $\mathbf{H}$ a non-trivial variety of finite groups and $\mathbf{V}$ a variety of finite monoids. Then $M \in \mathbf{H} ** \mathbf{V}$ if and only if there is a finite monoid N mapping onto M that admits a morphism $\varphi : N \to V \in \mathbf{V}$ such that $\mathsf{K}_\varphi \in \ell\mathbf{H}$.*

Notice that $\ell\mathbf{I}$ is a variety of finite categories [69], from which it easily follows that the collection of finite monoids M that are quotients of finite monoids N admitting a morphism to $\varphi : N \to V \in \mathbf{V}$ with $\mathsf{K}_\varphi \in \ell\mathbf{I}$ is a variety of finite monoids, which we denote $\ell\mathbf{I} \ast\ast \mathbf{V}$. This variety plays an important role in language theory [41], as we shall see below.

Let $\mathbf{Ab}(p)$ denote the variety of finite Abelian groups of exponent p, where p is a prime. Our goal is to prove the following two important cases of the results of Rhodes–Tilson–Weil [54, 55] (see also [41]).

Theorem 5.2. *Let $\mathbf{V}$ be a variety of finite monoids and p a prime. Then the smallest variety of finite monoids containing $\mathbf{V}$ and closed under the operations $\mathbf{W} \mapsto \ell\mathbf{I} \ast\ast \mathbf{W}$, respectively $\mathbf{W} \mapsto \mathbf{Ab}(p) \ast\ast \mathbf{W}$, is $\mathbf{LI} \,\textcircled{m}\, \mathbf{V}$, respectively $\mathbf{LG}_p \,\textcircled{m}\, \mathbf{V}$.*

The original proof of Theorem 5.2 is a case-by-case analysis using Rhodes's classification of maximal proper surjective morphisms [50, 55, 25]. We give a conceptual proof via representation theory. First we make some preliminary observations.

It is well known [54, 25, 52, 67] that a morphism is an $\mathbf{LI}$-morphism (respectively $\mathbf{LG}_p$-morphism) if and only if it is injective on two element semilattices and on subgroups (respectively on p'-subgroups). It follows immediately that $\mathbf{LI}$-morphisms (respectively $\mathbf{LG}_p$-morphisms) are closed under composition. Thus if K is a field,

$$\mathbf{LG}_K \,\textcircled{m}\, (\mathbf{LG}_K \,\textcircled{m}\, \mathbf{V}) = \mathbf{LG}_K \,\textcircled{m}\, \mathbf{V} \tag{5.1}$$

The following is well known (cf. [54, 55, 41]), but we include the proof for completeness.

Proposition 5.3. *Let $\mathbf{V}$ be a variety of finite monoids and $\varphi : M \to N$ be a morphism with K_φ locally in $\mathbf{V}$. Then φ is an $\mathbf{LV}$-morphism.*

Proof. Let $f \in E(N)$. Set $M_f = f\varphi^{-1}$ and let $m \in f\varphi^{-1}$. Then $[(f, m, f)] : (f, f) \to (f, f)$ is an arrow of K_φ. Let $e \in E(f\varphi^{-1})$ and define a map $\psi : eM_fe \to K_\varphi((f, f), (f, f))$ by $m \mapsto [(f, m, f)]$. Clearly this is a morphism; we show it is injective. Suppose $m\psi = m'\psi$. Then since $e \in f\varphi^{-1}$, this implies $m = eme = em'e = m'$. Thus $eM_fe \in \mathbf{V}$ and so $M_f \in \mathbf{LV}$, establishing that φ is a $\mathbf{LV}$-morphism. $\square$

Let $M_{m,r}(K)$ denote the collection of $m \times r$ matrices over a field K. The following lemma will afford us the decompositions needed for our proof of Theorem 5.2.

Lemma 5.4. *Let K be a ring and $M \leq M_n(K)$ be a finite monoid of block upper triangular matrices of the form*

$$\left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mid A \in M_m(K),\ B \in M_{m,r}(K),\ C \in M_r(K) \right\}.$$

Let N be the quotient of M obtained by projecting to the block diagonal and let φ be the projection. Then each local monoid of K_φ embeds in the additive

group of $M_{m,r}(K)$*. In particular, if* K *is a field, then*

$$\mathsf{K}_\varphi \in \begin{cases} \ell\mathbf{I} & char\ K = 0 \\ \ell\mathbf{Ab}(p) & char\ K = p. \end{cases}$$

Proof. Elements of N are certain pairs (A, C) with $A \in M_m(K)$ and $C \in M_r(K)$. Let $S = \mathsf{K}_\varphi(((X, Y), (U, V)))$. We define a map $\psi : S \to M_{m,r}(K)$ as follows. Given an arrow $a = [((X, Y), m, (U, V))] \in S$ with $m = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, define $a\psi = XBV$. To see that ψ is well defined, first observe that

$$XA = X,\ YC = Y,\ AU = U,\ CV = V. \tag{5.2}$$

Using this we calculate

$$\begin{pmatrix} X & Z \\ 0 & Y \end{pmatrix}\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\begin{pmatrix} U & W \\ 0 & V \end{pmatrix} = \begin{pmatrix} XU & XW + XBV + ZV \\ 0 & YV \end{pmatrix}. \tag{5.3}$$

Subtracting $XW + ZV$ (which doesn't depend on the choice of a representative of a) from the upper right hand corner shows that $a\psi$ is well defined. In fact it is evident from (5.3) that ψ is injective. We show that ψ is a morphism to the additive group of $M_{m,r}(K)$. It clearly sends the identity matrix to 0. Also if $a, b \in S$ with respective middle coordinates

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}, \begin{pmatrix} A' & B' \\ 0 & C' \end{pmatrix}$$

then $a\psi + b\psi = XBV + XB'V$. But the middle coordinate of ab is

$$\begin{pmatrix} AA' & AB' + BC' \\ 0 & CC' \end{pmatrix}.$$

So $(ab)\psi = X(AB' + BC')V = XAB'V + XBC'V = XB'V + XBV$ since $a, b \in S$ (cf. (5.2)). Hence S is isomorphic to a finite subgroup of the additive group of $M_{m,r}(K)$. In particular, if K is a field and char $K = 0$, then S must be trivial; if char $K = p$, then $S \in \mathbf{Ab}(p)$. The lemma follows. $\square$

Proof of Theorem 5.2. Let K be a field and $\mathbf{V}$ be a variety. Let $\mathbf{U}$ be the smallest variety containing $\mathbf{V}$ such $\ell\mathbf{I} \ast\ast \mathbf{U} = \mathbf{U}$ if char $K = 0$ or $\mathbf{Ab}(p) \ast\ast \mathbf{U} = \mathbf{U}$ if char $K = p$. Proposition 5.3 and (5.1) immediately implies $\mathbf{U} \subseteq \mathbf{LG}_K \textcircled{m} \mathbf{V}$. To prove the converse, we need the following.

Lemma 5.5. *Suppose M is a finite submonoid of $M_n(K)$ in block upper triangular form with diagonal block monoids $M_1, \ldots, M_k$ belonging to $\mathbf{V}$. Then $M \in \mathbf{U}$.*

Proof. We induct on k. If $k = 1$, then $M = M_1 \in \mathbf{V} \subseteq \mathbf{U}$. In general, note that we can repartition n into two blocks, one corresponding to the union of the first $k - 1$ of our original blocks and the other corresponding to the last block. We then obtain a block upper triangular matrix monoid with two diagonal block monoids M' and M_k. By induction $M' \in \mathbf{U}$ (being block upper triangular with $k - 1$ diagonal blocks $M_1, \ldots, M_{k-1}$ belonging to $\mathbf{V}$)

whilst $M_k \in \mathbf{V} \subseteq \mathbf{U}$. Therefore $M' \times M_k \in \mathbf{U}$. Lemma 5.4 shows that the kernel category of the projection to $M' \times M_k$ belongs to $\ell\mathbf{I}$, respectively $\ell\mathbf{Ab}(p)$, according to the characteristic of K. Hence $M \in \mathbf{U}$.   □

To complete the proof of Theorem 5.2, suppose $M \in \mathbf{LG}_K \, \textcircled{m} \, \mathbf{V}$. Consider the regular representation of M. By finding a composition series for M, we can put M in block upper triangular form where the diagonal blocks $M_1, \ldots, M_k$ are the action monoids of the irreducible representations of M over K. Since, by Theorem 3.8, $M/\mathrm{Rad}_K(M) \in \mathbf{V}$, the M_i belong to $\mathbf{V}$. The previous lemma then shows that $M \in \mathbf{U}$, establishing Theorem 5.2.

6. Applications to Formal Language Theory

Another application of the Rhodes radical is to Formal Language Theory, namely to unambiguous marked products and marked products with counter. Some of these results were announced in [3].

Recall that a word u over a finite alphabet Σ is said to be a *subword* of a word $v \in \Sigma^*$ if, for some $n \geq 1$, there exist words $u_1, \ldots, u_n, v_0, v_1, \ldots, v_n \in \Sigma^*$ such that $u = u_1 u_2 \cdots u_n$ and

$$v = v_0 u_1 v_1 u_2 v_2 \cdots u_n v_n. \tag{6.1}$$

The subword relation reveals interesting combinatorial properties and plays a prominent role in formal language theory, as well as in the theory of Coxeter groups via its relation to the Bruhat order [11]. For instance, recall that languages consisting of all words over Σ having a given word $u \in \Sigma^*$ as a subword serve as a generating system for the Boolean algebra of so-called *piecewise testable* languages. It was a deep study of combinatorics of the subword relation that led Simon [59,60] to his elegant algebraic characterization of piecewise testable languages. Further, the natural idea to put certain rational constraints on the factors $v_0, v_1, \ldots, v_n$ that may appear in a decomposition of the form (6.1) gave rise to the useful notion of a marked product of languages studied from the algebraic viewpoint by Schützenberger [57], Reutenauer [49], Straubing [64], Simon [61], amongst others.

Yet another natural idea is to count how many times a word $v \in \Sigma^*$ contains a given word u as a subword, that is, to count different decompositions of the form (6.1). Clearly, if one wants to stay within the realm of rational languages, one can only count up to a certain threshold and/or modulo a certain number. For instance, one may consider Boolean combinations of languages consisting of all words over Σ having t modulo p occurrences of a given word $u \in \Sigma^*$ (where p is a given prime number). This class of languages also admits a nice algebraic characterization, see [18, Sections VIII.9 and VIII.10] and also [66]. Combining modular counting with rational constraints led to the idea of marked products with modular counters explored, in particular, by Weil [71] and Péladeau [36].

The most natural version of threshold counting is formalized via the notion of an unambiguous marked product in which one considers words

$v \in \Sigma^*$ having exactly one decomposition (6.1) with a given subword u and given rational constraints on the factors $v_0, v_1, \ldots, v_n$. Such unambiguous marked products have been investigated by Schützenberger [58], Pin [38], Pin, Straubing, and Thérien [41], amongst others.

Many known facts on marked products rely on rather difficult techniques from finite semigroup theory, namely, on the bilateral semidirect product decomposition results of Rhodes *et al.* [54, 55] mentioned above. These results are proved using Rhodes's classification of maximal proper surmorphisms [50, 55, 25] via case-by-case analysis of the kernel categories of such maps [54, 55]. The aim of the present section is to give easier and — we hope — more conceptual proofs of several crucial facts about marked products by using matrix representations of finite semigroups as a main tool. In particular, we are able to prove the results of Péladeau and Weil in one step, without any case-by-case analysis and without using the machinery of categories. Rather we adapt Simon's analysis of the combinatorics of multiplying upper triangular matrices [61] from the case of Schützenberger products to block upper triangular matrices. We failed to obtain such a purely combinatorial argument for the case of unambiguous products; we still need to use a lemma on kernel categories. Nevertheless we have succeeded in avoiding the decomposition results and case-by-case analysis.

Recall that Eilenberg established [18, Vol. B, Chap. VII] a correspondence between varieties of finite monoids and so-called varieties of languages. If **V** is a variety of finite monoids and Σ a finite alphabet, then $\mathcal{V}(\Sigma^*)$ denotes the set of all languages over Σ that can be recognized by monoids in **V**. (Such languages are often referred to as **V**-*languages*.) The operator $\mathcal{V}$ that assigns each free monoid Σ^* the set $\mathcal{V}(\Sigma^*)$ is said to be *the variety of languages associated to* **V**. The syntactic monoid [18, loc. cit.] of a rational language L will be denoted M_L. It is known that L is a **V**-language if and only if $M_L \in$ **V**.

6.1. **Products with Counter.** Our first application is to prove the results of Péladeau and Weil [36, 71] on products with counter.

Let $L_0, \ldots, L_m \subseteq \Sigma^*$, $a_1, \ldots, a_m \in \Sigma$ and let n be an integer. Then the *marked product with modulo n counter* $L = (L_0 a_1 L_1 \cdots a_m L_m)_{r,n}$ is the language of all words $w \in \Sigma^*$ with r factorizations modulo n of the form $w = u_0 a_1 u_1 \cdots a_m u_m$ with each $u_i \in L_i$. One can show that L is rational [71] (see also the proof of Theorem 6.2 below). Using a decomposition result of Rhodes and Tilson [54] (see also [55]) based on case-by-case analysis of kernel categories of maximal proper surmorphisms (see [50, 55, 25]), Weil characterized the closure of a variety $\mathcal{V}$ under marked products with modulo p counter. This required iterated usage of the so-called "block product" principle. But Weil missed that the Boolean algebra generated by $\mathcal{V}(\Sigma^*)$ and marked products with modulo p counters of members $\mathcal{V}(\Sigma^*)$ is already closed under marked products with modulo p counters; this was later observed by Péladeau [36]. The difficulty arises because it is not so clear how to combine

marked products with modulo p counters into new marked products with modulo p counters.

We use representation theory to prove the result in one fell swoop. Our approach is inspired by a paper of Simon [61] dealing with marked products and the Schützenberger product of finite semigroups.

Lemma 6.1. *Let* **V** *be a variety of finite monoids,* $\varphi : \Sigma^* \to M$ *be a morphism with M finite. Let K be a field of characteristic p and suppose that M can be represented faithfully by block upper triangular matrices over K so that the monoids formed by the diagonal blocks of the matrices in the image of M all belong to* **V**. *Let $F \subseteq M$. Then $L = F\varphi^{-1}$ is a Boolean combination of members of $\mathcal{V}(\Sigma^*)$ and of marked products with modulo p counter $(L_0 a_1 L_1 \cdots a_n L_n)_{r,p}$ with the $L_i \in \mathcal{V}(\Sigma^*)$.*

Proof. Suppose $M \leq M_t(K)$ and $t = t_1 + \cdots + t_k$ is the partition of t giving rise to the block upper triangular form. Let M_i be the monoid formed by the $t_i \times t_i$ matrices over K arising as the ith diagonal blocks of the matrices in the image of M. Given $w \in \Sigma^*$ and $i, j \in \{1, \ldots, k\}$, define $\varphi_{i,j} : \Sigma^* \to M_{t_i, t_j}(K)$ by setting $w\varphi_{i,j}$ to be the $t_i \times t_j$ matrix that is the i, j-block of the block upper triangular form. So in particular $w\varphi_{i,j} = 0$ for $j < i$. Also $\varphi_{i,i}$ is a morphism $\varphi_{i,i} : \Sigma^* \to M_i$ for all i.

First we observe that we may take F to be a singleton $\{u\varphi\}$. For each $1 \leq i \leq j \leq k$, let
$$L_{i,j} = \{w \in \Sigma^* \mid w\varphi_{i,j} = u\varphi_{i,j}\}.$$
Then clearly
$$u\varphi\varphi^{-1} = \bigcap_{1 \leq i \leq j \leq k} L_{i,j}.$$
Since $L_{i,i}$ is recognized by M_i, it suffices to show $L_{i,j}$, where $1 \leq i < j \leq k$, can be written as a Boolean combination of marked products with modulo p counter of languages recognized by the M_l. Changing notation, it suffices to show that if $1 \leq i < j \leq k$ and $C \in M_{t_i, t_j}(K)$, then
$$L(C) = \{w \in \Sigma^* \mid w\varphi_{i,j} = C\} \tag{6.2}$$
is a Boolean combination of marked products with modulo p counter of languages recognized by the M_i.

The following definitions are inspired by [61], though what Simon terms an "object", we term a "walk". A *walk* from i to j is a sequence
$$\mathfrak{w} = (i_0, m_0, a_1, i_1, m_1, \ldots, a_r, i_r, m_r) \tag{6.3}$$
where $i = i_0 < i_1 < \cdots < i_r = j$, $a_l \in \Sigma$ and $m_l \in M_{i_l}$. There are only finitely many walks. The set of walks will be denoted $\mathfrak{W}$. Given a walk $\mathfrak{w}$, we define its *value* to be
$$\mathsf{v}(\mathfrak{w}) = m_0(a_1\varphi_{i_0,i_1})m_1 \cdots (a_r\varphi_{i_{r-1},i_r})m_r \in M_{t_i, t_j}(K).$$
If $\mathfrak{w}$ is a walk, we define the *language* of $\mathfrak{w}$ to be the marked product
$$L(\mathfrak{w}) = (m_0\varphi_{i_0,i_0}^{-1})a_1(m_1\varphi_{i_1,i_1}^{-1}) \cdots a_r(m_r\varphi_{i_r,i_r}^{-1}).$$

If $w \in \Sigma^*$ and $\mathfrak{w}$ is a walk of the form (6.3), we define $w(\mathfrak{w})$ to be the *multiplicity* of w in $L(\mathfrak{w})$, that is, the number of factorizations $w = u_0 a_1 u_1 \cdots a_r u_r$ with $u_l \varphi_{i_l, i_l} = m_l$; this number is taken to be 0 if there are no such factorizations. If $0 \leq n < p$, we establish the shorthand

$$L(\mathfrak{w})_{n,p} = \left((m_0 \varphi_{i_0,i_0}^{-1}) a_1 (m_1 \varphi_{i_1,i_1}^{-1}) \cdots a_r (m_r \varphi_{i_r,i_r}^{-1}) \right)_{n,p}.$$

Notice that $L(\mathfrak{w})_{n,p}$ consists of all words w with $w(\mathfrak{w}) \equiv n \bmod p$ and is a marked product with modulo p counter of $\mathcal{V}(\Sigma^*)$ languages.

The following is a variant of [61, Lemma 7].

Claim 1. *Let $w \in \Sigma^*$. Then*

$$w\varphi_{i,j} = \sum_{\mathfrak{w} \in \mathfrak{W}} w(\mathfrak{w}) \mathsf{v}(\mathfrak{w}). \tag{6.4}$$

Proof. Let $w = b_1 \cdots b_r$ be the factorization of w in letters. Then the formula for matrix multiplication gives

$$w\varphi_{i,j} = \sum (b_1 \varphi_{i_0,i_1})(b_2 \varphi_{i_1,i_2}) \cdots (b_r \varphi_{i_{r-1},i_r}) \tag{6.5}$$

where the sum extends over all i_l such that $i_0 = i$, $i_r = j$ and $i_l \in \{1, \ldots, k\}$ for $0 < l < r$. Since $v\varphi_{l,n} = 0$ for $l > n$, it suffices to consider sequences such that $i = i_0 \leq i_1 \leq \cdots \leq i_r = j$. For such a sequence, we may group together neighboring indices that are equal. Then since all the $\varphi_{n,n}$ are morphisms, we see that each summand in (6.5) is the value of a walk $\mathfrak{w}$ and that $\mathfrak{w}$ appears exactly $w(\mathfrak{w})$ times in the sum. $\qquad\square$

To complete the proof of Lemma 6.1, we observe that $L(C)$ (defined in (6.2)) is a Boolean combination of languages of the form $L(\mathfrak{w})_{n,p}$. Let X be the set of all functions $f : \mathfrak{W} \to \{0, \ldots, p-1\}$ such that

$$\sum_{\mathfrak{w} \in \mathfrak{W}} f(\mathfrak{w}) \mathsf{v}(\mathfrak{w}) = C.$$

It is then immediate from (6.4) and $\operatorname{char} K = p$ that

$$L(C) = \bigcup_{f \in X} \bigcap_{\mathfrak{w} \in \mathfrak{W}} L(\mathfrak{w})_{f(\mathfrak{w}),p}$$

completing the proof. $\qquad\square$

Theorem 6.2. *Let $L \subseteq \Sigma^*$ be a rational language, $\mathbf{V}$ be a variety of finite monoids and K be a field of characteristic p. Then the following are equivalent.*

(1) $M_L \in \mathbf{LG}_p \textcircled{m} \mathbf{V}$;
(2) $M_L / \operatorname{Rad}_K(M_L) \in \mathbf{V}$;
(3) *M_L can be faithfully represented by block upper triangular matrices over K so that the monoids formed by the diagonal blocks of the matrices in the image of M_L all belong to $\mathbf{V}$;*
(4) *L is a Boolean combination of members of $\mathcal{V}(\Sigma^*)$ and languages $(L_0 a_1 L_1 \cdots a_n L_n)_{r,p}$ with the $L_i \in \mathcal{V}(\Sigma^*)$.*

Proof. The equivalence of (1) and (2) was established in Theorem 3.8.

For (2) implies (3), take a composition series for the regular representation of M_L over K: it is then in block upper triangular form and, by (2) the monoids formed by diagonal blocks of matrices in the image of M_L all belong to **V**, being the action monoids from the irreducible representations of M_L over K.

(3) implies (4) is immediate from Lemma 6.1.

For (4) implies (1), it suffices to deal with a marked product with counter $L = (L_0a_1L_1\cdots a_nL_n)_{r,p}$. Let $\mathcal{A}_i$ be the minimal trim deterministic automaton [18, Vol. A] of L_i. Let $\mathcal{A}$ be the non-deterministic automaton obtained from the disjoint union of the $\mathcal{A}_i$ by attaching an edge labelled a_i from each final state of $\mathcal{A}_{i-1}$ to the initial state of $\mathcal{A}_i$. To each letter $a \in \Sigma$, we associate the matrix $a\varphi$ of the relation that a induces on the states. Since $a\varphi$ is a $0,1$-matrix, we can view it as a matrix over $\mathbb{F}_p$. In this way we obtain a morphism $\varphi : \Sigma^* \to M_k(\mathbb{F}_p)$ where k is the number of states of $\mathcal{A}$. Let $M = \Sigma^*\varphi$. Trivially, M is finite. We observe that M is block upper triangular with diagonal blocks the syntactic monoids M_{L_i} (the partition of k arises from taking the states of each $\mathcal{A}_i$). Notice that M recognizes L, since L consists of all words w such that $(w\varphi)_{s,f} = r$ where s is the start state of $\mathcal{A}_0$ and f is a final state of $\mathcal{A}_n$. Applying Lemma 3.1 to the projection to the diagonal blocks gives that M and its quotient M_L belong to $\mathbf{LG}_p \textcircled{m} \mathbf{V}$. $\square$

The proof of (4) implies (1) gives a fairly easy argument that marked products of rational languages with mod p counter are rational.

Since the operator $\mathbf{LG}_p \textcircled{m} (\ )$ is idempotent, we immediately obtain the following result of [36, 71].

Corollary 6.3. *Let* **V** *be a variety of finite monoids and* $\mathbf{W} = \mathbf{LG}_p \textcircled{m} \mathbf{V}$. *Let* $\mathcal{W}$ *be the corresponding variety of languages. Then*

(1) $\mathcal{W}(\Sigma^*)$ *is the smallest class of languages containing* $\mathcal{V}(\Sigma^*)$*, which is closed under Boolean operations and formation of marked products with modulo* p *counters.*

(2) $\mathcal{W}(\Sigma^*)$ *consists of all Boolean combinations of elements of* $\mathcal{V}(\Sigma^*)$ *and marked products with modulo* p *counters of elements of* $\mathcal{V}(\Sigma^*)$*.*

Some special cases are the following. If **V** is the trivial variety of monoids, then $\mathbf{LG}_p \textcircled{m} \mathbf{V} = \mathbf{G}_p$ and we obtain Eilenberg's result [18, Section VIII.10] that the $\mathbf{G}_p$ languages consist of the Boolean combinations of languages of the form $(\Sigma^*a_1\Sigma^* \cdots a_n\Sigma^*)_{r,p}$. Notice that $\mathbf{G}_p$ consists of the groups unitriangularizable over characteristic p. The languages over Σ^* associated to $\mathbf{LG}_p \textcircled{m} \mathbf{Sl}$ (as observed in [4] and Theorem 4.4, this variety consists of the unitriangularizable monoids over characteristic p) are the Boolean combinations of languages of the forms

$$\Sigma^*a\Sigma^* \quad \text{and} \quad (\Sigma_0^*a_1\Sigma_1^* \cdots a_n\Sigma_n^*)_{r,p}$$

where $\Sigma_i \subseteq \Sigma$.

We remark that Weil shows [71] that closing $\mathcal{V}(\Sigma^*)$ under marked products with modulo p^n counters, for $n > 1$, does not take you out of the $\mathbf{LG}_p \textcircled{m} \mathbf{V}$-languages.

6.2. **Unambiguous Products.** Our next application is to recover results of Schützenberger, Pin, Straubing, and Thérien concerning unambiguous products. Our proof of one direction is along the lines of [41] but our usage of representation theory allows us to avoid using results relying on case-by-case analysis of maximal proper surmorphisms.

Let Σ be a finite alphabet, $L_0, \ldots, L_n \subseteq \Sigma^*$ be rational languages and $a_1, \ldots, a_n \in \Sigma$. Then the *marked product* $L = L_0 a_1 L_1 \cdots a_n L_n$ is called *unambiguous* if each word $w \in L$ has exactly one factorization of the form $u_0 a_1 u_1 \cdots a_n u_n$, where each $u_i \in L_i$. We also allow the degenerate case $n = 0$.

We shall need to use a well-known and straightforward consequence of the distributivity of concatenation over union (cf. [41]), namely, if $L_0, \ldots, L_n$ are disjoint unions of unambiguous marked products of elements of $\mathcal{V}(\Sigma^*)$, then the same is true for any unambiguous product $L_0 a_1 L_1 \cdots a_n L_n$. We also need a lemma about languages recognized by finite monoids of block upper triangular matrices in characteristic 0.

Lemma 6.4. *Let $\mathbf{V}$ be a variety of finite monoids, $\varphi : \Sigma^* \to M$ be a morphism with M finite. Let K be a field of characteristic 0 and suppose that M can be represented faithfully by block upper triangular matrices over K so that the monoids $M_1, \ldots, M_k$ formed by diagonal blocks of matrices in the image of M all belong to $\mathbf{V}$. Let $F \subseteq M$. Then $L = F\varphi^{-1}$ is a disjoint union of unambiguous marked products $L_0 a_1 L_1 \cdots a_n L_n$ with the $L_i \in \mathcal{V}(\Sigma^*)$.*

Proof. We induct on the number k of diagonal blocks. If there is only one block we are done.

Now let $k > 1$. We can repartition n into two blocks, one corresponding to the union of the first $k-1$ of our original blocks and the other corresponding to the last block. The first diagonal block, call it N, is block upper triangular with diagonal blocks $M_1, \ldots, M_{k-1}$; the second is just M_k. By induction, any language recognized by N is a disjoint union of unambiguous marked products $L_0 a_1 L_1 \cdots a_r L_r$ with the $L_i \in \mathcal{V}(\Sigma^*)$. Since $M_k \in \mathbf{V}$ it is easy to check that any language recognized by $N \times M_k$ is also a disjoint union of unambiguous marked products $L_0 a_1 L_1 \cdots a_r L_r$ with the $L_i \in \mathcal{V}(\Sigma^*)$. Thus to prove the result, it suffices to show that L is a disjoint union of unambiguous marked products $L_0 a_1 L_1 \cdots a_n L_n$ with the L_i recognized by $N \times M_k$. By Lemma 5.4, the projection from M to $N \times M_k$ has locally trivial kernel category. Then [41, Proposition 2.2] shows us that L is a disjoint union of such unambiguous marked products. $\square$

We ask whether there is a simple combinatorial proof of this lemma that avoids the use of [41, Proposition 2.2] along the lines of the proof of Lemma 6.1.

Theorem 6.5. *Let $L \subseteq \Sigma^*$ be a rational language, $\mathbf{V}$ be a variety of finite monoids and K a field of characteristic 0. Then the following are equivalent.*

(1) *$M_L \in \mathbf{LI} \textcircled{m} \mathbf{V}$;*
(2) *$M_L/\mathrm{Rad}_K(M_L) \in \mathbf{V}$;*
(3) *M_L can be faithfully represented by block upper triangular matrices over K so that the monoids formed by the diagonal blocks of the matrices in the image of M_L all belong to $\mathbf{V}$.*
(4) *L is a disjoint union of unambiguous products $L_0 a_1 L_1 \cdots a_n L_n$ with the $L_i \in \mathcal{V}(\Sigma^*)$.*

Proof. The equivalence of (1) and (2) follows from Theorem 3.8.

For (2) implies (3), take a composition series for the regular representation of M_L over K: it is then in block upper triangular form and by (2) monoids formed by diagonal blocks of matrices in the image of M_L all belong to $\mathbf{V}$.

(3) implies (4) is immediate from Lemma 6.4.

For (4) implies (1), it suffices to deal with a single unambiguous marked product $L = L_0 a_1 L_1 \cdots a_n L_n$. Let $\mathcal{A}_i$ be the minimal trim deterministic automaton for L_i and let $\mathcal{A}$ be the non-deterministic automaton obtained from the disjoint union of the L_i by attaching an edge labelled a_i from each final state of $\mathcal{A}_{i-1}$ to the initial state of $\mathcal{A}_i$. To each letter $a \in A$, we associate the matrix $a\varphi$ of the relation that a induces on the states. In this way we obtain a morphism $\varphi : \Sigma^* \to M_k(\mathbb{Q})$ where k is the number of states of $\mathcal{A}$. Let $M = \Sigma^*\varphi$. We observe that M is block upper triangular with diagonal blocks the syntactic monoids M_{L_i} (the partition of k arises from taking the states of each $\mathcal{A}_i$). Notice that M recognizes L, since L consists of all words w such that $(w\varphi)_{s,f} > 0$ where s is the start state of $\mathcal{A}_0$ and f is a final state of $\mathcal{A}_n$. First we show that M is finite. In fact, we claim M contains only $0,1$-matrices (and hence must be finite). Indeed, suppose $(w\varphi)_{i,j} > 1$ some i, j. Since each M_{L_i} consists of $0,1$-matrices, we must have that i is a state of some $\mathcal{A}_l$ and j a state of some $\mathcal{A}_r$ with $l < r$. But $(w\varphi)_{i,j}$ is the number of paths labelled by w from i to j in $\mathcal{A}$. Thus if u, v are words reading respectively from the start state of $\mathcal{A}_0$ to i and from j to a final state of $\mathcal{A}_n$ (such exist since the $\mathcal{A}_i$ are trim), then uwv has at least two factorizations witnessing membership in L, contradicting that L was unambiguous. Since the collection of all block upper triangular matrices is an algebra over $\mathbb{Q}$, as is the collection of block diagonal matrices, an application of Lemma 3.1 to the projection to the diagonal blocks gives that $M \in \mathbf{LI} \textcircled{m} \mathbf{V}$ and so, since $M \twoheadrightarrow M_L$, we have $M_L \in \mathbf{LI} \textcircled{m} \mathbf{V}$. $\square$

Since the operator $\mathbf{LI} \textcircled{m} (\ )$ is idempotent, we immediately obtain the following result of [38, 41].

Corollary 6.6. *Let $\mathbf{V}$ be a variety of finite monoids and $\mathbf{W} = \mathbf{LI} \textcircled{m} \mathbf{V}$. Let $\mathcal{W}$ be the corresponding variety of languages. Then*

(1) *$\mathcal{W}(\Sigma^*)$ is the smallest class of languages containing $\mathcal{V}(\Sigma^*)$, which is closed under Boolean operations and formation of unambiguous marked products.*

(2) $\mathcal{W}(\Sigma^*)$ *consists of all finite disjoint unions of unambiguous marked*
     products of elements of $\mathcal{V}(\Sigma^*)$.

Recall that the Malcev product of the pseudovariety **LI** with the pseudovariety **Sl** of semilattices (idempotent commutative monoids) is equal to the famous pseudovariety **DA** of all finite monoids whose regular $\mathcal{D}$-classes are idempotent subsemigroups (see [65] for a nice survey of combinatorial, logical and automata-theoretic characterizations of **DA**). Applying the above corollary, one obtains the classical result of Schützenberger [58] that $\mathcal{DA}(\Sigma^*)$ consists of disjoint unions of unambiguous products of the form $\Sigma_0^* a_1 \Sigma_1^* \cdots a_n \Sigma_n^*$ with $\Sigma_i \subseteq \Sigma$ for all i. We saw in Corollary 4.6 that **DA** consists of precisely those finite monoids that can be faithfully represented by upper triangular matrices with zeroes and ones on the diagonal over $\mathbb{Q}$.

7. Černý's Conjecture for **DS**

A deterministic automaton $\mathcal{A} = (Q, A)$ is called *synchronizing* if there is a word $w \in A^*$ such that $|Qw| = 1$, that is w acts as a constant map on Q. Such a word w is called a *synchronizing word*. Černý raised the following question: how large can a minimal length synchronizing word for a synchronizing automaton be as a function of the number of states of the automaton? He showed that for each $n > 1$, there are n state synchronizing automata with minimal synchronizing words of size $(n-1)^2$ [14]. The best known upper bound, due to Pin [39], is $\frac{n^3 - n}{6}$. Černý conjectured that in fact $(n-1)^2$ is the exact answer. Many special cases of the conjecture have been proved (for instance, [37, 17, 23, 6]), but the conjecture in general remains wide open.

In this section we show, using representation theory, that Černý's conjecture is true for synchronizing automata with transition monoids in the variety **DS**. We begin by giving a representation theoretic rephrasing of the problem from the thesis of Steinberg's Master's student Arnold [7].

Let $\mathcal{A} = (Q, A)$ be a deterministic automaton and let M be its transition monoid. Set $n = |Q|$. Let V be the $\mathbb{Q}$-vector space with basis $B = \{e_q \mid q \in Q\}$. Then there is a faithful representation $\varphi : M \to \mathrm{End}_{\mathbb{Q}}(V)$ defined on the basis by

$$e_q m \varphi = e_{qm}.$$

We consider V with the usual inner product. Let

$$V_1 = \mathrm{Span}\{\sum_{q \in Q} e_q\} \text{ and } V_0 = V_1^{\perp}.$$

We claim that V_0 is M-invariant. Indeed, suppose $v \in V_0$ and $m \in M$. Let $v_1 = \sum_{q \in Q} e_q$. Then

$$\langle vm\varphi, v_1 \rangle = \langle v, v_1(m\varphi)^T \rangle$$

(where $()^T$ denotes transposition). With respect to the basis B, $m\varphi$ is a row monomial matrix (meaning each row has precisely one non-zero entry) and

hence $m\varphi^T$ is column monomial. On the other hand, in the basis B, the vector v_1 is the vector of all ones, hence v_1 is fixed by any column monomial matrix. Thus

$$\langle vm\varphi, v_1 \rangle = \langle v, v_1(m\varphi)^T \rangle = \langle v, v_1 \rangle = 0,$$

establishing that $vm\varphi \in V_0$, as desired. We conclude that V_0 is M-invariant. Let $\psi : M \to \mathrm{End}_{\mathbb{Q}}(V_0)$ be the associated representation.

Without loss of generality, let us assume $Q = \{1, \ldots, n\}$. Then V_0 has basis $B_0 = \{f_1, \ldots, f_{n-1}\}$ where $f_i = e_n - e_i$. Also $f_i m\psi = e_{n \cdot m} - e_{i \cdot m}$. In particular, $m\psi = 0$ if and only if $n \cdot m = i \cdot m$ for all $i = 1, \ldots, n-1$, that is, if and only if m is a constant map. Thus $w \in A^*$ is a synchronizing word if and only if $[w]\psi = 0$, where $[w]$ is the image of w in M. In particular, $\mathcal{A}$ is synchronizing if and only if $M\psi$ contains the zero matrix. Since V_0 has dimension $n - 1$, we will have proved Černý's conjecture for the case that M belongs to **DS** once we have proven the following theorem, which can be viewed as the "mortality problem" for **DS**.

Theorem 7.1. *Let K be a field and let A be a finite alphabet. Let M be a finite A-generated submonoid of $M_k(K)$ belonging to* **DS** *and suppose that $0 \in M$. Then there exists a word $w \in A^*$ of length at most k^2 such that w maps to 0 in M.*

Before proving this theorem, we need a lemma.

Lemma 7.2. *Let $S \in$ **DS** *be a non-trivial generalized group mapping semigroup with a zero element 0. Then $S \setminus \{0\}$ is a subsemigroup.*

Proof. By definition, S has a (0-)minimal ideal I on which it acts faithfully on both the left and right. Since S is non-trivial, I cannot be the ideal 0. Thus I is 0-minimal. Since I is regular, $I\setminus\{0\}$ is a regular $\mathcal{J}$-class J. Suppose $s, t \geq_{\mathcal{J}} J$. Then, since $S \in$ **DS**, we have $st \geq_{\mathcal{J}} J$, see [1, Section 8.1]. Since S acts faithfully on I, only 0 is not $\mathcal{J}$-above J. Thus $S \setminus \{0\}$ is indeed a subsemigroup. $\square$

Proof of Theorem 7.1. By choosing a composition series for the KM-module K^k, we can place M in block upper triangular form where the diagonal block monoids $M_1, \ldots, M_r$, with $1 \leq r \leq k$, are irreducible. Since each M_i is a homomorphic image of M, they each have a zero element and each belong to **DS**. Being irreducible, they are generalized group mapping monoids by Theorem 3.9. Thus $M_i \setminus \{0\}$ is a submonoid by Lemma 7.2. Let $\alpha : M \to M_1 \times \cdots \times M_r$ be the projection. Suppose $w \in A^*$ maps to zero in M, then $w\alpha = 0$ and hence, for each $i = 1, \ldots, r$, there is a letter $a_i \in A$ with the ith coordinate of $a_i \alpha$ equal to zero (using that the product of non-zero elements of M_i remains non-zero). Thus we can find a word $u \in A^*$ of length at most $r \leq k$ such that u represents an element m of M with zeroes on the diagonal blocks. But then m is nilpotent of index at most k since it is a $k \times k$ upper triangular matrix with zeroes on the diagonal. Thus u^k represents 0 and $|u^k| \leq k^2$. $\square$

We remark that the proof gives a bound of $\min\{|A|, r\} \cdot r$ where r is the number of irreducible constituents of M. This is because in forming u we do not need to repeat letters and because the nilpotency index is actually bounded by the number of zero blocks on the diagonal. Hence if either $|A|$ or r are small, then we can do better than k^2.

Applying the above theorem in the context of the representation ψ of the transition monoid of an automaton on V_0 discussed above, we obtain the following theorem, verifying Černý's conjecture for **DS**.

Theorem 7.3. *Every synchronizing automaton on n states with transition monoid in* **DS** *has a synchronizing word of length at most $(n-1)^2$.*

We do not know whether $(n-1)^2$ is sharp when restricted to automata with transition monoids in **DS**.

A further application of the representation theory to Černý's conjecture can be found in a recent paper by F. Arnold and the third author [8].

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