SEPARATRIZES FOR $\mathbb{C}^2$ ACTIONS ON 3-MANIFOLDS

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ABSTRACT. We prove the existence of a separatrix for the singular foliation induced by a rank 2 action of $\mathbb{C}^2$ on a 3-dimensional manifold.

1. INTRODUCTION

The problem of the existence of separatrizes is a central theme in the local theory of singular holomorphic foliations. On a neighborhood of a singularity in dimension 2, the existence of separatrizes was settled in [C-S] completing a classical work of Briot and Bouquet. Here a separatrix for $\mathcal{F}$ is by definition an (germ of) analytic curve passing through the singularity and invariant under $\mathcal{F}$. However, as we increase the dimension and consider foliations on manifolds having dimension equal to 3, it becomes necessary to distinguish between foliations of dimension 1 and foliations of dimension 2 (or of codimension 1). By using a local coordinate, we can place ourselves on a neighborhood of the origin in $\mathbb{C}^3$. Then, in the case of a 1-dimensional foliation, a separatrix still is an (germ of) analytic curve passing through the origin and invariant by the foliation. As to codimension 1 foliations, a separatrix in this context should be understood as a germ of surface (i.e. 2-dimensional analytic set) passing through the origin and invariant by the foliation. Unfortunately, the existence of separatrizes is no longer verified for all foliations as above. In [GM-L] the reader will find examples of 1-dimensional foliations without separatrizes. For codimension 1 foliations the existence of counterexamples goes back to Jouanolou [J-1]. By studying the existence of invariant curves for foliations in the complex projective plane (see for example [LN-S], [L-R]), it can now be shown that these foliations "almost never" have separatrizes as it will be discussed below. The existence of separatrizes for codimension 1 foliations was also the object of the remarkable papers [Ca], [C-C] where is proved in particular that a strictly non-dicritical codimension 1 foliation on a neighborhood of $\mathbb{C}^3$ always possesses a separatrix. As it follows from the preceding discussion, strictly non-dicritical foliations are, in a specific sense, very "non-generic". For completeness, we also mention the work of Sancho de Salas [S] concerning invariant sets for a vector field having a singularity of high codimension. Similarly Stolovitch has investigated normal forms for certain families of commuting vector fields having rank at least 2 and their applications to the existence of invariant sets [St].

In the present paper we consider codimension 1 foliations that are generated by an action of $\mathbb{C}^2$ of rank 2 on a complex manifold of dimension 3. More precisely we shall work on a neighborhood of the origin in $\mathbb{C}^3$. In this local setting, we consider the foliation spanned by two holomorphic vector fields that commute and are linearly independent at generic points. The main result of this paper is as follows.

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Main Theorem. *Consider holomorphic vector fields X, Y defined on a neighborhood of the origin of $\mathbb{C}^3$. Suppose that they commute and are linearly independent at generic points so that they span a codimension 1 foliation denoted by $\mathcal{D}$. Then $\mathcal{D}$ possesses a separatrix.*

Note that the foliation $\mathcal{D}$ can be much more degenerate than the vector fields X, Y themselves since their k-jets may coincide to an order higher than the first non-trivial homogeneous component of X, Y. This is a considerable source of difficulty in the proof of our theorem. Also, if we define $\mathcal{D}$ by the differential 1-form induced by the vector product of X, Y, this form may have a singular set of codimension 1 even though the singular sets of X, Y are of codimension ≥ 2.

An interesting application of the above theorem concerns the case of a $\mathbb{C}^2$-action having rank 2 on a complex manifold of dimension 3. By a rank 2 action, we simply mean that its orbits have dimension 2 at generic points. On a neighborhood of a singular point for this action, the vector fields X, Y can, in addition, be chosen semi-complete. A lot of information about these vector fields can then be derived from their restriction to the separatrix of $\mathcal{D}$. In fact, semi-complete vector fields in dimension 2 are very well understood cf. [G-R] for detailed recent results. A particularly remarkable example of this situation can be found in [G]. This example was first discovered by Lins-Neto [LN] in connection with the so-called Painlevé problem, its associated geometry and dynamics was described in [G]. It consists of choosing X, Y respectively as the vector fields

$$\begin{aligned} Z_\infty &= 2z_2(-z_1+z_3)\frac{\partial}{\partial z_1} + (3z_1^2 - z_2^2)\frac{\partial}{\partial z_2} + 2z_3 z_2 \frac{\partial}{\partial z_3} \\ Z_0 &= (-3z_1^2 + z_2^2 + 2z_1 z_3)\frac{\partial}{\partial z_1} + 2z_2(-3z_1 + 2z_3)\frac{\partial}{\partial z_2} + 2z_3(3z_1 - z_3)\frac{\partial}{\partial z_3}\,. \end{aligned}$$

These vector fields correspond to an action of $\mathbb{C}^2$ on a suitable 3-manifold. The family spanned by them is such that a generic element has an isolated singularity at the origin. Yet some elements, such as Z_∞, possesses a singular set with codimension 2.

Remark. The fact that the vector fields belonging to the above mentioned family have order equal to 2 at the origin is not an accident. Indeed, our theorem can be used to show that vector fields X, Y as above possess a "separatrix of dimension 1", i.e. that there exists an analytic curve passing through the origin and invariant by the vector field. Once the existence of one such curve was established, it follows that the second jet of a semi-complete vector field at an isolated singularity cannot vanish, cf. [Reb]. The reader will note that the generic element of the family of vector fields linearly spanned by X, Y has, indeed, an isolated singularity at the origin. In turn, the existence of the above mentioned analytic curve follows from combining our Main Theorem with results of [G-R]. In fact, consider the restriction of, say X, to a separatrix $\mathcal{S}$ of $\mathcal{D}$ whose existence in ensured by Main Theorem. Note that we cannot apply Camacho-Sad theorem, [C-S], to this restriction since $\mathcal{S}$ may be singular. Yet it is proved in [G-R] that the existence of separatrizes remains valid for 2-dimensional singular spaces supporting a semi-complete vector field. The initial claim then follows from the combination of these results.

To close this Introduction, let us summarize the structure of this paper. To begin with, let us indicate how to construct numerous examples of codimension 1 foliations on a neighborhood of $(0,0,0) \in \mathbb{C}^3$ without separatrizes. Consider a homogeneous polynomial vector field Z defined on $\mathbb{C}^3$ and having an isolated singularity at $(0,0,0) \in \mathbb{C}^3$. Unless Z is a multiple of the radial vector field R, it induces a 1-dimensional holomorphic foliation $\mathcal{F}_{\mathbb{C}P(2)}$ of dimension 1 on $\mathbb{C}P(2)$. Conversely every 1-dimensional foliation on $\mathbb{C}P(2)$ is induced by a homogeneous vector field on $\mathbb{C}^3$. Next we consider the 2-dimensional distribution of planes on $\mathbb{C}^3$ which is spanned by Z and by R. The Euler relation (Equation 6) shows that Z, R generates a Lie algebra isomorphic to the Lie algebra of the affine group. The corresponding distribution is therefore integrable and hence yields a codimension 1 foliation $\mathcal{D}$. Clearly the punctual blow-up of $\mathcal{D}$ does not leave the exceptional divisor $\pi^{-1}(0)$ invariant (for details see Lemma 1). In fact, the intersections of the leaves of $\mathcal{D}$ with $\pi^{-1}(0)$ coincide with the leaves of $\mathcal{F}_{\mathbb{C}P(2)}$.

The upshot of the preceding construction regarding existence of separatrizes for $\mathcal{D}$ is as follows: if $\mathcal{D}$ possesses a *smooth* separatrix, then this separatrix will intersect $\pi^{-1}(0)$ on an algebraic curve which has to be invariant under $\mathcal{F}_{\mathbb{C}P(2)}$. Nonetheless it is known that, in a very strong sense, most choices of Z leads to a foliation $\mathcal{F}_{\mathbb{C}P(2)}$ that does not leave any proper analytic set invariant (cf. for example [LN-S], [L-R]). Thus this allows us to obtain many examples of codimension 1 foliations without a smooth separatrix. We also note that, for these examples, no smooth separatrix can be produced by adding "higher order terms" to $\mathcal{D}$.

Finally to make sure that most foliations $\mathcal{D}$ do not possess singular separatrizes either, it suffices to choose Z slightly more "generic" so that the singularities of the blow-up of $\mathcal{D}$ are "simple", cf. below. In fact, with this extra-assumption, it follows easily that $\mathcal{D}$ can admit only smooth separatrizes so that the problem becomes reduced to the above discussion.

To show that this phenomenon cannot take place in our context, we shall consider the intersection of our codimension 1 foliation with a given component of the exceptional divisor. Unless this component is invariant by the codimension 1 foliation, this intersection defines a foliation of dimension 1 on it. We shall then prove that all leaves of the latter foliation are compact[1] unless we are in a very particular situation which is already "linear" in a suitable sense. When considering these "linear" situations, the existence of a separatrix can directly be established. An example of this would consist of a couple of vector fields X, Y with X linear and Y equal to the Radial vector field $x\partial/\partial x + y\partial/\partial y + z\partial/\partial z$. These two vector fields commute and span a codimension 1 foliation which does not leave the exceptional divisor invariant if blown-up at the origin. Furthermore the foliation induced on the corresponding exceptional divisor by the mentioned blown-up foliation coincides with the foliation induced on $\mathbb{C}P(2)$ by X. In particular X can be chosen so that the "generic" leaf is not compact. However, in this situation the foliation induced by X on $\mathbb{C}P(2)$ still has a compact leaf which "directly" leads to the existence of the desired separatrix. Apart from these so-called "linear situations", the fact that the above mentioned leaves are all compact will be obtained by exploiting the mutual symmetries of X, Y yielded by their commutativity and the fact that their proper transform should be singular over the whole exceptional divisor.

[1] a more accurate statement would be that these leaves are properly embedded since the exceptional divisor itself may be non-compact. In fact, the codimension 1 foliation in question may have a local curve of singularities that may be used as center for certain blow-ups, cf. below

A similar example concerning blow-ups over curves that was pointed out to us by D. Cerveau goes as follows. Consider the pair of vector fields X, Y given by

$$X = zy\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z} \text{ and } Y = x^2\frac{\partial}{\partial x} + axy\frac{\partial}{\partial y}\,.$$

These two vector fields commute and span a codimension 2 foliation denoted by $\mathcal{D}$. They also leave the axis $\{y = z = 0\}$ invariant. Consider the blow-up of $\mathcal{D}, X, Y$ centered over $\{y = z = 0\}$. The proper transform $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ does not leave the exceptional divisor invariant. Furthermore the generic leaf of the foliation induced on the exceptional divisor by intersecting it with the leaves of $\widetilde{\mathcal{D}}$ is non-compact. The explanation for this phenomenon is that the blow-up of X is regular at generic points of the exceptional divisor. Indeed, X is already regular at generic points of the axis $\{y = z = 0\}$. This case must thus be considered as linear (indeed even regular). As it will be clear in Section 3, the appropriate notion of order of a vector field relative to a curve is such that the resulting order for X as above is *zero*. This is totally coherent with the fact that X is regular at generic points of this axis. In this context, to be "non-linear" roughly means that the mentioned order has to be at least 2.

The organization of the paper is as follows. In Section 2 we consider the case of a single punctual blow-up. The condition for the proper transforms of X, Y to vanish over the whole exceptional divisor is equivalent to the triviality of their linear parts at the origin (ie. at the center of the blow-up). Under this condition we prove that, if the codimension 1 foliation spanned by X, Y does not leave the exceptional divisor invariant, then all (1-dimensional) leaves induced by it on the exceptional divisor are compact (Proposition 1). Section 3 is devoted to obtaining an analogue of Proposition 1 for the case of blow-ups centered at a (smooth, irreducible) curve. In particular, this will require a suitable analogue of the "linear parts" of X, Y which is adapted to the curve in question. This is going to raise some minor additional difficulties as indicated by the above example. After introducing the appropriate setting, the main result of Section 3 will be Proposition 2 which is a faithful analogue of Proposition 1.

Since it is hard to imagine a theorem about arbitrarily degenerate singularities being proved without resorting to a suitable "desingularization" theorem, the fundamental results of [C-C], [Ca] about reduction of singularities of codimension 1 foliations will play a role in this paper. They will be brought to bear in Section 4. First we shall prove that the desired separatrix must exist provided that the 1-dimensional foliations induced on the dicritical components of the (total) exceptional divisor have only compact leaves. In these cases, the existence of the separatrix will follow from the combination of the compactness of the mentioned (1-dimensional) leaves with the fact that the "reduced singularities" are simple enough to allow for a total understanding of their (local) separatrizes. To prove our main result, we are then led to discuss the effect of the blow-up procedure of [C-C], [Ca] on the initial vector fields X, Y. The outcome of this discussion is that, to a large extent, Propositions 1 and 2 can be applied to guarantee the compactness of the (1-dimensional) leaves in question. Thus, at this point, we shall have the existence of the separatrix established except for some "special" cases in which the assumptions of Propositions 1 and 2 are not fulfilled. Fortunately these remaining cases are simple enough to be amenable to more direct methods.

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2. On the dicritical character of $\mathcal{D}$, Part I: Blowing-up a point

Consider two commuting holomorphic vector fields, X, Y, defined on $(\mathbb{C}^3, 0)$. Throughout this section, the vector fields X, Y are supposed to satisfy the following conditions:

(1) X, Y are linearly independent at generic points.
(2) The linear parts of X, Y at the origin are trivial.

Next we let $X = \sum_{i=1}^{3} X_i \partial/\partial x_i$ and $Y = \sum_{i=1}^{3} Y_i \partial/\partial x_i$. Because X and Y commute, they define a codimension 1 singular foliation $\mathcal{D}$ which is represented by the holomorphic 1-form

$$\Omega = (X_2Y_3 - X_3Y_2)dx + (X_3Y_1 - X_1Y_3)dy + (X_1Y_2 - X_2Y_1)dz \tag{1}$$

As usual we can assume that $\mathrm{Sing}\,(\mathcal{D})$ has codimension greater than or equal to 2. In other words, we can eliminate all non-trivial common factors from the components of the form Ω considered above.

In this work we shall deal with the codimension 1 foliation $\mathcal{D}$ as well as with the foliations $\mathcal{F}_X$, $\mathcal{F}_Y$ associated respectively to the vector fields X, Y. Unlike $\mathcal{D}$, the foliations $\mathcal{F}_X$, $\mathcal{F}_Y$ have *dimension* 1. Nonetheless, their singular sets can also be supposed to have codimension 2 or greater.

Recall that a *separatrix* for a *foliation of dimension* 1 (such as $\mathcal{F}_X$, $\mathcal{F}_Y$) is a germ of analytic curve passing through the origin and invariant by the foliation in question. Note that this definition does not exclude the possibility of having a separatrix entirely contained in the singular set of the corresponding foliation.

On the other hand, if we have a *codimension* 1 *foliation* (such as $\mathcal{D}$), then a separatrix is a germ of *analytic surface* passing through the origin and invariant by the corresponding foliation. We can also say that, in the latter case, a separatrix is given by an irreducible germ of analytic function f that divides $\Omega \wedge df$. Since the singular set of any foliation has codimension at least 2, a separatrix for a codimension 1 foliation is always obtained from a regular leaf that accumulates on the singular set. This contrasts with the case of dimension 1 foliations.

The proof of the main result in this paper depends on a "topological" theorem for codimension 1 foliations not necessarily spanned by commuting vector fields. This theorem provides a sufficient condition to guarantee the existence of separatrizes for the foliation in question. It also shows that the examples of foliations without separatrizes given in the Introduction have a certain "universal character". The corresponding statement is as follows.

Theorem 1. *Let $\mathcal{D}$ be a codimension 1 foliation defined on a neighborhood of the origin in $\mathbb{C}^3$ and consider a reduction of singularities (cf. Section 4)*

$$\mathcal{D} = \mathcal{D}^0 \xleftarrow{\pi_1} \mathcal{D}^1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_k} \mathcal{D}^k$$

for $\mathcal{D}$. Suppose that the restriction of $\mathcal{D}^k$ to every non-invariant component E of the total exceptional divisor is such that the (1-dimensional) foliation induced on E by this restriction possesses a non-constant meromorphic first integral. Then $\mathcal{D}$ possesses a separatrix.

Naturally it is implicit in the statement of Theorem 1 the existence of a "reduction of singularities" as indicated which, in turn, is a fundamental result appearing in [C-C] and [Ca]. The accurate form of these results will be given in the first paragraph of Section 4 along with the proof of Theorem 1. We point out in particular that the mentioned proof can be read independently of the material discussed in this section and in Section 3.

Remark 1. We can now provide further details on our strategy to prove the theorem stated in the Introduction. In fact, to deduce Main Theorem from Theorem 1 we shall try to prove that, when $\mathcal{D}$ is spanned by vector fields X, Y as in the statement of this theorem then the assumption of Theorem 1 is always verified. Although this is not strictly true, we shall prove that foliations not satisfying this condition are "very close to linear foliations" in a sense that will fully be made clear in Section 4. Fortunately these "almost linear" foliations will turn out to be simple enough to allow for a direct verification of the existence of separatrizes for them.

The main tool to understand foliations that fail to satisfy the condition of Theorem 1 will be Propositions 1 and 2. In particular, in the case of a single punctual blow-up, Proposition 1 asserts that the absence of the mentioned meromorphic first integral can only occur if the proper transforms of X, Y do not vanish identically over the exceptional divisor. naturally, in this case, the linear parts of X, Y at the origin must be non-trivial. The analogous case for blow-ups centered over curves will be discussed in Section 3.

In the procedure of reducing the singularities of $\mathcal{D}$ as indicated in Theorem 1 two types of blow-ups are considered, namely those centered at a single point and those centered at a smooth (irreducible) curve. In either case the proper transform $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ is well-defined and consists of a singular holomorphic foliation defined on the new (blown-up) manifold. We also mention that a foliation is said to be *dicritical*, for a specific blow-up under consideration, if the proper transform of the foliation does not leave the corresponding exceptional divisor invariant.

To implement the strategy outlined in Remark 1 our next task is to find conditions ensuring that, in the dicritical case, the proper transform of $\mathcal{D}$, under a blow-up as above, induces on the exceptional divisor a foliation admitting a nontrivial meromorphic first integral. The remainder of this section and all of Section 3 are devoted to this question. Naturally in the course of this discussion we shall assume that $\mathcal{D}$ is spanned by two commuting vector fields X, Y that are linearly independent at generic points. The desired conditions are summarized by Propositions 1 and 2. Proposition 1 concerns the case of punctual blow-ups and will be the main result of this section. Proposition 2 is the analogue of Proposition 1 for the case of blow-ups centered over a curve. Section 3 will be entirely devoted to discussing this type of blow-up and to the proof of Proposition 2.

From now on we fix a holomorphic 1-form Ω defining $\mathcal{D}$ and having singular set of codimension at least 2. In particular the singular set $\mathrm{Sing}\,(\mathcal{D})$ of $\mathcal{D}$ consists of isolated points and analytic curves. In the remainder of this section we shall exclusively deal with the case of punctual blow-ups.

Suppose then that $\mathcal{D}$ is dicritical for the blow-up centered at the origin. We are going to prove, in particular, that the foliation induced on the resulting exceptional divisor by the proper transform of $\mathcal{D}$ possesses a non-constant meromorphic first integral provided that $\mathcal{D}$ is spanned by vector fields X, Y satisfying conditions 1 and 2 in the beginning of the section (cf. Proposition 1). In particular observe that condition 2 already gives some hint on the "linear nature" of foliations failing to satisfy the assumption of Theorem 1 as discussed in Remark 1.

To begin our approach to Main Theorem and, more precisely, to Proposition 1 ,we are going to give a characterization of foliations whose punctual blow-up does not leave the exceptional divisor invariant. With this purpose, let us denote by R the Radial vector field (Euler vector field)

$$R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

This vector field will play a major role in the subsequent discussion. Part of this role is related to dicritical foliations as indicated in the lemma below.

Lemma 1. *Suppose that $\mathcal{D}$ is a singular codimension 1 foliation whose punctual blow-up at the origin $\widetilde{\mathcal{D}}$ does not leave the resulting exceptional divisor $E \simeq \mathbb{C}P\,(2) = \pi^{-1}(0)$ invariant. Then there is a holomorphic vector field Z tangent to $\mathcal{D}$ such that the first non-trivial homogeneous component of Z at the origin is a multiple of the Radial vector field R.*

Proof. Let $\mathcal{D}$ be given by the holomorphic 1-form $\omega = F\,dx + G\,dy + H\,dz$ whose singular set has codimension 2 or greater. Denote by ω_d the first non-trivial homogeneous component of ω at $(0,0,0)$. Here d stands for the degree of ω_d. Furthermore set $\omega_d = F^d dx + G^d dy + H^d dz$. In standard affine coordinates (x,t,u) for $\widetilde{\mathbb{C}}^3$ in which $\pi(x,t,u) = (x,tx,ux)$, the blow-up of ω is represented by

$$[F(x,tx,ux) + tG(x,tx,ux) + uH(x,tx,ux)]\,dx + xG(x,tx,ux)\,dt + xH(x,tx,ux)\,du\;.$$

Since the order of ω at the origin is d, we conclude that $\widetilde{\mathcal{D}}$, the blow up of $\mathcal{D}$, is given by

$$\begin{aligned}(2)\quad [F^d(1,t,u) + tG^d(1,t,u) + uH^d(1,t,u) + xr_1]\,dx + x(G^d(1,t,u) + xr_2)\,dt + \\ + x(H^d(1,t,u) + xr_3)\,du = 0\;,\end{aligned}$$

where r_1, r_2, r_3 are holomorphic functions. The condition for $E = \pi^{-1}(0)$ not to be invariant by $\widetilde{\mathcal{D}}$ implies that $F^d(1,t,u) + tG^d(1,t,u) + uH^d(1,t,u) = 0$ (identically). In turn this means that the direction $(1,0,0)$ is tangent to the foliation $\widetilde{\mathcal{D}}^d$ (the blow-up of the foliation $\mathcal{D}^d$ induced by $\omega_d = 0$). Equivalently the Radial vector field R is tangent to the foliation defined by ω_d, i.e. we have

$$xF^d + yG^d + zH^d = 0 \tag{3}$$

To produce the desired vector field we proceed as follows. Recall that in $\mathbb{C}^3$ the exterior product of two vectors (a,b,c), (a',b',c') is defined as $(bc' - cb', ca' - ac', ab' - ba')$. First we defined a vector field v by letting $v(p) = R(p) \wedge (F(p), G(p), H(p))$. Then we set $Z(p) = v(p) \wedge (F(p), G(p), H(p))$. We claim that the vector field Z satisfies our conditions. Indeed Z is clearly tangent to the foliation $\mathcal{D}$. To show that the first homogeneous component of Z is a multiple of R, first note that Z is given in the coordinates (x,y,z) by the formula

$$Z = (zFH+yFG-x(H^2+G^2)\,, xFG+zHG-y(F^2+H^2)\,, yHG+xFH-z(G^2+F^2)) \tag{4}$$

In particular it is immediate from this formula to check that Z is tangent to $\mathcal{D}$. It also follows that the order of Z at the origin is at least $2d+1$. The homogeneous component of degree $2d+1$ is, in turn, given by

$$\begin{aligned}(zF^dH^d + yF^dG^d - x((H^d)^2+(G^d)^2)\,,\ xF^dG^d + zH^dG^d - y((F^d)^2+(H^d)^2)\,, \\ yH^dG^d + xF^dH^d - z((G^d)^2+(F^d)^2)).\end{aligned} \tag{5}$$

However this expression can be rewritten as

$$(xF^d+yG^d+zH^d)(F^d,G^d,H^d) - ((F^d)^2+(G^d)^2+(H^d)^2)(x,y,z)\ .$$

In view of Equation 3, we conclude that the component of degree $2d+1$ of u at the origin is given by $-((F^d)^2+(G^d)^2+(H^d)^2)R$.

To finish the proof of the lemma it suffices to check that the polynomial $(F^d)^2 + (G^d)^2 + (H^d)^2$ cannot vanish identically. This is however easy. Let us use the variables (x,t,u) for the blow-up so that $F^d(1,t,u)+tG^d(1,t,u)+uH^d(1,t,u)$ is identically zero by assumption. Now suppose for a contradiction that $(F^d)^2+(G^d)^2+(H^d)^2$ is also identically zero. Then we must have $(t^2+1)(G^d)^2 + 2tu(G^d)(H^d) + (u^2+1)(H^d)^2 = 0$ (identically). Solving this equation for G^d we obtain a contradiction with the fact that G^d is itself a polynomial in the variables t,u. The lemma is proved. □

Naturally we are going to need an analogue of Lemma 1 in the case where we have a blow-up centered over a smooth curve. This discussion will be the object of Section 3. For the time being let us return to our original setting where $\mathcal{D}$ is spanned by vector fields X, Y satisfying conditions 1 and 2.

Consider a homogeneous polynomial vector field Z of degree $d \geq 2$. The Euler relation then provides

$$[R,Z] = (d-1)Z\ . \tag{6}$$

In particular R and Z do not commute. Furthermore, if Z is not a multiple of R, then the same holds when R is replaced by a multiple hR. Indeed, we have

$$[hR,Z] = h[R,Z] - (Z.h)\,R = (d-1)hZ - \left(\frac{\partial h}{\partial Z}\right) R \tag{7}$$

It is then clear that $[hR,Z] \neq 0$ provided that R, Z are linearly independent.

Next we have a simple lemma concerning the vanishing of the Lie bracket for a special type of vector fields.

Lemma 2. *Let Z_1, Z_2 be vector fields defined around $(0,0,0) \in \mathbb{C}^3$. Suppose that the first non-trivial homogeneous component of Z_1 at $(0,0,0)$ is a multiple of R. Suppose also that the linear part of Z_2 at the origin is trivial and that $[Z_1,Z_2]=0$. Then Z_1, Z_2 are linearly dependent at every point.*

Proof. Consider the punctual blow-up $\widetilde{Z}_1$ (resp. $\widetilde{Z}_2$) of Z_1 (resp. Z_2) at the origin. Since the first non-trivial homogeneous component of Z_1 is a multiple of R, we set $Z_1^d = hR$ where h is a homogeneous polynomial of degree $d-1$. Then it follows that the foliation associated to $\widetilde{Z}_1$ is transverse to $\pi^{-1}(0)$ away from the proper transform of $\{h=0\}$. Thus we can fix local coordinates (x,t,u), $\pi^{-1}(0) \subset \{x=0\}$, around a generic point of $\pi^{-1}(0)$ in which the vector field $\widetilde{Z}_1$ becomes $f(x,t,u)\partial/\partial x$. In these coordinates, we set $\widetilde{Z}_2 = \widetilde{Z}_{2,1}\partial/\partial x + \widetilde{Z}_{2,2}\partial/\partial t + \widetilde{Z}_{2,3}\partial/\partial u$. The equation $[\widetilde{Z}_1, \widetilde{Z}_2] = 0$ then yields

$$\frac{\partial \widetilde{Z}_{2,2}}{\partial x} = \frac{\partial \widetilde{Z}_{2,3}}{\partial x} = 0\,.$$

Therefore $\widetilde{Z}_{2,2}$ and $\widetilde{Z}_{2,3}$ do not depend on the variable x. However, since the linear part of Z_2 vanishes identically at the origin, $\widetilde{Z}_2$ vanishes identically on $\pi^{-1}(0)$. Thus its components $\widetilde{Z}_{2,2}$, $\widetilde{Z}_{2,3}$ must vanish everywhere since they do not depend on x. It follows that Z_1, Z_2 are linearly dependent on an open set and therefore they are so everywhere. The lemma is proved. □

Remark 2. Note that the assumption concerning the vanishing of the linear part of Z_2 at the origin was used only to ensure that the blown-up vector field $\widetilde{Z}_2$ vanishes identically on the resulting exceptional divisor. Let us emphasize again that this condition will play the main role in the course of this section. However, since in this section only punctual blow-ups are studied, the mentioned condition is equivalent to the triviality of the linear part of Z_2 at the center of the blow-up.

Let us go back to the vector fields X, Y considered in the beginning of the section. Recall that $\mathcal{D}$ stands for the codimension 1 foliation spanned by X, Y. Suppose that the punctual blow-up of $\mathcal{D}$ at the origin gives rise to a foliation that does not leave the exceptional divisor $\pi^{-1}(0) \simeq \mathbb{C}P(2)$ invariant. Then, according to Lemma 1, there are holomorphic functions f, g and h such that

$$fX + gY = hZ\,, \tag{8}$$

where Z is a holomorphic vector field whose first non-trivial homogeneous component is a multiple of R. Let $\mathrm{ord}\,(fX)$ (resp. $\mathrm{ord}\,(gY), \mathrm{ord}\,(hZ)$) denote the order of the vector field fX (resp. gY, hZ) at the origin.

Lemma 3. *With the above notations we have the following alternative*

(1) $\mathrm{ord}\,(hZ) > \min\{\mathrm{ord}\,(fX)\,,\, \mathrm{ord}\,(gY)\}$.
(2) *the first homogeneous component of X, X^H, (resp. Y, Y^H) admits a non-constant meromorphic first integral.*

Proof. Suppose that $\mathrm{ord}\,(hZ) = \min\{\mathrm{ord}\,(fX)\,,\, \mathrm{ord}\,(gY)\}$. All we have to do is to show that X^H possesses a non-constant first integral. Denote respectively by f^H, g^H, h^H the first non-trivial homogeneous components of f, g, h. Analogously define X^H, Y^H, Z^H. Without loss of generality, we can assume that $\mathrm{ord}\,(fX) \leq \mathrm{ord}\,(gY)$. We claim that, indeed, we must have $\mathrm{ord}\,(fX) = \mathrm{ord}\,(gY)$. To check this claim, just note that otherwise $f^H X^H$ is a multiple of the radial vector field R as it can be seen by considering the first non-trivial homogeneous component in Equation (8). This is however impossible since it contradicts the generic linear independence of X, Y (cf. Lemma 2). Hence we have shown that $\mathrm{ord}\,(fX) = \mathrm{ord}\,(gY)$ as desired.

Therefore, by considering again the first non-trivial homogeneous component in Equation (8), it follows that

$$f^H X^H + g^H Y^H = h^H q^H R \tag{9}$$

where, by assumption, none of the two sides vanishes identically. Furthermore q^H is also a homogeneous polynomial.

Because X, Y commute, so do X^H, Y^H. Thus we have

$$\begin{aligned}
[X^H, Y^H] &= \left[X^H, \frac{h^H q^H}{g^H} R - \frac{f^H}{g^H} X^H\right] \\
&= \left[X^H . \left(\frac{h^H q^H}{g^H}\right)\right] R - \frac{h^H q^H}{g^H}[R, X^H] - \left[X^H . \left(\frac{f^H}{g^H}\right)\right] X^H \\
&= \left[X^H . \left(\frac{h^H q^H}{g^H}\right)\right] R - \left[(d-1)\frac{h^H q^H}{g^H} - X^H . \left(\frac{f^H}{g^H}\right)\right] X^H \\
&= 0
\end{aligned}$$

where d denotes de degree of X^H. In particular

$$\left[X^H . \left(\frac{h^H q^H}{g^H}\right)\right] R = \left[(d-1)\frac{h^H q^H}{g^H} - X^H . \left(\frac{f^H}{g^H}\right)\right] X^H$$

If the expression between brackets on the right hand side (i.e. the expression multiplying X^H) does not vanish identically then X^H is a multiple of the Radial vector field. Again this is impossible in view of Lemma 2. Thus we conclude this expression is always equal to *zero*. Then it follows that

$$X^H . \left(\frac{h^H q^H}{g^H}\right) = 0\,.$$

In other words, $h^H q^H / g^H$ is a meromorphic first integral for X^H.

It only remains to prove that $h^H q^H / g^H$ is non-constant. To do this note that, if this function were constant, then we can assume $h^H q^H / g^H = 1$ without loss of generality. Hence dividing (9) by g^H, it would follows that

$$\frac{f^H}{g^H} X^H + Y^H = R.$$

This last equation is however impossible since Y^H has degree at least 2 and the expression $f^H X^H / g^H$ is homogeneous. Therefore $h^H q^H / g^H$ cannot be constant. Since the argument is symmetric in the vector fields X, Y, the last assertion completes our proof. □

We are now able to prove the main proposition of this section. It summarizes the preceding results and clarifies the nature of the foliation induced by $\mathcal{D}$ over the exceptional divisor in the dicritical case.

Proposition 1. *Let X, Y be two commuting vector fields satisfying Conditions 1 and 2 at the beginning of the section. Denote by $\mathcal{D}$ the codimension 1 foliation spanned by X, Y and suppose that the punctual blow-up $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ at the origin does not leave the corresponding exceptional divisor $\pi^{-1}(0)$ invariant. Then one has:*

(1) *The foliations* $\widetilde{\mathcal{F}}_X$ *and* $\widetilde{\mathcal{F}}_Y$ *coincide in their restriction to* $\pi^{-1}(0)$*, where* $\widetilde{\mathcal{F}}_X$ *(resp.* $\widetilde{\mathcal{F}}_Y$*) stands for the proper transform of* $\mathcal{F}_X$ *(resp.* $\mathcal{F}_Y$*) by the blow-up map* π *in question.*
(2) *The restrictions to* $\pi^{-1}(0)$ *of* $\widetilde{\mathcal{F}}_X$ *and* $\widetilde{\mathcal{F}}_Y$ *also coincide with the foliation induced on* $\pi^{-1}(0)$ *by* $\widetilde{\mathcal{D}}$*.*
(3) *The restrictions of* $\widetilde{\mathcal{F}}_X, \widetilde{\mathcal{F}}_Y$ *to* $\pi^{-1}(0)$ *possess a meromorphic first integral. In other words, the foliation induced by* $\widetilde{\mathcal{D}}$ *on* $\pi^{-1}(0)$ *defines a pencil on* $\pi^{-1}(0) \simeq \mathbb{C}P(2)$*.*

Proof. Let X^H, Y^H denote the first nontrivial homogeneous components of respectively X, Y at the origin. We know that none of the vector fields is a multiple of the radial vector field R (Lemma 2). In particular each of them induces a non-trivial foliation on the exceptional divisor $\pi^{-1}(0) \simeq \mathbb{C}P(2)$. Let us first check that these two foliations actually coincide. The condition for these two foliations to coincide is that either the vector fields X^H, Y^H are parallel or they span a 2-dimensional plane containing the radial direction (away from a proper analytic set). If none of these possibilities hold, then a direct inspection in the 1-form Ω, cf. Equation 1, defining the foliation $\mathcal{D}$ and in Equation 3 will contradict the dicritical nature of $\mathcal{D}$. In fact, the coefficients (F, G, H) of Ω are given by the exterior product between X^H, Y^H and they cannot satisfy Equation 3 unless one of the two above possibilities above holds. This shows that the foliations induced by X^H, Y^H on the exceptional divisor are the same. Finally it becomes equally clear that these two foliations coincide also with the foliation induced by $\widetilde{\mathcal{D}}$ on $\pi^{-1}(0)$. This proves the first two conclusions in the above statement.

To complete the proof it suffices to show that X^H admits non-constant meromorphic first integral. Indeed, since X^H is not a multiple of the Radial vector field the mentioned first integral also yields a non-constant first integral for the foliation induced by the projection of X^H on $\mathbb{C}P(2)$.

To show the existence of the desired first integral for X^H, recall that Lemma 1 ensures the existence of a holomorphic vector field Z satisfying Equation 8 for suitable holomorphic functions f, g, h. Furthermore the first non-trivial homogeneous component of Z at the origin is a multiple of the Radial vector field R. In turn Lemma 3 allows us to suppose that $\mathrm{ord}\,(hZ) > \min\{\mathrm{ord}\,(fX)\,, \mathrm{ord}\,(gY)\}$ (strictly) without loss of generality. In particular, we must have $\mathrm{ord}\,(fX) = \mathrm{ord}\,(gY)$ and, in fact,

$$f^H X^H + g^H Y^H = 0\,.$$

Alternatively we write

$$\frac{f^H}{g^H} X^H + Y^H = 0\,.$$

Therefore

$$\left[X^H, \frac{f^H}{g^H} X^H + Y^H\right] = [X^H, 0] = 0\,.$$

However, since $[X^H, Y^H] = 0$, the above equation amounts to

$$\left[X^H. \left(\frac{f^H}{g^H}\right)\right] .X^H = 0$$

so that $X^H.(f^H/g^H)$ must vanish identically. In other words f^H/g^H is a first integral for X^H. The statement is then proved unless f^H/g^H is constant. Therefore we just need to consider this latter possibility. This means that X^H and Y^H differ by a multiplicative constant. Set $X^H = cY^H$ for some $c \in \mathbb{C}^*$. Now note that the order of the new vector field $Y' = X - cY$ must be strictly larger than the order of X. Besides Y' is not constant equal to *zero* since X, Y are linearly independent at generic points. In fact, Y' is itself linearly independent with X at generic points. Furthermore the vector fields X, Y' still commute and they span the same foliation $\mathcal{D}$ as the initial pair X, Y. Therefore we can repeat the argument using X, Y' instead of X, Y. By construction the first non-trivial homogeneous component of Y' cannot differ from X^H by a multiplicative constant. Therefore X^H must admit a non-constant meromorphic first integral. The proposition is proved. □

3. On the dicritical character of $\mathcal{D}$, Part II: Blowing-up a curve

The next step towards the proof of Theorem 1 consists of obtaining an analogue of Proposition 1 for the case of blow-ups centered over smooth (irreducible) curves contained in Sing $(\mathcal{D})$. Indeed this section is entirely devoted to discussing the effect of blowing up a smooth curve contained in the singular set of $\mathcal{D}$.

Consider a point p belonging to a smooth curve contained in Sing $(\mathcal{D})$. On a neighborhood of p, there are local coordinates (x, y, z) in which the curve in question coincides with the z-axis, i.e. it is given by $\{x = y = 0\}$. For the blow-up centered at $\{x = y = 0\}$, we can introduce affine coordinates (x, t, z) and (u, y, z) such that the resulting blow-up map π_z is given by $\pi_z(x, t, z) = (x, tx, z)$ (resp. $\pi_z(u, y, z) = (uy, y, z)$). In the context of blow-ups along a fixed curve, the expression "a generic point of $\{x = y = 0\}$" is a synonymous of "except for a finite set of points". If the neighborhood of $(0, 0, 0) \in \mathbb{C}^3$ can be reduced without affecting the generality of the discussion then "a generic point" becomes an expression equivalent to "for every point in $\{x = y = 0\}$ with possible exception of the origin". Finally let R_z denote the vector field defined by

$$R_z = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\,.$$

In the case of punctual blow-ups, the first non-trivial homogeneous component of X, Y played the main role in establishing the existence of a non constant meromorphic first integral for the foliation induced over $\mathbb{CP}(2)$ under appropriate conditions. We shall begin this section by introducing an analogue of these "homogeneous components" which are suitable to deal with the blow-up over the curve $\{x = y = 0\}$. In particular, the corresponding "homogeneous" vector fields will satisfy the same properties as those verified by the first components of X, Y in the case of punctual blow-up. Most notably, they will still commute and they will also encode the information determining the dicritical/non-dicritical nature of $\mathcal{D}$.

To motivate the definition, note that the first non-trivial homogeneous component of X can be recovered as

$$\lim_{\lambda \to 0} \frac{1}{\lambda^{d-1}} \Gamma_\lambda^* X$$

where $\Gamma_\lambda^* X$ denotes the pull-back of X by the homothety $\Gamma_\lambda : (x, y, z) \mapsto (\lambda x, \lambda y, \lambda z)$ and d stands for the degree of the first non-trivial homogeneous component of X. In the present case we shall consider an adapted notion of homothety, namely the one obtained through the family of automorphisms given by

$$\Lambda_\lambda : (x, y, z) \mapsto (\lambda x, \lambda y, z).$$

In this case, the pull-back of X by Λ_λ becomes

$$\Lambda_\lambda^* X = \frac{1}{\lambda}\left[X_1(\lambda x, \lambda y, z)\frac{\partial}{\partial x} + X_2(\lambda x, \lambda y, z)\frac{\partial}{\partial y}\right] + X_3(\lambda x, \lambda y, z)\frac{\partial}{\partial z}$$

Denote by k (resp. l) the maximal power of λ that divides $X_1(\lambda x, \lambda y, z)\partial/\partial x + X_2(\lambda x, \lambda y, z)\partial/\partial y$ (resp. $X_3(\lambda x, \lambda y, z)\partial/\partial z$). It corresponds to the degree of the first non-trivial homogeneous components relative to the variables x, y in each expression. Then

$$\begin{aligned}\Lambda_\lambda^* X \;=\; & \lambda^{k-1}\left[(X_1^k(x, y, z) + \lambda \bar{X}_{1,\lambda}(x, y, z))\frac{\partial}{\partial x} + (X_2^k(x, y, z) + \lambda \bar{X}_{2,\lambda}(x, y, z))\frac{\partial}{\partial y}\right] + \\ & + \lambda^l(X_3^l(x, y, z) + \lambda \bar{X}_{3,\lambda}(x, y, z))\frac{\partial}{\partial z}\end{aligned}$$

where X_i^j stands for the homogeneous component of degree j, of X_i, relative to the variables x, y. Since the powers of λ in the different components do not coincide, three cases must be considered according to the possibilities $l > k-1$, $l = k-1$ or $l < k-1$. The expression for the limit vector field $\bar{X}$ will change accordingly. Naturally, when it comes to blow-ups along a smooth curve, the vector field $\bar{X}$ is going to represent the desired analogue of the "first non-trivial homogeneous component" as considered in the preceding section.

a) Suppose $l > k-1$. In this case we consider the vector fields

$$X_\lambda = \frac{1}{\lambda^{k-1}}\Lambda_\lambda^* X\,.$$

and the limit

$$\bar{X} = \lim_{\lambda\to 0} X_\lambda.$$

Clearly $\bar{X}$ is the "first non-trivial homogeneous component in the variables x, y" and it has the form

$$\bar{X} = X_1^k(x, y, z)\frac{\partial}{\partial x} + X_2^k(x, y, z)\frac{\partial}{\partial y}.$$

b) If $l = k-1$ we still consider the vector field $X_\lambda = \Lambda_\lambda^* X/\lambda^{k-1}$ and define $\bar{X}$ as above. A similar argument shows that $\bar{X}$ has the form

$$\bar{X} = X_1^k(x, y, z)\frac{\partial}{\partial x} + X_2^k(x, y, z)\frac{\partial}{\partial y} + X_3^{k-1}(x, y, z)\frac{\partial}{\partial z}$$

c) Finally, if $l < k-1$ we let

$$X_\lambda = \frac{1}{\lambda^l}\Lambda_\lambda^* X$$

and consider $\bar{X} = \lim_{\lambda\to 0} X_\lambda$ which, in this case, is a "vertical" vector field

$$\bar{X} = X_3^l(x, y, z)\frac{\partial}{\partial z}.$$

As mentioned, the vector field $\bar{X}$ represents the analogue of the first non-trivial homogeneous component of X under the adapted homothety Λ_λ. Given a vector field X as above, throughout this section the *first non-trivial homogeneous component of* X *in the variables* x, y is, by definition, the vector field $\bar{X}$ constructed above. Naturally the order of X on the variables x, y will be the *degree w.r.t. the variables* x, y (or simply *degree* when no misunderstanding is possible) of $\bar{X}$. In turn the *degree w.r.t. the variables* x, y is by definition the minimum between k and $l + 1$. Finally note that $\bar{X}$ may also be viewed as a homogeneous polynomial vector field in the variables x, y with coefficients in $\mathbb{C}[z]$ (except that the degree of the third component may differ from the degree of the other two cf. below).

The vector field $\bar{Y}$ is analogously defined. The commutativity of $\bar{X}$ and $\bar{Y}$ follows easily from the commutativity between X and Y. In fact, $\Lambda_\lambda^* X$ commutes with $\Lambda_\lambda^* Y$ for every λ. The same being true when these two vector fields are multiplied by arbitrary constants. Hence, by taking a suitable limit, we conclude that the vector fields $\bar{X}$ and $\bar{Y}$ must commute as well.

A similar notion of "first non-trivial homogeneous component" is also natural for a codimension 1 foliation given by a holomorphic 1-form $Fdx+Gdy+Hdz$. Again this "first non-trivial homogeneous component in the variables x, y" is obtained as an appropriate limit of pull-backs of ω by the automorphisms $\Lambda_\lambda : (x, y, z) \mapsto (\lambda x, \lambda y, z)$. Details are left to the reader. We point out, however, that the resulting component can equally well be seen as a polynomial 1-form in the variables x, y with coefficients in $\mathbb{C}[z]$. Nonetheless the case analogous to the case "b)" (where $l = k - 1$) of vector fields in which this "first non-trivial homogeneous component in the variables x, y" may have non-trivial components in all the coordinates dx, dy, dz is now associated to the possibility $l = k + 1$.

Consider now the codimension 1 foliation $\mathcal{D}$ spanned by X, Y. The following lemma will play, in the context of blow-ups over curves, a role analogous to the role played by Lemma 1 in the preceding section.

Lemma 4. *Suppose that* $\mathcal{D}$ *is singular over* $\{x = y = 0\}$ *and denote by* $\widetilde{\mathcal{D}}$ *the corresponding blown-up foliation. Suppose also that* $\mathcal{D}$ *is dicritical, i.e. that the resulting exceptional divisor is not invariant by* $\widetilde{\mathcal{D}}$*. Then there exists a holomorphic vector field* Z *tangent to* $\mathcal{D}$ *and satisfying the following conditions:*

- Z *is singular over* $\{x = y = 0\}$*.*
- *The first non-trivial homogeneous component* $\bar{Z}$ *of* Z*, in the sense adapted to blow-ups over curves, is a multiple of* R_z *having the form* $P_z(x, y)R_z$ *where* P_z *stands for a homogeneous polynomial in the variables* x, y *with coefficients in* $\mathbb{C}[z]$*.*

Proof. To prove the statement let us consider a holomorphic 1-form $\omega = Fdx+Gdy+Hdz$ defining $\mathcal{D}$ and having singular set of codimension at least 2. Denote by $\bar{\omega}$ the first non-trivial homogeneous component of ω relative to the variables x, y (i.e. the first non-trivial homogeneous component adapted to the blow-up centered at the curve $\{x = y = 0\}$). The behavior of the foliation defined by $\bar{\omega}$ determines the behavior of $\widetilde{\mathcal{D}}$ over the exceptional divisor. In particular the former foliation is dicritical if and only if $\mathcal{D}$ is so.
Claim. The vector field R_z is tangent to the leaves of the foliation defined by $\bar{\omega}$.
Proof of the claim. As already seen, there are three possibilities for the 1-form $\bar{\omega}$, namely it has only a component dz, it has only components dx, dy and, finally, it has all the three

components dx, dy, dz. Recalling the existence of affine coordinates (x,t,z) for the blow-up of $\mathbb{C}^3$ over $\{x=y=0\}$ in which the blow-up map π_z becomes $\pi_z(x,t,z)=(x,tx,z)$, the blow-up $\pi_z^*(\bar{\omega})$ of $\bar{\omega}$ is respectively given by $h_{d,z}(1,t)dz$, $[f_{d,z}(1,t)+tg_{d,z}(1,t)]dx + xg_{d,z}(1,t)dt$ or

$$\pi_z^*(\bar{\omega}) = [f_{d,z}(1,t)+tg_{d,z}(1,t)]dx + xg_{d,z}(1,t)dt + xh_{d,z}(1,t)dz\ , \tag{10}$$

up to a multiplicative factor. Here we note that $f_{d,z}(x,t), g_{d,z}(x,t)$ (resp. $h_{d,z}(x,t)$) are homogeneous polynomials of degree d (resp. $d-1$) in the variables x,t with coefficients in $\mathbb{C}[z]$. In addition, the last case happens when the order (previously denoted by l) of H with respect to the variables x,y exceeds by one unit the order (previously denoted by k) of the form $Fdx+Gdy$, w.r.t. the variables x,y. The claim is equivalent to showing that the constant vector field $(1,0,0)$ is tangent to the leaves of $\pi_z^*(\bar{\omega})$. The proof of this will rely on the fact that the exceptional divisor is not invariant by the foliation defined by $\pi_z^*(\bar{\omega})$. As already noted the fact that the exceptional divisor is not invariant by $\pi_z^*(\bar{\omega})$ is indeed equivalent to our assumption that it is not invariant by $\widetilde{\mathcal{D}}$. Now, to prove that the leaves of $\pi_z^*(\bar{\omega})$ are all tangent to the vector $(1,0,0)$, we shall go through the three possibilities for $\pi_z^*(\bar{\omega})$. When $\pi_z^*(\bar{\omega})$ has the form $h_{d,z}(1,t)dz$ the statement is obvious and needs no further comment. If $\pi_z^*(\bar{\omega}) = [f_{d,z}(1,t)+tg_{d,z}(1,t)]dx + xg_{d,z}(1,t)dt$, then the condition of being dicritical implies that $f_{d,z}(1,t)+tg_{d,z}(1,t)$ must vanish identically. It is easy to deduce from this that $(1,0,0)$ is tangent to the leaves of $\pi_z^*(\bar{\omega})$. Finally in the third case, corresponding to Equation 10, all that has to be checked is that $f_{d,z}(1,t)+tg_{d,z}(1,t)$ vanishes identically. This is however obvious, otherwise the exceptional divisor (locally given by $\{x=0\}$) would be invariant by $\pi_z^*(\bar{\omega})$, hence contradicting the fact that $\mathcal{D}$ is dicritical. The claim is proved. □

To complete the proof of the lemma, we still need to construct the vector field Z. This however goes as in Lemma 1. Let $v(p)$ be given by the exterior product $R_z \wedge (F,G,H)$ and set $Z(p) = v(p) \wedge (F,G,H)$. Clearly Z is tangent to the leaves of $\mathcal{D}$. Besides a straightforward calculation shows that its first non-trivial component is a multiple of R_z. The lemma is proved. □

Next let $\mathcal{F}_X$ be the foliation associated to X. We assume that the axis $\{x=y=0\}$ is invariant by X, that is to say, either it is contained in the singular set of X or it constitutes a separatrix for X. Let k and l be as defined above (in connection with the first non-trivial homogeneous component in the variables x, y, $\bar{X}$, of X). A direct inspection in the formulas related to the preceding possibilities a), b) and c) for the nature of $\bar{X}$ makes it clear that the proper transform of $\mathcal{F}_X$ under π_z leaves the exceptional divisor invariant unless $l \geq k$ and the vector field $X_1(\lambda x, \lambda y, z)\partial/\partial x + X_2(\lambda x, \lambda y, z)\partial/\partial y$ is a multiple of R_z. In other words, the foliation is dicritical if and only if $\bar{X}$ is a multiple of R_z at points in $\{x=y=0\}$, i.e. if and only if $\bar{X}$ has the form $P_z(x,y)(x\partial/\partial x + y\partial/\partial y)$ for some homogeneous polynomial P_z in the variables x, y with coefficients in $\mathbb{C}[z]$.

Next we state:

Lemma 5. *Suppose that Z_1, Z_2 are two commuting vector fields defined on $(\mathbb{C}^3,0)$. Suppose that the first non-trivial homogeneous component of Z_1 in the variables x, y at points in $\{x=y=0\}$ has the form $P_z(x,y)(x\partial/\partial x + y\partial/\partial y)$ for a homogeneous polynomial P in the variables x, y with coefficients in $\mathbb{C}[z]$. Suppose also that the order of Z_2 relative* *to the variables x, y at points in $\{x = y = 0\}$ is at least 2. Then Z_1, Z_2 are linearly dependent everywhere.*

Proof. Keeping the preceding notations, consider the blow-up map π_z. The proper transforms $\widetilde{Z}_1, \widetilde{Z}_2$ of Z_1, Z_2 under π_z are holomorphic on a neighborhood of the corresponding exceptional divisor $\pi_z^{-1}(0)$. Besides the foliation associated to $\widetilde{Z}_1$ is transverse to $\pi_z^{-1}(0)$. Thus, on a neighborhood of a generic point of $\pi_z^{-1}(0)$, we can introduce coordinates (x, t, u) such that:

1. The foliation associated to $\widetilde{Z}_1$ is given by $\partial/\partial x$.
2. $\{x = 0\} \subset \pi_z^{-1}(0)$ is contained in the set of zeros of $\widetilde{Z}_2$.

It then follows that $\widetilde{Z}_1$ is given in these coordinates by $f\partial/\partial x$ for some holomorphic function f. The rest of the proof goes exactly as in the proof of Lemma 2. More precisely, the condition on the vanishing of the Lie bracket of $\widetilde{Z}_1$, $\widetilde{Z}_2$ ensures that the components of $\widetilde{Z}_2$ in the coordinates t, z do not depend on the variable x. Since $\widetilde{Z}_2$ equals zero over the exceptional divisor, locally given by $\{x = 0\}$, it follows that these components must be zero everywhere. In other words, $\widetilde{Z}_1$, $\widetilde{Z}_2$ must be parallel on an open set and hence everywhere. □

Clearly the assumption on the order of Z_2 cannot be removed by the above statement in view of the example given in the Introduction. Here it might be a good point to remind the reader that the order of the vector field $X = zy\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z}$ in the variables y, z over the axis $\{y = z = 0\}$ equals indeed *zero*. Similarly the blow-up of X over $\{y = z = 0\}$ yields a holomorphic vector field which does not vanish at generic points of the resulting exceptional divisor.

Remark 3. We note that the assumptions concerning the order of Z_2 and the first non-trivial homogeneous component of Z_1 in the variables x, y were used only in the items 1 and 2 in the above proof. The assumption on Z_1 translates into the fact that the blow-up of the foliation $\mathcal{F}_X$ associated to X is transverse to the exceptional divisor at generic points. As to vector field Z_2, the role played by the condition that its order w.r.t. the variables x, y (that was itself defined as the minimum between k and $l+1$ above) has to be at least 2 is totally encoded into the fact that the blow-up of Z_2 must vanish over the whole exceptional divisor. In other words, the assumptions in Lemma 5 can be replaced by the conditions above which appear more explicitly in its proof (whereas these conditions may seem more technical at first sight).

Before arriving to the desired analogue of Proposition 1, we are going to need the corresponding analogue of Lemma 3.

Again we go back to the vector fields X, Y that span the codimension 1 foliation $\mathcal{D}$. However now we suppose that $\{x = y = 0\}$ is contained in Sing $(\mathcal{D})$ and that the blow-up of $\mathcal{D}$ along $\{x = y = 0\}$ does not leave the exceptional divisor invariant. Then, according to Lemma 4, there are holomorphic functions f, g and h such that

$$fX + gY = hZ\,, \tag{11}$$

where Z is a holomorphic vector field whose first non-trivial homogeneous component $\bar{Z}$ in the variables x, y at points in $\{x = y = 0\}$ is a multiple of R_z. Let ord (fX) (resp.

$\mathrm{ord}\,(gY), \mathrm{ord}\,(hZ))$ denote the order of the vector field fX (resp. gY, hZ) in the variables x, y at a generic point in $\{x = y = 0\}$.

Lemma 6. *With the above notations, we have the following alternative:*

(1) $\mathrm{ord}\,(hZ) > \min\{\mathrm{ord}\,(fX)\,, \mathrm{ord}\,(gY)\}\,.$
(2) *the first homogeneous component in the variables x, y, $\bar{X}$, of X admits a non-constant meromorphic first integral.*

Proof. Suppose for a contradiction that the above estimate does not hold. For fX, gY and hZ we are going to consider their first homogeneous components in the variables x, y. Denote by $\bar{X}$, $\bar{Y}$ the respective non-trivial homogeneous components of X, Y and by $\bar{f}$, $\bar{g}$, $\bar{h}$ the homogeneous components of f, g, h (all these homogeneous components are to be understood as relative to the variables x, y). With these notations, one has:

$$\bar{f}\bar{X} + \bar{g}\bar{Y} = \bar{h}\bar{q}R_z\,, \tag{12}$$

where $\bar{q}$ is a homogeneous polynomial in the variables x, y with coefficients in $\mathbb{C}[z]$. Since X, Y commute, it follows that $\bar{X}, \bar{Y}$ commute as well. Therefore

$$\begin{aligned}
[\bar{X}, \bar{Y}] &= \left[\bar{X}, \frac{\bar{h}\bar{q}}{\bar{g}}R_z - \frac{\bar{f}}{\bar{g}}\bar{X}\right] \\
&= \left[\bar{X}.\left(\frac{\bar{h}\bar{q}}{\bar{g}}\right)\right] R_z - \frac{\bar{h}\bar{q}}{\bar{g}}[R_z, \bar{X}] - \left[\bar{X}.\left(\frac{\bar{f}}{\bar{g}}\right)\right] \bar{X} \\
&= 0\,.
\end{aligned}$$

The commutator $[R_z, \bar{X}]$ is given by

$$[R_z, \bar{X}] = (x\frac{\partial \bar{X}_1}{\partial x} + y\frac{\partial \bar{X}_1}{\partial y} - X_1)\frac{\partial}{\partial x} + (x\frac{\partial \bar{X}_2}{\partial x} + y\frac{\partial \bar{X}_2}{\partial y} - X_2)\frac{\partial}{\partial y} + (x\frac{\partial \bar{X}_3}{\partial x} + y\frac{\partial \bar{X}_3}{\partial y})\frac{\partial}{\partial z}.$$

As previously seen, the components $\bar{X}_1$, $\bar{X}_2$ are homogeneous of degree k in the variables x, y while $\bar{X}_3$ is homogeneous of degree $k-1$, if not identically zero. In fact, the remaining case in which $\bar{X}$ has only a component in the direction of $\partial/\partial z$ obviously admits a non-constant first integral so that it can be excluded. Therefore

$$x\frac{\partial \bar{X}_i}{\partial x} + y\frac{\partial \bar{X}_i}{\partial y} = kX_i$$

for $i = 1, 2$, while

$$x\frac{\partial \bar{X}_3}{\partial x} + y\frac{\partial \bar{X}_3}{\partial y} = (k-1)X_3$$

unless $\bar{X}_3$ is identically zero. In all cases, the equation $[R_z, \bar{X}] = (k-1)\bar{X}$ holds. Combined to the above equations, it follows that

$$\left[\bar{X}.\left(\frac{\bar{h}\bar{q}}{\bar{g}}\right)\right] R_z = \left[(k-1)\frac{\bar{h}\bar{q}}{\bar{g}} + \bar{X}.\left(\frac{\bar{f}}{\bar{g}}\right)\right] \bar{X}.$$

If the expression multiplying $\bar{X}$ on the right-hand side does not vanish identically, then $\bar{X}$ is a multiple of R_z. This is however impossible since it would imply that X and Y are linearly dependent at every point by virtue of Lemma 5. Therefore the mentioned expression is constant equal to *zero* and hence $\bar{X}.\left(\frac{\bar{h}\bar{q}}{\bar{g}}\right)$ is identically *zero* as well. This implies that $\bar{h}\bar{q}/\bar{g}$ is a meromorphic first integral for $\bar{X}$. The same argument of Lemma 3 proves that this first integral is not constant everywhere. $\square$

Thanks to the preceding lemmas, the desired analogue of Proposition 1 can finally be stated.

Proposition 2. *Let X, Y be two commuting vector fields that are linearly independent at generic points. Denote by $\mathcal{D}$ the codimension 1 foliation spanned by X, Y. The foliation $\mathcal{D}$ is supposed to be singular over the axis $\{x = y = 0\}$ and dicritical w.r.t. the blow-up centered at this axis. Finally, we suppose in addition that the order of X, Y in the variables x, y at points in $\{x = y = 0\}$ is at least equal to 2. Then one has:*

1. *The foliations $\widetilde{\mathcal{F}}_X$ and $\widetilde{\mathcal{F}}_Y$ coincide in their restriction to the exceptional divisor $\pi_z^{-1}(0)$, where $\widetilde{\mathcal{F}}_X$ (resp. $\widetilde{\mathcal{F}}_Y$) stands for the proper transform of $\mathcal{F}_X$ (resp. $\mathcal{F}_Y$) by the blow-up map π_z in question.*
2. *The restrictions to $\pi_z^{-1}(0)$ $\widetilde{\mathcal{F}}_X$ and $\widetilde{\mathcal{F}}_Y$ also coincide with the foliation induced on $\pi_z^{-1}(0)$ by $\widetilde{\mathcal{D}}$.*
3. *The restrictions of $\widetilde{\mathcal{F}}_X, \widetilde{\mathcal{F}}_Y$ to $\pi_z^{-1}(0)$ possess a non-constant meromorphic first integral.*

Proof. All the material was prepared so that the proof of Proposition 1 applies word-by-word in the present setting. It suffices to replace "first homogeneous component" (at a point) by "first homogeneous components in the variables x, y" (over the curve $\{x = y = 0\}$). $\square$

4. Existence of separatrizes

In this last section we are going to prove first Theorem 1 and then the Main Theorem already stated in the Introduction. We begin by noticing that, among codimension 1 foliation on $(\mathbb{C}^3, 0)$, those having a separatrix are relatively rare in the sense that "almost all homogeneous polynomial vector field in three variables" yields a codimension 1 foliation without separatrix by means of the construction explained in the Introduction. Not surprisingly the proof of our main theorems relies heavily on the fact that the corresponding foliations are spanned by two commuting vector fields, i.e. by two vector fields generating an Abelian Lie algebra. The examples presented in the Introduction also show that our results do not generalize to the case of vector fields generating a Lie algebra isomorphic to the Lie algebra of the affine group without further conditions.

Since a main ingredient in our proof of existence of separatrizes concerns the reduction of singularities for codimension 1 foliations on $\mathbb{C}^3$, let us begin the discussion with a brief review of the results in [C-C], [Ca].

4.1. Reduction of singularities of codimension 1 foliations and proof of Theorem 1.

Recall that Cano and Cerveau have proved a theorem of reduction of singularities for codimension 1 foliations on $(\mathbb{C}^3, 0)$ that are *strictly non-dicritical* [C-C]. More recently, Cano obtained a general reduction theorem for singularities of codimension 1 foliations on $(\mathbb{C}^3, 0)$ [Ca]. The latter theorem asserts the existence of a finite sequence of blowing-up maps along with proper transforms of $\mathcal{D}$,

$$\mathcal{D} = \mathcal{D}^0 \xleftarrow{\pi_1} \mathcal{D}^1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_k} \mathcal{D}^k \tag{13}$$

such that:

(1) The center of each blow-up map is invariant by the corresponding proper transform of $\mathcal{D}$ and has normal crossings with the corresponding exceptional divisor.
(2) $\mathcal{D}^k$ has only simple singularities.

First note that the foliation $\mathcal{D}$ is said to be *strictly non-dicritical* if, in the above procedure, all the irreducible components of the total exceptional divisor are invariant by $\mathcal{D}^k$. This implies that, for any sequence of blow-ups verifying condition 1 above, the resulting foliation leaves the exceptional divisor fully invariant.

The notion of simple singularities for a codimension 1 foliation defined on a complex 3-manifold was introduced in [C-C]. For the convenience of the reader, we briefly recall the possible models below referring to [C-C], [Ca] for further information.

Models of Type A: in this case, the foliation is locally given by a pair of commuting vector fields Z_1, Z_2, having the general form: $Z_1 = \partial/\partial z$, $Z_2 = x\partial/\partial x + a(x,y)\partial/\partial y$, $a(0,0) = 0$.

In this case, we can in addition assume that the eigenvalues of linear part of Z_2 at the origin are such that their quotient is not positive rational. The restriction of Z_2 to the invariant plane $\{z = 0\}$ is then a simple singularity in the usual sense of vector fields in dimension 2. In particular, if this singularity is not a saddle-node (i.e. if both eigenvalues are different from zero), then the vector field obtained as restriction of Z_2 to $\{z = 0\}$ has exactly 2 separatrizes. Considering the special form of the vector field Z_1, it becomes obvious that these separatrizes give rise to codimension 1 separatrizes for the corresponding foliation. Actually the foliation spanned by Z_1, Z_2 is nothing but the cylinder over the foliation induced by Z_2 on the plane $\{z = 0\}$.

Models of Type B: Here there are still locally defined commuting vector fields Z_1, Z_2 spanning the foliation and given by $Z_1 = x\partial/\partial x + a(x,y,z)\partial/\partial z$ and $Z_2 = y\partial/\partial y + b(x,y,z)\partial/\partial z$, with $a(0,0,0) = b(0,0,0) = 0$.

In the present case, we can assume that the eigenvalues of both Z_1, Z_2 possess a quotient lying in the complement of $\mathbb{Q}_+$.

Remark 4. Consider a singularity of Type B for a foliation $\mathcal{D}^k$ at a point p belonging to a dicritical component E of the total exceptional divisor. It is shown in [C-C] that the above coordinates (x,y,z) can be chosen so that, in addition, E locally coincides with one of the coordinate planes. Since the planes $\{x = 0\}$ and $\{y = 0\}$ are clearly invariant by Z_1, Z_2, we conclude that E must locally be given by $\{z = 0\}$. This means in particular that a or b is not divisible by z.

In this context, a standard argument involving 2-dimensional saddle-node singularities shows that the cylinders lying over the coordinates axes $\{y = z = 0\}$ and $\{x = z = 0\}$ define the two separatrizes of $\mathcal{D}^k$ at p. Note that this statement is not totally obvious as we shall explain. Fix for example the axis $\{y = z = 0\}$ and consider the saddle-node singularity on the plane $\{y = 0\}$ which is given by the vector field $x\partial/\partial x + a(x,0,z)\partial/\partial z$. The above statement then requires to know that, for these 2-dimensional saddle-node, the saturated of a small section transverse to the strong invariant manifold contains a neighborhood of the origin modulo taking its union with of a possible weak invariant manifold for the the singularity in question. In fact this condition ensures that the cylinder over the strong invariant manifold “is not pinched at the origin, i.e. it approaches 0 keeping a uniformly positive “vertical distance.

Having explained what is meant by “simple singularities” in condition 2, we need to say a few additional words about the nature of the centers of the above mentioned blow-ups. In particular, we would like to emphasize that it may be necessary to blow-up points lying away from the singular set of the foliation in question. The centers of a blow-up map that are not contained in the singular set of the corresponding foliation are however special. Namely, according to Condition 1 they are invariant by the foliations. In other words, the part of this center lying away from the singular set of the mentioned foliations is still contained in a single leaf of this foliation. This fact has an important consequence that will be used in our proofs. To explain it, let us suppose that π_j is a blow-up whose center in not fully contained in the singular set of $\mathcal{D}^{j-1}$. Clearly the center of π_j must be a (irreducible smooth) curve C_{j-1} and not a single point. This center also intersects non-trivially the singular set of $\mathcal{D}^{j-1}$ (otherwise this blow-up is not necessary). This intersection however consists of finitely many points $\{q_1, \ldots, q_r\}$. Since $\mathcal{D}^{j-1}$ is regular on $C_{j-1} \setminus \{q_1, \ldots, q_r\}$ we have:

Lemma 7. *With the preceding notations, the proper transform of $\mathcal{D}^{j-1}$ is non-dicritical with respect to the new divisor $\pi_j^{-1}(C_{j-1})$. Furthermore all the singularities of $\mathcal{D}^j$ lying over $\pi_j^{-1}(C_{j-1} \setminus \{q_1, \ldots, q_r\})$ are simple and, in fact, strictly non-dicritical.*

Proof. Note that there are coordinates (x, y, z) on a neighborhood of a generic point of C_{j-1} verifying two conditions: first $\mathcal{D}^{j-1}$ is locally given by $dy = 0$. Second C_{j-1} coincides with $\{x = y = 0\}$ (here we implicitly use the fact that C_{j-1} is contained in a single leaf of $\mathcal{D}^{j-1}$). The foliation $\mathcal{D}^{j-1}$ is therefore a cylinder over the foliation $dy = 0$ induced on the plane $\{z = 0\}$. The latter foliation is clearly non-dicritical and its blow-up possesses a unique (2-dimensional) singularity over the exceptional divisor which has eigenvalues $1, -1$. Clearly the blow-up of $\mathcal{D}^{j-1}$ is simply the cylinder over the mentioned foliation. The statement follows at once. □

The contents of Lemma 7 can be summarized as follows: dicritical components of the total exceptional divisor $\Pi^{-1}(0)$ arise only from the blow-ups centered at singular points of the corresponding foliations. In slightly more accurate terms, we first note that the “collapsing” of a dicritical divisor leads to a center contained in the singular set of the correspondent foliation. Next suppose that the resulting center is still contained in another component of the exceptional divisor. Then the collapsing of the mentioned component still leads to a center fully contained in the singular set of the blow-down foliation. Besides this procedure continues by induction. The following corollary is exactly what will be needed later in this section.

Corollary 1. *Suppose that Z is a holomorphic vector field tangent to $\mathcal{D} = \mathcal{D}^0$. Suppose also that $\pi_j^{-1}(C_{j-1})$ is a dicritical component appearing in* (13). *Then the proper transform of Z over the component $\pi_j^{-1}(C_{j-1})$ is obtained from Z by a finite number of blow-ups all of them centered at the singular sets of the corresponding proper transforms of $\mathcal{D}$.* □

To close this review on reduction of singularities let us make a remark concerning the compactness of the exceptional divisor arising from proceeding the reduction of singularities of $\mathcal{D} = \mathcal{D}^0$. In general this divisor is not compact due to the fact that the singularities of $\mathcal{D} = \mathcal{D}^0$ need not be isolated. In particular, in the local context of a neighborhood of the origin in $\mathbb{C}^3$, the exceptional divisor arising from blowing-up a curve of singularities of $\mathcal{D} = \mathcal{D}^0$ is already not compact. In turn, this non-compact component may lead to further non-compact components and this constitutes the source for the lack of compactness for the mentioned exceptional divisor. Yet observe that the preimage $\Pi^{-1}(0)$ of the origin under the total blow-up map Π is necessarily compact and constituted by components that have dimension either 1 or 2 depending on whether the center of the corresponding blow-up was a point or a curve.

Proof of Theorem 1. Since it was proved in [C-C] that a strictly non-dicritical foliation has a separatrix, we can assume that the total exceptional divisor appearing in (13) is not fully invariant by $\mathcal{D}^k$. Hence we can consider an irreducible component E of this divisor which is transverse to the leaves of $\mathcal{D}^k$ at generic points. Next we denote by $\mathcal{D}^k_{|E}$ the 1-*dimensional foliation* induced on E by intersecting E with the transverse leaves of $\mathcal{D}^k$. Recall that $\mathcal{D}^k_{|E}$ is supposed to have a non constant meromorphic integral. If the component E is compact, it then follows that all the leaves $L_{|E}$ of this foliation are compact. When E is not compact, these leaves are still properly embedded and this will be enough for our purposes (recall that the preimage of the origin by the total blow-up map Π is always compact).
Case 1. Suppose that the generic leaf of $\mathcal{D}^k_{|E}$ does not intersect the singular set of $\mathcal{D}^k$.

Note first that this case cannot take place if E is isomorphic to the projective plane. Indeed a compact curve contained in $\mathbb{C}P(2)$ cannot be a regular leaf of a foliation since the latter condition would imply the vanishing of the self-intersection of this curve. This situation however may be produced when E comes from blowing-up a center not reduced to a single point.

Suppose that E is compact. Pick one such leaf $L_{|E}$. Note that $L_{|E}$ is naturally contained in a leaf L of $\mathcal{D}^k$ which is not contained in the total exceptional divisor provided that $L_{|E}$ is "sufficiently generic". Now, since $L_{|E}$ is compact, L defines a germ of 2-dimensional analytic set containing $L_{|E}$. This analytic set does not intersect singularities of $\mathcal{D}^k$, in particular E is the only irreducible component of the exceptional divisor that meets L non-trivially. In other words, the analytic set defined by L is "complete" in the sense that its intersection with a closed tubular neighborhood of the total exceptional divisor yields a closed analytic set fully invariant by $\mathcal{D}^k$. This means that, away from the total exceptional divisor itself, this analytic set actually coincides with a leaf of $\mathcal{D}^k$ on the fixed closed tubular neighborhood of the total exceptional divisor. Since the blow-up projection is proper and analytic, the image of L must induce a separatrix for $\mathcal{D}$ as desired.

Suppose now that E is not compact. Note that the intersection of E with $\Pi^{-1}(0)$ is still a compact curve. If the curve is not contained in the singular set of $\mathcal{D}^k$, then the statement results at once provided that this curve is not invariant by $\mathcal{D}^k$. In fact, it suffices to consider a generic point q of this curve where it is transverse to the foliation. If q is sufficiently generic, the local leaf of $\mathcal{D}^k$ through q does not (locally) intersect any other component of the exceptional divisor. In particular, as in the preceding case, this local leaf is complete on a neighborhood of $\Pi^{-1}(0)$. Its projections yields then a separatrix for $\mathcal{D}$ at the origin.

Finally suppose that the curve in invariant by $\mathcal{D}^k$. Naturally the curve is allowed to contain singularities of $\mathcal{D}^k$ and it may even be entirely constituted by singularities of $\mathcal{D}^k$.
Claim. All these singularities possess at least one convergent separatrix which is not contained in E.

Before proving the claim, let us deduce the proof of Theorem 1 in this Case 1. This goes as follows. Because the curve in question, i.e. $E \cap \Pi^{-1}(0)$ is compact it follows that these separatrizes can be continued over this curve until they "close-up" forming a germ of analytic surface. Unless this germ is contained in $\Pi^{-1}(0)$, its projection gives the desired separatrix. On the other hand, if the mentioned germ is contained in a component of $\Pi^{-1}(0)$, then this component is obviously invariant (and hence nondicritical). This allows us to continue the discussion over the component of $\Pi^{-1}(0)$ in question. In other words, either we find another dicritical component or we shall be able to prove the existence of separatrix with the help of the argument in [C-C].

To finish the discussion of Case 1 let us provide the proof of the above claim.
Proof of the claim. Since these singularities are simple, they possess at least one separatrix which is not contained in E. At this point however this separatrix may be merely *formal*. Nonetheless in this case, the foliation is locally described by a pair of vector fields, one of them being regular and the other one representing a (2-dimensional saddle-node). In fact these singularities must be of Type A. Indeed, since E is dicritical, in the case of having singularities of Type B, E would locally be given by $\{z = 0\}$ in the sense of Remark 4. However in this case the separatrizes transverse to E are automatically convergent. Thus if the transverse separatrix is only formal the singularity is of Type A as claimed. Now, as it is shown in [C-C], the mentioned formal separatrix comes from a formal separatrix (weak invariant manifold) for a 2-dimensional "saddle-node". In particular, if one of these separatrizes is actually convergent , then they all must be convergent.

Suppose first that $E \cap \Pi^{-1}(0)$ contains only finitely many singularities. Since E is dicritical, at a generic point of this curve there must pass an actual leaf of $\mathcal{D}^k$. The above mentioned regular vector field then shows that all the, in principle, formal (weak manifolds) of the saddle-node singularities must thus be convergent. The claim then follows from the same argument employed above.

Finally suppose that $E \cap \Pi^{-1}(0)$ is entirely constituted by singularities of $\mathcal{D}^k$. Again the local structure of the foliation is given by a regular vector field (of which $E \cap \Pi^{-1}(0)$ is "an orbit") together with a (2-dimensional) saddle-node in the variables "transverse" to $E \cap \Pi^{-1}(0)$. Since the separatrix transverse to E should be formal, it follows that the actual separatrix (strong invariant manifold) of the saddle-node in question must be contained in E. However, in this case, E is invariant by both local vector fields and thus it is a leaf of $\mathcal{D}^k$. the claim follows since this contradicts the assumption that E is dicritical. $\square$

Case 2. Suppose that the generic leaf of $\mathcal{D}^k_{|E}$ intersects $\mathrm{Sing}\,(\mathcal{D}^k)$ and let P be an irreducible component of $\mathrm{Sing}\,(\mathcal{D}^k) \cap E$ that intersects the generic leaf of $\mathcal{D}^k_{|E}$.

We have to check that the effect of these singularities does not prevent us from employing once again the previous argument. This means the following. Suppose that $L_{|E}$ is a generic leaf of $\mathcal{D}^k_{|E}$ passing through P. Locally around P, this leaf is an analytic curve contained in the 2-plane induced by E. We need to show that the actual leaf of $\mathcal{D}^k$ which intersects E along the mentioned curve, defines a germ of analytic surface around P. In slightly vague words, we have to check that the leaf of $\mathcal{D}^k$ passes through P keeping a positive "vertical" distance of P itself.

Let p be a point in P and let $L_{p|E}$ be a leaf of $\mathcal{D}^k_{|E}$ through p. Note that there may exist more than a single leaf passing through p. For our purposes, it suffices to consider a chosen (irreducible branch of) separatrix which happen to be part of a (global) generic leaf of $\mathcal{D}^k_{|E}$.

Under the assumptions of normal crossing between the foliation $\mathcal{D}^k$ and the corresponding divisor, there are coordinates x, y, z, in a neighborhood of p, satisfying the following conditions:

(1) The corresponding irreducible (dicritical) component of the exceptional divisor is given by $\{z = 0\}$.
(2) $\widetilde{\mathcal{F}}$ is spanned by a pair of commuting vector fields Z_1, Z_2 which are either as in Type A (i.e. $Z_1 = \partial/\partial z$, $Z_2 = x\partial/\partial x + a(x, y)\partial/\partial y$, $a(0, 0) = 0$) or as in Type B (i.e. $Z_1 = x\partial/\partial x + a(x, y, z)\partial/\partial z$ and $Z_2 = y\partial/\partial y + b(x, y, z)\partial/\partial z$, with $a(0, 0, 0) = b(0, 0, 0) = 0$).

As to the singularities of Type A and Type B, we note that the coordinates (x, y, z) were already chosen so that $\{z = 0\}$ corresponds to the dicritical component E. In the case of a singularity of type A, we conclude in particular that $\partial/\partial z$ is tangent to the foliation $\mathcal{D}^k$. Hence the mentioned leaf of $\mathcal{D}^k$ passes through P keeping a positive "vertical" distance of P itself. When the singularity is of Type B, this vertical distance also exists as already discussed in Remark 4.

Finally the dicritical foliation $\mathcal{D}^k_{|E}$ is locally described by the pair of commuting vector fields Z_1, Z_2. Assume first that E is compact. Then the combination of its compactness with with the positive vertical "distance" allows us to apply once again the argument used in Case 1. The adaptations needed for the case in which E is not compact are clear and totally analogous to the discussion already carried out in Case 1. Basically they amount to discussing the case in which the curve $E \cap \Pi^{-1}(0)$ is singular and invariant by the foliation. The theorem is proved. □

4.2. **Proof of Main Theorem.** Let us begin this paragraph by further detailing the structure of our proof. The general idea is to reduce *as much as possible* this theorem to the statement of Theorem 1. Suppose that a reduction of the singularities of $\mathcal{D}$ as in (13) is fixed. To establish Main Theorem, it would suffice to check that the foliations induced by $\mathcal{D}^k$ on the dicritical components of the total exceptional divisor possess a nontrivial first integral. In turn, Propositions 1 and 2 can be used with this purpose modulo checking that the proper transform of X, Y vanish identically over the mentioned components of the exceptional divisor. Thus it is natural to investigate the structure of the proper transforms of these vector fields during the reduction procedure (13) for the singularities of $\mathcal{D}$. Here it should be emphasized that we shall be dealing with the vector fields X, Y rather than with their associated foliations $\mathcal{F}_X$, $\mathcal{F}_Y$. Indeed, the proper transforms of $\mathcal{F}_X$, $\mathcal{F}_Y$ are always 1-dimensional holomorphic foliations with singular sets of codimension at least 2. However, as to the vector fields X, Y, at least in principle their proper transforms may have poles as well as singular sets of codimension 1. The discussion of the proper transforms of X, Y is thus going to be carried out in the sequel.

The first simplifications in the discussion of the proper transforms of X, Y comes from Corollary 1. Namely, since we are interested only in dicritical components for the codimension 1 foliation, we can assume that all the blow-ups performed are centered at the singular set of the codimension 1 foliations in question (i.e. the initial $\mathcal{D}$ and its proper transforms). Precisely, suppose that we have a dicritical component E_s appearing at the blow-up π_s of center C_{s-1}. For $i = 1, \ldots, s$ denote by C_{i-1} the center of the blow-up π_i. Then we can suppose without loss of generality that, for every $i \in \{1, \ldots, s\}$, C_{i-1} is contained in the singular set of $\mathcal{D}^{i-1}$ ($\mathcal{D} = \mathcal{D}^0$). This condition will always be assumed in what follows.

Next we observe that the singular set of $\mathcal{D}^{i-1}$ is clearly invariant under the corresponding proper transforms $\widetilde{X}^{i-1}$, $\widetilde{Y}^{i-1}$ of X, Y since these vector fields form an Abelian algebra ($\widetilde{X}^0 = X$, $\widetilde{Y}^0 = Y$). Closely related to this observation, we have a very well-known and simple lemma.

Lemma 8. *Let Z be a holomorphic vector field and denote by $\widetilde{Z}$ the proper transform of Z with respect to a blow-up π of center C. If C is invariant by Z, then $\widetilde{Z}$ is holomorphic as well.*

Proof. C being invariant by Z, the local flow generated by Z naturally acts on the tangent bundle of C. It then follows that $\widetilde{Z}$ is holomorphic. □

Denote by $\mathrm{Sing}\,(\mathcal{D}^{i-1})$ the singular set of $\mathcal{D}^{i-1}$. Recall that the center C_{i-1} of the blow-up π_i is either a single point or a smooth (irreducible) curve.

Lemma 9. *Without loss of generality we can assume that the center C_{i-1} is invariant by $\widetilde{X}^{i-1}$, $\widetilde{Y}^{i-1}$.*

Proof. Recall that $C_{i-1} \subseteq \mathrm{Sing}\,(\mathcal{D}^{i-1})$ and that C_{i-1} is either a single point or a smooth irreducible curve. Since $\mathrm{Sing}\,(\mathcal{D}^{i-1})$ is invariant by $\widetilde{X}^{i-1}$, $\widetilde{Y}^{i-1}$, it follows that the only possibility for C_{i-1} to fail to be invariant by $\widetilde{X}^{i-1}$, $\widetilde{Y}^{i-1}$ occurs when C_{i-1} is reduced to a single point p with, say, $\widetilde{X}^{i-1}(p) \neq 0$. Note that this case implies that p belongs to an analytic curve contained in $\mathrm{Sing}\,(\mathcal{D}^{i-1})$ which is globally invariant by $\widetilde{X}^{i-1}$, $\widetilde{Y}^{i-1}$.

To treat the above case, we note that the foliation $\mathcal{F}_X^{i-1}$ associated to $\widetilde{X}^{i-1}$ is regular at p since $\widetilde{X}^{i-1}(p) \neq 0$. In particular, we can fix a neighborhood U of p in which $\mathcal{F}_X^{i-1}$ possesses two independent holomorphic first integrals. More precisely, we can fix coordinates (u, v, w) on U where the leaves of $\mathcal{F}_X^{i-1}$ are given by $v = \mathrm{cte}_1$ and $w = \mathrm{cte}_2$.

In the reduction procedure (13) of $\mathcal{D}$ consider the divisors lying above p ie., consider the sub-procedure consisting of reducing p as singularity of $\mathcal{D}^{i-1}$. Denote by Π_p the resolution map associated to this sub-procedure. All we need to check is that all dicritical divisors belonging to $\Pi_p^{-1}(p)$ are such that the foliations induced on them by $\mathcal{D}^k$ possess a meromorphic first integral. This goes as follows. Because the foliation $\mathcal{F}_X^{i-1}$ is regular at p, its proper transform $\Pi_p^* \mathcal{F}_X^{i-1}$ by Π_p is a foliation leaving the whole exceptional divisor $\Pi_p^{-1}(p)$ invariant. Besides, if E is a component of $\Pi_p^{-1}(p)$, then the foliation induced on

E by $\Pi_p^* \mathcal{F}_X^{i-1}$ possesses a non-trivial meromorphic first integral obtained by means of the two initial first integrals of $\mathcal{F}_X^{i-1}$ on U. Finally, if in addition E is not invariant by $\mathcal{D}^k$, then the intersection $\mathcal{D}^k_{|E}$ of $\mathcal{D}^k$ with E clearly coincides with the restriction to E of $\Pi_p^* \mathcal{F}_X^{i-1}$. Hence the existence of the desired first integral for $\mathcal{D}^k_{|E}$ follows immediately. $\square$

The preceding proof also yields the following:

Corollary 2. *If π_i is a punctual blow-up belonging to the procedure (13), then we can assume without loss of generality that its center C_{i-1} is a singular point for both foliations $\mathcal{F}_X^{i-1}$, $\mathcal{F}_Y^{i-1}$ associated respectively to the vector fields $\widetilde{X}^{i-1}$, $\widetilde{Y}^{i-1}$.* $\square$

Suppose now that $C_{i-1} \subseteq \mathrm{Sing}\,(\mathcal{D}^{i-1})$ is a smooth curve. As already observed, C_{i-1} is clearly invariant under the corresponding proper transforms of X, Y. Our next lemma shows that it is sufficient to keep track of those centers C_{i-1} that are invariant under $\mathcal{F}_X^{i-1}$, $\mathcal{F}_Y^{i-1}$.

Lemma 10. *Suppose that the center C_{i-1} consists of a smooth curve. Then without loss of generality we can assume that C_{i-1} is invariant by $\mathcal{F}_X^{i-1}$, $\mathcal{F}_Y^{i-1}$.*

Proof. Let us suppose that C_{i-1} is not invariant by $\mathcal{F}_X^{i-1}$. On a neighborhood of a generic point of C_{i-1} we can introduce coordinates (x, y, z) in which $\mathcal{F}_X^{i-1}$ is induced by the vector field $\partial/\partial x$. Furthermore, in these coordinates, C_{i-1} coincides with the axis $\{x = y = 0\}$. Obviously we have two (local) independent first integral σ_y, σ_z for $\mathcal{F}_X^{i-1}$ given respectively by the natural projections in the coordinates y and z. The argument is now similar to the one employed in Lemma 9 concerning punctual blow-ups. There is only one slightly difference with respect to the case of punctual blow-ups that lies in the fact that the proper transform of the above mentioned local first integrals by the blow-up map π_i are not defined on a neighborhood of the exceptional divisor $\pi_i^{-1}(C_{i-1})$. Therefore it is not immediate that the restriction to $\pi_i^{-1}(C_{i-1})$ of the foliation $\mathcal{F}_X^i = (\pi_i)^* \mathcal{F}_X^{i-1}$ admits a non-trivial first integral. To show that this is, in fact, the case we proceed as follows. Consider the open domain of definition U of the first integrals σ_y, σ_z. Fix also a sufficiently narrow tubular neighborhood $\mathcal{U}$ of the total exceptional divisor. Finally note that the leaves of $\mathcal{F}_X^{i-1}$, given by $\sigma_y = \mathrm{cte}_1$ and $\sigma_z = \mathrm{cte}_2$, have the following property: whenever they leave the neighborhood U they must intersect the boundary of the tubular neighborhood $\mathcal{U}$. In other words, these leaves induce closed leaves for $\mathcal{F}_X^i$ on $(\pi_i^{-1}(C_{i-1})) \cap (\pi_i^{-1}(U))$. Hence $\mathcal{F}_X^i$ restricted to $\pi_i^{-1}(C_{i-1})$ has infinitely many compact leaves. According, for example, to a theorem due to Jouanolou [J-2] this foliation must have only compact leaves and hence a non-trivial first integral. It is now easy to continue the argument to encompass the case of further blow-ups as it was carried out in Lemma 9. $\square$

Before starting the proof of Main Theorem, let us summarize the contents of the preceding lemmas. Recall that our aim is to reduce as far as possible the proof of the Main Theorem to the statement of Theorem 1. Consider again the resolution procedure (13). To conclude the existence of the separatrix, it would be sufficient to show that the foliation induced by $\mathcal{D}^k$ on each dicritical component of the total exceptional divisor possesses only compact leaves. Equivalently each of these restrictions admit a non-trivial first integral.

Consider the center C_{i-1} of the blow-up π_i. Let $\Pi_{C_{i-1}}$ denote the total blow-up map associated to the (sub)-procedure of (13) accounting for the resolution of the singularities of $\mathcal{D}^{i-1}$ (which is itself spanned by $\mathcal{F}_X^{i-1}$, $\mathcal{F}_Y^{i-1}$). The preceding lemmata then says that, unless C_{i-1} is invariant by both (1-dimensional) foliations $\mathcal{F}_X^{i-1}$, $\mathcal{F}_Y^{i-1}$, the restriction of $\mathcal{D}^k$ to every dicritical component of $\Pi_{C_{i-1}}$ possesses only compact leaves. Therefore we only need to keep track of those sequence of blow-ups starting from $\mathcal{D} = \mathcal{D}^0$ having centers invariant under the corresponding proper transforms of $\mathcal{F}_X, \mathcal{F}_Y$. In other words, to abridge notations we can assume without loss of generality that in the reduction procedure (13), the center C_{i-1} of π_i is invariant by $\mathcal{F}_X^{i-1}$, $\mathcal{F}_Y^{i-1}$ for every i. In particular C_{i-1} is contained in the singular set of $\mathcal{D}^{i-1}$. In what follows this condition will be assumed without further comments.

In particular we obtain the following useful lemma.

Lemma 11. *Suppose that C_{i-1} is contained in a codimension 1 component of the zero-set of X^{i-1} (resp. Y^{i-1}). Then the proper transform X^k of X vanishes identically over the whole of the divisor $\Pi_{C_{i-1}}^{-1}(C_{i-1})$.*

Proof. In view of the above assumption and of Lemma 8, we know that X^i is a holomorphic vector field. Next, by an induction argument , we see that it suffices to check that X^i must vanish identically on $\pi_i^{-1}(C_{i-1})$. Because C_{i-1} is contained in a codimension 1 component of the zero-set of X^{i-1}, on a neighborhood of each point of C_{i-1} we can write X^{i-1} in the form

$$X^{i-1} = f^{i-1} Z^{i-1}$$

where f^{i-1} is a holomorphic function that equals zero on C_{i-1}. Besides Z^{i-1} is a holomorphic vector field having singular set of codimension at least 2. In other words, Z^{i-1} generates the foliation $\mathcal{F}_X^{i-1}$. Therefore C_{i-1} is invariant under Z^{i-1} so that the proper transform $\pi_i^* Z^{i-1}$ of Z^{i-1} under π_i is holomorphic. Finally we have $X^i = (f^{i-1} \circ \pi_i).\pi_i^* Z^{i-1}$. Since $f^{i-1} \circ \pi_i$ is clearly equal to zero on $\pi_i^{-1}(C_{i-1})$ the statement follows at once. □

Let us close this paper with the proof of the central result stated in the Introduction.

Proof of Main Theorem. Let us begin with the reduction procedure (13). We want to understand those dicritical components of the total exceptional divisor on which the foliation induced by $\mathcal{D}^k$ may have non-compact leaves. Recall also that in the reduction procedure (13), all the centers C_{i-1} are invariant by the (1-dimensional) foliations $\mathcal{F}_X^{i-1}$, $\mathcal{F}_Y^{i-1}$.

Consider a center C_{i-1} along with the associated blow-up map π_i. Denote by $\Pi_{C_{i-1}}^{-1}(C_{i-1})$ the total exceptional divisor associated to the sub-procedure of (13) lying over C_{i-1}. Consider also the vector fields X^i, Y^i and suppose that they equal zero on all of the component $\pi_i^{-1}(C_{i-1}) \subseteq \Pi_{C_{i-1}}^{-1}(C_{i-1})$. It then follows from Lemma 11 that X^k, Y^k vanish on the whole divisor $\Pi_{C_{i-1}}^{-1}(C_{i-1})$. Therefore Proposition 1 and Proposition 2 can be used to guarantee that the foliation induced by $\mathcal{D}^k$ on each dicritical component of $\Pi_{C_{i-1}}^{-1}(C_{i-1})$ has only compact leaves.

The combination of the preceding with Theorem 1 essentially reduces the proof of Main Theorem to the discussion of the reduction procedures starting with a blow-up (denoted by π_1) such that the proper transforms of the initial vector fields X, Y are regular over the resulting exceptional divisor (at generic points). Denote by C_0 the center of π_1 and set $X^1 = \pi_1^* X$, $Y^1 = \pi_1^* Y$. If X^1 (resp. Y^1) is regular at generic points of $\pi_1^{-1}(C_0)$ then, since the center C_0 is invariant under the foliations $\mathcal{F}_X$, $\mathcal{F}_Y$, one of the following holds:

(1) If C_0 consists of a single point (identified with the origin $(0,0,0)$), then C_0 must be a singular point of $\mathcal{F}_X$, $\mathcal{F}_Y$. Furthermore the first jet of X or Y at $(0,0,0)$ must be non-trivial.
(2) If C_0 consists of a smooth curve, then the order of X or Y "in the variables x, y" belongs to $\{0,1\}$. In addition, if $\mathcal{F}_X$ (resp. $\mathcal{F}_Y$) is not singular over the whole C_0, then C_0 must be an invariant curve for this foliation.

In the sequel we shall only consider the case of punctual blow-ups. The adaptations needed for the case of blow-ups centered over curves are by now straightforward and hence left to the reader. According to Lemma 1, there are holomorphic functions f, g and h such that

$$fX + gY = hZ\,, \tag{14}$$

where Z is a holomorphic vector field whose first non-trivial homogeneous component is a multiple of R. The main point of the proof of Proposition 1 is the estimate

$$\mathrm{ord}\,(hZ) > \min\{\mathrm{ord}\,(fX)\,,\,\mathrm{ord}\,(gY)\}$$

proved in Lemma 3. In turn, the proof of Lemma 3 is based on Lemma 2. However Lemma 2 needs to be adapted to the new context. To do this, note that this lemma can be rewritten as follows:

Lemma 12. *Let Z_1, Z_2 be vector fields defined about $(0,0,0) \in \mathbb{C}^3$. Suppose that the linear part of Z_1 at $(0,0,0)$ is a multiple of R. Suppose, in addition, that $[Z_1, Z_2] = 0$. Then Z_1 and Z_2 can simultaneously be linearized.*

Proof of Lemma 12. Since the linear part of Z_1 is a multiple of the Radial vector field, in particular it is not trivial, there exists coordinates in which $X = R$. The assumption on the commutativity of Z_1, Z_2 together with the Euler relation (6) implies that the terms of order greater than or equal to 2 of Z_2 must vanish. Hence Z_2 is linear. This finishes the proof of Lemma 12. □

In Sections 2 and 3, the fact that the foliation induced by the blow-up of $\mathcal{D}$ over the dicritical exceptional divisor E must have only compact leaves was established with the help of the (strict) estimate

$$\mathrm{ord}\,(hZ) > \min\{\mathrm{ord}\,(fX)\,,\,\mathrm{ord}\,(gY)\}\,. \tag{15}$$

We note that, whenever this estimate holds, then the same proofs given in Propositions 1 and 2 guarantee the existence of a first integral for the foliation induced on a dicritical component of the (total) exceptional divisor. However this first integral needs no longer to exist if we have equality between both sides of 15. Fortunately the existence of a separatrix will directly be established in these special cases. We shall separate the discussion into two parts.

Case 1. Suppose that $\mathrm{ord}\,(fX) \neq \mathrm{ord}\,(gY)$.

Note that in the preceding sections we have initially proved the equation $\mathrm{ord}\,(fX) = \mathrm{ord}\,(gY)$ (cf. Lemma 3). This equation however no longer needs to hold in the present case.

Suppose first that $\mathrm{ord}\,(fX) < \mathrm{ord}\,(gY)$. Therefore the first non-trivial homogeneous component of fX is a multiple of the Radial vector field. Since the linear part of X at the origin is not trivial, it follows that $X = R$ in suitable coordinates. Now Lemma 12 ensures that Y is linear in the same coordinates so that the existence of the separatrix easily follows.

Suppose now that $\mathrm{ord}\,(fX) > \mathrm{ord}\,(gY)$. Then the first non-trivial homogeneous component of gY is a multiple of the Radial vector field R. If the linear part of Y at the origin is non-trivial, then X, Y are simultaneously linearizable as above. Hence we assume that $J^1(Y)(0,0,0) = 0$. To handle this case we shall prove the following:

Claim: Denoting by $\widetilde{\mathcal{D}}$ the (punctual) blow-up of $\mathcal{D}$ at the origin, the 1-dimensional foliation $\widetilde{\mathcal{D}}_{|E}$ induced on $E = \pi^{-1}(0)$ by the intersection of E with the leaves of $\widetilde{\mathcal{D}}$ has only compact leaves.

Proof of the Claim. Since the linear part X^1 of X at the origin is not trivial, the foliation $\widetilde{\mathcal{D}}_{|E}$ coincides with the foliation induced on E by the blow-up of X^1. It should be pointed out that X^1 cannot be radial since otherwise Y is linearizable what contradicts the fact that $J^1(Y)(0,0,0) = 0$. Therefore X^1 actually induces a foliation on E. Hence it suffices to show that X^1 admits a non-constant holomorphic first integral. To do this, we proceed as follows. Recall that the first homogeneous component of Y is a multiple of the Radial vector field R. Since $J^1(Y)(0,0,0) = 0$, this component must then have the form hR where h is a homogeneous polynomial of positive degree on the variables x, y, z. Besides X^1 clearly must commute with hR so that

$$0 = [hR, X^1] = h[R, X^1] + (X^1.h)R\,.$$

Since $[R, X^1] = 0$, it follows that $X^1.h = 0$, i.e. h is a non constant first integral for X^1. The claim is proved. $\square$

We can now move on to the second case.

Case 2. Suppose that $\mathrm{ord}\,(fX) = \mathrm{ord}\,(gY)$.

Here the discussion will further be divided into 3 possibilities.

a) $J^1(Y)(0,0,0) = 0$ - In this case we must have $\mathrm{ord}(f) \geq 1$, since the linear part of X at the origin is not trivial. Recalling that X^H, Y^H commute, the same calculations of Lemma 3 applied to Equation (14) lead to

$$0 = [X^H, Y^H] = \left[X^H.\left(\frac{h^H q^H}{g^H}\right)\right] R + \left[X^H.\left(\frac{f^H}{g^H}\right)\right] X^H$$

since the degree of X^H is now equal to 1. In particular

$$\left[X^H.\left(\frac{h^H q^H}{g^H}\right)\right] R = -\left[X^H.\left(\frac{f^H}{g^H}\right)\right] X^H\,.$$

Suppose first that the expression multiplying X^H does not vanish identically. Then X^H is a multiple of the Radial vector field and, since the degree of X^H is 1, it is a constant multiple of the Radial vector field. Therefore X, Y must simultaneously be linearizable (cf. Lemma 12).

Suppose now that the expression multiplying X^H vanishes identically. So $X^H.\left(\frac{h^H q^H}{g^H}\right)$ must vanish as well. In this case both $h^H q^H / g^H$ and f^H / g^H are meromorphic first integrals for X^H (whose blow-ups induce meromorphic first integrals for the restriction of $\widetilde{\mathcal{F}}_X$ to the exceptional divisor). It only remains to prove that at least one of these first integrals is not constant. For this we note that if $h^H q^H/g^H$ is constant, then we can arrange to have it equal to 1. Thus it would follow

$$\frac{f^H}{g^H}X^H + Y^H = R\,.$$

The above equation clearly contradicts the assumption that the linear part of Y is trivial at the origin.

b) $J^1(X)(0,0,0)$ and $J^1(Y)(0,0,0)$ are non-trivial and linearly dependent everywhere - If $J^1(X)(0,0,0)$ and $J^1(Y)(0,0,0)$ are a multiple of R, then X and Y are also linearly dependent everywhere. Indeed, there are coordinates in which $Y = R$ and the commutativity assumption implies that X is linear as well. Since the linear part of X is a multiple of R the claim follows. The resulting contradiction then shows that this possibility cannot occur.

   Assume now that their linear parts coincide but that they are not a multiple of R. The fact that the first homogeneous component of Z is a multiple of R implies that the first homogeneous components of fX and gY must cancel each other in $fX + gY$. The inequality $\mathrm{ord}\,(hZ) > \min\{\mathrm{ord}\,(fX)\,,\, \mathrm{ord}\,(gY)\}$ is then obvious.

c) $J^1(X)(0,0,0)$ and $J^1(Y)(0,0,0)$ are linearly independent and R belongs to the space generated by them - In this case the codimension 1 foliation $\mathcal{D}$ spanned by X, Y can also be viewed as the foliation spanned by X and by a vector field Z whose linear part at the origin is a constant multiple of R. From this it follows that all the singularities of $\widetilde{\mathcal{D}}$ over E are simple, i.e. they are of type A or of type B, cf. above or [C-C], [Ca]. In particular the reduction procedure consists of a single blow-up.

   On the other hand, note that the foliation $\widetilde{\mathcal{D}}_{|E}$ induced on $E = \pi^{-1}(0)$ by the blow-up $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ coincides with the foliation induced on E by the blow-up of X. This foliation possesses therefore a compact leaf since the linear part of X at the origin is not trivial (and it is not a multiple of R). Combining the existence of this compact leaf with the fact that the singularities of $\widetilde{\mathcal{D}}$ are all simple, the method already employed in the proof of Theorem 1 yields immediately the existence of a separatrix for $\mathcal{D}$ in the present case.

The remainder of the proof of the theorem is now totally straightforward and left to the reader. □

Motivated by the classical situation of vector fields in dimension 2, it is natural to ask whether there must exist infinitely many separatrizes for a foliation having dicritical components. Simple linear examples involving the Radial vector field R and another linear vector field X shows that this is not true in general.

Suppose however that we are in the context of Theorem 1, i.e. we begin with "sufficiently non-linear" vector fields X, Y so as to be able to ensure that the condition of the theorem in question is satisfied. Then a careful reading of the proof of Theorem 1 makes it clear that infinitely many separatrizes must always exist except in a specific case that will be detailed below. Resuming the setting of Theorem 1, it is the fact that $\Pi^{-1}(0)$ may consist of components having also dimension 1 that prevents us from concluding the existence of infinitely many separatrizes projecting over the origin. Indeed, if for example, E

is a (necessarily compact) 2-dimensional component of $\Pi^{-1}(0)$ that happens to be dicritical, then the reader will easily verify the existence of infinitely many separatrizes for $\mathcal{D}$ at the origin. The situation in which we may not have these infinitely many separatrizes is precisely the one portrayed in Case 1 appearing in the proof of the theorem in question. Precisely it happens when the intersection of $\Pi^{-1}(0)$ with a dicritical component E is reduced to a curve that is invariant by the restriction of $\mathcal{D}^k$ to E. Indeed, this curve should be contained in the singular set of $\mathcal{D}^k$. Here the existence of a separatrix for $\mathcal{D}$ may be obtained from the argument in [C-C] if there is no other dicritical component. In some sense this situation means that, although the exceptional divisor may contain dicritical components, its intersection with $\Pi^{-1}(0)$ is "essentially non-dicritical".

An interesting remark concerning the case where this situation actually takes place, so that in particular the separatrizes for $\mathcal{D}$ at the origin are obtained with the help of the method used in [C-C], is as follows: the separatrizes obtained through [C-C] do not pass through a "generic" point of a singular curve of $\mathcal{D}$ (note that this curve has to exist otherwise $\Pi^{-1}(0)$ will contain only 2-dimensional components). In particular, these "generic" singular points of $\mathcal{D}$ will themselves have separatrizes due to the preceding result even though X, Y have non-trivial linear parts at these latter singularities. With little extra effort, one can show the existence of *infinitely many (germs of) surfaces invariant by $\mathcal{D}$ and passing through "generic singular points of $\mathcal{D}$*. Naturally, in the above situation, the origin is not "generic among the singularities of $\mathcal{D}$.

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