IDEMPOTENT SUBMODULES

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Abstract. Bican, Jambor, Kepka and Nemec defined a product on the lattice of submodules of a module, making any module into a partially ordered groupoid. Submodules that are idempotent with respect to this product behave similar as idempotent ideals in rings. In particular jansian torsion theories can be described through idempotent submodules. Moreover so-called coclosed submodules, which are essentially closed elements in the dual lattice of submodules of a module, turn out to be idempotent in π-projective modules. The relation of strongly copolyform modules and the regularity of their endomorphism ring is discussed.

1. Introduction

1.1. Let M be a left R-module and $S := \mathrm{End}_R(M)$. We denote by $\mathcal{L}(M)$ the lattice of R-submodules of M. There exists a binary operation on $\mathcal{L}(M)$ making it a partially ordered groupoid: Set:

$$N \star L := N\mathrm{Hom}_R(M, L) = \sum\{(N)f \mid f : M \to L\}$$

for all $N, L \in \mathcal{L}(M)$. This product has been defined in [2] and had been studied by the author in [5].

Call a submodule N of M **idempotent** if $N \star N = N$. This definition generalises of course the definition of idempotent (left) ideals of rings.

1.2. Idempotent submodules N of M are obviously M-generated, since $N = N\mathrm{Hom}(M, N) \subseteq \mathrm{Tr}(M, N) \subseteq N$. On the other hand if N is an M-generated submodule such that $\mathrm{Hom}(M, N)$ is an idempotent left ideal of $\mathrm{End}(M)$, then N is idempotent.

1.3. Let k be a commutative ring and A a (not necessarily associative) k-algebra. Denote by $M(A)$ the multiplication algebra of A. Then the idempotent $M(A)$-submodules of A are precisely the idempotent ideals which are generated (as A-module) by central elements. We show this in more generality: Let B be a subalgebra of $\mathrm{End}_k(A)$ that contains $M(A)$. Let I be a B-stable ideal, then $\mathrm{Hom}_B(A, I)$ is k-isomorphic to an ideal I^B of the centre of A via $f \mapsto (1)f$. The elements of I^B are called B-invariant elements.

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Theorem. *The idempotent B-submodules of A are precisely the idempotent ideals of A which are generated as left A-module by B-invariant elements.*

Proof. Let I be an idempotent B-stable ideal of A. Hence $I = I\mathrm{Hom}_B(A, I) = II^B \subseteq I^2$ shows that I is idempotent and generated by the B-invariant elements of I^B. On the other hand assume I is idempotent and generated by some B-invariant elements, i.e. $I = AX$ where $X \subseteq I^B$. Thus $I = I^2 = AXAX = AX^2 \subseteq IX \subseteq II^B \subseteq I$ shows that I is idempotent as B-module. □

1.4. Let A be a k-algebra and $A^e = A \otimes A^{op}$ its envoloping algebra. Then setting $B = A^e$ we have that the B-submodules of A are precisely the two-sided ideals of A. Moreover the idempotent B-submodules I of A are precisely the idempotent ideals which are generated by central elements as A-module, i.e. $I^2 = I = A(Z(A) \cap I)$. Note that $Z(A) \cap I$ are the "B-invariants" of 1.3 in this case. Following [8] one says that A is an *ideal-algebra* if it is a self-generator as A^e-module. In other words, if any ideal of A is generated by central elements. Examples of ideal-algebras are Azumaya algebras. Hence idempotent A^e-submodules of an ideal-algebra A are precisely the idempotent ideals of A. Furthermore any A^e-submodule of A is idempotent if and only if A is an ideal-algebra and every ideal of A is idempotent.

1.5. Assume that A is a commutative G-graded algebra where G is a semigroup. Then each element $g \in G$ acts on A as projection on the component A_g and defines a k-linear endomorphism π_g. Let B denote the subalgebra of $\mathrm{End}_k(A)$ generated by $M(A)$ and the elements π_g. Note that B-submodules of A are the G-graded ideals of A and the B-invariant elements of A are precisely the homogeneous elements. Hence a G-graded ideal I is idempotent as B-submodule if and only if it is an idempotent ideal which is generated by homogeneous elements.

1.6. There exists a bijective correspondence between idempotent ideals of R and jansian torsion theories in R-Mod. We will see that a similar correspondence is true for jansian torsion theories in $\sigma[M]$. A class τ of modules in $\sigma[M]$ is called jansian if it is closed under factor modules, extensions and direct products in $\sigma[M]$.

Proposition. *Let τ be a hereditary jansian torsion theory in $\sigma[M]$. Then for any self-generator $N \in \sigma[M]$: $L_\tau(N) := /L)^{\mathrm{Rej}\,(N,\tau)}$ is an idempotent submodule of N.*

Proof. Let $L = L_\tau(N)$ and $\Lambda = \mathrm{Hom}(N, L)$. By hypothesis $N/L \in \tau$. Consider

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^{(\Lambda)} & \longrightarrow & N^{(\Lambda)} & \longrightarrow & (N/L)^{(\Lambda)} & \longrightarrow & 0 \\ & & \Phi\downarrow & & \Phi\downarrow & & \downarrow & & \\ 0 & \longrightarrow & L^2 & \longrightarrow & L & \longrightarrow & L/L^2 & \longrightarrow & 0 \end{array}$$

where $\Phi : N^{(\Lambda)} \to L$ is the evaluation map $((n_f)_{f\in\Lambda})\Phi = \sum_{f\in\Lambda}(n_f)f$. Hence Φ can be extended to a map $(N/L)^\Lambda \to L/L^2$. Note that all these maps are surjective since N is a self-generator. Thus $L/L^2 \in \tau$ and as $N/L \in \tau$ it follows $N/L^2 \in \tau$. Hence $L \subseteq L^2 \subseteq L$ showing that L is idempotent.

□

1.7. Let $L \subseteq G$ and X be modules in $\sigma[M]$. Set $L *_G X := L\mathrm{Hom}\,(G, X)$.

Theorem. *Let G be a projective generator of $\sigma[M]$. Then there exists a bijective correspondence between*

- *Hereditary jansian torsion theories in $\sigma[M]$ and*
- *idempotent fully invariant submodules L of G.*

Mapping a jansian torsion theory τ to the submodule $L_\tau := \mathrm{Rej}\,(G,\tau)$ and mapping an idempotent (fully invariant) submodule L of G to the class

$$\tau_L := \{X \in \sigma[M] \mid L *_G X = 0\} = \mathrm{Gen}\,(G/L).$$

Proof. We saw that $L := L_\tau = L_\tau(G) = \mathrm{Rej}\,(G,\tau)$ is idempotent. Since $\mathrm{Gen}\,(G/L) \subseteq \tau$, we have $L \subseteq \mathrm{Rej}\,(G, \mathrm{Gen}\,(G/L))$. As G is self-projective we get $L = \mathrm{Rej}\,(G, \mathrm{Gen}\,(G/L))$, i.e. L is fully invariant.

Let L be an idempotent submodule of L with $\mathrm{Rej}\,(G, \mathrm{Gen}\,(G/L)) = L$. Let $\tau_L := \{X \in \sigma[M] \mid L *_G X = 0\}$. Obviously $\tau_L \subseteq \mathrm{Gen}\,(G/L)$. We have to show that τ_L is a hereditary jansian torsion theory. First of all it is clear that τ_L is closed under submodules, since $L *_G Y \subseteq L *_G X$ for $Y \subseteq X$, and closed under direct products.

To show that τ_L is closed under extensions assume that X and Z are members of τ_L in the following short exact sequence:

$$0 \longrightarrow X \longrightarrow Y \xrightarrow{\ p\ } Z \longrightarrow 0$$

Let $f : G \to Y$ be any homomorphism. Then $(L)fp = 0$ as $L *_G Z = 0$. Hence $L *_G Y \subseteq X$. But since L is idempotent $L = L *_G L$ and we have

$$L *_G Y = (L *_G L) *_G Y \subseteq L *_G (L *_G Y) \subseteq L *_G X = 0,$$

i.e. $Y \in \tau_L$.

Let $X \in \tau_L$ and let Y be a factor module of X, then we have a surjective map

$$\mathrm{Hom}\,(G, X) \to \mathrm{Hom}\,(G, Y)$$

and hence $L *_G X = 0 \Rightarrow L *_G Y = 0$. Thus $Y \in \tau_L$.

Now let τ be a hereditary jansian torsion theory in $\sigma[M]$ and consider $\tau_{L_\tau} = \{X \mid L_\tau *_G X = 0\}$. Note that since G is a generator, G/L_τ is a generator for τ. As $G/L_\tau \in \tau$ we have $\tau = \mathrm{Gen}(G/L_\tau)$. Since $\mathrm{Rej}\,(G, G/L_\tau) = L_\tau$ it follows $L_\tau *_G G/L_\tau = 0$, i.e. $\tau \subseteq \tau_{L_\tau}$. On the other hand if $L_\tau *_G X = 0$ then any homomorphism $G \to X$ factors through G/L_τ, i.e. X is G/L_τ-generated. Thus $\tau = \tau_{L_\tau}$.

Let L be an idempotent submodule of G with $\mathrm{Rej}\,(G, G/L) = L$. Hence $L *_G (G/L) = 0$, i.e. $G/L \in \tau_L$. Thus $L_{\tau_L} := \mathrm{Rej}\,(G, \tau_L) \subseteq L$. On the other hand $L_{\tau_L} *_G (G/L) = \mathrm{Rej}\,(G, \tau_L)\mathrm{Hom}\,(G, G/L) = 0$, i.e. $L = \mathrm{Rej}\,(G, G/L) \subseteq L_{\tau_L}$. □

1.8. A k-algebra A is called Azumaya if it is a projective generator in A^e-Mod. The last theorem establishes a correspondence of hereditary jansian torsion theories in A^e-Mod and idempotent A^e-submodules of A. As mentioned before, the idempotent A^e-submodules of A are precisely the idempotent two-sided ideals of A in case of an Azumaya algebra. Thus there exists a correspondence between hereditary jansian torsion theories in A^e-Mod and A-Mod.

2. Coclosed submodules are idempotent

A closed submodule N of a module M has no proper essential extension in M. We say that an inclusion $N \subseteq L$ of submodules of M is *cosmall* if $L/N \ll M/N$. Golan introduced the following notion in [4]: A submodule N of an R-module M is said to be *coclosed* in M if N has no proper submodule K such that $K \subseteq N$ is cosmall in M.

Thus N is coclosed in M if and only if, for any submodule K properly contained in N, there is a submodule L of M such that $L + N = M$ but $L + K \neq M$. Consequently every direct summand is a coclosed submodule.

Cosemisimple modules can be characterized by coclosed submodules:

Proposition ([3, 3.8])**.** *Every submodule of a module M is coclosed in M if and only if M is cosemisimple.*

2.1. Recall that a module M is called π-projective if for any submodules $K, L \subseteq M$ with $M = K + L$ we have $\mathrm{End}\,(M) = \mathrm{Hom}\,(M, K) + \mathrm{Hom}\,(M, L)$ ([7]).

Proposition ([3, 4.16])**.** *Let M be a π-projective module. Then $N^2 \ll N$ is cosmall for $N \subseteq M$, i.e. $N/N^2 \ll M/N^2$.*

This immediately implies the

Corollary. *Any coclosed submodule of a π-projective module is idempotent.*

Hence any left ideal of a ring, which is coclosed as submodule, is idempotent.

2.2. Note that if N is coclosed in M then for any submodule $L \subseteq N$ we have: $L \ll M \Leftrightarrow L \ll N$. Since N/K is coclosed in M/K we conclude [3, 3.9]:

Lemma. *Let N be a coclosed submodule of M. For any $K \subset L \subset N$ we have:*

$$L/K \ll M/K \Leftrightarrow L/K \ll N/K.$$

In particular $\mathrm{Rad}\,(N/K) = \mathrm{Rad}\,(M/K) \cap N/K$ *holds and if* $\mathrm{Rad}\,(M/K) \ll M/K$ *then* $\mathrm{Rad}\,(N/K) \ll N/K$.

2.3. Recall that a module M is *coatomic* if every non-zero factor module of M contains a maximal submodule or equivalently if every proper submodule of M is contained in a maximal one.

Lemma. *A module M is coatomic if and only if* $\mathrm{Rad}\,(M/N) \ll M/N$ *for all* $N \subset M$.

Proof. Let M be coatomic and $N \subset M$. Assume $\mathrm{Rad}\,(M/N) + L/N = M/N$. Note that $\mathrm{Rad}(M/L) \supseteq (\mathrm{Rad}(M)+L)/L = M/L$. Hence $M/L = 0$ by hypothesis and $M = L$. On the contrary if $N \subset M$ and $\mathrm{Rad}\,(M/N) \ll M/N$, then $\mathrm{Rad}\,(M/N) \neq M/N$ and there exists a maximal submodule in M/N. □

2.4. With the last two Lemmas we can conclude.

Corollary. *Any coclosed submodule of a coatomic module is coatomic.*

Hence for any ring R, a left ideal I of R is coclosed implies that I is idempotent and a left coatomic module. For left duo rings, i.e. every left ideal is a two-sided ideal this necessary condition is already sufficient for I to be coclosed.

2.5. The next theorem characterises coclosed ideals in left duo rings.

Theorem. *Let R be a left duo ring and I an ideal of R. Then I is coclosed as a left ideal in R if and only if I is idempotent and a left coatomic module.*

Proof. We just need to prove the necessity. Assume $I = I^2$ and I is a coatomic module. Let $K \subset I$ and choose a maximal submodule N/K of I/K. Then $P = Ann(I/N)$ is a maximal ideal of R. Suppose $N + P = R$, then

$$I = RI = NI + PI \subseteq N$$

yields a contradiction. Hence $N \subseteq P$. Suppose $I \subseteq P$, then

$$I = I^2 \subseteq PI \subseteq N \subset I$$

leads also to a contradiction. Thus $I + P = R$ and hence $I/K + P/K = R/K$ with $P/K \neq R/K$, i.e. $K \subseteq I$ is not cosmall in R. Hence I is coclosed. □

2.6. Over a commutative noetherian ring the sets of coclosed ideals, idempotent ideals and direct summands coincide. Note that in any commutative von Neumann regular ring which is not semisimple, there are coclosed ideals which are not direct summands (take for example a direct product of fields for the ring).

3. Idempotents and copolyform modules

Non-singular modules and polyform modules have their dualisations which we are going to recall here. A module M is called *copolyform* if $\mathrm{Hom}\,(M, N/L) = 0$ for all small submodules $N \ll M$ and $L \subset N$. The rings R that are copolyform as modules over themselves are precisely the semiprimitve rings, i.e. $\mathrm{Jac}\,(R) = 0$.

A module M is called *non-small* if it has no small factor module M/N, i.e. M/N is not small in any left R-module. It is not difficult to see that non-small modules are copolyform, but the converse is not true. The non-small rings R are precisely the left V-rings.

In between those two classes of modules one has those modules whose factor modules are copolyform. Call a module M *strongly copolyform* if every factor module of M is copolyform. This notion had been introduced by Vanaja and Talebi in [6].

3.1. A ring is called *left fully idempotent* if every left ideal is idempotent. The next theorem characterises the endomorphism ring of projective strongly copolyform modules as left fully idempotent ones.

Theorem ([3, 9.25])**.** *Let P be finitely generated self-projective module and $S = \mathrm{End}\,(P)$. If P is strongly copolyform then S is left fully idempotent.*

We include the proof for the reader's sake.

Proof. Consider a nonzero $f \in S$. Since P is finitely generated and self-projective $\mathrm{Hom}\,(P, Pf) = Sf$ and $Pf\mathrm{Hom}\,(P, Pf) = PfSf$. As P is π-projective, $PfSf$ is a cosmall submodule of Pf in P by 2.1. Now P is strongly copolyform implies that $\mathrm{Hom}\,(P, Pf/PfSf) = 0$ and hence $Pf = PfSf$. Thus

$$Sf = \mathrm{Hom}\,(P, Pf) = \mathrm{Hom}\,(P, PfSf) = \mathrm{Hom}\,(P, P(Sf)^2) = (Sf)^2,$$

where the last equality follows since P is finitely generated and self-projective. Hence cyclic left ideals of S are idempotent and this implies that every left ideal is idempotent. □

3.2. It now follows that a ring R which is strongly copolyform as a left R-module must be left fully idempotent. Hence any ring R with $\mathrm{Jac}\,(R) = 0$ which is not left fully idempotent gives an example of a module which is copolyform but not strongly copolyform.

3.3. Call a module M *fully idempotent* if all submodules of M are idempotent. As mentioned in 1.4 a ring R is a fully idempotent R^e-module if and only if it is an ideal-algebra and all ideals are idempotent. In the next theorem we examine when strongly copolyform π-projective modules are fully idempotent.

Theorem. *A π-projective strongly copolyform module M is a self-generator if and only if M is fully idempotent. Moreover in this case* $\mathrm{Rad}\,(M/N) = 0$ *for all fully invariant submodules N of M.*

Proof. Suppose that N is a submodule of a self-generator M which is strongly copolyform and π-projective. By 2.1 $N/N^2 \ll M/N^2$. Take any $f \in \mathrm{Hom}(M, N)$, then

$$(N^2)f = N\mathrm{Hom}\,(M, N)f \subset N\mathrm{Hom}\,(M, N) = N^2.$$

Hence f lifts to $\overline{f} : M/N^2 \to N/N^2$. By hypothesis $\overline{f} = 0$, i.e. $\mathrm{Im}\,(f) \subseteq N^2$. Since N is M-generated, $N = M\mathrm{Hom}\,(M, N) = N^2$ is idempotent.

The contrary is clear, since any idempotent submodule is M-generated.

If N is a fully invariant submodule of M and $C/N \ll M/N$, then since N is fully invariant, any $f : M \to C$ can be lifted to some $\overline{f} : M/N \to C/N$. Since M is strongly copolyform, $\overline{f} = 0$, i.e. $\text{Im}\,(f) \subseteq N$. As M is a self-generator, $C = N$. □

3.4. From [8, 30.10] it follows that a ring R has the property $\text{Rad}\,(R/I) = 0$ for all ideals I if and only if R is a cosemisimple R^e-module. Thus any ring R which is strongly copolyform as left R-module is left fully idempotent (by 3.2) and cosemisimple as R^e-module (by 3.3). For rings which are finitely generated over their centre we have the following:

Theorem. *Let R be a ring which is finitely generated as a module over its centre $Z(A)$. Then R is left strongly copolyform if and only if it is von Neumann regular.*

Proof. Let R be left strongly copolyform. As shown before, R is cosemisimple as R^e-module. Since R is finitely generated as $Z(R)$-module, also R^e is finitely generated as $Z(R)$-module and [8, 30.12] applies which says that $Z(R)$ is von Neumann regular, R is biregular (i.e. any principal ideal is generated by a central idempotent) and an Azumaya algebra. By [1, Theorem 2] R is von Neumann regular. □

We are left with the three following questions:
Question 1: Are left strongly copolyform ring V-rings ?
Question 2: Are left strongly copolyform ring biregular ?
Question 3: Is the centre of a left strongly copolyform ring regular?

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