DUALITY FOR PARTIAL GROUP ACTIONS

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Abstract. Given a finite group G acting as automorphisms on a ring $\mathcal{A}$, the skew group ring $\mathcal{A} * G$ is an important tool for studying the structure of G-stable ideals of $\mathcal{A}$. The ring $\mathcal{A}*G$ is G-graded, i.e.G coacts on $\mathcal{A}*G$. The Cohen-Montgomery duality says that the smash product $\mathcal{A} * G\#k[G]^*$ of $\mathcal{A} * G$ with the dual group ring $k[G]^*$ is isomorphic to the full matrix ring $M_n(\mathcal{A})$ over $\mathcal{A}$, where n is the order of G. In this note we show how much of the Cohen-Montgomery duality carries over to partial group actions in the sense of R.Exel. In particular we show that the smash product $(\mathcal{A} *_\alpha G)\,\#k[G]^*$ of the partial skew group ring $\mathcal{A} *_\alpha G$ and $k[G]^*$ is isomorphic to a direct product of the form $K \times \mathbf{e}M_n(\mathcal{A})\mathbf{e}$ where $\mathbf{e}$ is a certain idempotent of $M_n(\mathcal{A})$ and K is a subalgebra of $(\mathcal{A} *_\alpha G)\,\#k[G]^*$. Moreover $\mathcal{A}*_\alpha G$ is shown to be isomorphic to a separable subalgebra of $\mathbf{e}M_n(\mathcal{A})\mathbf{e}$. We also look at duality for infinite partial group actions and for partial Hopf actions.

1. Introduction

Let k be a commutative unital ring and $\mathcal{A}$ a untial k-algebra. Given a finite group G acting as k-linear automorphisms on $\mathcal{A}$, Cohen and Montgomery showed in [3] that the smash product $\mathcal{A} * G\#k[G]^*$ of the skew group ring $\mathcal{A} * G$ and the dual group ring $k[G]^* = \mathrm{Hom}(k[G], k)$ is isomorphic to the full matrix ring $M_n(\mathcal{A})$ over $\mathcal{A}$, where n is the order of G.

R.Exel introduced in [6] the notion of a partial group action on a k-algebra: G acts partially on $\mathcal{A}$ by a family $\{\alpha_g : D_{g^{-1}} \to D_g\}_{g\in G}$ if for all $g \in G$, D_g is an ideal of $\mathcal{A}$ and α_g is an isomorphism of k-algebras such that for all $g, h \in G$:

(i) $D_e = \mathcal{A}$ and α_e is the identity map of $\mathcal{A}$;
(ii) $\alpha_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$;
(iii) $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$ for all $x \in D_{h^{-1}} \cap D_{(gh)^{-1}}$.

The partial skew group ring of $\mathcal{A}$ and G is defined to be the projective left $\mathcal{A}$-module $\mathcal{A} *_\alpha G = \bigoplus_{g\in G} D_g$ with multiplication

$$(a\overline{g})(b\overline{h}) = \alpha_g(\alpha_{g^{-1}}(a)b)\overline{gh}$$

for all $a \in D_g$ and $b \in D_h$ and where $\overline{g}$ is the placeholder for the gth component of $\bigoplus_{g\in G} D_g$. Since $\mathcal{A} *_\alpha G$ is naturally G-graded, the question arises how much of the Cohen-Montgomery duality carries over to partial group actions.

As in [5] we will assume that the ideals D_g are generated by central idempotents, i.e. $D_g = \mathcal{A}1_g$ with central idempotent $1_g \in \mathcal{A}$ for all $g \in G$. For any $g \in G$ we define the following endomorphism $\beta_g : A \to A$ of $\mathcal{A}$ by

$$\beta_g(a) = \alpha_g(a1_{g^{-1}}) \quad \forall a \in \mathcal{A}$$

This map gives rise to a k-linear map $k[G] \otimes A \to A$ with

$$g \otimes a \mapsto g \cdot a := \beta_g(a) = \alpha_g(a1_{g^{-1}})$$

for all $g \in G, a \in \mathcal{A}$.

Lemma 1.1. *With the notation above we have that*

(1) β_g *are k-algebra endomorphisms of $\mathcal{A}$ for all $g \in G$, i.e.*

$$g \cdot (ab) = (g \cdot a)(g \cdot b) \quad \forall a, b \in \mathcal{A}.$$

(2) $g \cdot (h \cdot a) = ((gh) \cdot a)1_g$ *for all $g, h \in G$ and $a \in \mathcal{A}$.*

(3) $(g \cdot a)b = g \cdot (a(g^{-1} \cdot b))$ *for all $a, b \in \mathcal{A}$ and $g \in G$.*

Proof. (1) follows since the α_g are algebra homomorphisms and the idempotents 1_g are central, i.e. for all $a, b \in \mathcal{A}$:

$$\beta_g(ab) = \alpha_g(ab1_{g^{-1}}) = \alpha_g(a1_{g^{-1}}b1_{g^{-1}}) = \alpha_g(a1_{g^{-1}})\alpha_g(b1_{g^{-1}}) = \beta_g(a)\beta_g(b).$$

(2) follows from [5, 2.1(ii)]:

$$\alpha_g(\alpha_h(a1_{h^{-1}})1_{g^{-1}}) = \alpha_{gh}(a1_{h^{-1}g^{-1}})1_g$$

what expressed by β yields the statement of (2).

(3) Using (1), (2) and the fact that $\beta_e = id$ and that the image of β_g is $D_g = A1_g$ we have that

$$g \cdot (a(g^{-1} \cdot b)) = (g \cdot a)(g \cdot (g^{-1} \cdot b)) = (g \cdot a)b1_g = (g \cdot a)b.$$

□

Obviously we also have $g \cdot 1 = \alpha_g(1_{g^{-1}}) = 1_g$ and $g \cdot (g^{-1} \cdot a) = ((gg^{-1}) \cdot a)1_g = a1_g$ for all $a \in \mathcal{A}$ and $g \in G$ using property (2). Moreover using the fact that α_g is bijective and 1_g central we have for all $a \in \mathcal{A}$ and $g \in G$ that $g \cdot a = 0$ if and only if $a \in \mathcal{A}(1 - 1_g)$.

2. Grading of the partial skew group ring

The partial skew group ring is the projective left $\mathcal{A}$-module $\mathcal{A} *_\alpha G = \bigoplus_{g \in G} D_g$. We will write an element of $\mathcal{A} *_\alpha G$ as a finite sum of elements $\sum_{g \in G} a_g \overline{g}$ where $a_g \in D_g = A1_g$ and $\overline{g}$ is a placeholder for the g-th component. $\mathcal{A} *_\alpha G$ becomes an associative k-algebra by the product:

$$(a\overline{g})(b\overline{h}) = \alpha_g(\alpha_{g^{-1}}(a)b)\overline{gh}$$

for all $g, h \in G$ and $a \in D_g$ and $b \in D_h$. Using our '$\cdot$'-notation we see easily

$$(a\overline{g})(b\overline{h}) = a(g \cdot b)\overline{gh}.$$

The algebra $\mathcal{A} *_\alpha G$ is naturally G-graded where the homogeneous elements are those in $\{D_g\}_{g \in G}$, i.e. $D_g D_h \subseteq D_{gh}$ by definition of the multiplication in $\mathcal{A} *_\alpha G$. Thus $\mathcal{A} *_\alpha G$ becomes a $k[G]$-comodule algebra. Note that the G-grading is strong, in the sense that $D_gD_h = D_{gh}$ if and only if $D_g = \mathcal{A}$ for all $g \in G$, i.e. the G-action is global (since if $D_gD_h = D_{gh}$ for all $g, h \in G$, then

$$\mathcal{A}1_g1_{g^{-1}} = D_gD_{g^{-1}} = D_{gg^{-1}} = D_e = \mathcal{A},$$

thus 1_g is an invertible central idempotent and hence equals 1, i.e. $D_g = \mathcal{A}$). Known results on graded rings can be applied to the G-grading of $\mathcal{A} *_\alpha G$.

3. Duality for partial actions of finite groups

Assume G to be finite, then $k[G]^*$ becomes a Hopf algebra with projective basis $p_g \in k[G]^*$ where $p_g(h) = \delta_{g,h}$ for all $g, h \in H$. The multiplication is defined as $p_g * p_h = \delta_{g,h}p_g$ and the identity element of $k[G]^*$ is $1 = \sum_{h\in H} p_h$. Now $\mathcal{A} *_\alpha G$ becomes a $k[G]^*$-module algebra by

$$p_h \triangleright (a\overline{g}) = \delta_{g,h}a\overline{g}$$

for all $g, h \in G$ and $a_g \in D_g$. The multiplication of the smash product $(\mathcal{A} *_\alpha G)\#k[G]^*$ is defined as

$$(a\overline{g}\#p_h)(b\overline{k}\#p_l) = \sum_{s\in G}(a\overline{g})[p_s\triangleright(b\overline{k})]\#p_{s^{-1}h}*p_l = (a\overline{g})(b\overline{k})\#p_{k^{-1}h}*p_l = a(g\cdot b)\overline{gk}\#\delta_{h,kl}p_l.$$

The identity element of $\mathcal{B} = \mathcal{A}*_\alpha G\#k[G]^*$ is $\sum_{h\in G} 1\overline{e}\#p_h$. In the case of global actions Cohen and Montgomery proved in [3] that $\mathcal{A} * G\#k[G]^* \simeq M_n(\mathcal{A})$ where $n = |G|$ and $M_n(\mathcal{A})$ denotes the ring of $n\times n$-matrizes over $\mathcal{A}$. We will index the matrizes of $M_n(\mathcal{A})$ by elements of G and denote by $E_{g,h}$ the elementary matrix that has the value 1 in the g-th row and the h-th column and zero elsewhere.

Proposition 3.1. *Let G be a finite group of n elements, acting partially on an k-algebra $\mathcal{A}$ and consider the k-algebra $\mathcal{B} = (\mathcal{A} *_\alpha G)\#k[G]^*$. The map*

$$\Phi : \mathcal{B} \longrightarrow M_n(\mathcal{A}) \quad \textit{with}$$

$$\sum_{g,h} a_{g,h}\overline{g}\#p_h \mapsto \sum_{g,h} h^{-1}\cdot(g^{-1}\cdot a_{g,h})E_{gh,h}$$

is a k-algebra homomorphism.

Proof. First note that for any $g, h, k \in G$ and $a \in D_g, b \in D_h$ we have, using Lemma 1.1(2) in the 2nd, 4th and 6th line and Lemma 1.1(1) in the 3rd line:

$$\begin{aligned}
k^{-1}\cdot((gh)^{-1}\cdot(a(g\cdot b))) &= k^{-1}\cdot\big(((gh)^{-1}\cdot a)((gh)^{-1}\cdot(g\cdot b))\big)\\
&= k^{-1}\cdot\big[((gh)^{-1}\cdot a)(h^{-1}\cdot b)1_{(gh)^{-1}}\big]\\
&= \big[k^{-1}\cdot((gh)^{-1}\cdot a)\big]\big[k^{-1}\cdot(h^{-1}\cdot b)\big]\\
&= ((ghk)^{-1}\cdot a)((hk)^{-1}\cdot b)1_{k^{-1}}\\
&= ((ghk)^{-1}\cdot a)1_{(hk)^{-1}}((hk)^{-1}\cdot b)1_{k^{-1}}\\
&= ((hk)^{-1}\cdot(g^{-1}\cdot a))(k^{-1}\cdot(h^{-1}\cdot b))
\end{aligned}$$

Thus we showed:

$$k^{-1} \cdot ((gh)^{-1} \cdot (a(g \cdot b))) = ((hk)^{-1} \cdot (g^{-1} \cdot a))(k^{-1} \cdot (h^{-1} \cdot b)) \tag{1}$$

For any $a\overline{g}\#p_h, b\overline{k}\#p_l \in (\mathcal{A} *_\alpha G)\#k[G]^*$ we have, using equation (1):

$$\begin{aligned}
\Phi((a\overline{g}\#p_h)(b\overline{k}\#p_l)) &= \Phi(a(g \cdot b)\overline{gk}\#\delta_{h,kl}p_l) \\
&= l^{-1} \cdot ((gk)^{-1} \cdot (a(g \cdot b)))E_{gkl,l}\delta_{h,kl} \\
&= ((kl)^{-1} \cdot (g^{-1} \cdot a))(l^{-1} \cdot (k^{-1} \cdot b))E_{gh,h}E_{kl,l}\delta_{h,kl} \\
&= (h^{-1} \cdot (g^{-1} \cdot a))E_{gh,h}\,(l^{-1} \cdot (k^{-1} \cdot b))E_{kl,l} \\
&= \Phi(a\overline{g}\#p_h)\Phi(b\overline{k}\#p_l)
\end{aligned}$$

Hence Φ is an algebra homomorphism. □

Note that Φ restricted to $\mathcal{A} *_\alpha G$ is injective, i.e. $\mathcal{A} *_\alpha G$ can be considered a subalgebra of $M_n(\mathcal{A})$. In general $\mathrm{Ker}(\Phi)$ is non-trivial, unless the partial action is a global action.

Proposition 3.2. $\mathrm{Ker}(\Phi) = \bigoplus_{g,h\in G} \mathcal{A}(1 - 1_{gh})1_g\overline{g}\#p_h$.

Proof. Suppose $\gamma = \sum_{g,h} a_{g,h}\overline{g}\#p_h \in \mathrm{Ker}(\Phi)$, then $h^{-1} \cdot (g^{-1} \cdot a_{g,h}) = 0$ for all $g, h \in G$. Thus $(g^{-1} \cdot a_{g,h}) \in \mathcal{A}(1 - 1_h) \cap D_{g^{-1}} = \mathcal{A}(1 - 1_h)1_{g^{-1}}$. Hence

$$a_{g,h} = g \cdot (g^{-1} \cdot a_{g,h}) \in \mathcal{A}g \cdot (1 - 1_h) = \mathcal{A}(1_g - 1_g 1_{gh}),$$

i.e. $\gamma \in \bigoplus_{g,h} \mathcal{A}(1 - 1_{gh})1_g\overline{g}\#p_h$. The other inclusion follows because

$$\Phi\left((g \cdot (1 - 1_h))\overline{g}\#p_h\right) = h^{-1} \cdot (g^{-1} \cdot (g \cdot (1 - 1_h)))E_{gh,h} = h^{-1} \cdot ((1 - 1_h)1_g)E_{gh,h} = 0.$$

□

Note that the inclusion of $\mathcal{A} *_\alpha G$ into $(\mathcal{A} *_\alpha G)\#k[G]^*$ is given by $a\overline{g} \mapsto \sum_{h\in G} a\overline{g}\#p_h$ for all $g \in G$ and $a \in D_g$. If $\sum_{h\in G} a\overline{g}\#p_h \in \mathrm{Ker}(\Phi)$, then $a \in \mathcal{A}(1 - 1_{gh})1_g$ for all $h \in G$. In particular for $h = e$ we have $a \in \mathcal{A}(1 - 1_g)1_g = 0$. Hence Φ restricted to $\mathcal{A} *_\alpha G$ is injective.

We will describe the image of Φ. By definition of Φ, the image of an arbitrary element $\gamma = \sum_{g,h} a_{g,h}\overline{g}\#p_h$ is

$$\Phi(\gamma) = \sum_{g,h} ((gh)^{-1} \cdot a_{g,h})1_{(gh)^{-1}}1_{h^{-1}}E_{gh,h} = \left(b_{r,s}1_{r^{-1}}1_{s^{-1}}\right)_{r,s\in G}$$

with $b_{r,s} = r^{-1} \cdot a_{rs^{-1},s}$ for all $r, s \in G$.

Proposition 3.3. *The image of* Φ *consists of all matrices of the form* $\left(b_{g,h}1_{g^{-1}}1_{h^{-1}}\right)_{g,h\in G}$ *for any matriz* $(b_{g,h})$ *of elements of* $\mathcal{A}$. *In particular* $\mathrm{Im}(\Phi) = \mathbf{e}M_n(A)\mathbf{e}$, *where* $\mathbf{e}$ *is the idempotent* $\sum_{g\in G} 1_{g^{-1}}E_{g,g}$.

Proof. We saw already that an element of the image of Φ is of the given form. Note that by definition of partial group action we have

$$D_g \cap D_{gh} = \alpha_g(D_{g^{-1}} \cap D_h)$$

for all $g, h \in G$. Hence also

$$D_{g^{-1}} \cap D_{h^{-1}} = \alpha_{g^{-1}}(D_g \cap D_{gh^{-1}})$$

holds for all $g, h \in G$. Thus for all $b \in \mathcal{A}$ there exists $a \in \mathcal{A}$ such that

$$b1_{g^{-1}}1_{h^{-1}} = \alpha_{g^{-1}}(a1_{gh^{-1}}1_g) = g^{-1} \cdot (a1_{gh^{-1}}).$$

This implies that

$$\begin{aligned} \Phi(a1_g1_{gh^{-1}}\overline{gh^{-1}}\#p_h) &= h^{-1} \cdot ((hg^{-1}) \cdot (a1_g1_{gh^{-1}}))E_{g,h} \\ &= g^{-1} \cdot (a1_g1_{gh^{-1}}))1_{h^{-1}}E_{g,h} \\ &= b1_{g^{-1}}1_{h^{-1}}E_{g,h} \end{aligned}$$

Hence given any matrix $(b_{g,h})$ there are elements $a_{g,h}$ such that

$$\Phi\left(\sum_{g,h} a_{g,h}1_g1_{gh^{-1}}\overline{gh^{-1}}\#p_h\right) = \sum_{g,h} b_{g,h}1_{g^{-1}}1_{h^{-1}}E_{g,h} = \left(b_{g,h}1_{g^{-1}}1_{h^{-1}}\right)_{g,h\in G}.$$

This shows that $\mathrm{Im}(\Phi)$ consists of all matrizes of the given form and hence is equal to $\mathbf{e}M_n(A)\mathbf{e}$. Note that $\mathbf{e}$ is the image of the identity element of $\mathcal{B}$. □

The last Propositions yield our main result in this section

Theorem 3.4. $(\mathcal{A} *_\alpha G)\#k[G]^* \simeq \mathrm{Ker}(\Phi) \times \mathbf{e}M_n(\mathcal{A})\mathbf{e}$.

Proof. The kernel of Φ is an ideal and a direct summand of $\mathcal{B} = (\mathcal{A} *_\alpha G)\#k[G]^*$. To see this we first show that the left $\mathcal{A}$-module $I = \bigoplus_{g,h\in G} \mathcal{A}1_{gh}1_g\overline{g}\#p_h$ is a two-sided ideal of $\mathcal{B}$. For any $x\overline{k}\#p_l \in \mathcal{B}$ and $a1_{gh}1_g\overline{g}\#p_h \in I$ we have

$$\begin{aligned} (a1_{gh}1_g\overline{g}\#p_h)(b\overline{k}\#p_l) &= a1_{gh}1_g(g \cdot b1_k)\overline{gk}\#\delta_{h,kl}p_l = a(g \cdot b)\delta_{h,kl}1_{gkl}1_{gk}\overline{gk}\#p_l \in I. \\ (b\overline{k}\#p_l)(a1_{gh}1_g\overline{g}\#p_h) &= b(k \cdot a1_{gh}1_g)\overline{kg}\#\delta_{k,gh}p_h = b(g \cdot a)\delta_{h,kl}1_{kgh}1_{kg}\overline{kg}\#p_h \in I. \end{aligned}$$

Since $I \oplus \mathrm{Ker}(\Phi) = \mathcal{B}$ and both direct summands are two-sided ideals we have $\mathcal{B} = I \times \mathrm{Ker}(\Phi)$ (ring direct product). Moreover $\Phi(I) = \mathbf{e}M_n(\mathcal{A})\mathbf{e} = \mathrm{Im}(\Phi)$. This implies $\mathcal{B} \simeq \mathrm{Ker}(\Phi) \times \mathbf{e}M_n(\mathcal{A})\mathbf{e}$.

□

Note that Φ embedds $\mathcal{A} *_\alpha G$ into the Pierce corner $\mathbf{e}M_n(\mathcal{A})\mathbf{e}$.

Corollary 3.5. *$\mathcal{A} *_\alpha G$ is isomorphic to a separable subalgebra of* $\mathbf{e}M_n(\mathcal{A})\mathbf{e}$.

Proof. Recall that the subalgebra $\mathcal{A} *_\alpha G$ sits into $\mathcal{B}$ by $a\overline{g} \mapsto \sum_{h\in G} a\overline{g}\#p_h$. The right action of $\mathcal{A} *_\alpha G$ on $\mathcal{B}$ is given by

$$(x\overline{k}\#p_l) \cdot a\overline{g} = (x\overline{k}\#p_l)\left(\sum_{h\in G} a\overline{g}\#p_h\right) = (x\overline{k})(a\overline{g})\#p_{g^{-1}l}$$

The left action is given by

$$a\overline{g} \cdot (x\overline{k}\#p_l) = \left(\sum_{h\in G} a\overline{g}\#p_h\right)(x\overline{k}\#p_l) = (a\overline{g})(x\overline{k})\#p_l$$

The element

$$f = \sum_{g\in G} \overline{e}\#p_g \otimes \overline{e}\#p_g \in \mathcal{B} \otimes_{\mathcal{A}*_\alpha G} \mathcal{B}$$

is $\mathcal{A} *_\alpha G$-centralising, i.e. for all $a\overline{h} \in \mathcal{A} *_\alpha G$ we have

$$fa\overline{h} = \sum_{g\in G} \overline{e}\#p_g \otimes a\overline{h}\#p_{h^{-1}g} = \sum_{g\in G} a\overline{h}\#p_{h^{-1}g} \otimes \overline{e}\#p_{h^{-1}g} = a\overline{h}f$$

Since also $\mu(f) = \overline{e}\# \sum_{g\in G} p_g = 1_{\mathcal{B}}$ we have that f is a seperability idempotent for $\mathcal{B}$ over $\mathcal{A} *_\alpha G$. Hence $\mathbf{e}M_n(\mathcal{A})\mathbf{e} \simeq \Phi(\mathcal{B})$ is separable over $\Phi(\mathcal{A} *_\alpha G) \simeq \mathcal{A} *_\alpha G$. $\square$

4. Trivial partial actions

The easiest example of partial actions arise from (central) idempotents in a k-algebra $\mathcal{A}$. Suppose that $\mathcal{A}$ admits a non-zero central idempotent, i.e. there exist algebras R, S such that $\mathcal{A} = R \times S$ as algebras. For any group G set $D_g = R \times 0$ and $\alpha_g = id_{D_g}$for all $g \neq e$ and $D_e = \mathcal{A}$ and $\alpha_e = id_{\mathcal{A}}$. Then $\{\alpha_g \mid g \in G\}$ is a partial action of G on $\mathcal{A}$. The partial skew group ring turns out to be $\mathcal{A} *_\alpha G \simeq R[G] \times S$, where $R[G]$ denotes the group ring of R and G. Note that $0 \times S$ is in the zero-componente of the G-grading on $\mathcal{A} *_\alpha G$. If G is finite, say of order n, then a short calculation (using Cohen-Montgomery duality and Theorem 3.4) shows that $\mathcal{B} = (\mathcal{A} *_\alpha G)\#k[G]$ is isomorphic to $M_n(R) \times S^n$ where S^n denotes the direct product of n copies of S. Depending on the rings R and S, $\mathcal{B}$ might or might not be Morita equivalent to $\mathcal{A}$. For instance if $R = S = F$ is a field, then any progenerator P for $\mathcal{A}$ has the form $F^k \times F^m$ for numbers $k, m \geq 1$. Thus $\mathrm{End}_k(P) \simeq M_k(F) \times M_m(F)$, whose center is isomorphic to $F^2 = \mathcal{A}$. On the other hand $\mathcal{B} = (\mathcal{A} *_\alpha G)\#k[G] \simeq M_n(F) \times F^n$ has center F^{n+1}, i.e. $\mathcal{B}$ will be Morita equivalent to $\mathcal{A}$ if and only if G is trivial.

On the other hand, there are algebras which satisfy (as algebras) $\mathcal{A}^n \simeq \mathcal{A} \simeq M_n(\mathcal{A})$ for any n. To give an example, let R be the ring of sequences of elements of a field k, i.e. $R = k^{\mathbb{N}}$. The function χ with $\chi(2n) = 1$ and $\chi(2n+1) = 0$ for all n defines an idempotent of R. The map $\Psi : \chi R \to R$ with $\Psi(\chi f)(n) = f(2n)$ is a ring isomorphism. Analogosuly we can show that $(1-\chi)R \simeq R$. Hence $R^2 \simeq R$. Now take $\mathcal{A} = \mathrm{End}_k(F)$, where $F = R^{(\mathbb{N})}$ denotes the countable infinite free R-module. Using again χ we have that

$$\mathcal{A} = (\chi\mathcal{A}) \times ((1-\chi)\mathcal{A}) \simeq \mathcal{A} \times \mathcal{A} \simeq \cdots \simeq \mathcal{A}^n$$

for any $n \geq 2$. Moreover for any partition of $\mathbb{N}$ into n infinite disjoint subsets $\Lambda_1, \ldots, \Lambda_n$, we have that

$$F = R^{(\mathbb{N})} \simeq R^{(\Lambda_1)} \oplus \cdots \oplus R^{(\Lambda_n)} \simeq F^n.$$

Hence $\mathcal{A} = \mathrm{End}_k(F) \simeq \mathrm{End}_k(F^n) \simeq M_n(\mathcal{A})$. Applying the double skew group ring construction again we conclude that

$$\mathcal{B} = (\mathcal{A} *_\alpha G)\#k[G] \simeq M_n(\chi\mathcal{A}) \times ((1-\chi)\mathcal{A})^n \simeq \mathcal{A} \times \mathcal{A} \simeq \mathcal{A}.$$

5. Infinite partial group action

Following Quinn [8] we define Φ in case of G being infinite as a map from $\mathcal{A} *_\alpha G$ to the ring of row and column finite matrizes. Let $M_G(\mathcal{A})$ be the subring of $\mathrm{End}_k(\mathcal{A}^{|G|})$ consisting of row and column finite matrizes $(a_{g,h})_{g,h\in G}$ indexed by elements of G with entries in $\mathcal{A}$, i.e. for any $g \in G$ the sets $\{a_{gh} | h \in G\}$ and $\{a_{hg} | h \in G\}$ are finite. Let $E_{g,h}$ be, as above, those matrizes that are 1 in the (g,h)th component and zero elsewhere. Note that $E_{g,h}E_{r,s} = \delta_{h,r}E_{g,s}$. Then define $\Phi : \mathcal{A} *_\alpha G \to M_G(\mathcal{A})$ by

$$a\overline{g} \mapsto \sum_{h\in G} h^{-1} \cdot (g^{-1} \cdot a)E_{gh,h}$$

for any $a\overline{g} \in \mathcal{A} *_\alpha G$. Note that the (infinite) sum on the right side makes sense in $M_G(\mathcal{A})$. As above one checks that Φ is an algebra homomorphism.

Proposition 5.1. *Let G be any group acting partially on $\mathcal{A}$. Then $\mathcal{A} *_\alpha G$ is isomorphic to a subalgebra of $\mathbf{e}M_G(A)\mathbf{e}$ where $M_G(\mathcal{A})$ denotes the ring of row and column finite matrizes indexed by elements of G and with entries in $\mathcal{A}$. The element $\mathbf{e}$ is the idempotent $\sum_{g\in G} 1_{g^{-1}}E_{g,g}$.*

Proof. For all $a\overline{g}, b\overline{h} \in \mathcal{A} *_\alpha G$ we have using equation (1) in the 4th line:

$$\begin{aligned}
\Phi(a\overline{g})\Phi(b\overline{h}) &= \left(\sum_{k\in G} k^{-1} \cdot (g^{-1} \cdot a)E_{gk,k}\right)\left(\sum_{l\in G} l^{-1} \cdot (h^{-1} \cdot b)E_{hl,l}\right)\\
&= \sum_{k,l\in G} (k^{-1} \cdot (g^{-1} \cdot a))(l^{-1} \cdot (h^{-1} \cdot b))E_{gk,k}E_{hl,l}\\
&= \sum_{l\in G} ((hl)^{-1} \cdot (g^{-1} \cdot a))(l^{-1} \cdot (h^{-1} \cdot b))E_{ghl,l}\\
&= \sum_{l\in G} l^{-1} \cdot ((gh)^{-1} \cdot (a(g \cdot b)))E_{ghl,l}\\
&= \Phi(a(g \cdot b)\overline{gh})\\
&= \Phi((a\overline{g})(b\overline{h}))
\end{aligned}$$

Hence Φ is an algebra homomorphism. Since

$$\Phi(a\overline{g}) = 0 \Leftrightarrow (\forall h \in G) : h^{-1} \cdot (g^{-1} \cdot a) = 0 \Rightarrow g \cdot (g^{-1} \cdot a) = a1_g = 0 \Rightarrow a = 0,$$

we have that Φ is injective. Moreover $\Phi(a\overline{g}) \in \mathbf{e}M_G(A)\mathbf{e}$ as above. $\square$

6. Partial Hopf action

In [2] Caenepeel and Janssen defined the notion of a partial Hopf action as follows: Let H be a Hopf algebra, with comultiplication Δ, counit ϵ and antipode S, and let $\mathcal{A}$ be a k-algebra such that there exists a k-linear map

$$\cdot : H \otimes A \to A$$

sending $h \otimes a \mapsto h \cdot a$. The action $\cdot$ is called a *partial Hopf action* if for all $h, g \in H$ and $a, b \in \mathcal{A}$:

(1) $h \cdot (ab) = \sum_{(h)} (h_1 \cdot a)(h_2 \cdot b)$;
(2) $1_H \cdot a = a$;
(3) $h \cdot (g \cdot a) = \sum_{(h)} (h_1 \cdot 1)((h_2 g) \cdot a)$;

Let H be a Hopf algebra which is finitely generated and projective as k-module with dual basis $\{(b_i, p_i) \in H \times H^* \mid 1 \leq i \leq n\}$. Then there exist structure constants $c^i_{k,l}$ and $m^i_{k,l}$ in k such that $\Delta(b_i) = \sum_{k,l=1}^n c^i_{k,l} b_k \otimes b_l$ and $b_k b_l = \sum_{i=1}^n m^i_{k,l} b_i$ for all $1 \leq i, k, l \leq n$. It is well-known that H^* becomes a Hopf algebra with comultiplication and multiplication defined on the generators $\{p_i \mid 1 \leq i \leq n\}$ as follows: $\Delta_{H^*}(p_i) = \sum_{k,l=1}^n m^i_{k,l} p_k \otimes p_l$ and $p_k * p_l = \sum_{i=1}^n c^i_{k,l} p_i$. The counit of H^* is given by $\epsilon_{H^*}(f) = f(1)$.

Recall that H^* acts on H from the left by $f \rightharpoonup h = \sum_{(h)} h_1 f(h_2)$, such that the smash product $H \# H^*$ can be considered whose multiplication is given by

$$(h\#f)(k\#g) = \sum_{(f)} h(f_1 \rightharpoonup k)\# f_2 * g$$

for all $h, k \in H$ and $f, g \in H^*$. The smash product yields a left module action on H, i.e. an algebra homomorphism

$$\lambda : H\#H^* \to \mathrm{End}_k(H) \quad h\#f \mapsto [k \mapsto h(f \rightharpoonup k)].$$

The smash product $H\#H^*$ is sometimes called the Heisenberg double of H and in case H is free of finite rank isomorphic to $\mathrm{End}_k(H)$ (see [7, 9.4.3]).

Analougosly we have a right action of H^* on H by $h \leftharpoonup f = \sum_{(h)} h_2 f(h_1)$ for all $f \in H^*$ and $h \in H$, turning H into a right H^*-module algebra. The smash product $H^*\#H$ yields a right module action on H, i.e. an algebra anti-homomorphism

$$\rho : H^*\#H \to \mathrm{End}_k(H) \quad f\#h \mapsto [k \mapsto (k \leftharpoonup f)h]$$

As in [7, 9.4.10] one shows that for all $h, k \in H$ and $f, g \in H^*$:

$$\lambda(h\#f)\rho(g\#1) = \sum_{(g)} \rho(g_2\#1)\lambda((h \leftharpoonup S(g_1))\#f) \tag{2}$$

Now assume that H acts partially on $\mathcal{A}$, then the map $\Delta_{\mathcal{A}} : A \to A \otimes H^*$ with

$$\Delta(a)_A = \sum_{i=1}^n (b_i \cdot a) \otimes p_i$$

for all $a \in \mathcal{A}$ defines a *partially coaction*. The map $\Delta_{\mathcal{A}}$ satisfies:

$$\begin{aligned}
\Delta_{\mathcal{A}}(ab) &= \Delta_{\mathcal{A}}(a)\Delta_{\mathcal{A}}(b) \\
(1 \otimes \epsilon_{H^*})\Delta_{\mathcal{A}}(a) &= id_A(a) \\
(\Delta_{\mathcal{A}} \otimes 1)\Delta_{\mathcal{A}}(a) &= (\Delta_{\mathcal{A}}(1) \otimes 1)(1 \otimes \Delta_{H^*})\Delta_{\mathcal{A}}(a)
\end{aligned}$$

The last equation shows that in general this coaction does not make $\mathcal{A}$ into a right H-comodule. It can be deduced using the structre constants and property (3) from above

$$\begin{aligned}
\sum_{i,j=1}^{n} b_j \cdot (b_i \cdot a) \otimes p_j \otimes p_i &= \sum_{i,j,k,l,r=1}^{n} c_{k,l}^{j} m_{l,i}^{r} (b_k \cdot 1)(b_r \cdot a) \otimes p_j \otimes p_i \\
&= \sum_{i,k,l,r=1}^{n} m_{l,i}^{r} (b_k \cdot 1)(b_r \cdot a) \otimes p_k p_l \otimes p_i \\
&= \sum_{k,r=1}^{n} (b_k \cdot 1)(b_r \cdot a) \otimes (p_k \otimes 1)\Delta_{H^*}(p_r) \\
&= \left(\sum_{k=1}^{n} (b_k \cdot 1) \otimes p_k\right)\left(\sum_{r=1}^{n} (b_r \cdot a) \otimes \Delta(p_r)\right)
\end{aligned}$$

With the above notation we define a homomorphism $\phi : \mathcal{A} \to \mathcal{A} \otimes \mathrm{End}_k(H)$ by

$$\phi(a) = \sum_{i=1}^{n} (b_i \cdot a) \otimes \rho(S^{-1}(p_i)\#1).$$

Then ϕ is an algebra homomorphism, because

$$\begin{aligned}
\phi(ab) &= \sum_{i=1}^{n} (b_i \cdot (ab) \otimes \rho(S^{-1}(p_i)\#1) \\
&= \sum_{i=1}^{n} ((b_i)_1 \cdot a)(((b_i)_2 \cdot b) \otimes \rho(S^{-1}(p_i)\#1) \\
&= \sum_{k,l=1}^{n} (b_k \cdot a)((b_l \cdot b) \otimes \rho(S^{-1}(c_{k,l}^{i} p_i)\#1) \\
&= \sum_{k,l=1}^{n} (b_k \cdot a)((b_l \cdot b) \otimes \rho(S^{-1}(p_l)S^{-1}(p_k)\#1) \\
&= \sum_{k,l=1}^{n} (b_k \cdot a)((b_l \cdot b) \otimes \rho(S^{-1}(p_k)\#1)\rho(S^{-1}(p_l)\#1) \\
&= \phi(a)\phi(b).
\end{aligned}$$

where we use in the line before the last the fact that ρ is an anti-homomorphism.

The partial smash product of $\mathcal{A}$ and H is defined as a certain submodule of $\mathcal{A}\otimes H$. On $\mathcal{A}\otimes H$ we define a new (associative) multiplication by

$$(a\otimes h)(b\otimes g) := \sum_{(h)} a(h_1\cdot b)\otimes h_2 g.$$

for all $a,b\in\mathcal{A}$, $h,g\in H$. Note that $\mathcal{A}\otimes H$ is naturally an $\mathcal{A}$-bimodule given by

$$x(a\otimes h)y = (x\otimes 1)(a\otimes h)(y\otimes 1) = \sum_{(h)} xa(h_1\cdot y)\otimes h_2$$

The partial smash product is defined to be $\underline{\mathcal{A}\#H} = (\mathcal{A}\otimes H)1_{\mathcal{A}}$ and is spanned by the elements of the form $\sum_{(h)} a(h_1\cdot 1_{\mathcal{A}})\otimes h_2\}$ for all $a\in\mathcal{A}, h\in H$. The partial smash product becomes naturally a right H-comodule algebra by

$$\rho = 1\otimes\Delta : \mathcal{A}\otimes H\to\mathcal{A}\otimes H\otimes H,\quad a\otimes h\mapsto\sum_{(h)} a\otimes h_1\otimes h_2$$

and for all $(a\otimes h)1_{\mathcal{A}}\in\underline{\mathcal{A}\#H}$ we have

$$\rho((a\otimes h)1_{\mathcal{A}}) = \sum_{(h)} a(h_1\cdot 1_{\mathcal{A}})\otimes h_2\otimes h_3,$$

making $\underline{\mathcal{A}\#H}$ into a right H-comodule algebra. Moreover $\underline{\mathcal{A}\#H}$ becomes a left H^*-module algebra, where the action is defined by

$$f\rhd((a\#h)1_{\mathcal{A}}) = \sum_{(h)}(a(h_1\cdot 1_{\mathcal{A}})\#(f\rightharpoonup h_2) = (a\#(f\rightharpoonup h))1_{\mathcal{A}},$$

for all $f\in H^*, h\in H, a\in\mathcal{A}$. The classical Blattner-Montgomery duality ([1] says that the double smash product $\mathcal{A}\#H\#H^*$ is isomorphic to $M_n(\mathcal{A})$ where n is the rank of H over k.

Lemma 6.1. *Let $\psi : H\#H^*\to\mathcal{A}\otimes\mathrm{End}_k(H)$ be the map defined by $h\#f\mapsto 1\otimes\lambda(h\#f)$ for all $h\in H, f\in H^*$. Then for all $a\in\mathcal{A}, h\in H, f\in H^*$ we have*

$$\phi(1)\psi(h\#f)\phi(a) = \sum_{(h)}\phi(h_1\cdot a)\psi(h_2\#f).$$

Proof.

$$\begin{aligned}
\sum_{(h)} \phi(h_1 \cdot a)\psi(h_2\#f) &= \sum_{(h),i} p_i(h_1)\phi(b_i \cdot a)\psi(h_2\#f) \\
&= \sum_{i,j} b_j \cdot (b_i \cdot a) \otimes \rho(S^{-1}(p_j)\#1)\lambda(h \leftharpoonup p_i\#f) \\
&= \sum_{k,r} (b_k \cdot 1)(b_r \cdot a) \otimes \rho(S^{-1}(p_k(p_r)_1)\#1)\lambda(h \leftharpoonup (p_r)_2\#f) \\
&= \sum_{k,r} (b_k \cdot 1)(b_r \cdot a) \otimes \rho(S^{-1}(p_k)\#1)\rho(S^{-1}((p_r)_1)\#1)\lambda(h \leftharpoonup (p_r)_2\#f) \\
&= \phi(1)\sum_{r} (b_r \cdot a) \otimes \rho(S^{-1}(p_r)_2\#1)\lambda(h \leftharpoonup (S(S^{-1}(p_r)_1)\#f) \\
&= \phi(1)\sum_{r} (b_r \cdot a) \otimes \lambda(h\#f)\rho(S^{-1}(p_r)\#1) \\
&= \phi(1)\psi(h\#f)\phi(a)
\end{aligned}$$

where we use equation (2) in the third line from below. □

Theorem 6.2. *Suppose that H is a Hopf algebra, finitely generated projective over k, which partially actions on $\mathcal{A}$. Then $\Phi : A \otimes H\#H^* \to \mathcal{A} \otimes \mathrm{End}_k(H)$ with*

$$a \otimes h\#f \mapsto \phi(a)\psi(h\#f)$$

is an algebra homomorphism. The image of the restriction to $\underline{\mathcal{A}\#H}\#H^*$ lies inside $\mathbf{e}\,(A \otimes \mathrm{End}_k(H))\,\mathbf{e}$ where $\mathbf{e}$ is the idempotent

$$\mathbf{e} = \sum_{i=1}^{n} (b_i \cdot 1) \otimes \rho(S^{-1}(p_i) \otimes 1).$$

Proof. For any $a, b \in \mathcal{A}, h, k \in H$ and $f, g \in H^*$ we have

$$\begin{aligned}
\Phi(a \otimes h\#f)\Phi(b \otimes k\#g)) &= \phi(a)\psi(h\#f)\phi(b)\psi(k\#g) \\
&= \phi(a)\phi(1)\psi(h\#f)\phi(b)\psi(k\#g) \\
&= \sum_{(h)} \phi(a)\phi(h_1 \cdot b)\psi(h_2\#f)\psi(k\#g) \\
&= \sum_{(h,f)} \phi(a(h_1 \cdot b))\psi(h_2(f_1 \rightharpoonup k)\#f_2 * g) \\
&= \Phi\left(\sum_{(h,f)} a(h_1 \cdot b) \otimes h_2(f_1 \rightharpoonup k)\#f_2 * g\right) \\
&= \Phi\left((a \otimes h\#f)(b \otimes k\#g)\right).
\end{aligned}$$

Hence Φ is an algebra homomorphism. Since the image of the identity $\mathbf{1} = 1_{\mathcal{A}}\#1_H\#1_{H^*}$ of $\underline{\mathcal{A}\#H}\#H^*$ under the map Φ is $\mathbf{e}$, $\mathbf{e}$ is an idempotent. Moreover

$$\Phi(\gamma) = \Phi(\mathbf{1}\gamma\mathbf{1}) \in \mathbf{e}(\mathcal{A} \otimes \mathrm{End}_k(H))\mathbf{e},$$

for all $\gamma \in \underline{\mathcal{A}\#H}\#H^*$. $\square$

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