AROUND q-APPELL POLYNOMIAL SEQUENCES

Ana F. Loureiro [1]
(anafsl@fc.up.pt)

Inst. Sup. de Engenharia de Coimbra & CMUP
Rua Pedro Nunes - Quinta da Nora, 3030 - 199 Coimbra, Portugal

Pascal Maroni
(maroni@ann.jussieu.fr)

Lab. Jacques-Louis Lions – CNRS, Université Pierre et Marie Curie
Bôite courrier 187, 75252 Paris cedex 05, France

Abstract. The quadratic decomposition of the Appell polynomials with respect to the q-divided difference operator is supplied by two other Appell sequences with respect to a new operator $\mathscr{L}_{q;\varepsilon,\mu}$. While seeking all the orthogonal polynomial sequences invariant under the action of $\mathscr{L}_{\sqrt{q};\varepsilon,\mu}$ (the $\mathscr{L}_{\sqrt{q};\varepsilon,\mu}$-Appell), only the Little q-Laguerre polynomials are achieved, up to a linear transformation, whereas the q-Laguerre polynomials are characterized as orthogonal $\mathscr{L}_{\sqrt{q^{-1}};\varepsilon,-1}$-Appell. This brings a new characterization of these polynomial sequences.

Keywords. orthogonal polynomials; Appell sequences; lowering operators; q-derivative; Hahn's operator; quadratic decomposition

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1. Introduction and preliminaries

This work deals with Appell sequences with respect to lowering (or annihilating) operators involving the **Hahn's operator**, often called as q-**derivative operator**, here denoted as H_q:

$$\left(H_q f\right)(x) := \frac{f(qx)-f(x)}{(q-1)x}, \quad f \in \mathscr{P},$$

where q belongs to the set $\tilde{\mathbb{C}} := \mathbb{C} - \bigcup_{n\geqslant 0} U_n$, where

$$U_n = \begin{cases} \{0\} & , \quad n=0 \\ \{z\in\mathbb{C} : z^n = 1\} & , \quad n \geqslant 1. \end{cases}$$

After giving preliminary results and notations in use, in section 2 the definition and characterization of the so-called H_q-Appell polynomial sequences (with particular emphasis to the orthogonal ones) will be recalled, but also some new results will be achieved. The main goal of this work is to obtain information concerning the quadratic decomposition of these H_q-Appell sequences, which amounts to the same as characterizing the four polynomial sequences associated to the description of the even and odd terms of an H_q-Appell sequence. This brings us to section 3. Insofar as such quadratic decomposition gives rise to Appell sequences with respect to a new operator (which is of second order in H_{q^2}), in the last section, these new Appell sequences will be characterized, stressing those possessing orthogonality. Finally, we will prove that these latter polynomial sequences must be the *Little*

[1] corresponding author: anafsl@fc.up.pt

q^2-*Laguerre* polynomials, up to a linear change of variable. The q-Laguerre polynomials will be characterized as well as Appell sequences with respect to a precise operator.

We denote by $\mathscr{P}$ the vector space of the polynomials with coefficients in $\mathbb{C}$ (the field of complex numbers) and by $\mathscr{P}'$ its dual space, whose elements are called *forms* (or linear functionals). The action of $u \in \mathscr{P}'$ on $f \in \mathscr{P}$ is denoted as $\langle u, f\rangle$. In particular, we denote by $(u)_n := \langle u, x^n\rangle, n \geqslant 0$ the moments of u. Recall that a linear operator $T : \mathscr{P} \to \mathscr{P}$ has a transpose ${}^tT : \mathscr{P}' \to \mathscr{P}'$ defined by

$$\langle {}^tT(u), f\rangle = \langle u, T(f)\rangle, \quad u \in \mathscr{P}', f \in \mathscr{P}. \tag{1.1}$$

For example, for any form u and any polynomial g, let $Du = u'$ and gu be the forms defined as usually by

$$\langle u', f\rangle := -\langle u, f'\rangle \quad , \quad \langle gu, f\rangle := \langle u, gf\rangle,$$

where D is the derivative operator. Thus, D on forms is minus the transpose of the differentiation operator D on polynomials. We will denote by $h_a f$ the homothecy of a polynomial f with $a \in \mathbb{C} - \{0\}$, precisely, we have $\big(h_a f\big)(x) := f(ax)$. In accordance with (1.1), by duality we define the homothecy of any form u by

$$\langle h_a u, f\rangle := \langle u, h_a f\rangle = \langle u, f(ax)\rangle, \quad f \in \mathscr{P},\, a \in \mathbb{C} - \{0\}.$$

The q-derivative operator may be also defined as follows

$$H_q = \frac{1}{q-1}\, \vartheta_0 \circ (h_q - \mathrm{I}_{\mathscr{P}}), \tag{1.2}$$

where $\mathrm{I}_{\mathscr{P}}$ represents the identity operator in $\mathscr{P}$ and ϑ_c with $c \in \mathbb{C}$ is the linear application

$$\begin{array}{llll} \vartheta_c & : \mathscr{P} & \longrightarrow & \mathscr{P} \\ & \quad p & \longmapsto & \big(\vartheta_c p\big)(x) = \dfrac{p(x) - p(c)}{x - c}\,. \end{array}$$

This linear application ϑ_c allows to define the division of form u by a first degree polynomial, that is $(x-c)^{-1}u$, so that, by transposition, we merely have $\langle (x-c)^{-1}u, p\rangle := \langle u, \vartheta_c p\rangle$, for all $p \in \mathscr{P}$.

The linear operator H_q has a transpose ${}^tH_q$, from $\mathscr{P}'$ into $\mathscr{P}'$, defined by duality according to (1.1):

$${}^tH_q = \frac{1}{q-1}\,(h_q - \mathrm{I}_{\mathscr{P}'})x^{-1}$$

We can define the q-derivative operator H_q on $\mathscr{P}'$ as minus the transpose of the q-derivative operator on $\mathscr{P}$, that is, ${}^tH_q := -H_q$, so that

$$\langle H_q u, f\rangle := -\langle u, H_q f\rangle\,, \quad f \in \mathscr{P},\, u \in \mathscr{P}', \tag{1.3}$$

In particular, this yields

$$\big(H_q u\big)_n = -[n]_q\,(u)_{n-1},\; n \geqslant 0,$$

with the convention $(u)_{-1} = 0$, and

$$[n]_q := \frac{q^n - 1}{q - 1}, \quad n \geqslant 0.$$

Next, we formally list some properties of this operator H_q, either on $\mathscr{P}$ or on $\mathscr{P}'$, essential for the sequel:

Lemma 1.1. *The following properties hold* [8, 17]

$$\big(H_q f_1 f_2\big)(x) = \big(h_q f_1\big)(x)\ \big(H_q f_2\big)(x) + f_2(x)\ \big(H_q f_1\big)(x), \quad f_1, f_2 \in \mathscr{P}, \tag{1.4}$$
$$\big(h_a f_1 f_2\big)(x) = \big(h_a f_1\big)(x)\ \big(h_a f_2\big)(x), \quad f_1, f_2 \in \mathscr{P},\ a \in \mathbb{C} - \{0\}, \tag{1.5}$$
$$h_a\big(gu\big) = \big(h_{a^{-1}} g\big)\ \big(h_a u\big), \quad g \in \mathscr{P}, u \in \mathscr{P}'\ a \in \mathbb{C} - \{0\}, \tag{1.6}$$
$$H_q\big(gu\big) = g\ H_q u + \Big(H_{q^{-1}} g\Big) h_q u, \quad g \in \mathscr{P}, u \in \mathscr{P}' \tag{1.7}$$
$$H_q\big(gu\big) = \big(h_{q^{-1}} g\big)\ H_q u + q^{-1}\Big(H_{q^{-1}} g\Big) u, \quad g \in \mathscr{P}, u \in \mathscr{P}' \tag{1.8}$$
$$H_q \circ h_{q^{-1}} = q^{-1} H_{q^{-1}} \quad \textit{in } \mathscr{P} \tag{1.9}$$
$$h_{q^{-1}} \circ H_q = H_{q^{-1}} \quad \textit{in } \mathscr{P} \tag{1.10}$$
$$H_q \circ h_a = a\, h_a \circ H_q \quad \textit{in } \mathscr{P} \quad (\textit{with } a \in \mathbb{C} - \{0\}), \tag{1.11}$$
$$H_q \circ H_{q^{-1}} = q^{-1}\, H_{q^{-1}} \circ H_q \quad \textit{in } \mathscr{P} \tag{1.12}$$
$$\textit{The operator } H_q \textit{ is injective in } \mathscr{P}'. \tag{1.13}$$

Let $\{B_n\}_{n\geqslant 0}$ be a sequence of monic polynomials with $\deg B_n = n$, $n \geqslant 0$ (monic polynomial sequence: MPS) and let $\{u_n\}_{n\geqslant 0}$ be the corresponding dual sequence, $u_n \in \mathscr{P}'$, defined by $\langle u_n, P_k\rangle := \delta_{n,k}$, $n,k \geqslant 0$. We recall from [13], that any form $u \in \mathscr{P}'$ may be represented through

$$u = \sum_{n\geqslant 0} \langle u, B_n\rangle\ u_n\ . \tag{1.14}$$

A monic orthogonal polynomial sequence - hereafter MOPS - $\{B_n\}_{n\geqslant 0}$ is characterized by

$$\begin{cases} B_0(x) = 1 \quad ; \quad B_1(x) = x - \beta_0 \\ B_{n+2}(x) = (x - \beta_{n+1}) B_{n+1}(x) - \gamma_{n+1} B_n(x) \quad , \quad n \geqslant 0, \end{cases} \tag{1.15}$$
$$u_{n+1} = \big(\langle u_0, B_{n+1}^2\rangle\big)^{-1}\ B_{n+1} u_0 \tag{1.16}$$

where $\{u_n\}_{n\geqslant 0}$ represents the corresponding dual sequence and $(\beta_n, \gamma_{n+1})_{n\geqslant 0}$ are known as the recurrence coefficients with $\gamma_{n+1} \neq 0$, $n \geqslant 0$.

2. **Appell sequences with respect to lowering operators**

Consider $\mathscr{O}$ to be a lowering operator, that is, a linear operator that decreases in one unit the degree of a polynomial and such that $\mathscr{O}(1) = 0$.

Given a MPS $\{B_n\}_{n\geqslant 0}$, we construct another MPS $\{B_n^{[1]}(\cdot;\mathscr{O})\}_{n\geqslant 0}$ by setting

$$B_n^{[1]}(x;\mathscr{O}) = \frac{1}{\rho_{n+1}(\mathscr{O})}\ \mathscr{O}\big(B_{n+1}(x)\big) \quad , \quad n \geqslant 0,$$

where $\{\rho_{n+1}(\mathscr{O})\}_{n\geqslant 0}$ represents a sequence of nonzero constants realizing the condition $\mathscr{O}(x^{n+1}) = \rho_{n+1}(\mathscr{O})\ x^n + b_n(x)$ with $\deg b_n < n$, for any integer $n \geqslant 0$.

Definition 2.1. *A MPS $\{B_n\}_{n\geqslant 0}$ is called an Appell sequence with respect to a lowering (annihilating) operator $\mathscr{O}$ or simply $\mathscr{O}$-Appell sequence if $B_n^{[1]}(\cdot,\mathscr{O}) = B_n(\cdot)$ for all integers $n \geqslant 0$,* [4, 5].

Lemma 2.1. *Let $\mathscr{O}$ represent a lowering operator such that $\Big(\mathscr{O}\zeta^{n+1}\Big)(x) = \rho_{n+1}\Big\{x^n + \eta_n x^{n-1} + \chi_{n-1} x^{n-2} + \dots\Big\}$ with $\eta_0 = \chi_{-1} = \chi_0 = 0$ and supose that $\{B_n\}_{n\geqslant 0}$ is an $\mathscr{O}$-Appell sequence. Expressing the terms of $\{B_n\}_{n\geqslant 0}$ as*

$$B_n(x) = x^n + b_n x^{n-1} + c_{n-1} x^{n-2} + \dots \quad , \quad n \geqslant 0, \tag{2.1}$$

with $b_0 = c_{-1} = c_0 = 0$*, we then have*

$$\begin{cases} b_{n+1} = \rho_{n+1}\left(\dfrac{b_1}{\rho_1} - \displaystyle\sum_{\nu=1}^{n}\dfrac{\eta_\nu}{\rho_\nu}\right) \\ c_{n+1} = \rho_{n+1}\rho_{n+2}\left(\dfrac{c_1}{\rho_1\rho_2} - \displaystyle\sum_{\nu=1}^{n}\dfrac{\chi_\nu + \eta_\nu\, b_{\nu+2}\,\frac{\rho_{\nu+1}}{\rho_{\nu+2}}}{\rho_\nu\,\rho_{\nu+1}}\right) \end{cases} \quad , \quad n \geqslant 1. \tag{2.2}$$

Proof. The $\mathscr{O}$-Appell character of the sequence $\{B_n\}_{n\geqslant 0}$ ensures the equalities

$$\rho_{n+1}B_n(x) = \Big(\mathscr{O}B_{n+1}\Big)(x)\,, \quad n \geqslant 0,$$

which, under the assumptions, become

$$\begin{aligned} &\rho_{n+1}\Big(x^n + \eta_n x^{n-1} + \chi_{n-1}x^{n-2} + \dots\Big) + b_{n+1}\rho_n\Big(x^{n-1} + \eta_{n-1}x^{n-2} + \chi_{n-2}x^{n-3} + \dots\Big) \\ &+c_n\rho_{n-1}\Big(x^{n-2} + \eta_{n-2}x^{n-3} + \chi_{n-3}x^{n-4} + \dots\Big) = \rho_{n+1}\Big(x^n + b_n x^{n-1} + c_{n-1}x^{n-2} + \dots\Big) \quad , \quad n \geqslant 0, \end{aligned}$$

with the convention $\rho_{-1} = \chi_{-k} = \eta_{-k} = 0$ for $k \geqslant 0$. The comparison of the coefficients in the previous identities provides

$$\begin{cases} \dfrac{b_n}{\rho_n} - \dfrac{b_{n+1}}{\rho_{n+1}} = \dfrac{\eta_n}{\rho_n} \\ \dfrac{c_n}{\rho_n\,\rho_{n+1}} - \dfrac{c_{n+1}}{\rho_{n+1}\,\rho_{n+2}} = \dfrac{\chi_n + \eta_n\, b_{n+2}\,\frac{\rho_{n+1}}{\rho_{n+2}}}{\rho_n\,\rho_{n+1}} \end{cases} \quad , \quad n \geqslant 1,$$

yielding the relations (2.2). □

For instance, if $\mathscr{O} = H_q$ or $\mathscr{O} = D$, then we have $\big(H_q\zeta^{n+1}\big)(x) = [n+1]_q x^n$ and $\big(D\zeta^{n+1}\big)(x) = (n+1)x^n$, $n \geqslant 0$, respectively. In both cases, $\eta_n = \chi_n = 0$, $n \geqslant 0$. More generally, we have:

Corollary 2.1. *Let* $\mathscr{O}$ *be a lowering operator such that* $\Big(\mathscr{O}\zeta^{n+1}\Big)(x) = \rho_{n+1}x^n$ *for any integer* $n \geqslant 0$*. If an* $\mathscr{O}$*-Appell sequence* $\{B_n\}_{n\geqslant 0}$ *is orthogonal, then the corresponding recurrence coefficients are given by*

$$\beta_n = \frac{\rho_{n+1} - \rho_n}{\rho_1}\,\beta_0 \quad , \quad n \geqslant 0, \tag{2.3}$$

$$\gamma_{n+1} = \frac{\rho_{n+1}}{{\rho_1}^2\,\rho_2}\left\{\rho_1\left(\rho_{n+2} - \rho_n\right)\gamma_1 + \Big(\rho_1\left(\rho_{n+2} - \rho_n\right) - \rho_2\left(\rho_{n+1} - \rho_n\right)\Big){\beta_0}^2\right\} \quad , \quad n \geqslant 0, \tag{2.4}$$

with the notation $\rho_{n+1} := \rho_{n+1}(\mathscr{O})$ *for any integer* $n \geqslant 0$*.*

Proof. Let us express any element of $\{B_n\}_{n\geqslant 0}$ as in (2.1). By equating the highest powers of x in the second order recurrence relation (1.15), the standard relations are derived

$$\begin{aligned} &\beta_n = b_n - b_{n+1} \quad , \quad n \geqslant 0 \\ &\gamma_{n+1} = c_n - c_{n+1} - \beta_{n+1}b_{n+1} \quad , \quad n \geqslant 0, \end{aligned}$$

with $b_0 = c_{-1} = c_0 = 0$. Under the assumptions, based on the previous lemma 2.1, we achieve the result, inasmuch as $\quad b_1 = -\beta_0 \quad$ and $\quad c_1 = \beta_0^2\left(\dfrac{\rho_2}{\rho_1} - 1\right) - \gamma_1$. □

Example. Consider a MPS $\{B_n\}_{n\geqslant 0}$ as above and let $\{B_n^{[1]}(\cdot;q)\}_{n\geqslant 0}$ be the MPS defined by

$$B_n^{[1]}(x;H_q) := \frac{1}{[n+1]_q}\big(H_q B_{n+1}\big)(x), n \geqslant 0. \tag{2.5}$$

The corresponding dual sequence $\{u_n^{[1]}(q)\}_{n\geqslant 0}$ of $\{B_n^{[1]}(\cdot;q)\}_{n\geqslant 0}$ is related to the dual sequence of $\{B_n\}_{n\geqslant 0}$ through (see [8])

$$H_q\Big(u_n^{[1]}(q)\Big) = -[n+1]_q\, u_{n+1}, \quad n \geqslant 0. \tag{2.6}$$

An **$\mathbf{H_q}$-Appell sequence** $\{B_n\}_{n\geqslant 0}$ is defined by the condition $B_n(\cdot) = B_n^{[1]}(\cdot, H_q)\,,\ n \geqslant 0.$

The existence of orthogonal H_q-Appell polynomial sequences is well known and it was studied from different points of view, see for instance [1, 8].

Under the assumption that the H_q-Appell polynomial sequence $\{B_n\}_{n\geqslant 0}$ is orthogonal, on the strength of corollary 2.1 with the natural replacement $\rho_{n+1} = [n+1]_q$, $n \geqslant 0$, the corresponding recurrence coefficients are given by

$$\beta_n = \beta_0\, q^n\,, \quad n \geqslant 0, \tag{2.7}$$

$$\gamma_{n+1} = q^n [n+1]_q\, \gamma_1\,, \quad n \geqslant 0, \tag{2.8}$$

where β_0 and $\gamma_1 \neq 0$ are two arbitrary constants. In addition, the H_q-Appell character of $\{B_n\}_{n\geqslant 0}$ supplies the equalities $u_n^{[1]} = u_n$, $n \geqslant 0$, whence from (1.16) and (2.6), follows the relation

$$H_q\big(B_n u_0\big) = -\frac{[n+1]_q}{\gamma_{n+1}}\, B_{n+1}\, u_0, \quad n \geqslant 0.$$

The particular choice $n = 0$, yields

$$H_q(u_0) + \gamma_1^{-1}\, B_1\, u_0 = 0. \tag{2.9}$$

Thus, $\{B_n\}_{n\geqslant 0}$ represents the Al-Salam and Carlitz polynomial sequence [1], up to a linear transformation. For further reading see the book of Ismail [7, Ch.18]. $\Diamond$

3. Quadratic Decomposition of H_q-Appell sequences

It is always possible to consider the quadratic decomposition (QD) of a given a MPS $\{B_n\}_{n\geqslant 0}$, through the association of two other MPS $\{P_n\}_{n\geqslant 0}$, $\{R_n\}_{n\geqslant 0}$ and two sequences of polynomials $\{a_n\}_{n\geqslant 0}$ and $\{b_n\}_{n\geqslant 0}$, with $\deg a_n, \deg b_n \leqslant n$, according to [12, 15]

$$B_{2n}(x) = P_n(x^2) + x\, a_{n-1}(x^2), \quad n \geqslant 0, \tag{3.1}$$

$$B_{2n+1}(x) = b_n(x^2) + x\, R_n(x^2), \quad n \geqslant 0, \tag{3.2}$$

with the convention $a_{-1}(\cdot) = 0$. Providing the characteristics of $\{B_n\}_{n\geqslant 0}$, it is possible to infer properties about $\{P_n\}_{n\geqslant 0}$, $\{R_n\}_{n\geqslant 0}$, $\{a_n\}_{n\geqslant 0}$ and $\{b_n\}_{n\geqslant 0}$ and consequently, to get more acquainted from the original MPS $\{B_n\}_{n\geqslant 0}$. However this procedure is sometime quite hard to solve, specially when the sequence $\{B_n\}_{n\geqslant 0}$ is not symmetric, since the symmetry of $\{B_n\}_{n\geqslant 0}$ implies $a_n(\cdot) = b_n(\cdot) = 0$, $n \geqslant 0$.

Under the assumption of the H_q-Appell character over the MPS $\{B_n\}_{n\geqslant 0}$ we are able to derive properties concerning the four associated sequences, as stated in the next lemma.

Lemma 3.1. *Consider the quadratic decomposition of the MPS $\{B_n\}_{n\geqslant 0}$ according to (3.1)-(3.2). If $\{B_n\}_{n\geqslant 0}$ is H_q-Appell, then the sequences $\{P_n\}_{n\geqslant 0}$ and $\{R_n\}_{n\geqslant 0}$ are Appell sequences with respect to another q-differential operator. Moreover,*

$$R_n(x) = \frac{1}{q^{-1}\,[n+1]_{q^2}\,[2n+3]_q}\ \mathscr{M}_q^{(+1)}\big[R_{n+1}\big](x)\,, \quad n \geqslant 0, \tag{3.3}$$

$$P_n(x) = \frac{1}{q\,[n+1]_{q^2}\,[2n+1]_q}\ \mathscr{M}_q^{(-1)}\big[P_{n+1}\big](x)\,, \quad n \geqslant 0. \tag{3.4}$$

$$b_n(x) = \frac{1}{q^{-1}\,[n+1]_{q^2}\,[2n+3]_q}\ \mathscr{M}_q^{(-1)}\big[b_{n+1}\big](x)\,, \quad n \geqslant 0, \tag{3.5}$$

$$a_n(x) = \frac{1}{q\,[n+2]_{q^2}\,[2n+3]_q}\ \mathscr{M}_q^{(+1)}\big[a_{n+1}\big](x)\,, \quad n \geqslant 0. \tag{3.6}$$

with

$$\mathscr{M}_q^{(\varepsilon)} = (q+1)\, H_{q^2}\, x\, H_{q^2} - [-\varepsilon]_q\, H_{q^2} \tag{3.7}$$

Proof. Representing by $\{B_n\}_{n\geqslant 0}$ an H_q-Appell sequence, we proceed to its quadratic decomposition in accordance with (3.1)-(3.2). Operating with H_q on both sides of (3.1), with n replaced by $n+1$, and on (3.2), we respectively obtain

$$[2n+2]_q\, B_{2n+1}(x) = \Big(H_q P_{n+1}(\xi^2)\Big)(x) + \Big(H_q \xi\; a_n(\xi^2)\Big)(x), \quad n\geqslant 0, \tag{3.8}$$

$$[2n+1]_q B_{2n}(x) = \Big(H_q b_n(\xi^2)\Big)(x) + \Big(H_q \xi\; R_n(\xi^2)\Big)(x), \quad n\geqslant 0, \tag{3.9}$$

since the H_q-Appell character of $\{B_n\}_{n\geqslant 0}$ provides $\big(H_q B_{n+1}\big)(x) = [n+1]_q\, B_n(x)$, $n\geqslant 0$. The substitution of B_{2n} and B_{2n+1} on the left hand side of the equalities (3.8)-(3.9) by their expressions given in (3.1)-(3.2), permits to obtain two new relations, none of them depending on the elements of $\{B_n\}_{n\geqslant 0}$, which are:

$$[2n+2]_q\Big\{b_n(x^2) + x\, R_n(x^2)\Big\} = \Big(H_q P_{n+1}(\xi^2)\Big)(x) + \Big(H_q \xi\; a_n(\xi^2)\Big)(x), \quad n\geqslant 0, \tag{3.10}$$

$$[2n+1]_q\Big\{P_n(x^2) + x\, a_{n-1}(x^2)\Big\} = \Big(H_q b_n(\xi^2)\Big)(x) + \Big(H_q \xi\; R_n(\xi^2)\Big)(x)\,, \quad n\geqslant 0. \tag{3.11}$$

If $\sigma : \mathscr{P} \to \mathscr{P}$ represents the linear operator defined by $(\sigma f)(x) := f(x^2)$, for any $f \in \mathscr{P}$, then the identity

$$H_q \circ \sigma = (q+1)\, x\, \sigma \circ H_{q^2} \quad \text{in } \mathscr{P}\,. \tag{3.12}$$

holds. Combining the latter with

$$H_q\, x = q\, x\, H_q + \mathrm{I}_{\mathscr{P}} \quad \text{in } \mathscr{P} \tag{3.13}$$

obtained from (1.4), we get

$$H_q\, x \circ \sigma = \sigma\Big(\, q(q+1)\, x\, H_{q^2} + \mathrm{I}_{\mathscr{P}}\Big) \quad \text{in } \mathscr{P}\,. \tag{3.14}$$

Based on (3.12) and (3.14) the relations (3.10)-(3.11) become, respectively, as follows:

$$\begin{aligned}[2n+2]_q\Big\{\sigma\, b_n(x) + x\,\sigma\, R_n(x)\Big\} &= (q+1)\, x\, (\sigma \circ H_{q^2})[P_{n+1}](x)\\ &\quad + \sigma \circ \Big(\, q(q+1)\, H_{q^2} + \mathrm{I}_{\mathscr{P}}\Big)[a_n](x)\,, \quad n\geqslant 0,\end{aligned} \tag{3.15}$$

$$\begin{aligned}[2n+1]_q\Big\{\sigma\, P_n(x) + x\,\sigma\, a_{n-1}(x)\Big\} &= (q+1)\, x\, (\sigma \circ H_{q^2})[b_{n+1}](x)\\ &\quad + \sigma \circ \Big(\, q(q+1)\, H_{q^2} + \mathrm{I}_{\mathscr{P}}\Big)[R_n](x)\,, \quad n\geqslant 0.\end{aligned} \tag{3.16}$$

Equating the even and odd terms in (3.15) and in (3.16), we respectively have:

$$[2n+2]_q\, R_n(x) = (q+1)\, H_{q^2}[P_{n+1}](x)\,, \quad n\geqslant 0, \tag{3.17}$$

$$[2n+2]_q\, b_n(x) = \Big(q(q+1)\, x\, H_{q^2} + \mathrm{I}_{\mathscr{P}}\Big)[a_n](x)\,, \quad n\geqslant 0, \tag{3.18}$$

$$[2n+1]_q\, P_n(x) = \Big(q(q+1)\, x\, H_{q^2} + \mathrm{I}_{\mathscr{P}}\Big)[R_n](x)\,, \quad n\geqslant 0, \tag{3.19}$$

$$[2n+1]_q\, a_{n-1}(x) = (q+1)\, H_{q^2}[b_n](x)\,, \quad n\geqslant 1. \tag{3.20}$$

The relations (3.17) and (3.19), provide

$$[2n+2]_q\, [2n+3]_q\, R_n(x) = (q+1) H_{q^2}\Big(q(q+1)\, x\, H_{q^2} + \mathrm{I}_{\mathscr{P}}\Big)[R_{n+1}](x)\,, \quad n\geqslant 0, \tag{3.21}$$

$$[2n+1]_q\, [2n+2]_q\, P_n(x) = \Big(q(q+1)\, x\, H_{q^2} + \mathrm{I}_{\mathscr{P}}\Big)(q+1) H_{q^2}[P_{n+1}](x)\,, \quad n\geqslant 0. \tag{3.22}$$

By taking into account the identity (3.13), with q^2 instead of q, and also the identities $(q+1)^{-1}\,[2n+2]_q = [n+1]_{q^2}$, with $n \in \mathbb{N}$, and $q^{-1} = -[-1]_q$, the achievement of the relations (3.3)-(3.4), under the definition in (3.7), comes as consequence of the previous obtained relations (3.23)-(3.24).

The other two relations (3.5) and (3.6) that remain to be proved, may be obtained through a procedure as simple as the previous one. Indeed, by virtue of (3.18) and (3.20), it follows

$$[2n+2]_q\, [2n+3]_q\, b_n(x) = \Big(q(q+1)\, x\, H_{q^2} + \mathrm{I}_{\mathscr{P}}\Big)(q+1) H_{q^2}[b_{n+1}](x)\,, \quad n\geqslant 0, \tag{3.23}$$

$$[2n+3]_q\, [2n+4]_q\, a_n(x) = (q+1) H_{q^2}\Big(q(q+1)\, x\, H_{q^2} + \mathrm{I}_{\mathscr{P}}\Big)[a_{n+1}](x)\,, \quad n\geqslant 0. \tag{3.24}$$

which, because of (3.13), may be transformed into the relations (3.5) and (3.6), respectively. □

The arisen operator $\mathscr{M}_q^{(\varepsilon)}$ is another example of a lowering operator. The relations (3.3)-(3.4) ensure that the two MPS $\{R_n\}_{n\geqslant 0}$ and $\{P_n\}_{n\geqslant 0}$ associated to QD of an H_q-Appell sequence are, according to definition 2.1, $\mathscr{M}_q^{(+1)}$ and $\mathscr{M}_q^{(-1)}$-Appell sequences. Therefore, the characterization of these arisen $\mathscr{M}_q^{(\varepsilon)}$-Appell sequences is now the issue. We will follow an algebraic approach operating on the dual space $\mathscr{P}'$. In order to accomplish this we need to determine the transpose of $\mathscr{M}_q^{(\varepsilon)}$ denoted by ${}^t\mathscr{M}_q^{(\varepsilon)}$, which, according to (1.3) and (3.7) is given by ${}^t\mathscr{M}_q^{(\varepsilon)} = (q+1)\,H_{q^2}\,x\,H_{q^2} + [-\varepsilon]_q\,H_{q^2}$ in $\mathscr{P}'$. We introduce the operator

$$\mathscr{L}_{q;\varepsilon,\mu} := (q+1)\,H_{q^2}\,x\,H_{q^2} + \mu[-\varepsilon]_q\,H_{q^2} \qquad \text{in } \mathscr{P} \tag{3.25}$$

and thereby $\mathscr{M}_q^{(\varepsilon)} = \mathscr{L}_{q;\varepsilon,-1}$. Thus, the transpose of $\mathscr{L}_{q;\varepsilon,\mu}$ is ${}^t\mathscr{L}_{q;\varepsilon,\mu} := \mathscr{L}_{q;\varepsilon,-\mu}$ in $\mathscr{P}'$ leaving out a slight abuse of notation without consequence, and it is injective in $\mathscr{P}'$ since $\mathscr{L}_{q;\varepsilon,\mu}$ is surjective in $\mathscr{P}$.

Henceforth, we will deal with the characterization of the $\mathscr{L}_{q;\varepsilon,\mu}$-Appell sequence and, thereafter, we will seek all the MOPS that are invariant under the action of this lowering operator.

4. The arisen $\mathscr{L}_{q;\varepsilon,\mu}$-Appell polynomial sequence

From a given a MPS $\{P_n\}_{n\geqslant 0}$ we construct another MPS $\{P_n^{[1]}(\cdot;\mathscr{L}_{q;\varepsilon,\mu})\}_{n\geqslant 0}$ defined through

$$P_n^{[1]}(x;\mathscr{L}_{q;\varepsilon,\mu}) := \frac{1}{\rho_{n+1}(\varepsilon,\mu;q)}\,\big(\mathscr{L}_{q;\varepsilon,\mu}P_{n+1}\big)(x)\,, \quad n\geqslant 0, \tag{4.1}$$

where the operator $\mathscr{L}_{q;\varepsilon,\mu}$ is given by (3.7) and $\{\rho_{n+1}(\varepsilon,\mu;q) := \rho_{n+1}\}_{n\geqslant 0}$ represents a sequence of nonzero numbers conveniently chosen in order to have

$$\mathscr{L}_{q;\varepsilon,\mu}[\zeta^{n+1}](x) = \rho_{n+1}\,x^n\,, \quad n\geqslant 0.$$

Thereby,

$$\rho_{n+1} = q^{-\varepsilon}\,[n+1]_{q^2}\,\Big([2n+2+\varepsilon]_q - (\mu+1)\,[\varepsilon]_q\Big), \quad n\geqslant 0, \tag{4.2}$$

where necessarily

$$(\varepsilon,\mu) \in \Big\{(x,y)\in\mathbb{C}^2 \,:\, x\neq -2(n+1) \quad \wedge \quad [-x]_q\,y \neq -[2n+2]_q\,,\, n\geqslant 0\Big\}. \tag{4.3}$$

For the sake of simplicity, we will loosely write $P_n^{[1]}(\cdot) := P_n^{[1]}\big(\,\cdot\,;\mathscr{L}_{q;\varepsilon,\mu}\big)\,,\, n\geqslant 0.$

Let us denote by $\{u_n\}_{n\geqslant 0}$ the dual sequence of $\{P_n\}_{n\geqslant 0}$ and by $\{u_n^{[1]}\}_{n\geqslant 0}$ the dual sequence of $\{P_n^{[1]}\}_{n\geqslant 0}$.

Lemma 4.1. *The elements of the sequence $\{u_n^{[1]}\}_{n\geqslant 0}$ fulfil*

$$\mathscr{L}_{q;\varepsilon,-\mu}\left(u_n^{[1]}\right) = \rho_{n+1}\,u_{n+1}\,, \quad n\geqslant 0. \tag{4.4}$$

Proof. The dual sequence $\{u_n^{[1]}\}_{n\geqslant 0}$ is defined by $\langle u_n^{[1]}\,,\,P_m^{[1]}\rangle = \delta_{n,m}$, $n,m\geqslant 0$. The definition of the transpose of $\mathscr{L}_{q;\varepsilon,\mu}$ enables the identities

$$\rho_n\,\delta_{n,m} = \langle u_n^{[1]}\,,\,\mathscr{L}_{q;\varepsilon,\mu}(P_m)\,\rangle = \langle\,\mathscr{L}_{q;\varepsilon,-\mu}(u_n^{[1]})\,,\,P_m\,\rangle \quad , \quad n,m\geqslant 0,$$

and, according to (1.14), the result is attained. □

There are a few other properties that ought to be determined in order to seek the $\mathscr{L}_{q;\varepsilon,\mu}$-Appell orthogonal sequences, such as the action of $\mathscr{L}_{q;\varepsilon,\mu}$ over the product of two polynomials or the action of $\mathscr{L}_{q;\varepsilon,-\mu}$ over the product of a polynomial by a form.

Lemma 4.2. *For any $f,\ g\in\mathscr{P}$ and $u\in\mathscr{P}'$, the following relations hold*

$$\mathscr{L}_{q;\varepsilon,\mu}\Big(f(x)\,g(x)\Big)=f(x)\,\mathscr{L}_{q;\varepsilon,\mu}\Big(g(x)\Big)+\mathscr{L}_{q;\varepsilon,\mu}\Big(f(x)\Big)\,h_{q^4}\Big(g(x)\Big)+c_q\,x\,\Big(H_{q^2}\circ h_{q^2}\,g(x)\Big)\,H_{q^2}\Big(f(x)\Big)\,, \tag{4.5}$$

and

$$\begin{aligned}\mathscr{L}_{q;\varepsilon,-\mu}(\,g\,u\,) &= h_{q^{-4}}(g)\ \mathscr{L}_{q;\varepsilon,-\mu}(\,u\,)-q^{-4}\Big(h_{q^{-4}}\circ\mathscr{L}_{q;\varepsilon,\mu}(g)\Big)\,u\\ &\quad+q^{-4}\,c_q\,\Big\{\Big(x\,h_{q^{-2}}\circ H_{q^{-2}}(g)\Big)\,H_{q^2}(u)+\Big(H_{q^{-2}}\,x H_{q^{-2}}\,(g)\Big)\,u\Big\}\quad,\quad g\in\mathscr{P}\,,\ u\in\mathscr{P}'\end{aligned} \tag{4.6}$$

where

$$c_q:=c_{q;\varepsilon,\mu}=2[2]_q-\mu[-\varepsilon]_q(q^2-1)=[2]_q\Big(2-\mu(q^{-\varepsilon}-1)\Big) \tag{4.7}$$

Proof. From (1.4) and after some computations according to (1.5), (1.11) and $h_{q^2}-I_{\mathscr{P}}=(q^2-1)H_{q^2}$ the relation (4.5) comes out, that holds for any two polynomials f and g. With the substitution of g by $\big(h_{q^{-4}}g\big)$, (4.5) becomes like

$$\begin{aligned}\mathscr{L}_{q;\varepsilon,\mu}\Big(f\ h_{q^{-4}}(g)\Big) &= f\,\Big(\mathscr{L}_{q;\varepsilon,\mu}\circ h_{q^{-4}}(g)\Big)+\Big(\mathscr{L}_{q;\varepsilon,\mu}(f)\Big)\,g\\ &\quad+c_q\,\Big\{H_{q^2}\Big(f\,\big(xH_{q^2}\circ h_{q^{-4}}g\big)\Big)-q^{-2}f\,\Big(H_{q^{-2}}\,xH_{q^2}\circ h_{q^{-2}}g\Big)\Big)\Big\}\end{aligned}$$

which, because of (1.5) and (1.9)-(1.12), amounts to the same as

$$\begin{aligned}\mathscr{L}_{q;\varepsilon,\mu}\Big(f\ h_{q^{-4}}(g)\Big) &= q^{-4}\,\Big(h_{q^{-4}}\circ\mathscr{L}_{q;\varepsilon,\mu}(g)\Big)\,f\,+\,g\,\Big(\mathscr{L}_{q;\varepsilon,\mu}(f)\Big)\\ &\quad+c_q\,q^{-4}\,\Big\{H_{q^2}\Big(f\,\big(x\,h_{q^{-2}}\circ H_{q^{-2}}g\big)\Big)-f\,\Big(H_{q^{-2}}\,xH_{q^{-2}}g\Big)\Big)\Big\}\end{aligned} \tag{4.8}$$

This permits to infer the action of $\mathscr{L}_{q;\varepsilon,\mu}$ over the product of a polynomial by a form. Indeed, by duality we define the product of any $g\in\mathscr{P}$ by $u\in\mathscr{P}'$:

$$\big\langle\mathscr{L}_{q;\varepsilon,-\mu}(gu)\,,\,f\big\rangle\ =\ \big\langle u\,,\,g\,\mathscr{L}_{q;\varepsilon,\mu}(f)\big\rangle$$

and, because of (4.8) it follows,

$$\begin{aligned}\big\langle\mathscr{L}_{q;\varepsilon,-\mu}(gu)\,,\,f\big\rangle &= \Big\langle\,u\,,\,\mathscr{L}_{q;\varepsilon,\mu}\Big(f\ h_{q^{-4}}(g)\Big)-q^{-4}\,\Big(h_{q^{-4}}\circ\mathscr{L}_{q;\varepsilon,\mu}(g)\Big)\,f\\ &\qquad-c_q\,q^{-4}\,\Big\{H_{q^2}\Big(f\,\cdot\,x\,h_{q^{-2}}\circ H_{q^{-2}}(g)\Big)-\,f\cdot H_{q^{-2}}\,xH_{q^{-2}}(g)\Big\}\,\Big\rangle\\ &= \Big\langle\,h_{q^{-4}}(g)\ \mathscr{L}_{q;\varepsilon,-\mu}(\,u\,)\,,\,f\Big\rangle-\Big\langle\,q^{-4}\,\Big(h_{q^{-4}}\circ\mathscr{L}_{q;\varepsilon,\mu}(g)\Big)\,u\,,\,f\Big\rangle\\ &\qquad+c_q\,q^{-4}\Big\langle\,\Big(x\,h_{q^{-2}}\circ H_{q^{-2}}(g)\Big)\,H_{q^2}(u)+\Big(H_{q^{-2}}\,xH_{q^{-2}}\,(g)\Big)\,u\,,\,f\Big\rangle\quad,\quad f\in\mathscr{P}\end{aligned}$$

ensuring the identity (4.6). □

We have fulfilled the necessary requirements to seek all the orthogonal polynomial sequences invariant under the action of $\mathscr{L}_{q;\varepsilon,\mu}$ to state the result:

Theorem 4.1. *If $\{P_n\}_{n\geqslant0}$ is an $\mathscr{L}_{q;\varepsilon,\mu}$-Appell MOPS, then the corresponding regular form u_0 fulfils*

$$H_{q^2}\Big(x\,u_0\Big)-\vartheta^{-1}\,(x-\beta_0)\,u_0=0\qquad\textit{with}\qquad\vartheta=\frac{q^{2+\varepsilon}\,(q^2-1)\,\beta_0}{(\,(\mu+1)q^{\varepsilon}-q^{\varepsilon+2}-\mu)}\neq0 \tag{4.9}$$

and the recurrence coefficients are given by

$$\beta_n=\frac{q^{2n}\,\big(1+q^2(\mu q^{-2}+q^{-2}-\mu q^{-\varepsilon-2})-(q^2+1)\,q^{2n}\big)}{q^2(q^2-1)}\;\vartheta\quad,\qquad n\geqslant0, \tag{4.10}$$

$$\gamma_{n+1}=\frac{q^{4n+2}(q+1)\beta_0^2\rho_{n+1}}{\rho_1^2} \tag{4.11}$$

where the range for the complex parameters ε,μ is set in (4.3).

Proof. Under the assumption of the existence of a MOPS $\{P_n\}_{n\geqslant 0}$ that is $\mathscr{L}_{q;\varepsilon,\mu}$-Appell, proposition 2.1 ensures that the recurrence coefficients of $\{P_n\}_{n\geqslant 0}$ would be given by (2.3)-(2.4) with ρ_{n+1} given by (4.2). In particular, (2.3) corresponds to (4.10). Now the goal is to simplify the expression (2.4) in order to ensure $\gamma_{n+1}\neq 0$ and, concomitantly, to characterize and identify the MOPS $\{P_n\}_{n\geqslant 0}$. Following (4.4), the $\mathscr{L}_{q;\varepsilon,\mu}$-Appell character of $\{P_n\}_{n\geqslant 0}$ yields the relation $\mathscr{L}_{q;\varepsilon,\mu}\big(u_n\big)=\rho_{n+1}\;u_{n+1},\; n\geqslant 0,$ where $\{u_n\}_{n\geqslant 0}$ represents the corresponding dual sequence, whose elements may be expressed as in (1.16), because of the orthogonality of $\{P_n\}_{n\geqslant 0}$. Consequently, we have

$$\mathscr{L}_{q;\varepsilon,-\mu}\big(P_n(x)\,u_0\big)=\lambda_n\,P_{n+1}(x)\,u_0 \quad , \quad n\geqslant 0, \tag{4.12}$$

where

$$\lambda_n=\rho_{n+1}\frac{\langle u_0,P_n^2\rangle}{\langle u_0,P_{n+1}^2\rangle}=\frac{\rho_{n+1}}{\gamma_{n+1}} \quad , \quad n\geqslant 0 \tag{4.13}$$

When $n=0$, the relation (4.12) becomes

$$\mathscr{L}_{q;\varepsilon,-\mu}\big(u_0\big)=\lambda_0\,P_1(x)\,u_0. \tag{4.14}$$

After the substitution $n\rightarrow n+1$, and according to the product rule (4.6), the relation (4.12) may be rewritten like

$$\begin{aligned} &h_{q^{-4}}\left(P_{n+1}\right)\;\mathscr{L}_{q;\varepsilon,-\mu}\left(u_0\right)-q^{-4}\;h_{q^{-4}}\Big(\mathscr{L}_{q;\varepsilon,\mu}\left(P_{n+1}\right)\Big)u_0\\ &+q^{-4}\;c_q\;\Big\{x\,h_{q^{-2}}\circ H_{q^{-2}}\Big(P_{n+1}\Big)\;H_{q^2}(u_0)+H_{q^{-2}}\,x\,H_{q^{-2}}\Big(P_{n+1}\Big)\;u_0\Big\}=\lambda_{n+1}\,P_{n+2}\,u_0 \quad , \quad n\geqslant 0 \end{aligned} \tag{4.15}$$

By virtue of (4.14) and also because $\mathscr{L}_{q;\varepsilon,\mu}\big(P_{n+1}\big)=\rho_{n+1}P_n\,,\, n\geqslant 0$, the equality (4.15) becomes

$$q^{-4}\;c_q\,x\,\Big(h_{q^{-2}}\circ H_{q^{-2}}\big(P_{n+1}(x)\big)\Big)\,H_{q^2}(u_0)=A_{n+2}(x)\,u_0 \quad , \quad n\geqslant 0, \tag{4.16}$$

where

$$A_{n+2}(x)=\lambda_{n+1}P_{n+2}-\lambda_0\,P_1\,h_{q^{-4}}\big(P_{n+1}\big)+q^{-4}\big\{\,\rho_{n+1}\,h_{q^{-4}}\big(P_n\big)-H_{q^{-2}}xH_{q^{-2}}\big(P_{n+1}\big)\big\} \quad , \quad n\geqslant 0. \tag{4.17}$$

The particular choice of $n=0$ in (4.16) results in another simple functional relation fulfilled by u_0:

$$q^{-4}\;c_q\,x\,H_{q^2}(u_0)=A_2(x)\,u_0\;. \tag{4.18}$$

Depending on the polynomial A_2, the latter functional equation is expected to describe the regular form u_0. Before accomplishing the computation of A_2 we will firstly derive more accurate conditions over the λ's (and consequently over the γ's) which will indeed provide that $\deg A_{n+1}\leqslant n$ for any integer $n\geqslant 1$. To begin with, we will derive a new expression for the polynomials A_{n+2} and afterwards we will show that necessarily $\deg A_2\leqslant 1$ implying the claimed condition. Between (4.16) and (4.18), and on account of the regularity of u_0, a q-differential-difference equation fulfilled by the sequence $\{P_n\}_{n\geqslant 0}$ is achieved:

$$A_{n+2}(x)=\Big(h_{q^{-2}}\circ H_{q^{-2}}\big(P_{n+1}(x)\big)\Big)A_2(x) \quad , \quad n\geqslant 0, \tag{4.19}$$

(which can be transformed into a q-differential equation, upon the replacements $P_{n+2}=(x-\beta_{n+1})P_{n+1}-\gamma_{n+1}P_n$ and also $P_n=\rho_{n+1}^{-1}\,\mathscr{L}_{q;\varepsilon,\mu}(P_{n+1})$). By equating the coefficients of x^{n+2} in (4.19), we derive the relation

$$\lambda_{n+1}-\lambda_0 q^{-4(n+1)}=q^{-2n}\,[n+1]_{q^{-2}}\,(\lambda_1-\lambda_0 q^{-4})\,, \quad n\geqslant 0. \tag{4.20}$$

On the other hand, the action of H_{q^2} over the relation (4.18) provides

$$q^{-4}\,c_q\,H_{q^2}\,x\,H_{q^2}(u_0)=h_{q^{-2}}(A_2)H_{q^2}(u_0)+q^{-2}H_{q^{-2}}(A_2)\,u_0\;.$$

and the comparison with (4.14) gives rise to new equation

$$\Big\{\frac{q^{-4}c_q\mu[-\varepsilon]_q}{q+1}-h_{q^{-2}}(A_2)\Big\}H_{q^2}(u_0)=\Big\{q^{-2}H_{q^{-2}}(A_2)-\frac{q^{-4}c_q}{q+1}\lambda_0P_1\Big\}u_0 \tag{4.21}$$

Between (4.18) and (4.21) and by taking into account the regularity of u_0, we derive polynomial the relation

$$q^{-4}c_q\,x\,\Big\{q^{-2}H_{q^{-2}}(A_2)-\frac{q^{-4}c_q}{q+1}\lambda_0P_1\Big\}=\Big\{\frac{q^{-4}c_q\mu[-\varepsilon]_q}{q+1}-h_{q^{-2}}(A_2)\Big\}A_2$$

which implies $\deg A_2 \leqslant 1$. From the definition (4.17) with $n=0$, it follows $\lambda_1 = q^{-4}\lambda_0$ and therefore (4.20) becomes like

$$\lambda_n = q^{-4n}\lambda_0 \quad , \quad n \geqslant 0.$$

Now, recalling the definition of λ_n in (4.13), the previous relation provides

$$\gamma_{n+1} = q^{4n}\frac{\rho_{n+1}}{\rho_1}\,\gamma_1 \quad , \quad n \geqslant 0.$$

Comparing this expression for the γ coefficients with the one previously obtained in (2.4), leads us to a new relation between β_0 and γ_1:

$$q^{4n}\,\frac{\rho_{n+1}}{\rho_1}\,\gamma_1 = \frac{\rho_{n+1}}{\rho_1^2\,\rho_2}\Big\{\rho_1(\rho_{n+2}-\rho_n)\,\gamma_1 + \Big(\rho_1\,(\rho_{n+2}-\rho_n) - \rho_2\,(\rho_{n+1}-\rho_n)\Big)\beta_0{}^2\Big\} \quad , \quad n \geqslant 0,$$

yielding

$$\gamma_1 = \frac{\rho_1\,(\rho_{n+2}-\rho_n) - \rho_2\,(\rho_{n+1}-\rho_n)}{\rho_1\big(q^{4n}\,\rho_2 - \rho_{n+2}+\rho_n\big)}\,\beta_0{}^2 \quad , \quad n \geqslant 0,$$

which can only be admissible if $\beta_0 \neq 0$ (rejecting the existence of symmetric sequences). Considering the definition of the ρ coefficients in (4.2), we have

$$\rho_1\,(\rho_{n+2}-\rho_n) - \rho_2\,(\rho_{n+1}-\rho_n) = q^2(q+1)\big(q^{4n}\,\rho_2 - \rho_{n+2}+\rho_n\big) \quad , \quad n \geqslant 0,$$

whence

$$\gamma_1 = \frac{q^2(q+1)}{\rho_1}\,\beta_0{}^2$$

providing (4.11). Consequently, $\quad \lambda_n = \dfrac{q^{-4n-2}\rho_1^2}{(q+1)\beta_0^2} \quad$ and

$$\begin{aligned} A_2(x) &= \frac{\rho_1\left(\left(q^4+1\right)\rho_1-\rho_2\right)}{q^6(q+1)\beta_0}\;x + \frac{\rho_1\left(\rho_2-\left(q^4+1\right)\rho_1\right) - c_q q^2(q+1)}{q^6(q+1)} \\ &= -\frac{\left(q^{\varepsilon+2}+\mu-(\mu+1)q^{\varepsilon}\right)\left(q^{\varepsilon}(\mu+2)-\mu\right)}{q^{2(2+\varepsilon)}(q-1)\beta_0}\;x - \mu\,\frac{\left(q^{\varepsilon}-1\right)\left(q^{\varepsilon}(\mu+2)-\mu\right)}{q^{2(2+\varepsilon)}(q-1)} \end{aligned}$$

Based on (1.7), the functional equation (4.18) may be rewritten like

$$H_{q^2}\Big(\;x\;u_0\;\Big) - \frac{q^2}{c_q}\,\Big(q^{-4}c_q + A_2(x)\Big)\;u_0 = 0\,.$$

which corresponds to (4.9). □

From this latter we may read that an $\mathscr{L}_{q;\varepsilon,\mu}$-Appell orthogonal sequence $\{P_n\}_{n\geqslant 0}$ is necessarily H_{q^2}-classical [8] and the corresponding regular form u_0 is H_{q^2}-classical. Hence, we will follow the work of [8] in order to be acquainted with this sequence within the context of the already known sequences. Consider the sequence $\{\widehat{P}_n\}_{n\geqslant 0}$ obtained from the original $\mathscr{L}_{q;\varepsilon,\mu}$-Appell orthogonal sequence through $\widehat{P}_n(\cdot) := A^{-n}\,P_n(Ax)$ for $n \geqslant 0$, where

$$A = \vartheta\,\frac{\alpha_1}{q^2-1}$$

in which ϑ is given by (4.9) and $\alpha_1 := q^{-2}(\mu+1-\mu\,q^{-\varepsilon})$. Let us set

$$\alpha := \left(\alpha_1\,q^2\right)^{-1} = \Big(\mu+1-\mu\,q^{-\varepsilon}\Big)^{-1} \quad .$$

Consequently, the regular form associated to the $\mathscr{L}_{q;\varepsilon,\mu}$-Appell MOPS $\{\widehat{P}_n\}_{n\geqslant 0}$ is $\widehat{u}_0 := h_{A^{-1}}u_0$ and fulfils

$$H_{q^2}\Big(x\,\widehat{u}_0\Big) - (q^2-1)^{-1}(\alpha\;q^2)^{-1}\left(x-1+\alpha\;q^2\right)\,\widehat{u}_0 = 0\,. \tag{4.22}$$

The corresponding recurrence coefficients are then given by

$$\widehat{\beta}_n := \widehat{\beta}_n(q^2|\alpha) = q^{2n}\left(\alpha+1-\alpha\,\left(q^2+1\right)q^{2n}\right)\,, \quad n \geqslant 0,$$

and

$$\widehat{\gamma}_{n+1} := \widehat{\gamma}_{n+1}(q^2|\alpha) = \alpha\,q^{4n+2}\,\left(1-q^{2n+2}\right)\left(1-\alpha\,q^{2n+2}\right)\,, \quad n \geqslant 0.$$

Following the discussion carried out on [8] or on the report [9], we conclude that an $\mathscr{L}_{q;\varepsilon,\mu}$-Appell orthogonal sequence corresponds, up to a linear transformation, to the *Little* q^2*-Laguerre* polynomials that are also H_{q^2}-classical. The MOPS $\{\widehat{P}_n\}_{n\geqslant 0}$ actually depends on two parameters: q^2 and α, which in turn depends on ε and μ. This compels us to more accurately write $\{\widehat{P}_n(\cdot;q^2|\alpha)\}_{n\geqslant 0}$. The corresponding regular form $\widehat{u}_0 := \widehat{u}_0(q^2|\alpha)$ is regular if and only if $\alpha \neq q^{-2(n+1)}$, $n \geqslant 0$ and it is positive definite for

$$0<q^2<1 \quad \text{and} \quad 0<\alpha<q^{-2} \qquad \text{OR} \qquad q^2>1 \quad \text{and} \quad \alpha\in\left]-\infty,0\right[\bigcup\left]-q^{-2},\infty\right[$$

Replacing q^2 by q, the MOPS $\{\widehat{P}_n(\cdot;q|\alpha)\}_{n\geqslant 0}$ is now H_q-classical, $\mathscr{L}_{\sqrt{q};\varepsilon,\mu}$-Appell and it corresponds to the *Little q-Laguerre* polynomial sequence. With the substitution of q by q^{-1}, the MOPS $\{\widehat{P}_n(\cdot;q^{-1}|\alpha)\}_{n\geqslant 0}$ is $H_{q^{-1}}$-classical, $\mathscr{L}_{q^{-1/2};\varepsilon,\mu}$-Appell and it corresponds to the *q-Charlier I* polynomial sequence [10].

Based on the discussion in [8, pp.96-97], the form $\widehat{u}_0(q^2|\alpha)$ may be expressed according to

$$\widehat{u}_0(q^2|\alpha)=\begin{cases}(\alpha q^2;q^2)_\infty\displaystyle\sum_{k\geqslant 0}\frac{(\alpha q^2)^k}{(q^2;q^2)_k}\;\delta_{q^{2k}} & ,\quad 0<q^2<1\ ,\ |\alpha|<q^{-2}\\[2ex] \dfrac{1}{(\alpha;q^{-2})_\infty}\displaystyle\sum_{k\geqslant 0}\frac{q^{-k(k-1)}}{(q^{-2};q^{-2})_k}\;(-\alpha)^k\,\delta_{q^{2k}} & ,\quad q^2>1\ ,\ |\alpha|<1 \ \text{ or } \ \alpha<0\end{cases}$$

therefore, we get the integral representations for any $f\in\mathscr{P}$ and with $\alpha:=q^{2\tau}$, it holds

$$\langle\,\widehat{u}_0(q^2|\alpha)\ ,\ f\,\rangle=\begin{cases}\dfrac12(\alpha q^2;q^2)_\infty\displaystyle\sum_{k\geqslant 0}\frac{(\alpha q^2)^k}{(q^2;q^2)_k}\;\langle\,\delta_{q^{2k}}\ ,\ f\,\rangle \\ \qquad +\dfrac12\,K_1\displaystyle\int_0^{q^{-2}}x^\tau\;(q^2x;q^2)_\infty\;f(x)\;dx & ,\quad 0<q^2<1\ ,\ 0<\alpha<q^{-2}\\[2ex] \dfrac{1}{2\;(\alpha;q^{-2})_\infty}\displaystyle\sum_{k\geqslant 0}\frac{q^{-k(k-1)}}{(q^{-2};q^{-2})_k}\;(-\alpha)^k\,\langle\,\delta_{q^{2k}}\ ,\ f\,\rangle \\ \qquad +\dfrac12\,K_2\displaystyle\int_{-\infty}^0\frac{|x|^\tau}{(-|x|;q^{-2})_\infty}\;f(x)\;dx & ,\quad q^2>1\ ,\ q^{-2}<\alpha<1\end{cases}$$

where

$$K_1=\left(q^{-2(\tau+1)}\int_0^1 t^\tau\;(t;q^2)_\infty\;dt\right)^{-1}$$

and

$$K_2=\left(\int_0^\infty\frac{t^\tau}{(-t;q^{-2})_\infty}\;dt\right)^{-1}$$

with the notations

$$(a;q)_n=\begin{cases}1 & ,\quad n=0\\ \displaystyle\prod_{\nu=0}^{n-1}(1-aq^\nu) & ,\quad n\geqslant 1\end{cases}\qquad \text{and} \qquad (a;q)_\infty=\prod_{\nu\geqslant 0}(1-aq^\nu)\ ,\ |q|<1\ .$$

The choice of $\mu=-1$ provides $\alpha=q^{-\frac{\varepsilon}{2}}$ and $\mathscr{L}_{q^{-\frac12};\varepsilon,-1}=\mathscr{M}_{q^{-\frac12};\varepsilon}$. Hitherto, from the MOPS $\left\{\widehat{P}_n(\cdot;q^{-1}|q^{-\frac{\varepsilon}{2}})\right\}_{n\geqslant 0}$ we may construct another one $\left\{\widetilde{P}_n\left(x;q|\frac{\varepsilon}{2}\right)\right\}_{n\geqslant 0}$ by defining

$$\widetilde{P}_n\left(x;q|\ \tfrac{\varepsilon}{2}\right):=(-1)^n\,\widehat{P}_n(-x;q^{-1}|q^{-\frac{\varepsilon}{2}})\quad ,\quad n\geqslant 0.$$

which is orthogonal with respect to the form $\widetilde{u}_0=h_{-1}\widehat{u}_0=h_{-A^{-1}}u_0$ that fulfills the equation

$$H_{q^{-1}}\left(x\,\widetilde{u}_0\right)+q^{\frac{\varepsilon}{2}+1}(q^{-1}-1)^{-1}\left(x+1-q^{-\frac{\varepsilon}{2}-1}\right)\widetilde{u}_0=0$$

The corresponding recurrence coefficients are

$$\widetilde{\beta}_n := \widetilde{\beta}_n(q|-\tfrac{\varepsilon}{2}) = -\widehat{\beta}_n(q^{-1}|q^{-\frac{\varepsilon}{2}}) = q^{-2n-1-\frac{\varepsilon}{2}}\Big(1-q^{n+1}+(1-q^{n+\frac{\varepsilon}{2}})\ q\ \Big)\,,\quad n\geqslant 0,$$

$$\widetilde{\gamma}_{n+1} := \widetilde{\gamma}_{n+1}(q|-\tfrac{\varepsilon}{2}) = \widehat{\gamma}_{n+1}(q^{-1}|q^{-\frac{\varepsilon}{2}}) = q^{-4n-3-\varepsilon}(1-q^{n+1})(1-q^{n+1+\frac{\varepsilon}{2}})\,,\quad n\geqslant 0.$$

Hence the MOPS $\{\widetilde{P}_n\left(x;q|\frac{\varepsilon}{2}\right)\}_{n\geqslant 0}$ corresponds to the *q-Laguerre polynomials* of parameter $\frac{\varepsilon}{2}$ and it is indeed $\mathscr{L}_{\sqrt{q^{-1}},\varepsilon,-1}$-Appell (which amounts to the same as $\mathscr{M}_{\sqrt{q^{-1}},\varepsilon}$-Appell) [18].

The moment and integral representations of the aforementioned forms may, as well, be found in [8].

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