Hysteresis in a tatonnement process

Sofia B. S. D. Castro [1]
Centro de Matemática da Universidade do Porto [2]
and
Faculdade de Economia do Porto - Universidade do Porto
Rua Dr. Roberto Frias
4200-464 Porto
Portugal.
E-mail: sdcastro@fep.up.pt

Abstract

Hysteresis is associated to the permanence of effects from a temporary stimulus and to the idea of path dependence. Here, we perturb a simple model of an economy with two agents and two goods and obtain a continuous-time tatonnement process exhibiting hysteresis. This means equilibrium price is sensitive to both, changes in any parameter describing the preferences, and the way these occur. We use the fact that different parameter values of the utility function correspond to different preferences to explain hysteresic behaviour in terms of the substitution and income effects associated to different parameter values of the utility function.

Keywords: hysteresis, continuous-time tatonnement, equilibrium price

AMS classification: 37G40, 91B50

[1] I am grateful to C. Hervés-Beloso and A. Brandão for fruitful conversations.

[2] Work (partially) supported by the Centro de Matemática da Universidade do Porto (CMUP), financed by FCT (Portugal) through the programmes POCTI (Programa Operacional "Ciência, Tecnologia, Inovação") and POSI (Programa Operacional Sociedade da Informação), with national and European Community structural funds.

1 Introduction

Various economic systems exhibit the phenomenon of hysteresis. It has been observed by several authors (the references provided here are merely illustrative and far from exhaustive) in models that vary from, for instance, models of unemployment (Roed, 1997) to investment (Dixit,1992) through problems of trade (Amable, 1995). See Göcke (2002) and Katzner (1999) for short surveys and the bibliography therein for particular applications. These two authors also provide a discussion of the various ways in which the term hysteresis has been used in the economic context. Hysteresis is associated to the permanence of effects from a temporary stimulus and to the idea of path dependence. See Saunders (1986) for a mathematical explanation of the phenomenon of hysteresis.

Ljungqvist (1994) considers a general equilibrium analysis of hysteresis in international trade arising from a temporary change in the exchange rate. The effect of this change lingers after the exchange rate has returned to its initial value.

Here, we establish the equilibrium price of an economy with two agents and two goods where the preferences of the agents vary according to a parameter $\mu \in (0, 1)$. In this setting, hysteresis means that the equilibrium price attained in the economy is sensitive, not only to the change in value of the parameter μ, but also to the way this change occurs — increasing and decreasing the parameter does not produce the same result. This is the third type of hysteresis referred to in Katzner (1999), which requires the model to contemplate the possibility of a change in its parameters. In this way, the model considers a changing reality. This issue is formulated in terms of the continuous-time tatonnement process parametrized by the parameter μ of the agents' preferences. This parameter appears in the model through the utility function and is closely related to the balance between substitution and income effects. Substitution effects dominate for small values of μ whereas, for μ close to unity, income effects are dominant. We show that the changes in preferences induce a change in equilibrium price exhibiting hysteresis. In this way, the hysteresic behaviour we observe has its origin endogenously in the model. We do not address the cause of the change in preferences. As mentioned by Katzner (1999) this may be caused by changes in reality or in knowledge.

Our model extends and is based on that of Bala (1997) who presented a model of a class of exchange economies with two agents and two commodities, parametrized by a parameter in the utility function of the agents. The continuous time tatonnement process for this class of economies exhibits a *pitchfork bifurcation*, that is, there exists a parameter value for which the unique stable equilibrium price loses stability and two new stable equilibrium prices arise. Bala shows that this situation is robust under a symmetric perturbation of the initial endowments.

We show, in section 3, that if we want to consider non-symmetric perturbations of the endowments, the tatonnement process behaves in a qualitatively different way, exhibiting hysteresis for a wide class of perturbations. Furthermore, we show that in order to contemplate all possible perturbations of Bala's model we must include perturbations of the utility function. We show that these perturbations of the utility function correspond to perturbations in the agent's preferences. In this way, the use of a very particular utility function, as is the case in Bala's model, is not restrictive in terms of the preferences of the agents.

In section 4, we describe the behaviour observed in the tatonnement process of the most generic family of perturbations. This, of course, includes Bala's example as the origin of the space of the perturbation parameters.

Section 5 is devoted to the analysis of the hysteresic behaviour and the last section contains the conclusions.

We start out, in the next section, with the description of the utility function used by Bala and the effect of its parameter in the preferences of the agents.

2 The utility function

The utility function used by Bala (1997) is a CES such that agents are symmetric and as follows

$$\begin{aligned} u^1(x_1, x_2, \mu) &= -x_1^{\frac{\mu}{\mu-1}} - 2^{\frac{1}{\mu-1}} x_2^{\frac{\mu}{\mu-1}} \\ u^2(x_1, x_2, \mu) &= -x_2^{\frac{\mu}{\mu-1}} - 2^{\frac{1}{\mu-1}} x_1^{\frac{\mu}{\mu-1}}, \end{aligned}$$

where x_i, $i = 1, 2$ denotes the quantity of good i and $\mu \in (0, 1)$ is the parameter. An increase in μ corresponds to an increase in the income effect relative to the substitution effect. Regardless of this duality between income and substitution effects, it is easy to see that the law of demand holds for all values of μ.

The corresponding tatonnement process is defined by

$$\dot{p} = z_1(p, \mu) = \frac{2p^\mu}{2p^\mu + 1} + \frac{p^{\mu-1}}{p^\mu + 2} - 1,$$

where p is the price of good 1 and the price of good 2 has been normalized to unity. We note that $p = 1$ is always an equilibrium price.

To understand how the variation in income and substitution effect relates to the values of the parameter μ, we calculate the total effect (TE) as a function of μ at the equilibrium price in Bala's model. This will be positive if and only if the substitution effect is bigger than the income effect. We have

$$TE = \frac{-2\mu p^{\mu}(2+p^{\mu})^2 + (p^{\mu} - 2\mu + 2)p^{\mu-1}(2p^{\mu}+1)^2}{(2p^{\mu}+1)^2(2+p^{\mu})^2},$$

which is positive if and only if $\mu < \mu^* = 3/4$ at the equilibrium price $p = 1$. It is for this value of $\mu = \mu^*$ that a pitchfork bifurcation occurs in the tatonnement process in Bala's model. Hence, as stated by Bala in his paper, the balance between substitution and income effects are the cause of the bifurcation in the tatonnement process.

We draw attention to the fact that, with the above utility functions, we are assuming that the sensitivity to income and substitution effect is the same for both agents. That is, the expression of income and substitution effects, via the parameter μ, is the same for both agents corresponding to total symmetry of the agents' utility in this respect.

We show below that a change in the parameter of either utility function representing each agent's preferences actually corresponds to a change in the agent's preference.

It is worthwhile stressing that, as shown by a straightforward calculation which we omit, monotonicity is preserved by the change in preferences, thus guaranteeing that the bifurcation cannot be explained by risk aversion as Quah's (2003) work might suggest. Therefore, our explanation of the behaviour observed in this model will rely solely on the balance between substitution and income effects.

To guarantee that a change in the parameter of either utility function corresponds to a change in the agent's preference, we use a result by Kannai (1970) establishing the uniqueness of a utility function which, in the diagonal of the space of preferences, is equal to the norm of the vector of commodities. It is easily shown that this utility depends on the parameter μ. Because it is unique for representing a given preference, we conclude that a change in μ corresponds to a change of preference. We note also that this utility function defines a metric inducing a topology on the set of preference orders (see Kannai, 1970, section 3). Its behaviour when μ tends to either 0 or 1 is as described by Bala (1997).

Recall that Bala (1997) refers explicitly the restriction imposed by the use of a particular CES utility function. This restriction in genericity is thus overcome.

3 Generic perturbations

Consider the following perturbation of Bala's (1997) model given by a perturbation in the initial endowment of agent 1 and a perturbation in the preference of agent 2, through the parameter in this agent's utility function, as below

$$\omega^1 = (1, \theta) \quad \text{and} \quad u^1(x_1, x_2, \mu, \varepsilon) = -x_2^{\frac{\rho}{\rho-1}} - 2^{\frac{1}{\rho-1}} x_1^{\frac{\rho}{\rho-1}}, \tag{1}$$

with $\rho = \mu - \varepsilon$ and $\varepsilon,\ \theta > 0$. Notice that for $\varepsilon = \theta = 0$ we recover Bala's unperturbed case, since the utility of agent 1, $u^1(x_1, x_2, \mu)$, and the endowment of agent 2, $\omega^2 = (0, 1)$, remain unchanged.

For the family of perturbations described by (1) we have the following equation for the tatonnement

$$\dot{p} = Z_1(p, \mu, \varepsilon, \theta),$$

with

$$Z_1(p, \mu, \varepsilon, \theta) = \frac{p^{\rho-1}}{2 + p^\rho} - \frac{1}{2p^\mu + 1} + \frac{2\theta p^{\mu-1}}{2p^\mu + 1}. \tag{2}$$

Using results in equivariant singularity theory (see both Golubitsky and Schaeffer, 1985 and Golubitsky *et. al.*, 1988), it is easy to establish that Bala's tatonnement equation is equivalent to a pitchfork bifurcation problem, as expected. It is then well-known that such a problem is robust uniquely under perturbations that preserve the initial symmetry of the problem. We say that a bifurcation problem described by a map g is symmetric under the action of a group Γ if and only if for all points x in the domain of g

$$g(\gamma.x) = \gamma.g(x), \quad \forall \gamma \in \Gamma.$$

In the case of a pitchfork bifurcation the group of symmetries is $\mathbf{Z}_2$ acting on the real numbers as $x \mapsto -x$. The particular perturbation introduced by Bala (1997) preserves the invariance of the map describing the tatonnement and hence, the qualitative nature of the results.

Furthermore, proposition III, 4.4 in Golubitsky and Schaeffer (1985) guarantees that equation (2) is a universal unfolding of the unperturbed tatonnement process of Bala. A universal unfolding is a parametrized family of maps containing all possible perturbations of a given map where the number of parameters is minimal. This means that any perturbation of Bala's original problem can be transformed, through a change of coordinates, into Z_1 and two is the minimum number of parameters necessary to describe all perturbations.

4 Qualitative bifurcation diagrams

In this section, we do the local bifurcation analysis leading to the description of the qualitative behaviour of the unfolded (generic) problem in terms of the unfolding parameters ε and θ. Since we are interested uniquely in small (local) perturbations of the original symmetric problem around the bifurcation point $(p^*, \mu^*) = (1, 3/4)$, we can, without loss of generality, consider the Taylor polynomial of Z_1 at (p^*, μ^*). Furthermore, since a pitchfork is 3-determined (that is, equivalent to its Taylor polynomial of degree 3), it is sufficient to consider Taylor expansion up to order 3. This can be further simplified if we consider the weighted homogeneous setting.[3] The weighted homogenous, with respect to weights $(1, 2)$, Taylor polynomial of weighted degree 3 at (p^*, μ^*) of Z_1 is given by

$$G(x, \lambda, \varepsilon, \theta) = a_1 + a_2 x + a_3 x^2 + a_4 x\lambda + a_5 x^3,$$

where $x = p - 1$, $\lambda = \mu - 3/4$ and the coefficients are functions of the unfolding parameters ε and θ as follows:

$$\begin{aligned}
a_1 &= \frac{2}{3}\theta \\
a_2 &= -\frac{1}{2}\theta - \frac{2}{9}\varepsilon \\
a_3 &= \frac{19}{24}\theta + \frac{5}{9}\varepsilon + \frac{2}{27}\varepsilon^2 \\
a_4 &= \frac{2}{9}\theta + \frac{4}{9} \\
a_5 &= -\frac{187}{96}\theta - \frac{1}{48} - \frac{119}{72}\varepsilon - \frac{11}{18}\varepsilon^2 + \frac{2}{27}\varepsilon^3.
\end{aligned}$$

Hence, locally, the tatonnement problem described by Z_1 is equivalent to $\dot{x} = G(x, \lambda, \varepsilon, \theta)$ and the equilibrium price depends on the values of all three parameters μ (through λ), ε and θ. The qualitative perturbations of the original tatonnement process are captured in distinguished regions of the (ε, θ)-plane. These regions are separated by distinguised curves called the bifurcation and the hysteresis sets. Again see Golubitsky and Schaeffer (1985) chapter III, in particular definition 5.1, for more detail.

Let $\mathcal{B}$ and $\mathcal{H}$ denote, respectively, the bifurcation and hysteresis sets,

[3] A function $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is weighted homogeneous of weight r with respect to a set of weights (α_1, α_2) if $f(t^{\alpha_1} x_1, t^{\alpha_2} x_2) = t^r f(x_1, x_2)$, for all $t \in \mathbf{R}$.

defined below

$$
\begin{aligned}
\mathcal{B} &= \{(\varepsilon,\theta)\in \mathbf{R}^2:\ \exists(x,\lambda):\ G=G_x=G_\lambda=0\}=\{(\varepsilon,\theta)\in\mathbf{R}^2:\ \theta=0\}\\
\mathcal{H} &= \{(\varepsilon,\theta)\in \mathbf{R}^2:\ \exists(x,\lambda):\ G=G_x=G_{xx}=0\}\\
&= \{(\varepsilon,\theta)\in \mathbf{R}^2:\ 27a_1a_5^2-a_3^3=0\}.
\end{aligned}
$$

We shall restrict our attention to the positive half ($\theta > 0$) of the plane since θ represents the quantity of good 2 in the initial endowment of agent 1. In order to draw these sets we have to solve $27a_1a_5^2 - a_3^3 = 0$. This we do in an approximate way by approximating the coefficients a_i to first degree in θ and ε and then applying the Implicit Function Theorem near the origin to guarantee that the set of points described by this equation is similar to a cubic near the origin. Note that we are not interested in the exact shape of this curve but rather in the qualitatively different behaviour observed in the connected components defined by this curve in the half-plane.

The bifurcation and hysteresis sets are drawn in figure 1 together with the qualitative bifurcation diagrams observed in the connected components of the positive half-plane that these sets define. A *bifurcation diagram* is a low-dimension qualitative representation of all equilibria as the parameter varies. Here, bifurcation diagrams are drawn in the (μ, p)-plane and branches of equilibria represented by solid lines correspond to stable equilibria whereas dotted lines correspond to unstable values of the equilibrium price.

5 Hysteresis in the tatonnement process

In this section we concentrate on the behaviour observed in the parameter region on the right-hand side of $\mathcal{H}$ in figure 1. In this region, we observe the existence of hysteresis in the tatonnement process.

In figure 2, we show the variation in equilibrium price as we vary the parameter μ (through λ — in the following we shall omit the dependence of μ on λ and refer only to μ). We distinguish two values, $\mu_1 < \mu_2 \in (0,1)$, for which the equilibrium price changes drastically. If we increase μ, the change in equilibrium price occurs at $\mu = \mu_2$, whereas if we decrease μ this change occurs for the lower value $\mu = \mu_1$. Furthermore, assuming the values of the unfolding parameters ε and θ fixed in the right-hand side region of figure 1 and take μ such that $\mu_1 < \mu - \varepsilon < \mu < \mu_2$. Without any additional information about the variation in μ that placed the parameter in this position, it is impossible to determine the actual equilibrium price: it could be either the lower or the higher value of the price. Note that, for parameter values on the left-hand side of $\mathcal{H}$ in figure 1, this unclear situation does not occur and the variation in equilibrium price is smooth with μ.

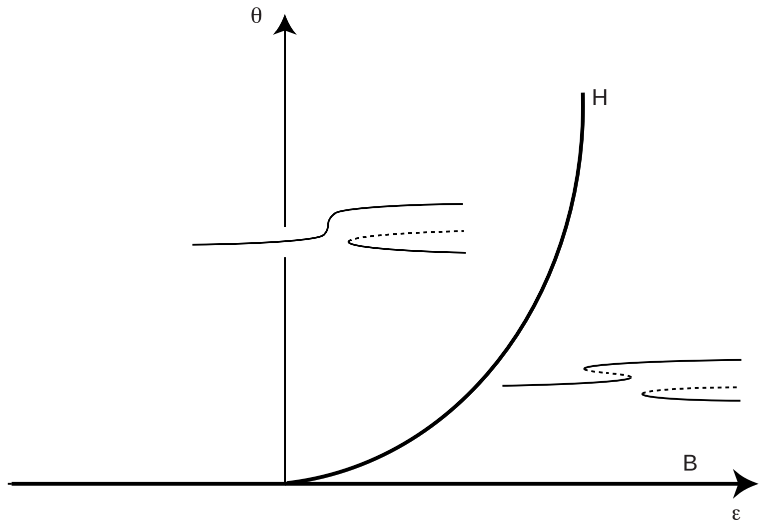

Figure 1: Bifurcation and hysteresis sets (in bold) in the positive half-plane. The bifurcation diagrams drawn (on the (μ, p)-plane) remain qualitatively equivalent in each connected component. A solid line in a bifurcation diagram represents stability while a dotted line means the branch consists of unstable equilibria.

In order to understand the causes of the delay in response to parameter changes, recall that low values of μ indicate prevalence of the substitution effect over the income effect whereas the reverse happens for high values of μ. The delay, relative to μ, introduced in the preference of agent 2 by ε corresponds to a desynchronization of substitution and income effects in the two agents. Suppose that μ is close to 0 as a starting point and it increases towards 1, that is, both agents have dominant substitution effect in their preferences. As μ increases, agent 1 will swap the dominance of the substitution for the dominance of the income effect while agent 2 (whose preference has been delayed) still feels the dominance of the substitution effect. If μ varies in the opposite direction (decreasing from 1) then substitution effect is felt by agent 2 earlier than by agent 1. Hence, the choice of equilibrium price is not symmetric with respect to the direction of variation of the parameter μ, since each agent prefers a different good.

The perturbation parameter for the economy, θ, contributes to the same effect for different reasons and is as essential for the existence of hysteresis. In fact, $\theta = 0$ corresponds to the bifurcation set where this phenomenon does not occur. Recall that θ is the amount of good 2 in the initial endowment of agent 1 and that agent 1 has a stronger preference for good 1 than for good 2. When income effect is dominant, the possession of good 2 by agent 1 contributes

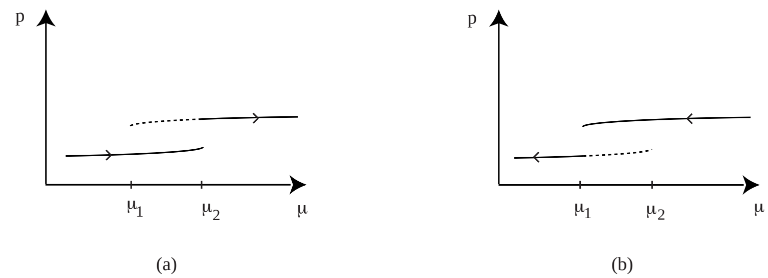

Figure 2: Variation of equilibrium price with the parameter μ: (a) as μ increases; (b) as μ decreases. The price variation follows the solid line – the dotted line is used merely as a reference.

to the maintenance of a higher price thus supporting the equilibrium price at p_+ for longer than otherwise justifiable. On the other hand, θ is small enough to not counteract the dominance of the substitution effect when it is high ($\mu \sim 0$).

The quantities of perturbation parameters ε and θ that produce an effect capable of this hysteresic behaviour are determined by the curve $\mathcal{H}$ in figure 1.

6 Conclusions

We have shown that the model proposed by Bala (1997), while not robust to all reasonable perturbations, is in fact generic in the sense that the particular family of utility functions used is indeed representative of a family of preferences. As such, it provides a way of studying the way in which a change of preference induces a change in equilibrium price for a continuous-time tatonnement process. We have found all perturbations of the original tatonnement process in an optimal way, that is, with the minimal number of perturbation parameters and provided a qualitative description of equilibrium price caused by these perturbations. The results can be explained using the dominance of income or substitution effect associated with the utility function at a given parameter value. Changes, and the way they occur, in equilibrium price are explained by both perturbation parameters. If the starting point of the economy is one where income effect dominates then a higher equilibrium price is sustained through a change in preferences towards a dominance of substitution effect. If, on the contrary, substitution effect dominates initially, a lower equilibrium price prevails for longer as preferences change towards income effect dominance.

References

[1] Amable, B., 1995, Weak and strong hysteresis: an application to foreign trade, Economic Notes **24** (2), 353 –374.

[2] Bala, V., 1997, A pitchfork bifurcation in the tatonnement process, Economic Theory **10**, 521 – 530.

[3] Dixit, A., 1992, Investment and Hysteresis, Journal of Economic Perspectives **6** (1), 107 – 132.

[4] Göcke, M., 2002, Various concepts of hysteresis applied in economics, Journal of Economic Surveys **16** (2), 167 – 188.

[5] Golubitsky, M. and D. Schaeffer, 1985, Singularities and groups in bifurcation theory, vol. I (Springer, New York).

[6] Golubitsky, M., I. Stewart and D. Schaeffer, 1988, Singularities and groups in bifurcation theory, vol. II (Springer, New York).

[7] Kannai, Y., 1970, Continuity properties of the core of a market, Econometrica **38** (6), 791 – 815.

[8] Katzner, D. W., 1999, Hysteresis and the modeling of economic phenomena, Review of Political Economy **11** (2), 171 – 181.

[9] Ljungqvist, L., 1994, Hysteresis in international trade: a general equilibrium analysis, Journal of International Money and Finance **13**, 387 – 399.

[10] Quah, J. K.-H., 2003, The law of demand and risk aversion, Econometrica **71** (2), 713 – 721.

[11] Roed, K., 1997, Hysteresis in Unemployment, Journal of Economic Surveys **11** (4), 389 – 418.

[12] Saunders, P. T., 1986, An introduction to catastrophe theory (Cambridge University Press).